Crossing Symmetric Dispersion Relations

Aninda Sinha
Centre for High Energy Physics,
Indian Institute of Science,
Bangalore, India.

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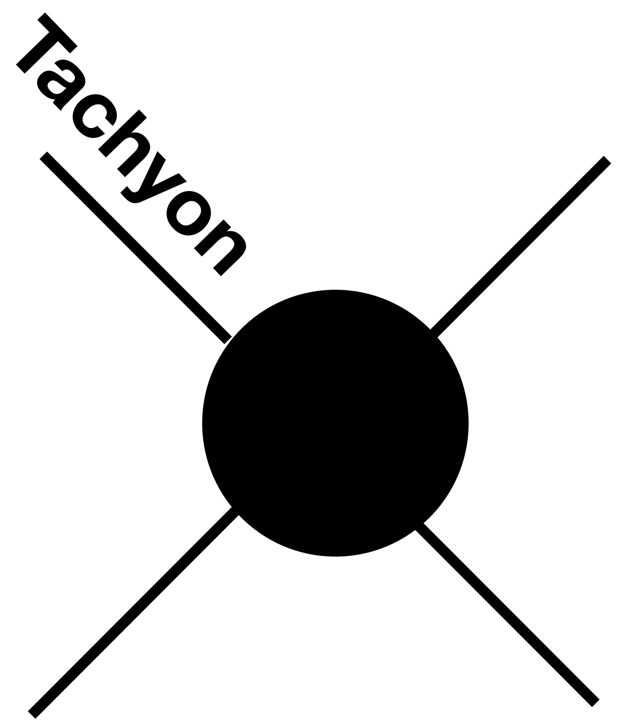
Work done in collaboration with students: Ahmadullah Zahed, Parthiv Haldar (IISc); postdocs: Prashanth Raman (IISc), Subham Dutta Chowdhury (Chicago), Kausik Ghosh (ENS), Sudip Ghosh (IISc); friends: Rajesh Gopakumar (ICTS), Agnese Bissi (Uppsala)

—Job market this year

A string theory motivation

2-2 scattering in string theory



$$M_{VS}(s,t) = \frac{\Gamma(-1-\frac{s}{4})\Gamma(-1-\frac{t}{4})\Gamma(-1-\frac{u}{4})}{\Gamma(2+\frac{s}{4})\Gamma(2+\frac{t}{4})\Gamma(2+\frac{u}{4})}$$

$$s+t+u=-16$$

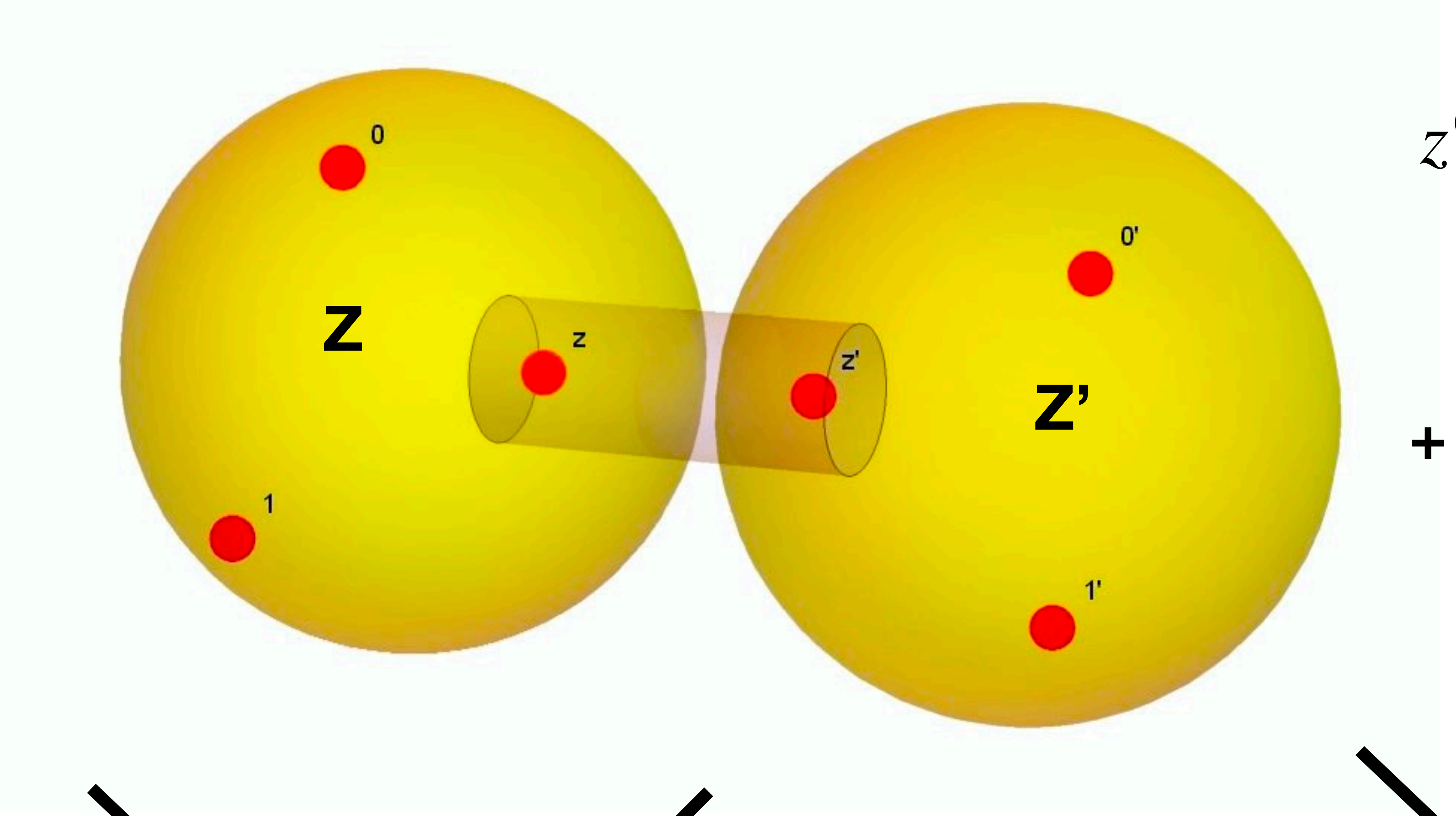
$$= -\frac{1}{2\pi i}\int d^2\sigma\,|\sigma|^{-\frac{u}{2}-4}\,|\sigma-1|^{-\frac{s}{2}-4}$$

Needs analytic continuation

Using string field theory insights, Sen in 2019 showed how to calculate this amplitude without needing analytic continuation, leading to a 3-channel Feynman diagram like picture + contact diagrams.

Polchinski vol I ;
Witten '13;
Sen '19;
AS, Zahed

Plumbing Fixture



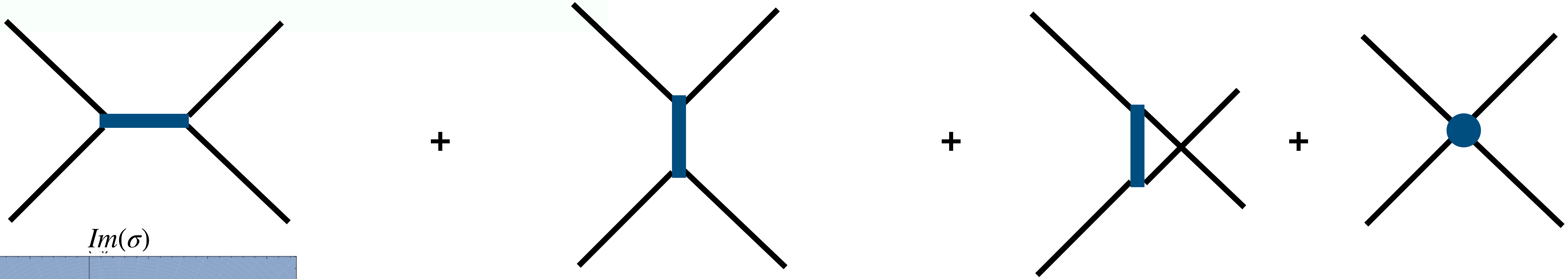
$$z^{(0)} = \frac{\lambda z}{z-2}, z^{(1)} = \frac{\lambda(z-1)}{(z+1)}, z^{(\infty)} = \frac{\lambda}{2z-1}$$

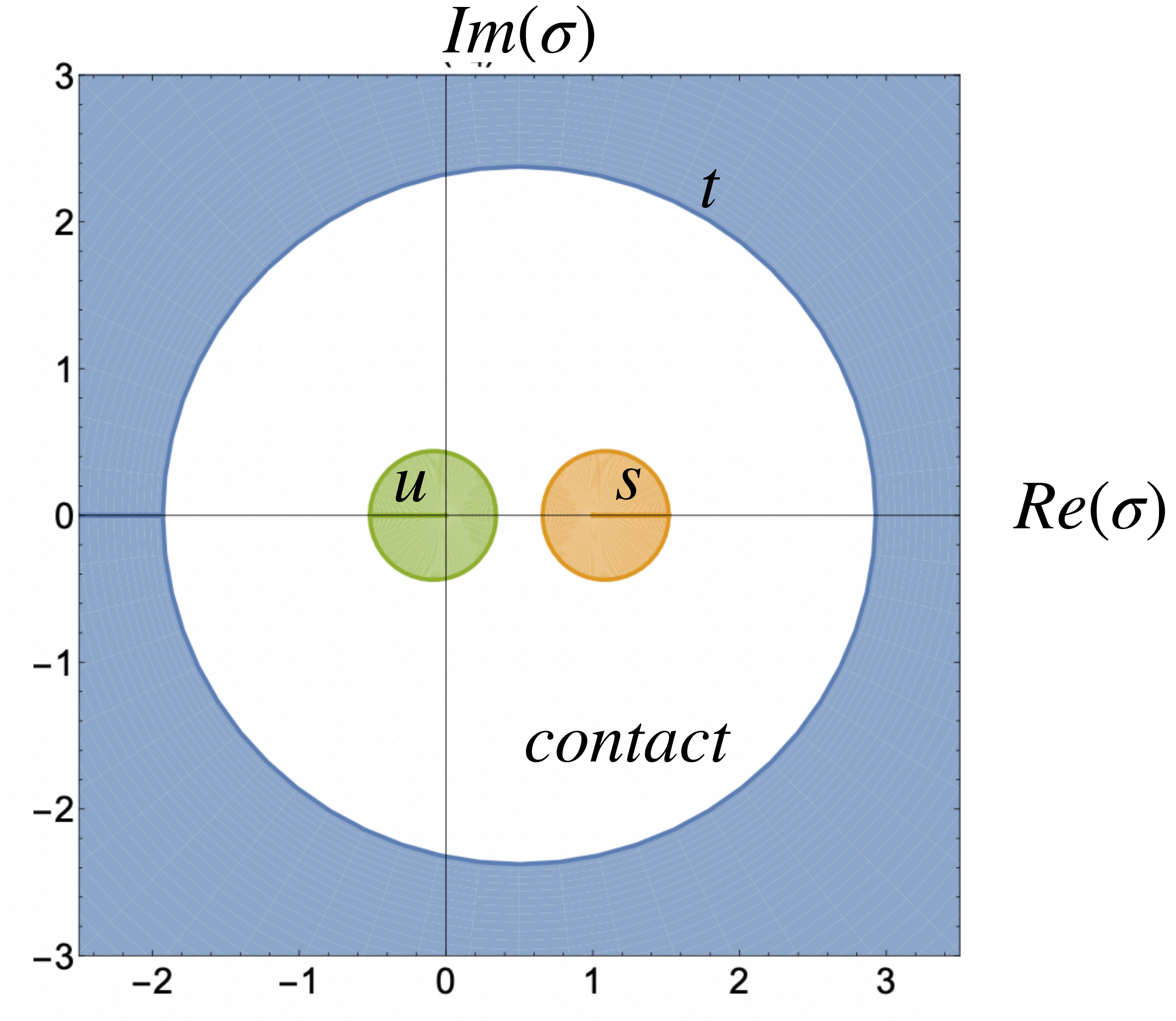
$$z^{(\infty)} z'^{(0)} = q\,, \quad 0 \le |q| \le 1$$

$$\xrightarrow{\textbf{Mobius}} (0,1,\infty,\sigma(q))$$

+ 2 other ways of gluing

λ : Free parameter > 3





- Entire moduli space not covered
- Singularities at $\sigma = 0,1,\infty$
- White region needs contact terms
- Sen, '19 showed how the 3 channels emerge as boundary terms.
- Numerically verified.

Permutation invariant s, t, u channels with poles

$$B_s + B_t + B_u = \sum_k \frac{(\frac{u}{4}+2)_k^2 + (\frac{t}{4}+2)_k^2}{2(k!)^2(k-1-\frac{s}{4})} + \text{crossed} + f_k(\lambda, s, t, u)$$

λ dependent, will cancel

Bulk integral/ Contact terms

$$C = \int_{white} d^2\sigma |\sigma|^{-\frac{u}{2}-4} |\sigma - 1|^{-\frac{s}{2}-4} = -\sum_k f_k(\lambda, s, t, u) + \text{Polynomial}^{(C)}{}_k(s, t, u)$$

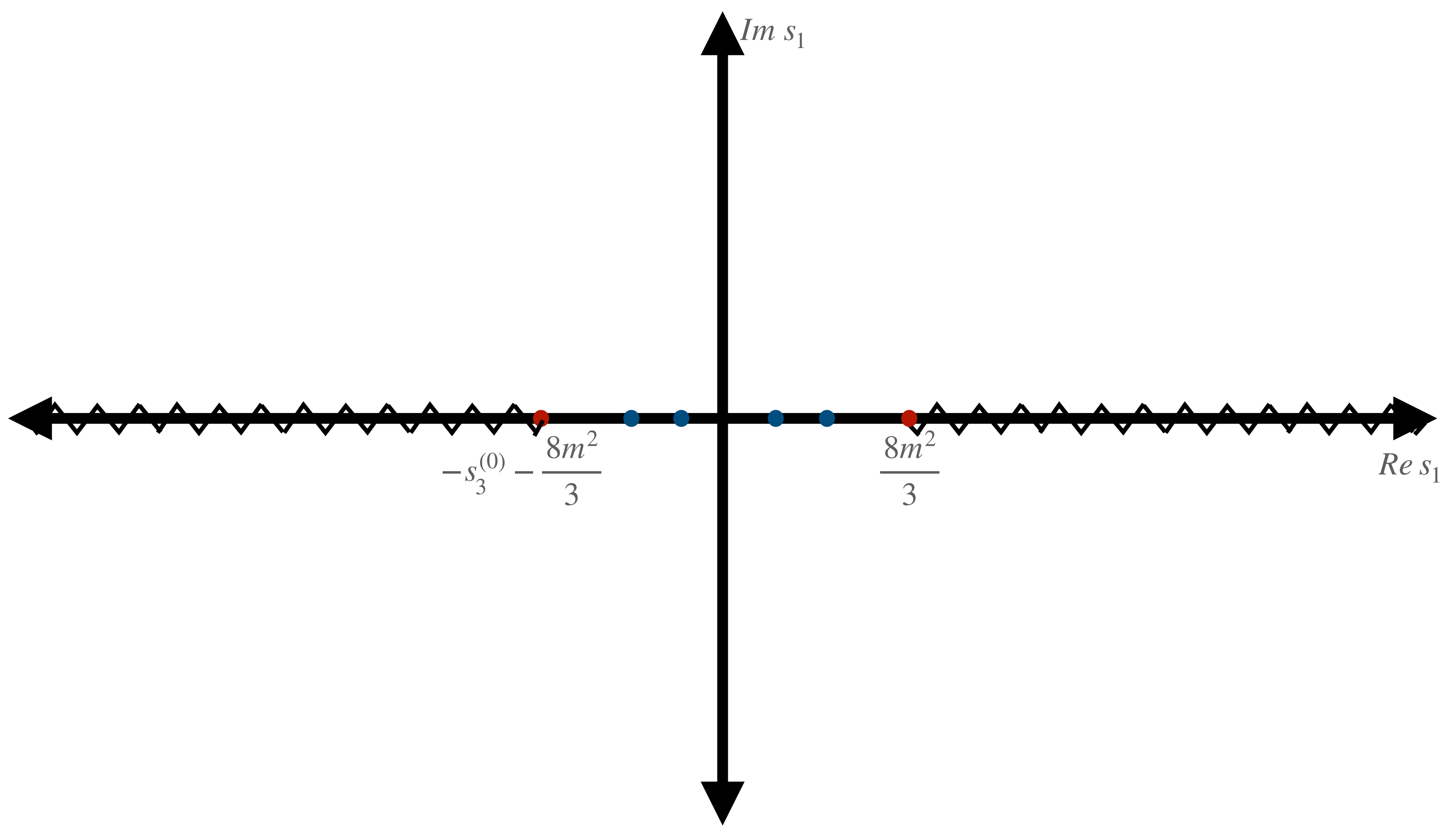
- λ dependence is due to field redefinition ambiguity in string field theory, final answer is λ-independent.
- Bulk integral not known analytically but numerical checks work perfectly (Sen, '19) for any s,t,u.
- *Given just the s-poles can we fix the rest?*
- *We will bootstrap the answer analytically using the CSDR.*

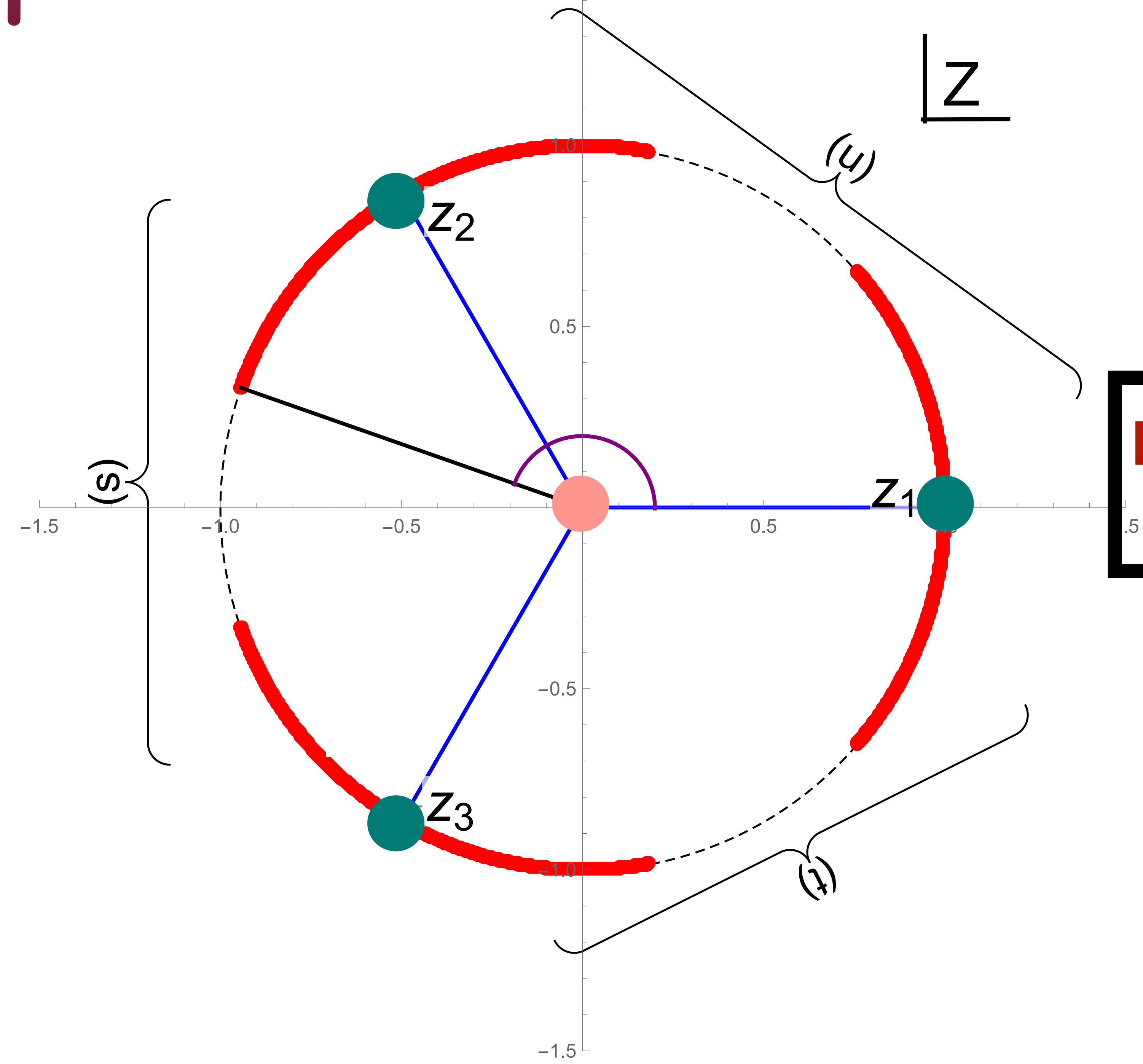
CSDR

- I will talk about a crossing symmetric dispersion relation (CSDR) which not only makes the connection with Feynman diagrams transparent but also has surprising connections with an area of mathematics called Geometric Function Theory.

- A key point to note in what follows: To have manifest crossing symmetry, we will have to sacrifice manifest locality in the basis.

Crossing Symmetric Dispersion Relation (CSDR)





New variables z, a Instead of s, t

Shifted Mandelstam variables $s_1 + s_2 + s_3 = 0$

Hold fixed $a \equiv \dfrac{s_1 s_2 s_3}{s_1 s_2 + s_2 s_3 + s_1 s_3}$

$$s_k = a - a\frac{(z - z_k)^3}{z^3 - 1}$$

z_k : Cube root of unity

$$-\sigma_3 \equiv s_1 s_2 s_3 = \frac{27 a^3 z^3}{(z^3 - 1)^2}$$

$$-\sigma_2 \equiv s_1 s_2 + s_2 s_3 + s_1 s_3 = \frac{27 a^2 z^3}{(z^3 - 1)^2}$$

"Extremal functions"

Auberson, Khuri, 1972
AS, Zahed, 2020: QFT
Gopakumar, AS, Zahed, 2021: Mellin-CFT

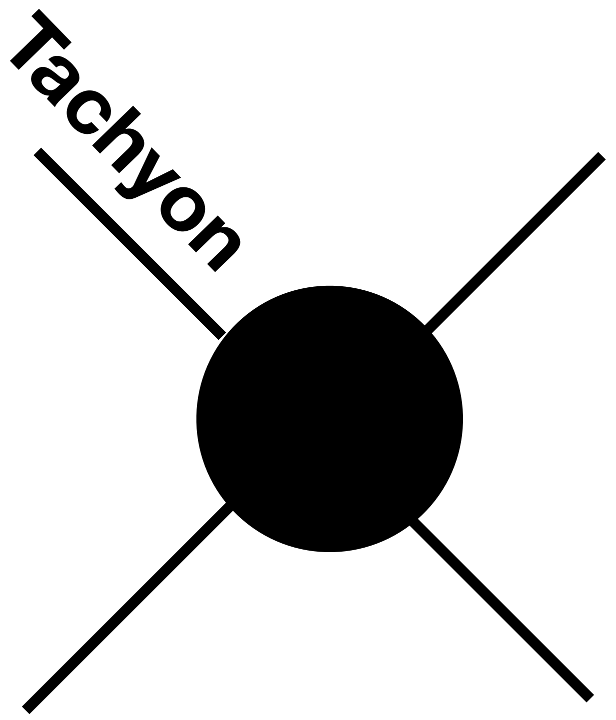
For any gapped theory $|M(s,t)| = o(|s|^2)$ for large $|s|$

Manifestly crossing symmetric

$$M(s_1,s_2) = M(0,0) + \frac{1}{\pi}\int_{\delta}^{\infty}\frac{dm^2}{m^2}\underbrace{\mathscr{A}\left(m^2, -\frac{m^2}{2}\left[1-\sqrt{\frac{m^2+3a}{m^2-a}}\right]\right)}_{\text{Absorptive part}}\left[\underbrace{\frac{m^2}{m^2-s_1}+\frac{m^2}{m^2-s_2}+\frac{m^2}{m^2-s_3}}_{\phi^3}-\underbrace{3}_{\phi^4}\right]$$

$\equiv H$

- Notice the second argument in $\mathscr{A}$ in the above expression.
- The kernel is as if we are integrating over ϕ^3+ϕ^4 theories with specific relative coefficients.
- Convergence in the Lehmann-Martin ellipse enables us to expand around $a = 0$.
- Absorptive part in the dispersive integral can have arbitrary $a = \dfrac{s_1s_2s_3}{s_1s_2+s_2s_3+s_1s_3}$ powers.
- In a local theory with a gap, we do not expect negative powers of $s_1s_2+s_2s_3+s_1s_3$. These must cancel. This gives rise to the LOCALITY CONSTRAINTS.
- We can find a decomposition in terms of poles+contact but owing to the locality constraints one can find different decompositions where the contact terms change. This is analogous to the field redefinition ambiguity in string field theory.



$$M_{VS}^{(pole)} = \frac{\Gamma(1/3)^3}{\Gamma(2/3)^3} + \sum_{k=0}^{\infty} \left[\frac{1}{k - \frac{s}{4} - 1} + \frac{1}{k - \frac{t}{4} - 1} + \frac{1}{k - \frac{u}{4} - 1} - \frac{3}{k + \frac{1}{3}} \right] \frac{\left(\left(\Lambda^{(k)} \right)_k \right)^2}{(k!)^2}$$

$$Re(a) < \frac{16}{3}$$

Where $$\Lambda^{(k)} = \frac{1}{6} \left((3k+1) \left(\sqrt{\frac{12a}{-3a + 12k + 4} + 1} - 1 \right) + 4 \right)$$

Locality restored after summing over all levels

s	u	Exact	$k_{\max}$=30	$k_{\max}$=40	$k_{\max}$=50
$-\frac{31}{5}$	$-\frac{17}{5}$	-4.38207	-4.38208 -4.38207	-4.38208 -4.38207	-4.38208 -4.38207
$-\frac{36}{5}$	$-\frac{1}{2}$	0.157462	0.157461 0.157462	0.157461 0.157462	0.157461 0.157462
$-3+i$	-3	$-5.59108 + 2.70761i$	$-5.59108 + 2.70761i$ $-5.59205 + 2.70562i$	$-5.59108 + 2.70761i$ $-5.59161 + 2.7064i$	$-5.59108 + 2.70761i$ $-5.59141 + 2.70679i$

$$\lambda_{SFT} = 11$$

Convergence after summing over 3 channels and contact terms
—SFT, —CSDR

Properties of the Kernel

$$H \equiv \left[\frac{m^2}{m^2-s_1}+\frac{m^2}{m^2-s_2}+\frac{m^2}{m^2-s_3}-3\right] \xrightarrow{\tilde{z}\equiv z^3, a} (2\alpha\partial_\alpha \xi)\frac{\tilde{z}}{1-2\xi\tilde{z}+\tilde{z}^2}\Bigg|_{\alpha=\frac{a}{m^2}}, \qquad 2\xi = 2-27\alpha^2+27\alpha^3$$

Typically real/ Herglotz

$$Im\,\tilde{z}\; Im\,\frac{\tilde{z}}{\tilde{z}^2-2\xi\tilde{z}+1} \geq 0 \;\; \text{inside } |\tilde{z}|<1\,, \quad |\xi|\leq 1$$

→ **Geometric Function Theory**

$$\frac{\tilde{z}}{\tilde{z}^2-2\xi\tilde{z}+1} = \sum_{n=0}^{\infty} \underbrace{U_n(\xi)}_{\text{Chebyshev 2nd kind}} \tilde{z}^{n+1} = \sum_{n=0}^{\infty} \underbrace{[n+1]_q}_{\text{q-number}} \tilde{z}^{n+1}$$

→ **q-deformed algebras**

AS, '22

$$\frac{\tilde{z}(1-\tilde{z})}{\tilde{z}^2-2\xi\tilde{z}+1} = \sum_{n=0}^{\infty} \underbrace{\mathscr{A}^{(2,2n+1)}(q)}_{\text{Alexander polynomial torus knot}} \tilde{z}^{n+1}\,, \qquad 2\xi = q+\frac{1}{q}$$

→ **Knot Theory**

- **Same kernel makes appearance in many places!**

Haldar, AS, Zahed '20; Raman, AS '21; AS, '22

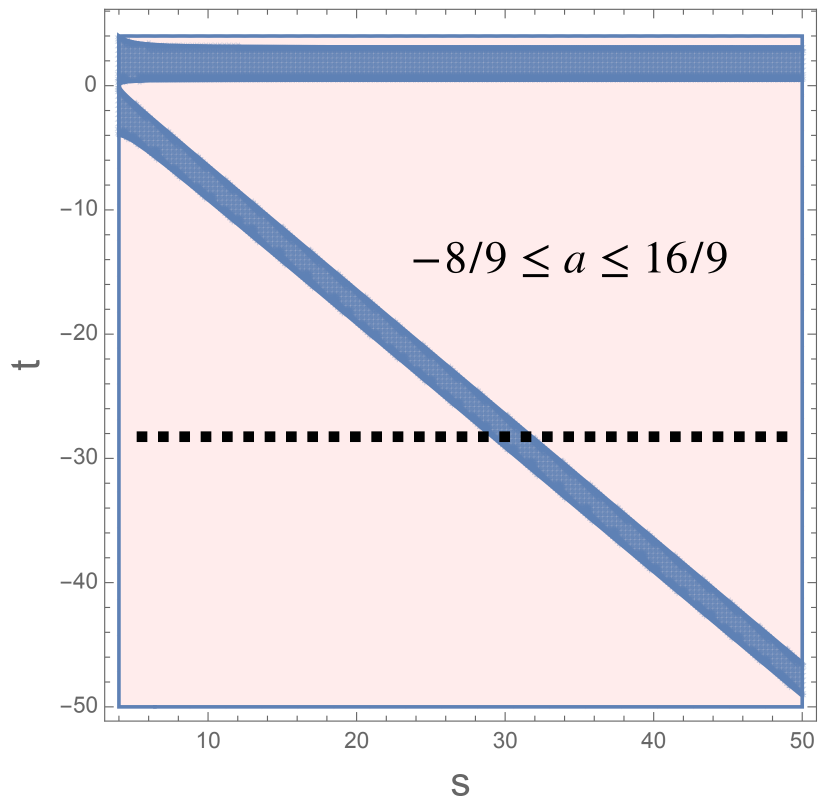
QFT Applications

EFT bounds

- Consider first for concreteness no massless exchanges so that

$$M(s_1, s_2) = \sum \mathscr{W}_{pq}\sigma_2^p\sigma_3^q \qquad \sigma_2 \sim (s_1s_2 + s_1s_3 + s_2s_3), \quad \sigma_3 \sim s_1s_2s_3$$

- Recent research has put two-sided bounds on the Wilson coefficients ('EFT-hedron'). In fixed-t dispersion relation, this makes use of crossing symmetry constraints and numerical (SDPB) techniques.
- CSDR provides a simpler explanation for the 2-sided bounds making use of the typically real-ness of the dispersive kernel.[Haldar, AS, Zahed, '21; Raman, AS. '21; Chowdhury, Ghosh, Haldar, Raman, AS '21]



$$M = const - \frac{1}{\pi}\int_\delta^\infty \frac{dm^2}{m^2} \underbrace{\overline{\mathscr{A}}(m^2, a)}_{>0,\, -8/9 \le a \le 16/9} \frac{\tilde{z}}{\tilde{z}^2 - 2\xi\tilde{z} + 1}$$

No singularity inside $|\tilde{z}| < 1$

For a range of "a", we have a convex sum of typically real functions. This is again typically real: Robertson representation.

de Rham, Kundu, Reece, Tolley, Zhou '22; Arkani-Hamed, Huang, Huang, '20; Tolley, Wang, Zhou '20; Caron-Huot, van Duong, '20; Bellazzini, Miro, Rattazzi, Riembau, Riva, '20; AS, Zahed, '20; Alberte, de Rham, Jaitley, Tolley '21; Caron-Huot, Mazac, Rastelli, Simmons-Duffin, '21; Chowdhury, Ghosh, Haldar, Raman, AS, '21; Chiang, Huang, Rodina, Weng '21; many, many more

- Typically real functions f(z) which are analytic inside the unit disc obey the famous 1916 Bieberbach conjecture (proven for this limited case by Rogosinski in 1932).

$$f(z) = z + \sum_{n=2}^{\infty} c_n z^n$$

$$\left.\begin{array}{ll} -n \leq c_n \leq n, & n \text{ even} \\ -\kappa_n \leq c_n \leq n, & n \text{ odd} \end{array}\right\}$$

$$\frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots$$

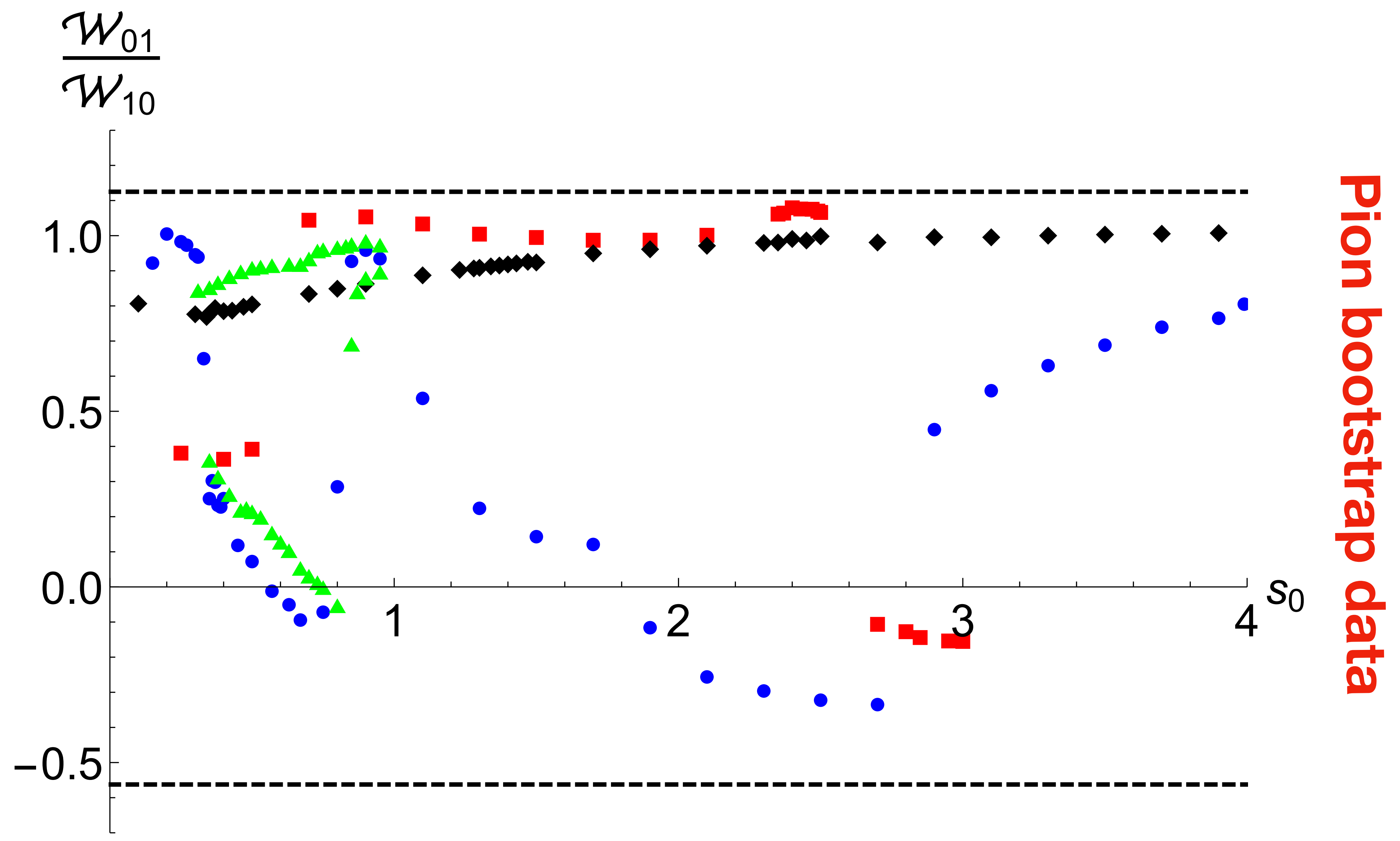
Koebe function

- In the CSDR approach, two sided bounds on Wilson coefficients arise from these Bieberbach-Rogosinski bounds.

- In some cases the bounds can be found analytically unlike other approaches. For instance

Dimension independent!

$$-\frac{9}{16m_\pi^2} < \frac{\mathscr{W}_{0,1}}{\mathscr{W}_{1,0}} < \frac{9}{8m_\pi^2}$$

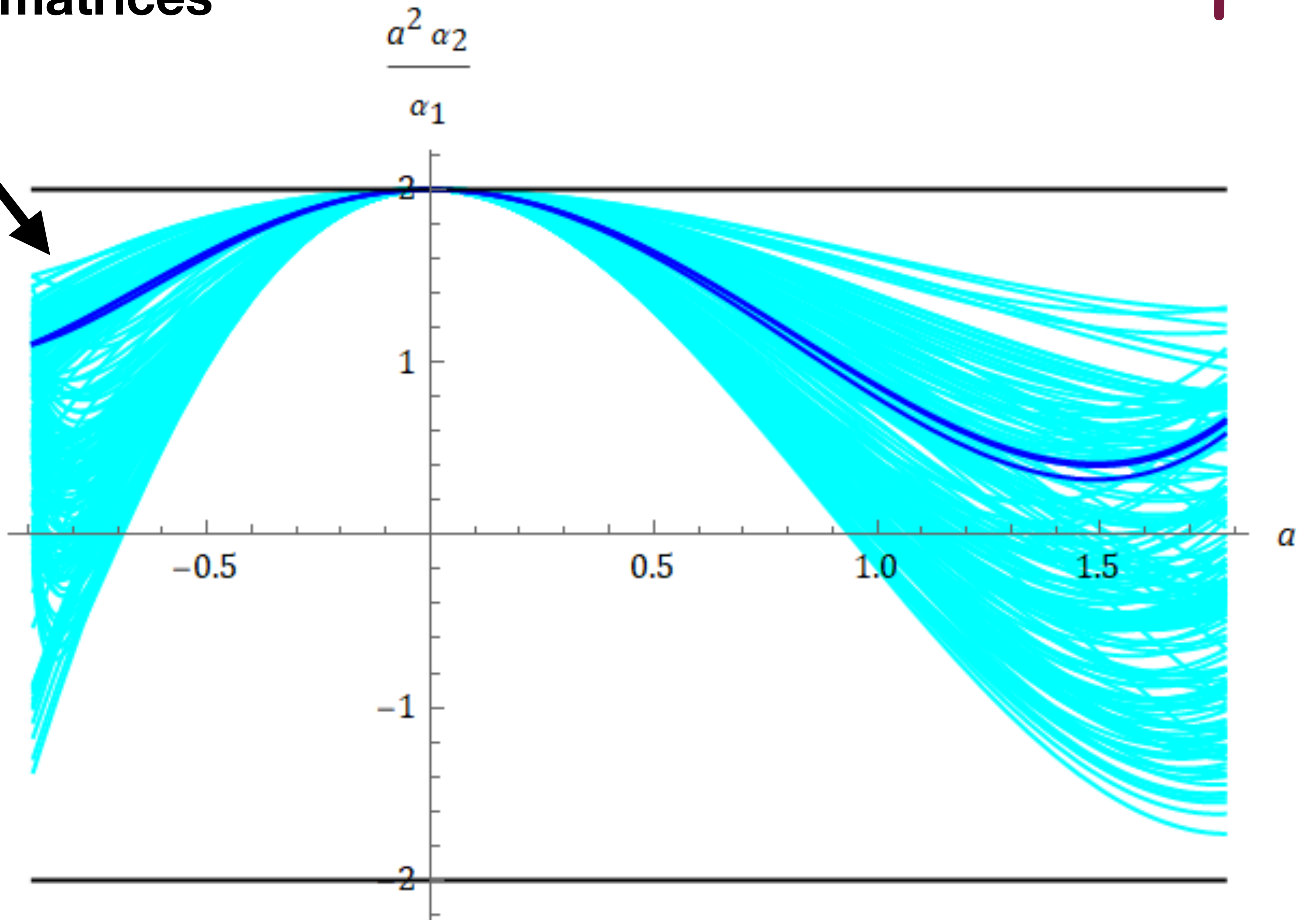


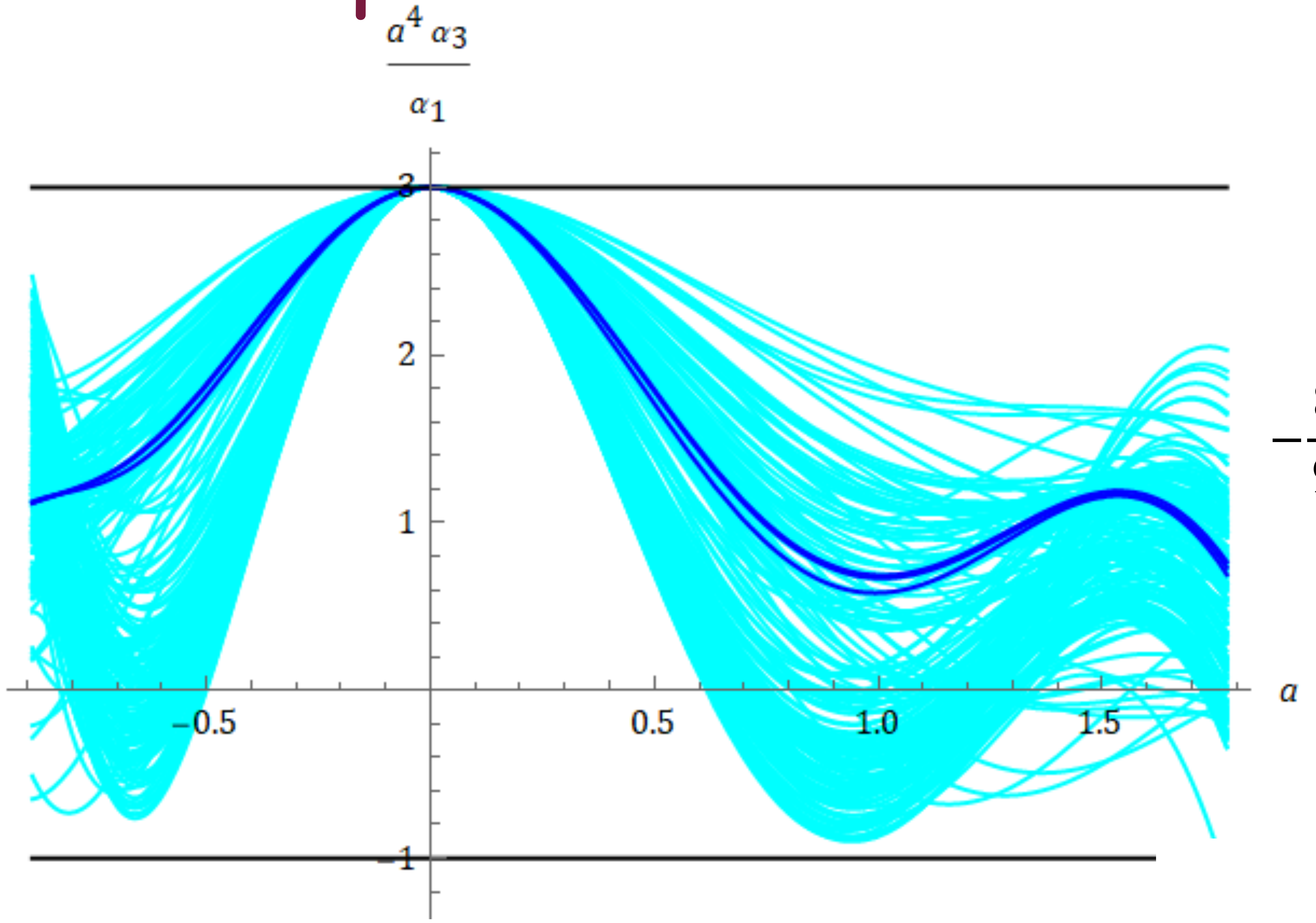
2103.12108

Extended to external spinning particles in 2112.11755
w S.D. Choudhury, K. Ghosh, P. Haldar, P. Raman

Bieberbach conjecture/de Branges theorem from pion bootstrap!

200 S-matrices





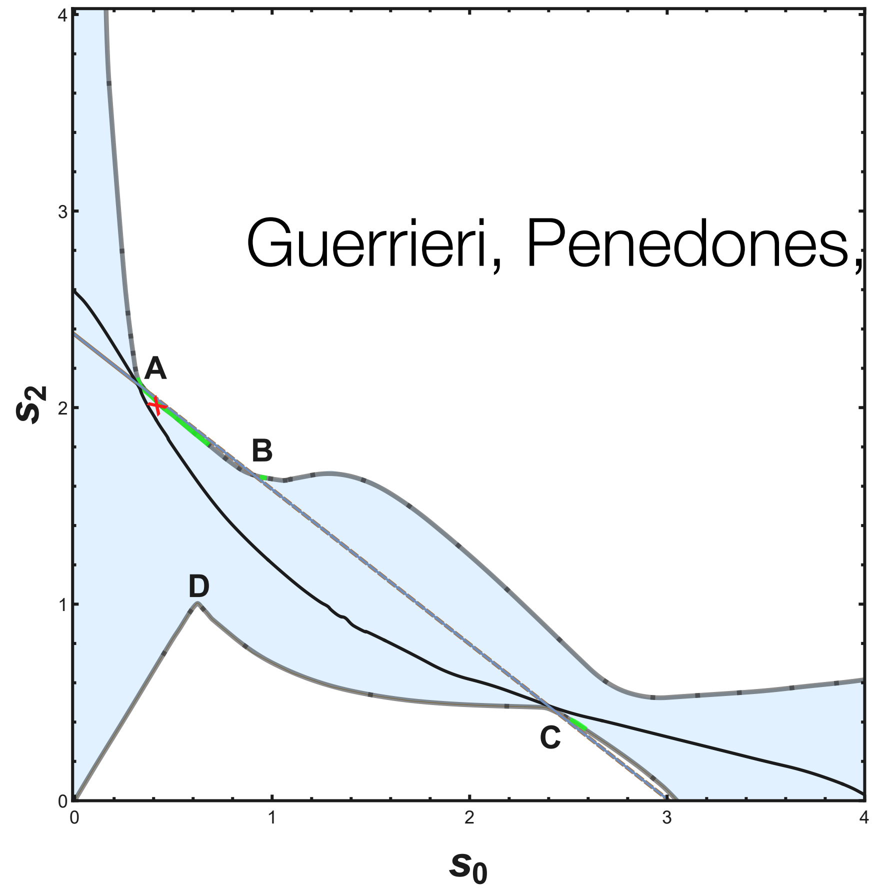
$$-\frac{8}{9} \le a \le \frac{16}{9}$$

$$m^2 = 1$$

$$\alpha_1(a) = a\mathscr{W}_{01} + \mathscr{W}_{10}$$

$$a^2\alpha_2(a) = 2\alpha_1(a) - 27a^2(\mathscr{W}_{20} + a\mathscr{W}_{11} + a^2\mathscr{W}_{02})$$

$$a^4\alpha_3(a) = -5\alpha_1(a) + 4a^2\alpha_2(a) + 729a^4(\mathscr{W}_{30} + a\mathscr{W}_{21} + a^2\mathscr{W}_{12})$$



Guerrieri, Penedones, Vieira ’18

2103.12108 with P. Haldar and A. Zahed; 2107.06559 with P. Raman

Bounds in terms of knot polynomials

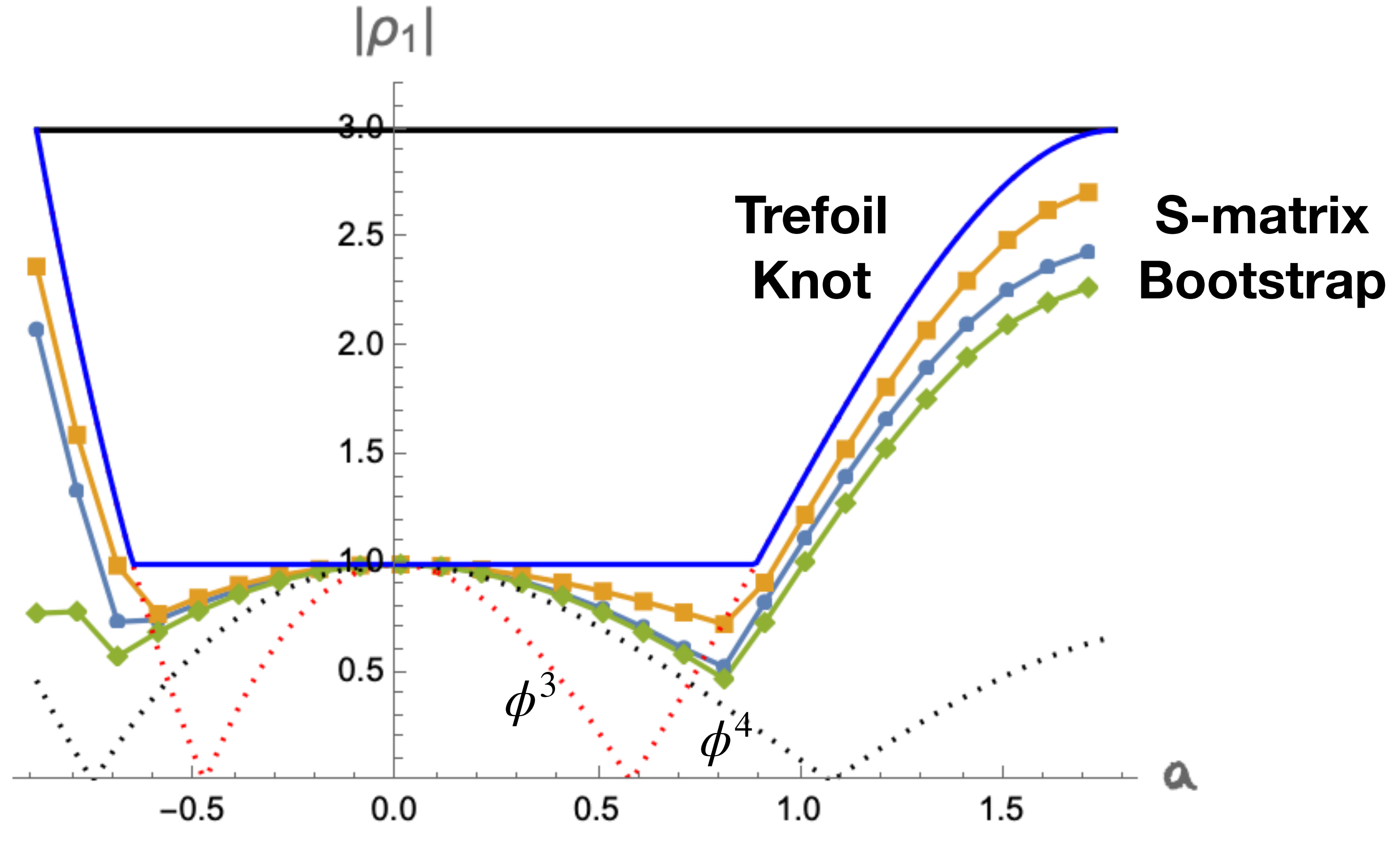
$$\rho_n = \frac{1}{\partial_{\tilde{z}} M(\tilde{z}, a)} \left(\frac{\partial_{\tilde{z}}^{n+1} M(\tilde{z}, a)}{(n+1)!} - \frac{\partial_{\tilde{z}}^{n} M(\tilde{z}, a)}{n!} \right) \Bigg|_{\tilde{z}=0}$$

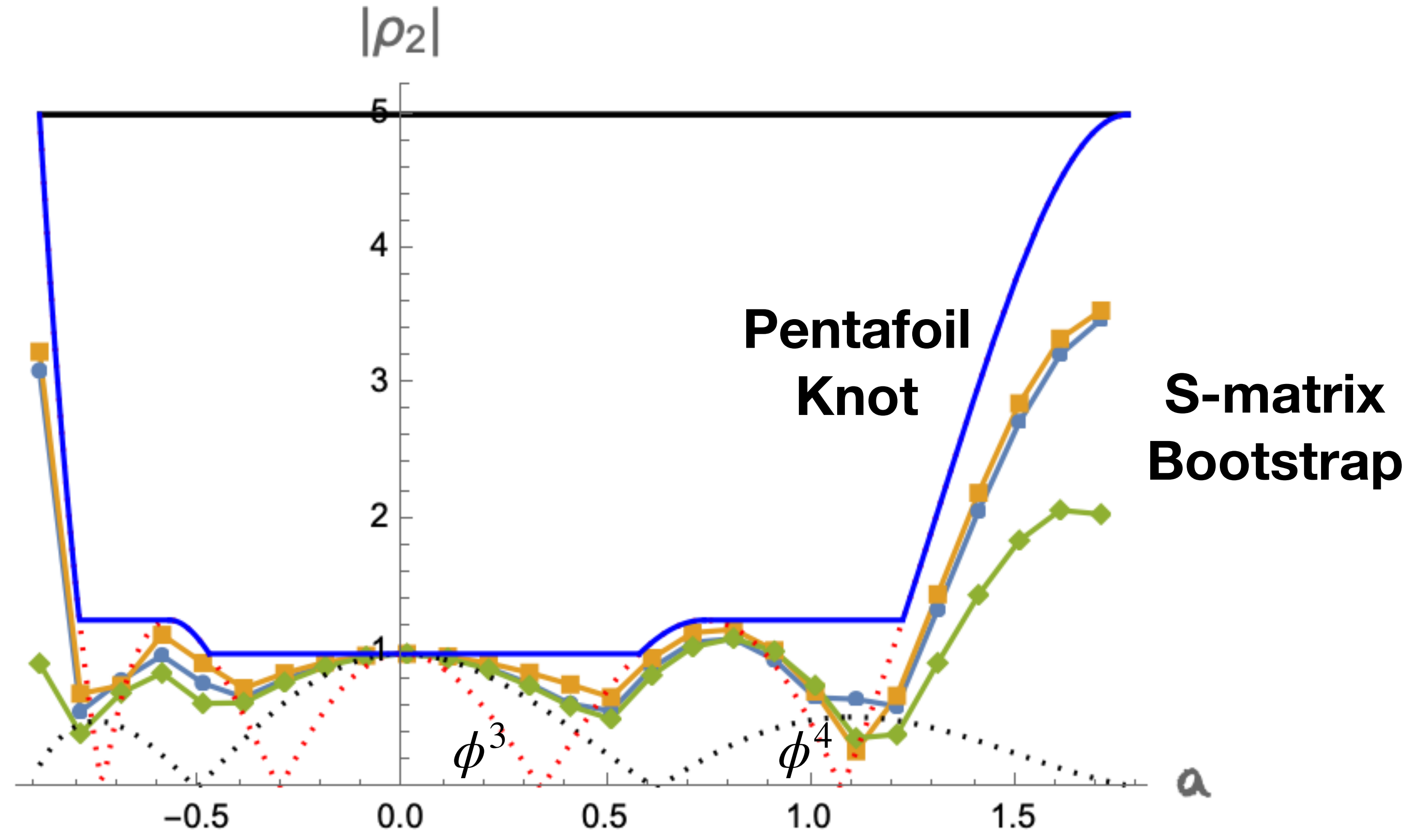
Using CSDR and the triangle inequality one can easily derive

$$|\rho_n| \leq \sup_{\xi_{min} \leq \xi \leq \xi_{max}} \left| \mathscr{A}^{(2,2n+1)}(q) \right| \leq \underbrace{\left| \mathscr{A}^{(2,2n+1)}(-1) \right|}_{=2n+1}$$

$\mathscr{A}^{(2,2n+1)}(q)$: Alexander polynomial for torus (2,2n+1) knot

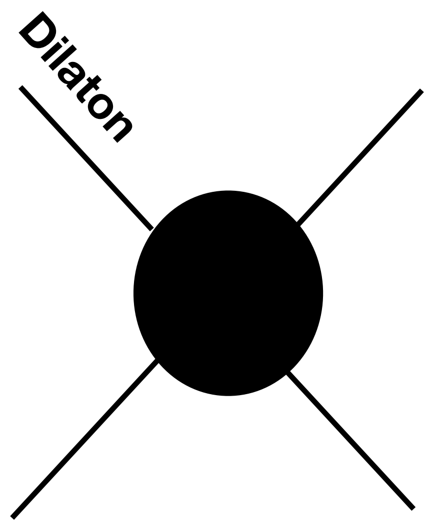
Knot det, Crossing #





AS, '22

Massless exchanges



$$M = 8\pi G\left(\frac{(s_1s_2+s_1s_3+s_2s_3)^2}{s_1s_2s_3} + 2\hat{g}_4(s_1s_2+s_1s_3+s_2s_3)^2 + \cdots\right), \quad \hat{g}_4|_{II-string} = \zeta(3) \approx 1.202$$

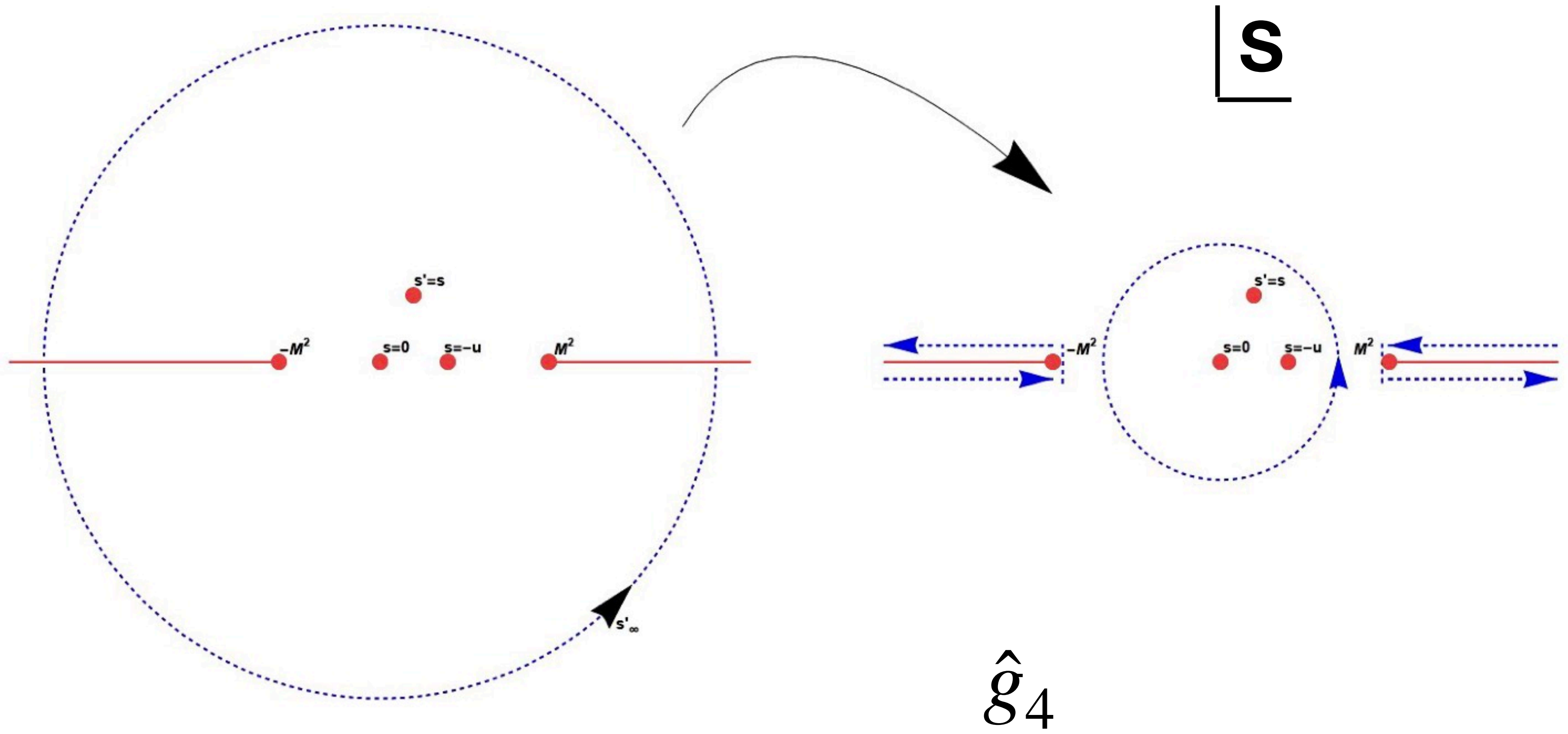
$$= 8\pi G a^2\left(\frac{1}{a}\frac{27\tilde{z}}{(1-\tilde{z})^2} + 2\hat{g}_4 a^2\left(\frac{27\tilde{z}}{(1-\tilde{z})^2}\right)^2 + \cdots\right)$$

$$= 216\pi G a\left(\tilde{z} + (54a^3\hat{g}_4+2)\tilde{z}^2 + O(\tilde{z}^3)\right)$$

NB

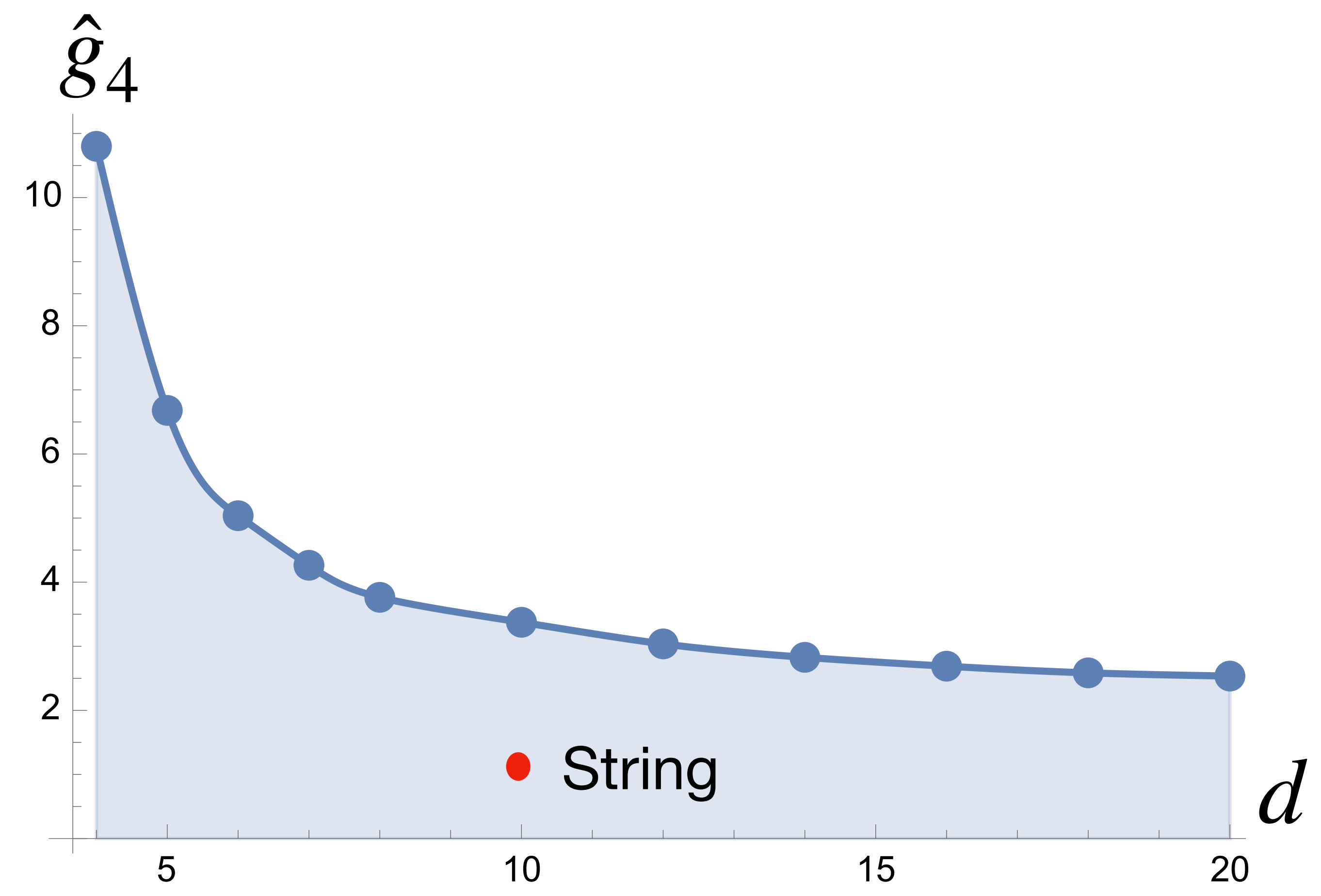
$\sigma_2, \sigma_3, \underbrace{\frac{\sigma_2^2}{\sigma_3}}_{\text{graviton exchange}}$

are all Bieberbach extremal (Koebe fn)



Bieberbach $\Longrightarrow -2 \leq 54a^3\hat{g}_4 + 2 \leq 2$

$$\Longrightarrow 0 \leq \hat{g}_4 \leq \frac{2}{27|a|^3}, \quad a < 0$$



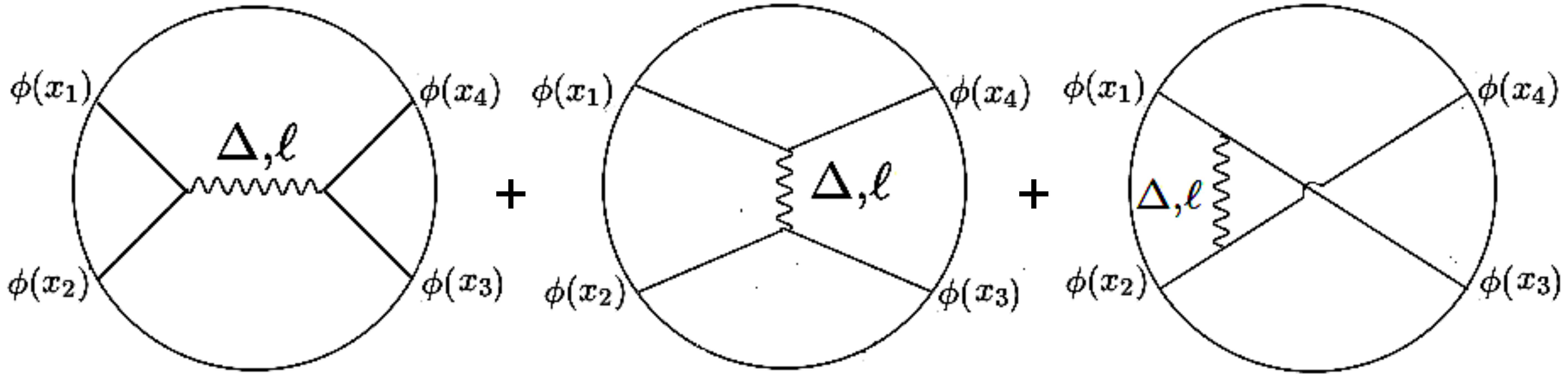
- Argument based on GFT gives bounds in a few lines without need for numerics.
- Flipside: In what I have presented above, we need an independent argument to determine the range of a where typically-realness holds. Locality gives max $|a|$. Large d asymptotes to $0 \leq \hat{g}_4 \leq 2$.

Caron-Huot, Mazac, Rastelli, Simmons-Duffin, '21; Bern, Kosmopoulos, Zhiboedov, '21; de Rham, Jaitley, Tolley, '22 (to appear); work in progress with F. Bhat and A. Zahed; argument using Celestial variables in 2204.07617 w S. Ghosh and P. Raman

CFT Applications

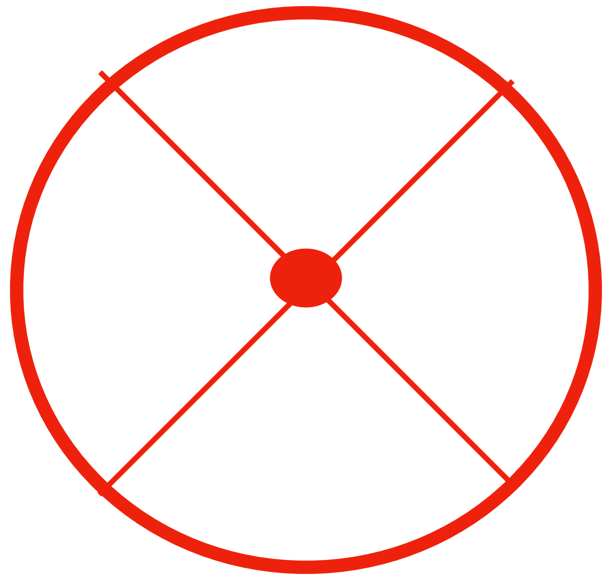
Polyakov Bootstrap

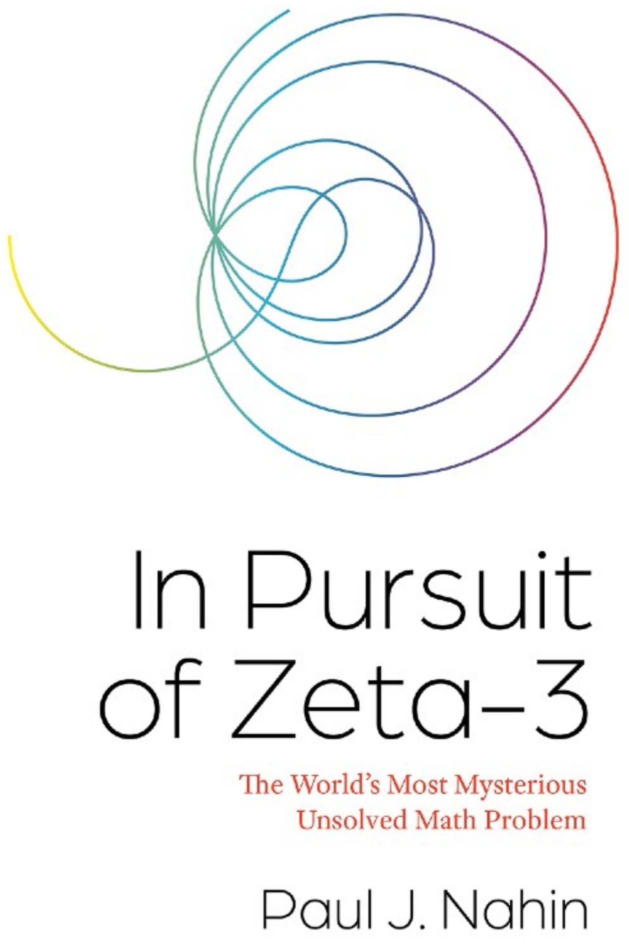
$$\mathcal{M}(s,t) \overset{??}{=} \sum_{\Delta,\ell} c_{\Delta,\ell}$$



$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$

In Mellin space

+  ??



- **CSDR for Mellin amplitude fixes the contact terms in the basis of "Witten diagrams". Logic is similar to the QFT discussion. Assumes existence of Mellin amplitude*. Provides a derivation of the Polyakov bootstrap.**

Polyakov '74; Sen, AS '15; Gopakumar, Kaviraj, Sen, AS '16; Dey, Kaviraj AS '16; Gopakumar, AS '18; Gopakumar, AS, Zahed, 2021
***Penedones, Silva, Zhiboedov, '19.**

CFT$_d$ in position space

u, v: **Conformal cross ratios**

$$\underbrace{v^{\Delta_\phi} f(u,v)}_{\equiv F(u,v)} = \underbrace{u^{\Delta_\phi} f(v,u)}_{F(v,u)}$$

2 channel symmetry

Numerical bootstrap we expand around $u = \frac{1}{4}, v = \frac{1}{4}$

so $u = s_1 + \frac{1}{4}, v = s_2 + \frac{1}{4}$

$$F(s_1, s_2) = F(0,0) + \frac{1}{\pi}\int_{-\infty}^{-\frac{1}{4}} \frac{dm^2}{m^2} \mathscr{A}(m^2, \frac{am^2}{m^2 - a})\left[\frac{s_1}{m^2 - s_1} + \frac{s_2}{m^2 - s_2}\right]$$

2107.06559 w P. Raman

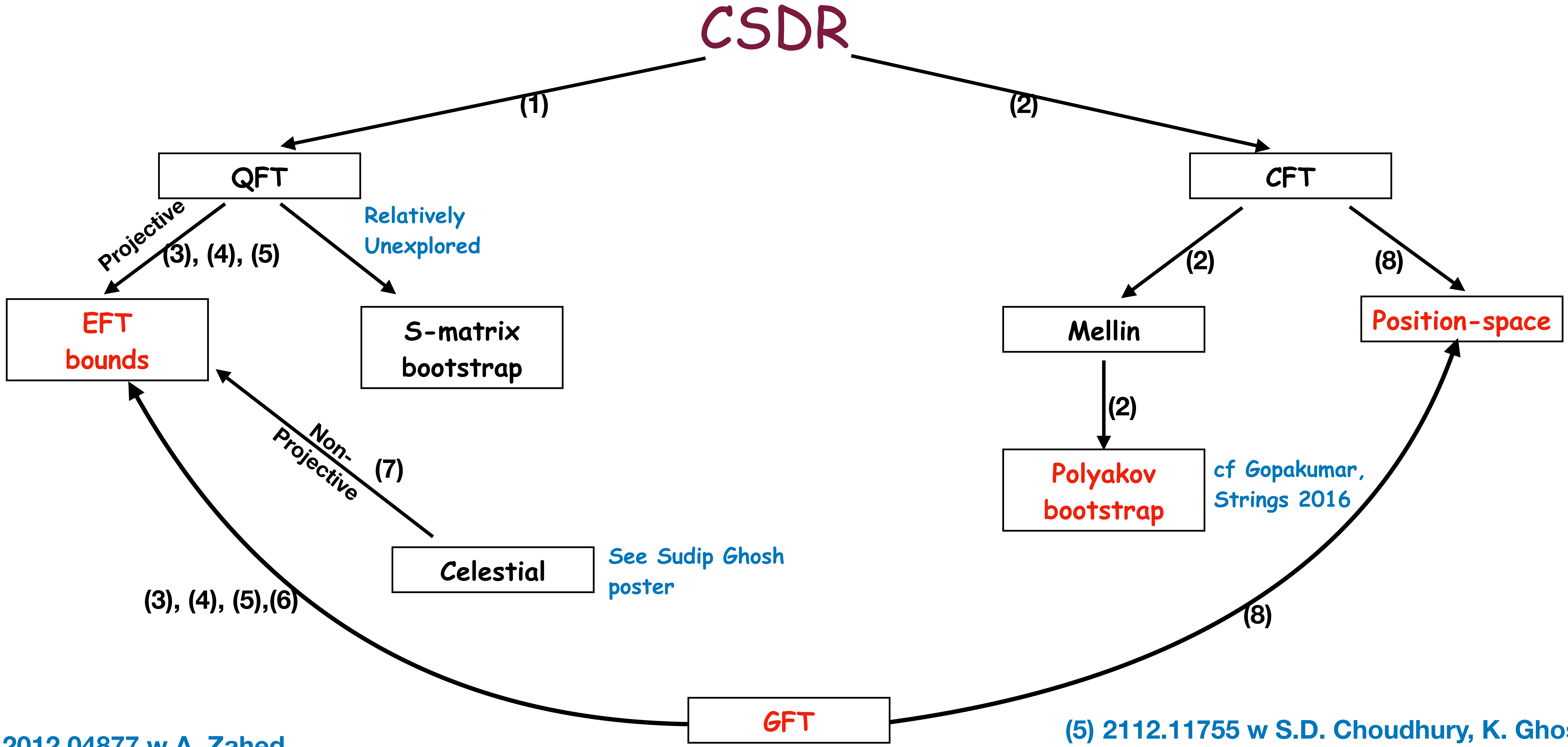
$$s_k = a(1 - \frac{(z + z_k)^2}{(z - z_k)^2})$$

z_k : Square root of unity

$$a = \frac{s_1 s_2}{s_1 + s_2}$$

The kernel is the same as the 3-channel case in the $\tilde{z} \equiv z^2$ variable

Work in progress with A. Bissi; 1d/diagonal limit: Mazac '18; Mazac, Paulos '18; Paulos '20



(1) 2012.04877 w A. Zahed
(2) 2101.09017 w R. Gopakumar, A. Zahed
(3) 2103.12108 w P. Haldar, A. Zahed
(4) 2107.06559 w P. Raman
(5) 2112.11755 w S.D. Choudhury, K. Ghosh, P. Haldar, P. Raman
(6) 2204.13986
(7) 2204.07617 w S. Ghosh, P. Raman
(8) In progress with A. Bissi

Outlook

- Lessons from String Field Theory: When can we talk about a world-sheet in the CSDR? Can we prove better convergence when locality in the partial-wave basis is inbuilt? Hard scattering limit? Recall that in the String Field Theory—Feynman diagram story, there was no need for analytic continuation in momenta.

- Suggestions for numerical bootstrap: The current formulation of the numerical S-matrix bootstrap cannot tackle questions where the amplitude grows like the Froissart bound. The CSDR suggests how to improve this.

- Ubiquitousness: The CSDR kernel appears in higher subtracted dispersion relations, Mellin space CFT, CFT position space dispersion relation as well as in recent work on defect CFTs. The z-variable in the CSDR is related by a SL(2,C) transformation to the celestial variable. Similar techniques may be applicable in all these situations.