Non-perturbative Amplituhedron geometry

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work with **Nima Arkani-Hamed** and **Johannes Henn**
2112.06956 + work in progress

related work in progress: Nima Arkani-Hamed, Johannes Henn, Dmitry Chicherin, JT;
Taro Brown, Umut Oktem, Shruti Paranjape, JT

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Outline

- Scattering amplitudes in planar N=4 SYM theory: playground for new theoretical ideas
- Amplituhedron formulation for the loop integrand in the perturbation theory: want to do an all-loop resummation
- Define an IR finite quantity $\mathcal{F}(g,z)$ derived from the 4pt amplitude: new expansion (exact in g) natural from perspective of Amplituhedron geometry — "**loops of loops expansion**"
- Connection to the cusp anomalous dimension

Introduction

Planar N=4 SYM amplitudes

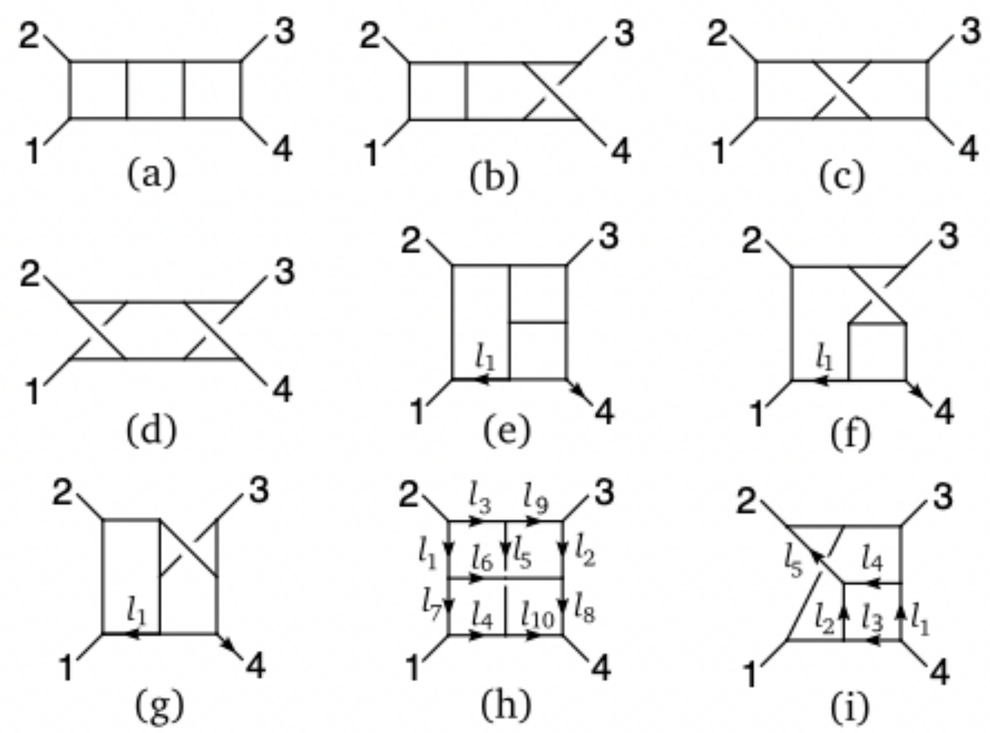
- Amplitudes in planar (large N) limit of maximally supersymmetric Yang-Mills theory in four dimensions
- Symmetries: conformal + dual conformal symmetry, amplitudes simpler — good toy model for gauge theories
- Amplitudes are UV finite, perturbative expansion exact, finite radius of convergence
- IR divergencies - regularization (breaking of symmetries)

Perturbative expansion

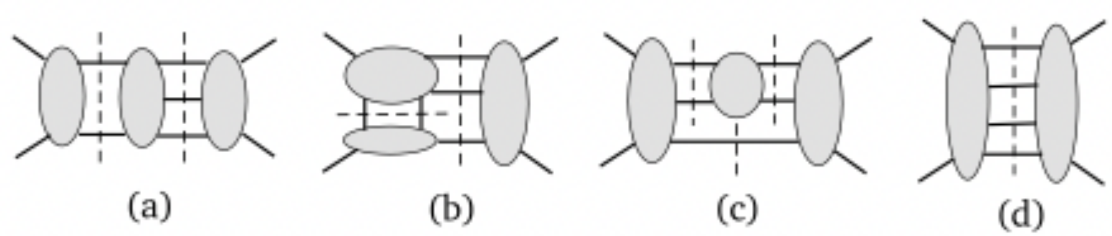
- Loop expansion: construct L-loop scattering amplitude

Unitarity methods (Bern, Dixon, Kosower, Roiban, Carrasco, Johansson, Dunbar,...)

(i) Construct basis of scalar integrals



(ii) Fix coefficients using cuts



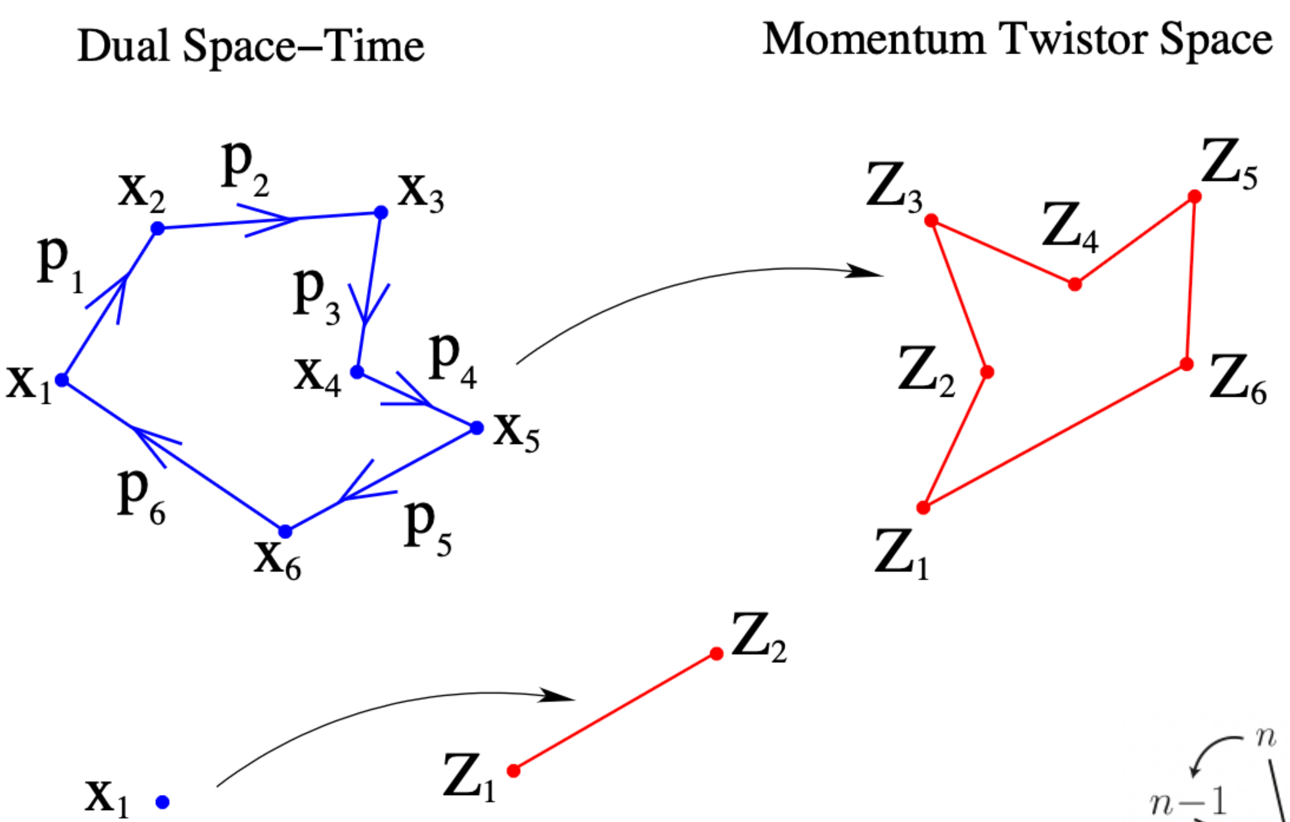
(iii) Loop integration

- Feynman parameters
- IBP relations
- Differential equations
- ....

(Chetyrkin, Tkachov, Smirnov, Gehrmann, Melnikov, Duhr, Gluza, Kosower, Remiddi, Henn,...)

Loop integrand

- Planar limit: global dual variables, momentum twistors



loop momenta:

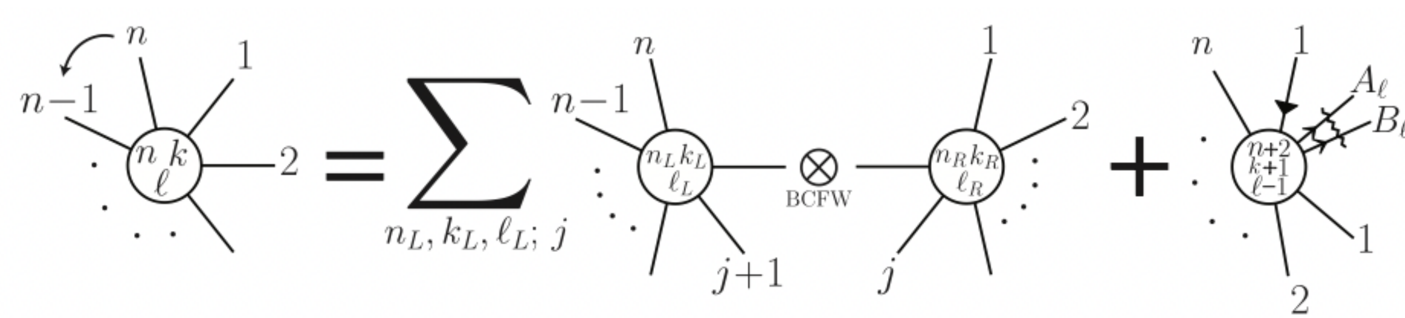
- points y_j in dual space
- lines $(AB)_j$ in mom twistor space

Loop integrand

$$\mathcal{I}_n^{(L)}(x_i, y_j) = \mathcal{I}_n^{(L)}[Z_i, (AB)_j]$$

unique rational function

- satisfies all cuts
- dual conformal invariant
- recursion relations



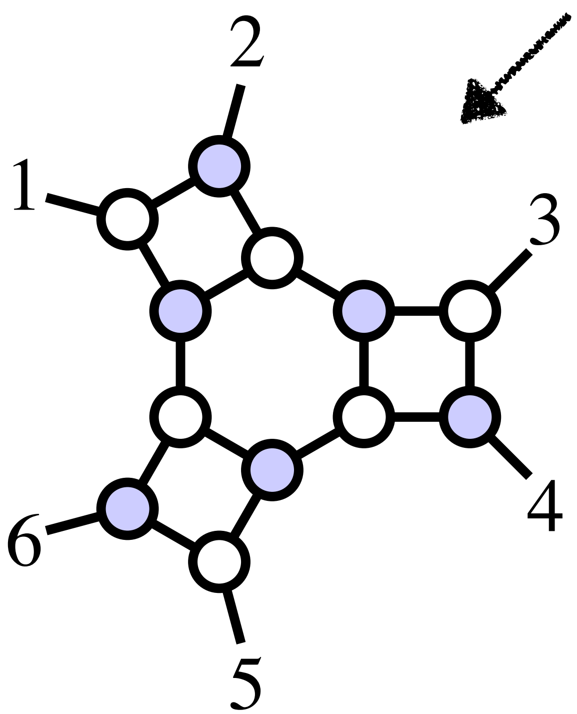
(Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, JT, 2010)

Positive Grassmannian

(Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, JT 2012)

- New gauge invariant building blocks for amplitudes:

On-shell diagrams



Products of 3pt on-shell amplitudes, cuts of loop integrands

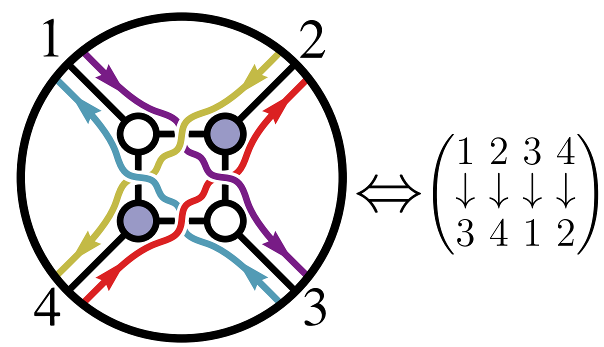
Positive Grassmannian

$$C = \begin{pmatrix} * & * & * & \dots & * \\ * & * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * \end{pmatrix}$$

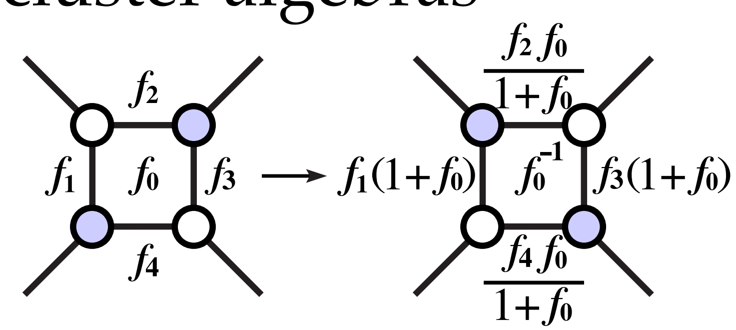
dlog form reproduces on-shell diagram

$$\int \frac{df_1}{f_1}\frac{df_2}{f_2}\dots\frac{df_m}{f_m}\,\delta(C\cdot Z)$$

permutations



cluster algebras



"gluing" together -> Amplituhedron

Bootstrap methods

(Goncharov, Spradlin, Vergu, Volovich, Dixon, Drummond, Henn, McLeod, von Hippel, Caron-Huot, Papathanasiou,...)

- Skip the integrand step, using the knowledge of the function space to construct the amplitude directly
  - Determine **symbol** of an amplitude

$$\mathcal{S}(f^{(k)}) = \sum_{\vec{\alpha}} \underbrace{\phi_{\alpha_1} \otimes \ldots \otimes \phi_{\alpha_k}}$$ ← encodes branch cuts

$$\{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}$$ ← symbol letters

impose physical and mathematical conditions

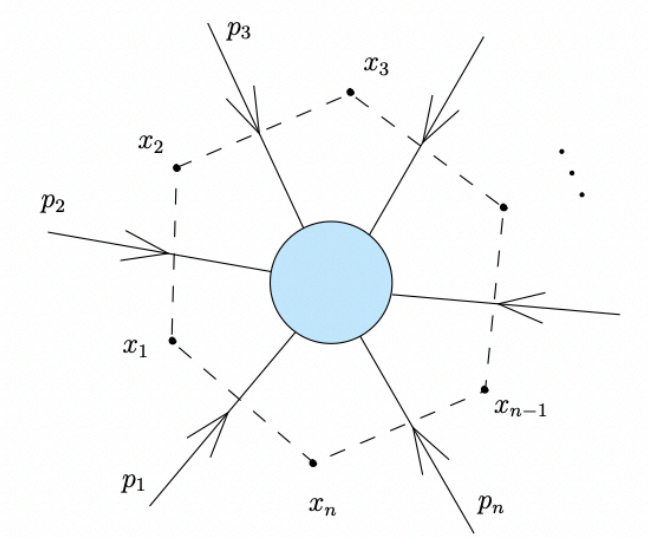
$$\mathcal{S}(R_6^{(2)}) = -\frac{1}{8}\Big\{\Big[u \otimes (1-u) \otimes \frac{u}{(1-u)^2} + 2\,(u \otimes v + v \otimes u) \otimes \frac{w}{1-v} + 2\,v \otimes \frac{w}{1-v} \otimes u\Big] \otimes \frac{u}{1-u} + \Big[u \otimes (1-u) \otimes y_u y_v y_w - 2\,u \otimes v \otimes y_w\Big] \otimes y_u y_v y_w\Big\} + \text{ permutations}\,,$$

  - From symbol to function: more constraints, fix constants

Amplitudes-Wilson loops duality

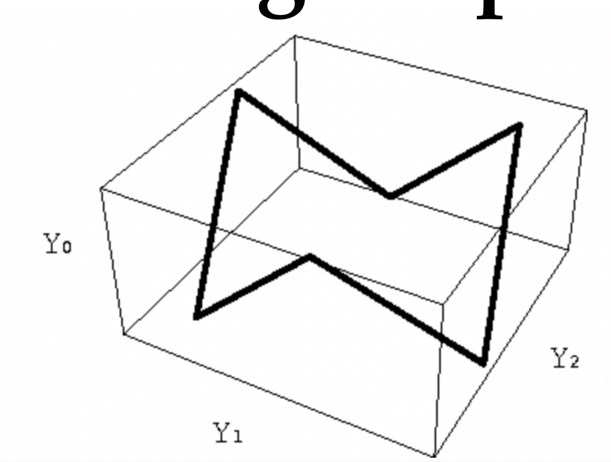
(Drummond, Henn, Korchemsky, Sokatchev, Alday, Maldacena)

- Duality: amplitudes and null polygonal Wilson loops, positions ~ momenta



Dual conformal symmetry of amplitude
= conformal symmetry of Wilson loops

Strong coupling

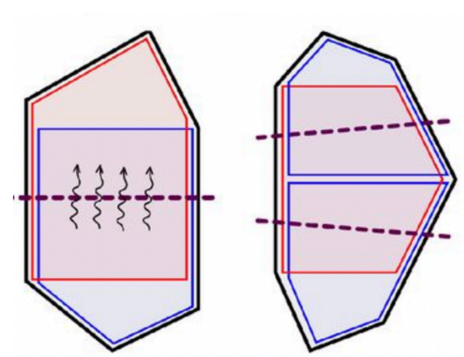


AdS/CFT correspondence
- minimal surface in AdS

(Alday, Maldacena, 2007)

Flux tube at finite coupling

- near collinear limit kinematics
- pentagons fixed by integrability
- 4d S-matrix: OPE in # of excitations



expansion around collinear limit, **exact** in coupling

(Alday, Gaiotto, Maldacena, Sever, Vieira, Basso,...)

Logarithm of the amplitude and IR finiteness

BDS Ansatz

(Bern, Dixon, Smirnov, 2005)

- Dual conformal symmetry is extremely restrictive

- The kinematical structure of 4pt and 5pt amplitudes are **fixed to all loops** up to constant factors

$$M_n = \sum_{L=0}^{\infty} g^L M_n^{(L)}(\epsilon) = \exp\left[\sum_{\ell=1}^{\infty} g^\ell \left(\underbrace{f^{(\ell)}(\epsilon) M_n^{(1)}(\ell\epsilon)} + C^{(\ell)} + \mathcal{O}(\epsilon)\right)\right]$$

n>6: dual conformal invariant IR finite remainder function $R_n^{(\ell)}$

leading $\frac{1}{\epsilon^2}$ behavior

mild divergence for logarithm

$$\ln M = \frac{h(g)}{\epsilon^2} + \mathcal{O}\left(\frac{1}{\epsilon}\right)$$

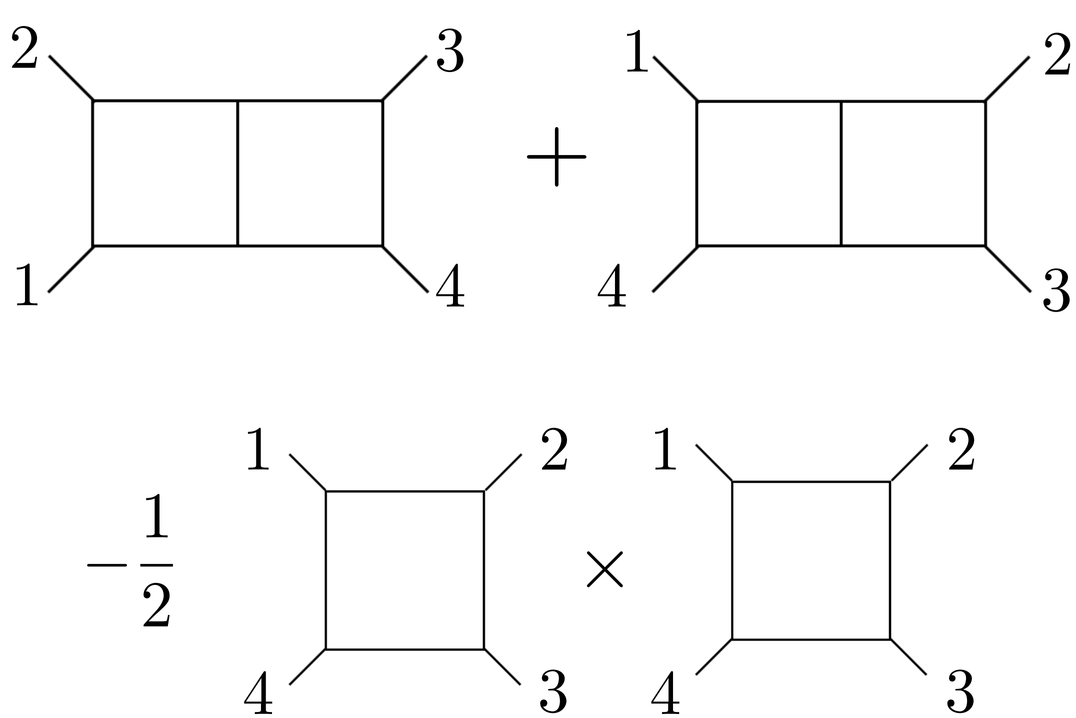
't Hooft coupling $g^2 \equiv g_{\rm YM}^2 N/(16\pi^2)$

Logarithm of the amplitude

- Expand logarithm in terms of combinations of amplitudes

$$\ln M = \ln\left(1 + gM^{(1)} + g^2 M^{(2)} + g^3 M^{(3)} + \ldots\right)$$

$$= gM^{(1)} + g^2\Big[\underbrace{M^{(2)} - \frac{1}{2}(M^{(1)})^2}_{\ln M_2}\Big] + g^3\Big[M^{(3)} - \overbrace{M^{(2)}M^{(1)} + \frac{1}{3}(M^{(1)})^3}^{\ln M_3}\Big] + \ldots$$



each term $1/\epsilon^6$ divergent,
combination only $1/\epsilon^2$ divergent

$\mathcal{I}_L$: integrand for $\ln M_L$

cut structure encodes
mild IR divergencies

Integrating the logarithm

- Property of $\mathcal{I}_L$ in collinear regions: to generate IR divergence need to integrate **over all loop momenta**

$$\mathcal{I}_L(AB_1\ldots AB_L) \longleftarrow \text{rational function}$$

$\downarrow$ integrate over AB_L

$$\mathcal{I}_L(AB_1\ldots AB_{L-1})$$

$\downarrow$ integrate over AB_{L-1}

$\vdots \qquad \vdots$

$\downarrow$ integrate over AB_2

$$\mathcal{I}_L(AB_1)$$

$\downarrow$ integrate over AB_1

$$\ln M_L \longleftarrow \text{IR divergent}$$

all are IR finite

Ideal object to study

$$\boxed{\mathcal{F}_L(z) \equiv \mathcal{I}_L(AB)}$$

$\uparrow$

interesting trascendental function depends on a single cross-ratio

$$z = \frac{\langle AB12\rangle\langle AB34\rangle}{\langle AB23\rangle\langle AB14\rangle}$$

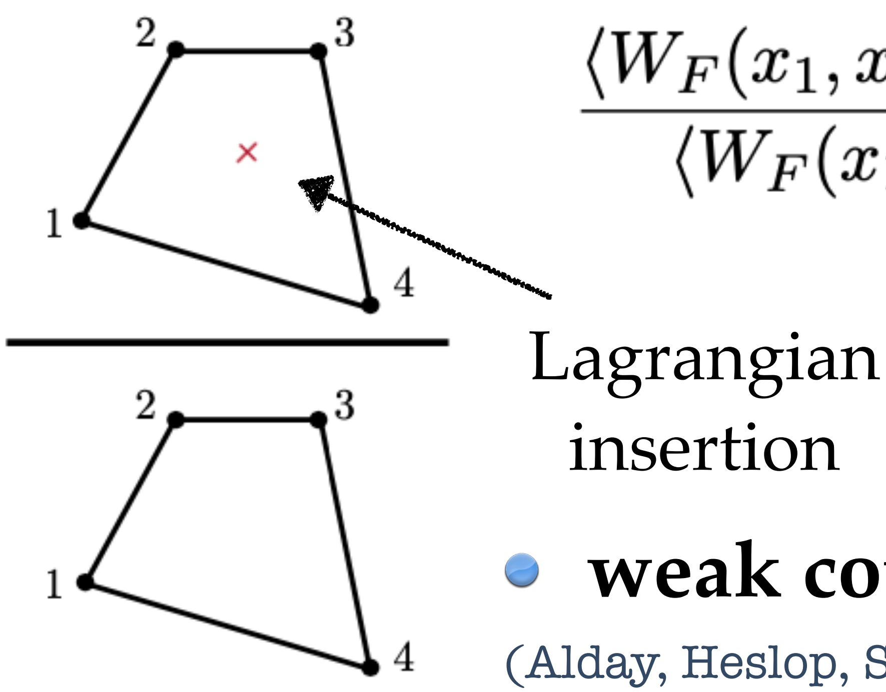
Non-perturbative finite object

$$\boxed{\mathcal{F}(g,z) = \sum_{L=1}^{\infty} g^{2L}\mathcal{F}_L(z)}$$

Relation to Wilson loops

(Englund, Roiban, 2011) (Alday, Buchbinder, Tseytlin, 2011)

- Same object appeared earlier in the context of Wilson loops



$$\frac{\langle W_F(x_1,x_2,x_3,x_4)\mathcal{L}(x_0)\rangle}{\langle W_F(x_1,x_2,x_3,x_4)\rangle} = \frac{1}{\pi^2}\frac{x_{13}^2 x_{24}^2}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2} F(g;z)$$

$$F(g;z) = -g^2\mathcal{F}(g,z) \qquad z = \frac{x_{20}^2 x_{40}^2 x_{13}^2}{x_{10}^2 x_{30}^2 x_{24}^2}$$

$$x_{ij} = (x_i - x_j)^2$$

- **weak coupling**: expansion in g^2

(Alday, Heslop, Sikorowski, 2012) (Alday, Henn, Sikorowski, 2013)

$$\mathcal{F}(g,z) = 1 + g^2(\log^2 z + \pi^2) + \ldots$$

- **strong coupling**: expansion in $1/g$ (string tension)

(Alday, Buchbinder, Tseytlin, 2011)

$$\mathcal{F}(g,z) = g\frac{z}{(z-1)^3}\left[2(1-z) + (z+1)\log z\right] + \cdots$$

Cusp anomalous dimension

- After integrating $\mathcal{F}(g,z)$ we recover $\ln M$

leading IR divergence: **cusp anomalous dimension**

$$\ln M = -\sum_{L\geq 1} g^{2L} \frac{\Gamma_{\text{cusp}}^{(L)}}{(L\epsilon)^2} + \mathcal{O}(1/\epsilon) \longrightarrow \boxed{\Gamma_{\text{cusp}}(g) = \sum_{L\geq 1} g^{2L}\Gamma_{\text{cusp}}^{(L)}}$$

exact result from integrability

(Beisert, Eden, Staudacher, 2006)

We can extract $\Gamma_{\text{cusp}}(g)$ from $\mathcal{F}(g,z)$

(Alday, Henn, Sikorowski, 2013)

(Henn, Korchemsky, Mistlberger, 2019)

(Arkani-Hamed, Henn, JT, 2021)

$$g\frac{\partial}{\partial g}\Gamma_{\text{cusp}}(g) = -\frac{1}{\pi}\int_{-\pi}^{\pi} d\phi\, F(e^{i\phi})$$

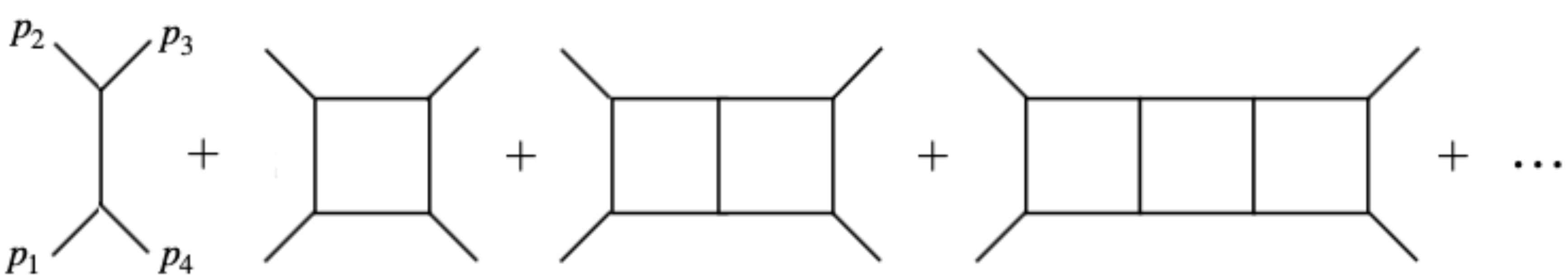
Our goal

- Focus on $\mathcal{F}(g, z)$: motivated by the Amplituhedron picture we find a new expansion
- This expansion will be exact in g, z but will organize contributions differently, based on the "number of loops in the loop space"
- We will be mainly interested in the strong coupling, so far completely inaccessible using our methods

Ladder resummation

- Our simplest case we will study later is somewhat analogous to the ladder resummation

(Broadhurst, Davydychev, 2010)



- Integral representation: strong coupling expansion exponentially suppressed

$$F(g, z) = \mathcal{O}(e^{-g})$$

Negative Amplituhedron geometry

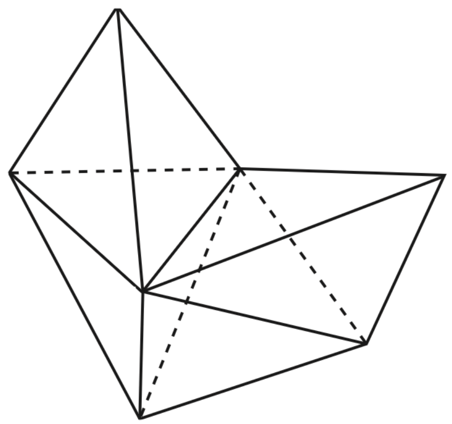
(Arkani-Hamed, Henn, JT 2021)

Positive geometry

- Geometric space defined using a set of inequalities

$$P_k(x_i) \geq 0$$

polynomials   parametrize kinematics



- Define a canonical **differential form** on this space $\Omega(x_i)$
  - Special form: logarithmic singularities on the boundaries

$$\Omega(x_i) \sim \frac{dx_i}{x_i} \quad \text{near boundary } x_i = 0$$

Amplituhedron: tree-level amplitudes and loop integrands

(Arkani-Hamed, JT 2013)

Four-point Amplituhedron

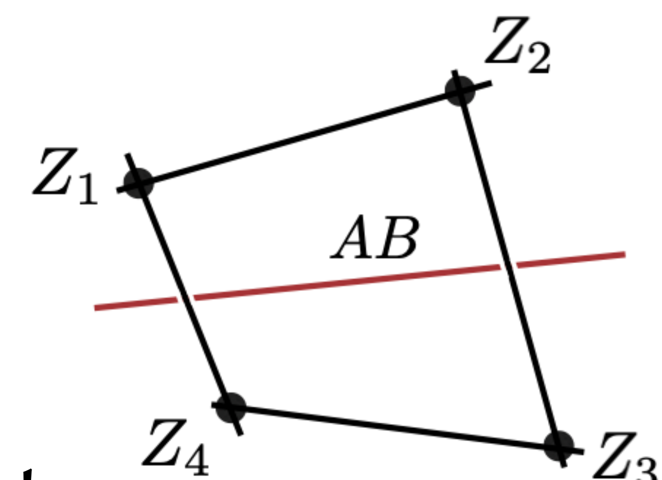
(Arkani-Hamed, JT 2013) (Arkani-Hamed, Thomas, JT 2017)

- Configuration of L lines in momentum twistor space

Each line: 4 positive coordinates

$$x_i, y_i, z_i, w_i > 0$$

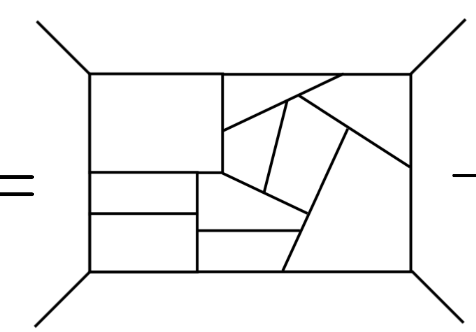
(parametrize loop momentum)



For any pair we get a quadratic constraint

$$\langle (AB)_i (AB)_j \rangle = -(x_i - x_j)(w_i - w_j) - (y_i - y_j)(z_i - z_j) > 0$$

- The canonical form on this space is a 4pt L-loop integrand

$\Omega =$  $+ \dots$

Graphical notation

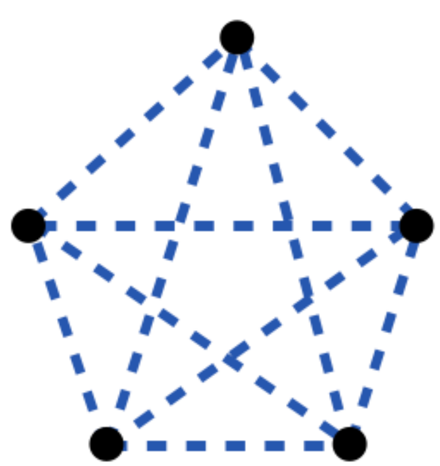
- We introduce a graphical notation:
  - vertex: loop line $(AB)_i$
  - blue dashed link: mutual positivity condition $\langle (AB)_i (AB)_j \rangle > 0$
- We denote the dlog form on the two-loop space

$(AB)_1$ $(AB)_2$

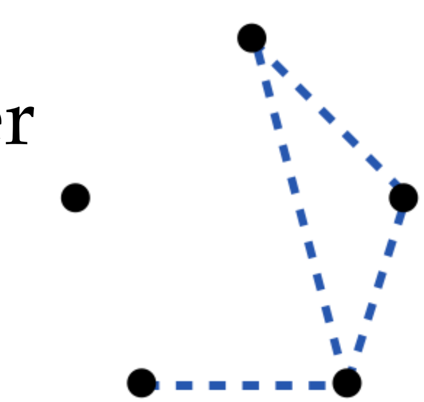
$\equiv \Omega(\mathcal{A}_2)$

- The L-loop integrand dlog form: complete graph

$\Omega(\mathcal{A}_L) =$

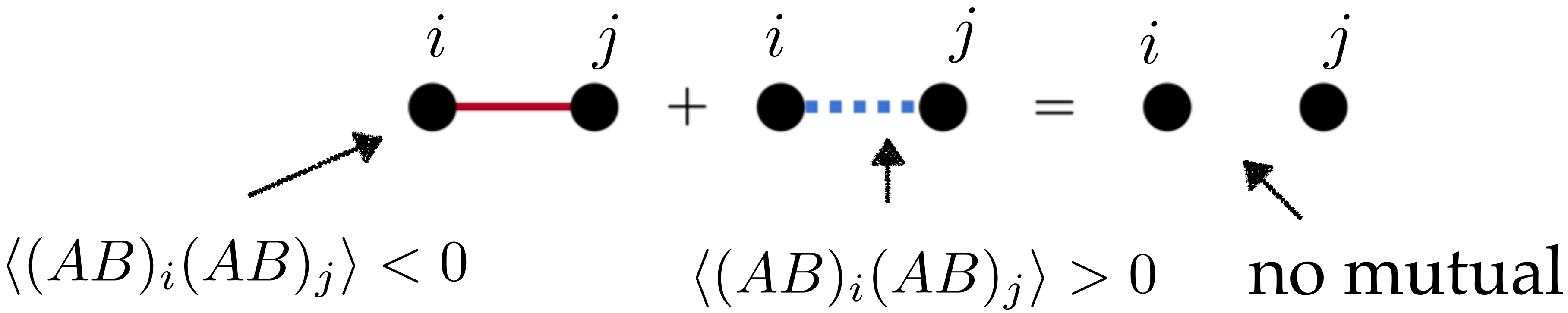


we can have a simpler positive geometry with fewer links

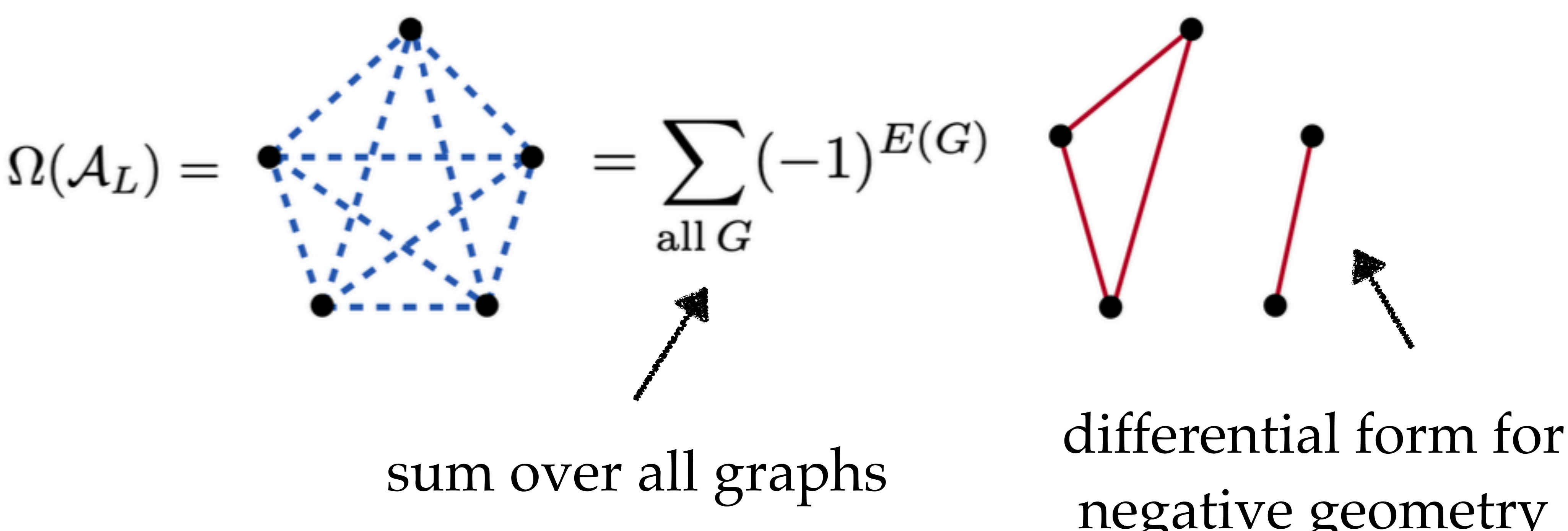


Negative geometries

- Replace positive "links" by negative



- New formula for L-loop integrand dlog form:



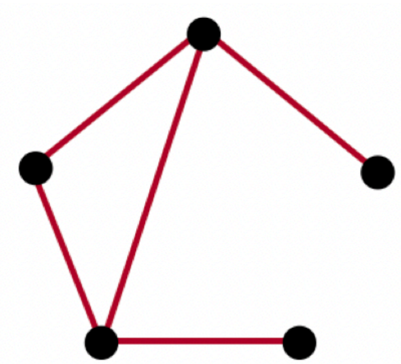
Exponentiation

- We can now write a formal sum over all loops

$$\Omega(g) = \sum_{L=0}^{\infty} (-g^2)^L \Omega(\mathcal{A}_L) \qquad \text{where } \Omega(\mathcal{A}_0) = 1$$

- The formula for $\Omega(g)$ exponentiates

$$\Omega(g) = \exp\left\{ \sum_{\substack{\text{all connected} \\ \text{graphs } G}} (-1)^{E(G)} (g^2)^L \quad \right\}$$

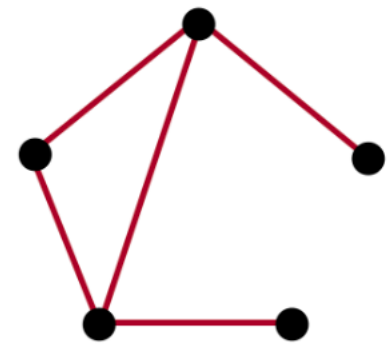


- We take the logarithm of both sides and expand in g

Geometry of $\mathcal{F}(g, z)$

- The L-loop logarithm: **connected graphs** with L vertices

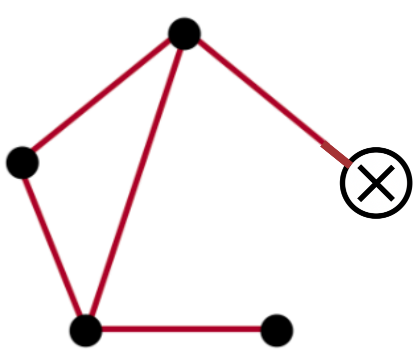
$$\log \Omega(g)\Big|_{(-g^2)^L} = \widetilde{\Omega}_L = \sum_{\substack{\text{all connected} \\ \text{graphs } G \\ \text{with } L \text{ vertices}}} (-1)^{E(G)}$$



Manifest IR behavior: each term only $\frac{1}{\epsilon^2}$ divergent after integration

- Freezing one of the loops:

$$\mathcal{F}_L(g, z) = \sum_{\substack{\text{all connected} \\ \text{graphs } G \\ \text{with } L \text{ vertices}}} (-1)^{E(G)} \frac{1}{S}$$



Integrate over all loops except one

↓

Manifest IR finiteness

Low loop examples

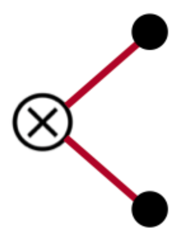
Tree-level

 $= 1$

One-loop

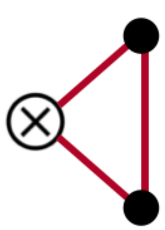
 $= \left[\pi^2 + \log^2 z\right]$

Two-loops

 $= -\frac{1}{2}\left[\pi^2 + \log^2 z\right]^2$

 $= -\frac{1}{12}\left[\pi^2 + \log^2 z\right] \times \left[5\pi^2 + \log^2 z\right]$

} simple “tree” geometries

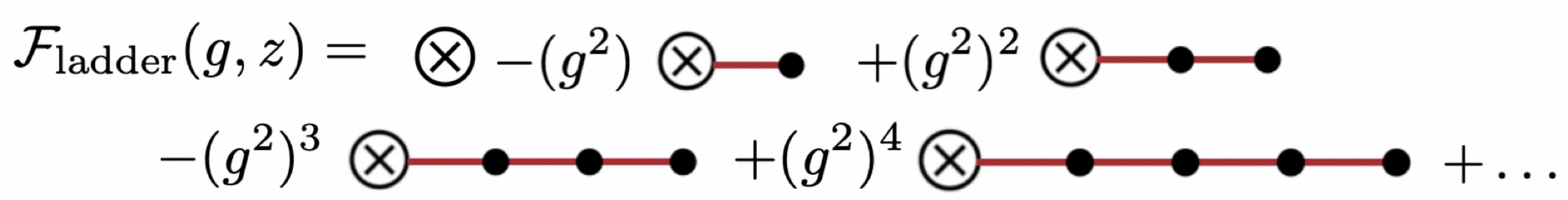


$$= -\frac{1}{6}\log^4 z + \log^2 z\left[-\frac{2}{3}\mathrm{Li}_2\left(\frac{1}{z+1}\right) - \frac{2}{3}\mathrm{Li}_2\left(\frac{z}{z+1}\right) + \frac{\pi^2}{9}\right]$$
$$+ \log z\left[4\mathrm{Li}_3\left(\frac{z}{z+1}\right) - 4\mathrm{Li}_3\left(\frac{1}{z+1}\right)\right] - \frac{2}{3}\left[\mathrm{Li}_2\left(\frac{1}{z+1}\right) + \mathrm{Li}_2\left(\frac{z}{z+1}\right) - \frac{\pi^2}{6}\right]^2$$
$$- \frac{8}{3}\pi^2\left[\mathrm{Li}_2\left(\frac{1}{z+1}\right) + \mathrm{Li}_2\left(\frac{z}{z+1}\right) - \frac{\pi^2}{6}\right] - 8\mathrm{Li}_4\left(\frac{1}{z+1}\right) - 8\mathrm{Li}_4\left(\frac{z}{z+1}\right) - \frac{\pi^4}{18}$$

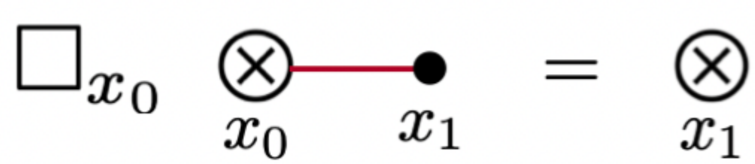
} complicated “one-loop” geometry

“Ladder” expansion

- To determine the integrand and integrate an arbitrary negative geometry is complicated
- We approximate $\mathcal{F}(g, z)$ by the simplest geometries



- Property of the integrand: Laplace operator acting on subgraph



in dual variables

Closed formula for "ladders"

- Applying this operation on $\mathcal{F}_{\text{ladder}}(g,z)$ we get

$$\frac{1}{2}(z\partial z)^2\mathcal{F}_{\text{ladder}}(g,z)+g^2\mathcal{F}_{\text{ladder}}(g,z)=0$$

which is solved by

$$\mathcal{F}_{\text{ladder}}(g,z)=\frac{\cos(\sqrt{2}g\log z)}{\cosh(\sqrt{2}g\pi)}$$

satisfying certain boundary conditions

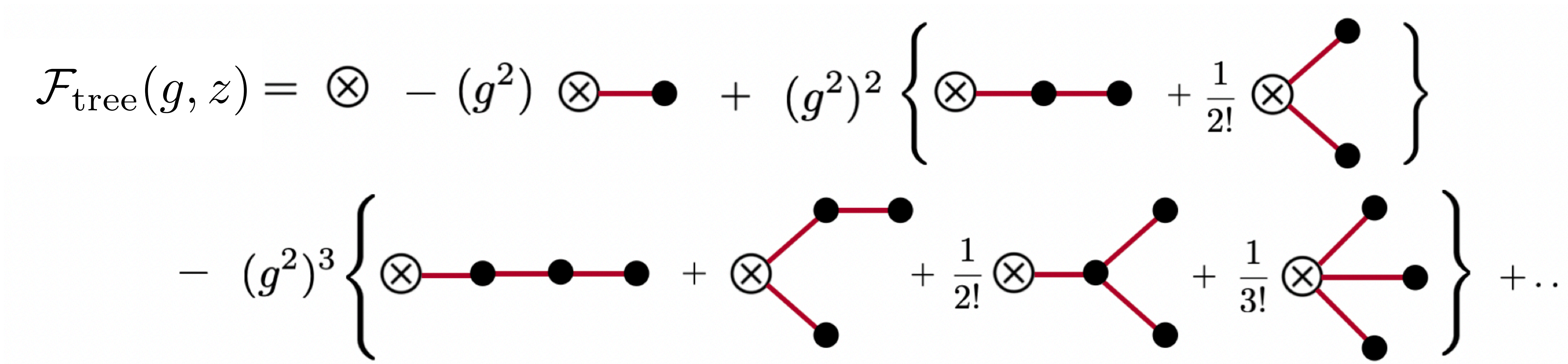
- Weak coupling: only a small subset of diagrams

- Strong coupling: $|\mathcal{F}_{\text{ladder}}(g;z)|\leq\frac{1}{\cosh(\sqrt{2}g\pi)}\leq 2e^{-\sqrt{2}g\pi}$

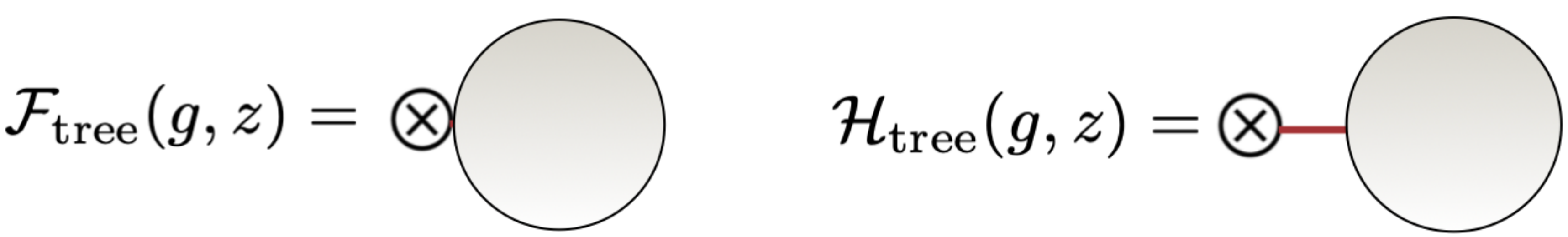
  but the exact result behaves as $\mathcal{F}(g,z)\sim g+1+\mathcal{O}\left(\frac{1}{g}\right)$

“Tree” expansion

- Consider a collection of all “tree” negative geometries



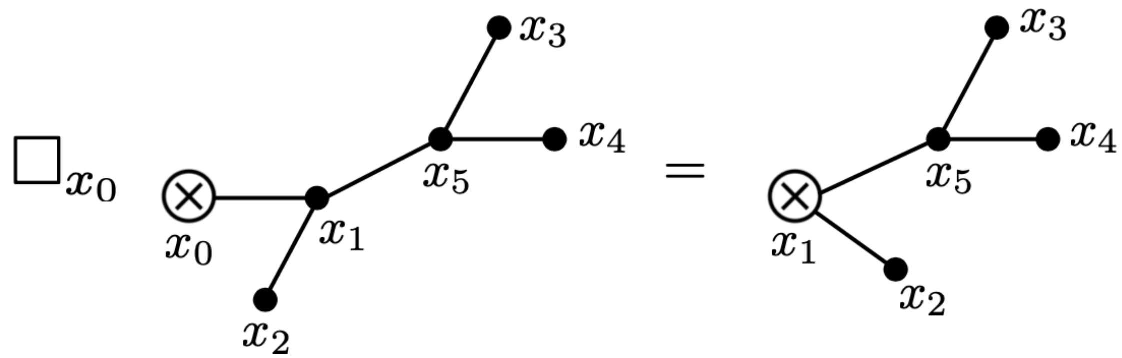
- Generating function



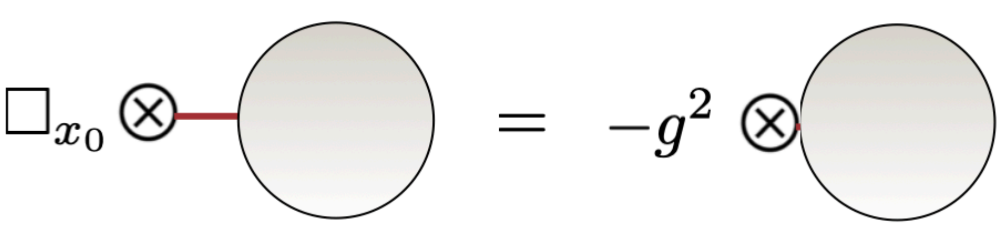
with the relation $\mathcal{F}_{\text{tree}}(g,z) = e^{\mathcal{H}_{\text{tree}}(g,z)}$

Closed formula for “trees”

- Apply the same differential operator on any tree graph



- Differential equation for $\mathcal{H}_{\text{tree}}(g, z)$



$$\longrightarrow \frac{1}{2}(z\partial_z)^2\mathcal{H}_{\text{tree}}(g,z) + g^2 e^{\mathcal{H}_{\text{tree}}(g,z)} = 0$$

- Solve: $\mathcal{F}_{\text{tree}}(g,z) = \dfrac{A^2}{g^2}\dfrac{z^A}{(z^A+1)^2}$, where $\dfrac{A}{2g\cos\frac{\pi A}{2}} = 1$

Strong coupling expansion

- Expand the result at strong coupling

$$\mathcal{F}_{\text{tree}}(g,z) = -\frac{z}{(1+z)^2} + \mathcal{O}\left(\frac{1}{g}\right)$$

in comparison to the exact result

$$\mathcal{F}(g,z) = g\frac{z}{(z-1)^3}\left[2(1-z) + (z+1)\log z\right] + \cdots + \mathcal{O}\left(\frac{1}{g}\right)$$

Our "tree" approximation misses the leading term but does have $1/g$ expansion at strong coupling

Gamma cusp

- Gamma cusp controls the leading IR divergence

$$\log M = -\sum_{L\geq 1} g^{2L} \frac{\Gamma^{(L)}_{\text{cusp}}}{(L\epsilon)^2} + \mathcal{O}(1/\epsilon) \quad \longrightarrow \quad \Gamma_{\text{cusp}}(g) = \sum_{L\geq 1} g^{2L}\Gamma^{(L)}_{\text{cusp}}$$

Exact result from integrability

(Beisert, Eden, Staudacher, 2006)

$$\Gamma_{\text{cusp}}(g) \to \begin{cases} 4g^2 - 8\zeta_2 g^4 + \cdots & g \ll 1 \\ 2g - \frac{3\log 2}{2\pi} + \cdots & g \gg 1 \end{cases}$$

radius of convergence $g_* = 0.25$

- Can be also extracted from $\mathcal{F}(g,z)$

(Alday, Henn, Sikorowski, 2013)
(Henn, Korchemsky, Mistlberger 2019)
(Arkani-Hamed, Henn, JT, 2021)

qualitatively correct behavior at strong coupling ⟵

$$\Gamma_{\text{tree}}(g) \to \begin{cases} 4g^2 - 8\zeta_2 g^4 + \ldots & g \ll 1 \\ \frac{8}{\pi} g - 1 + \ldots & g \gg 1 \end{cases}$$

radius of convergence $g_* = 0.35$

Summary & Outlook

Summary

- Our first attempt to use the Amplituhedron picture for planar N=4 SYM amplitudes for resummation and strong coupling.
- We considered an IR finite object $\mathcal{F}(g,z)$ derived from the logarithm of the amplitude (freezing one loop, integrate others).
- New expansion in terms of negative geometries and organization using "loop of loops". We solved for $\mathcal{F}(g,z)$ at for "trees"
- Extracted gamma cusp, expanded at strong coupling and compared to the exact result $\Gamma_{\rm cusp}(g)$: surprisingly good behavior

Outlook

- How do we restore the leading $\mathcal{F}(g, z) \sim g$ behavior at strong coupling? Subleading "one-loop" negative geometries: integrands known, integrate or find a differential equation…

  (Arkani-Hamed, Henn, JT, in progress)

- Higher points, richer analytic structure even at lower loops

  (Arkani-Hamed, Chicherin, Henn, JT, in progress)

- Deformed negative geometries: relation to integrability?

  (Arkani-Hamed, Henn, JT, in progress)

- General negative geometries, IR divergencies, applications

  (Brown, Oktem, Paranjape, JT, in progress)

- Strong coupling geometry? Emergence of strings?

Thank you!