Adiabatic continuity, anomaly preserving compactifications, and confinement in Yang-Mills theory

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Adiabatic continuity

Adiabatic continuity is the idea that in (non-)supersymmetric gauge theories, Yang-Mills, QCD, etc non-perturbative strong coupling phenomena - e.g. confinement, chiral symmetry breaking, mass gap, multi-branched vacuum structure can be continuously connected to arbitrarily weak coupling regimes!

Since mid-70s, all of these phenomena were believed to take place at strong coupling. This belief is still common in contemporary literature, but not true.

In the last 15 years, it is understood that adiabatic continuity can be achieved by judiciously chosen circle compactifications on $R_3 \times S_1$, matter content, b.c. But $R_3 \times S_1$ is not the subject of my talk today.

Rather, I will introduce another realization of adiabatic continuity, which provides new insights.

Two challenging questions

⇒Can we continuously connect 4d physics (in generic non-susy theories) with 2d physics? (without intervening phase transitions?)

⇒Technical, but equally important: How do we formulate semi-classics in 't Hooft flux background in a thermodynamic limit?

Adiabatic continuity on $R_2 \times T_2$?

- At large $T_2 \times R_2$, SU(N) gauge theory, SU(N) with the insertion of 't Hooft flux, and PSU(N) theory possess identical local dynamics. Local correlation functions (e.g mass gap) are the same.

- But at small $T_2 \times R_2$, studying the SU(N) with the insertion of 't Hooft flux is far more useful than standard periodic b.c. The reason for this is that flux stabilizes 0-form part of the center symmetry which is otherwise broken at small T_2. (We will show this.)

- This may sound strange, but it is well-known in a small sub-community in the context of lattice gauge theory. TEK model. (Gonzalez-Arroyo and Okawa 83).

Similarly, one can see imprints of it in classic calculation of supersymmetric index in SU(N) SYM theory. (Witten 82).

•The ’t Hooft flux background is commonly used to provide kinematical constraints. Seiberg et.al. used it to demonstrate mixed anomalies involving 1-form symmetries, constraints on possible IR-phases, but not to study semi-classical dynamics.

This talk:

⇒Yang-Mills theory : Multi-branch structure, string tensions

⇒QCD with fundamental fermions. (Derivation of chiral Lagrangian at small $T_2 \times R_2$ and matching to large $T_2 \times R_2$)

⇒N=1 SYM

⇒QCD with (S/AS) rep. fermions (S/AS), chiral theories

Yang-Mills on R2 x T2

•Take symmetric T2, size smaller than strong scale. Compactified directions x_3, x_4.

•Classical minima given by flat connections, $F_{34}=0$. Let us name holonomies in the compact directions P3, P4. Classically, each holonomy takes values in the maximal torus T_N.

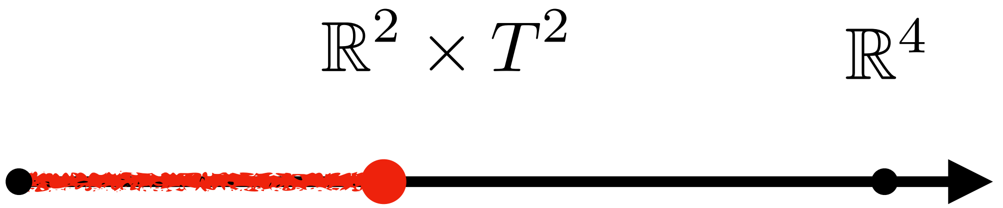
$$P_3 = \mathcal{P}\exp\left(\mathrm{i}\int_0^L a_3 \mathrm{d}x_3\right), \quad P_3 = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\alpha_1}, \mathrm{e}^{\mathrm{i}\alpha_2}, \ldots, \mathrm{e}^{\mathrm{i}\alpha_N}),$$

•Classical moduli space: $\mathcal{M}_{\mathrm{cl}} = (\mathbf{T}_N)^2/S_N.$

•Q: What happens to this moduli space quantum mechanically? What happens in the presence of 't Hooft flux classically and quantum mechanically?

Yang-Mills on R2 x T2 without and with 't Hooft flux

1-form symmetry: $\left(\mathbb{Z}_N^{[1]}\right)_{4\text{d}} \xrightarrow{T^2 \text{ compact.}} \left(\mathbb{Z}_N^{[1]}\right)_{2\text{d}} \times \mathbb{Z}_N^{[0]} \times \mathbb{Z}_N^{[0]}.$



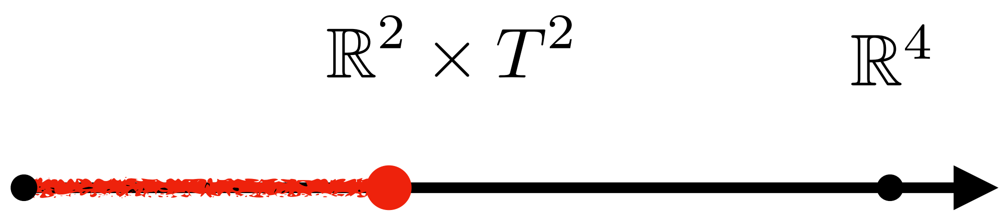
Phase transition

$\mathbb{Z}_N^{[0]} \times \mathbb{Z}_N^{[0]}$ broken $a)$

$$V_{1-\text{loop}} = -\frac{2}{\pi^2 L^4} \sum_{(n_3,n_4)\in\mathbb{Z}^2\setminus\mathbf{0}} \frac{|\text{tr}(P_3^{n_3} P_4^{n_4})|^2}{(n_3^2+n_4^2)^2}$$

Yang-Mills on R2 x T2 without and with 't Hooft flux

1-form symmetry: $\left(\mathbb{Z}_N^{[1]}\right)_{4\mathrm{d}} \xrightarrow{T^2 \text{ compact.}} \left(\mathbb{Z}_N^{[1]}\right)_{2\mathrm{d}} \times \mathbb{Z}_N^{[0]} \times \mathbb{Z}_N^{[0]}.$



Phase transition

a)

$\mathbb{Z}_N^{[0]} \times \mathbb{Z}_N^{[0]}$ broken

$$V_{1-\text{loop}} = -\frac{2}{\pi^2 L^4} \sum_{(n_3,n_4)\in\mathbb{Z}^2\setminus\mathbf{0}} \frac{|\mathrm{tr}(P_3^{n_3} P_4^{n_4})|^2}{(n_3^2+n_4^2)^2}$$

$\mathbb{R}^2 \times T^2$ with 't Hooft flux

Adiabatic continuity

b)

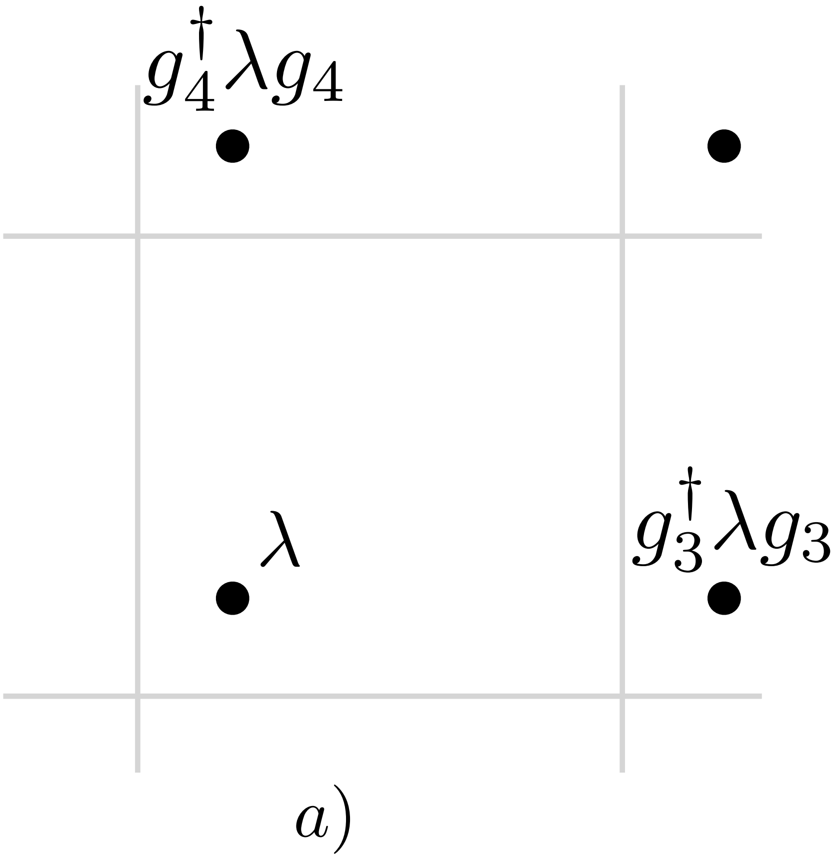
$\mathbb{Z}_N^{[0]} \times \mathbb{Z}_N^{[0]}$ unbroken

- Opposite to periodic case
- Why does it occur?

Activating a fixed 't Hooft flux= Activating a fixed $B^{(2)}$

't Hooft 78

YM, SYM,
QCD(adj),
N=2 SYM



λ : adjoint matter

$$g_3(L_4)^\dagger g_4(0)^\dagger = g_4(L_3)^\dagger g_3(0)^\dagger \exp\left(\frac{2\pi \mathrm{i}}{N} n\right).$$

Co-cycle or consistency condition

Classical minima is again in terms of flat connections, and the Polyakov loops are dictated by transition matrices. With flux, classical minima is given by non-commuting Polyakov loops.

$$P_3 = g_3\, \mathcal{P} \mathrm{e}^{\mathrm{i} \int_0^L a_3 \mathrm{d}x_3} = S, \qquad P_4 = g_4\, \mathcal{P} \mathrm{e}^{\mathrm{i} \int_0^L a_4 \mathrm{d}x_4} = C.$$

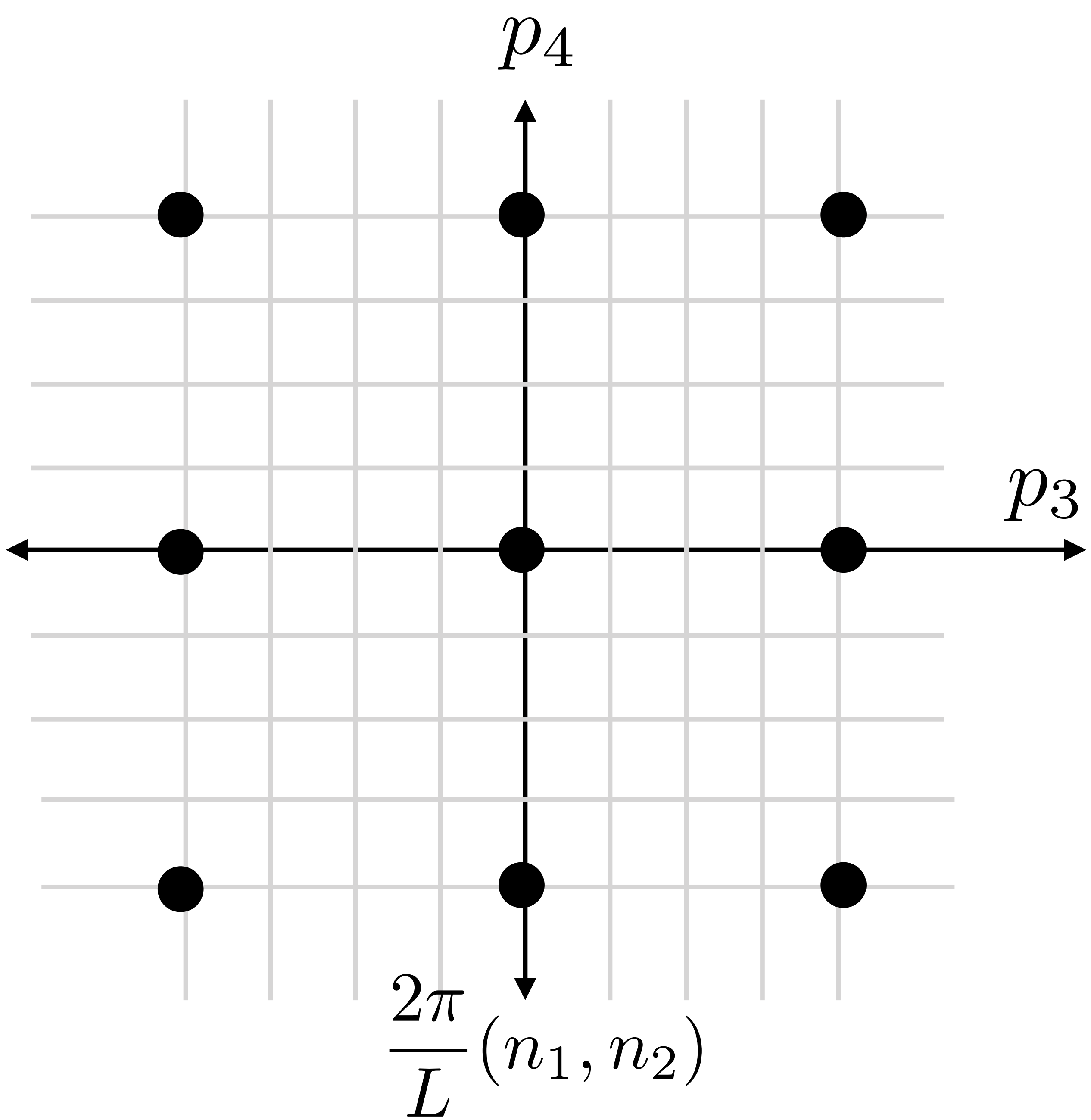
$$C \propto \mathrm{diag}(1, \omega, \ldots, \omega^{\mathrm{N}-1}), \qquad (S)_{i,j} \propto \delta_{i+1,j} \text{ with } \omega = e^{2\pi i/N}$$

Think of these as two non-commuting adjoint Higgs field. Hence

$$SU(N) \xrightarrow{\text{Higgsing}} \mathbb{Z}_N$$

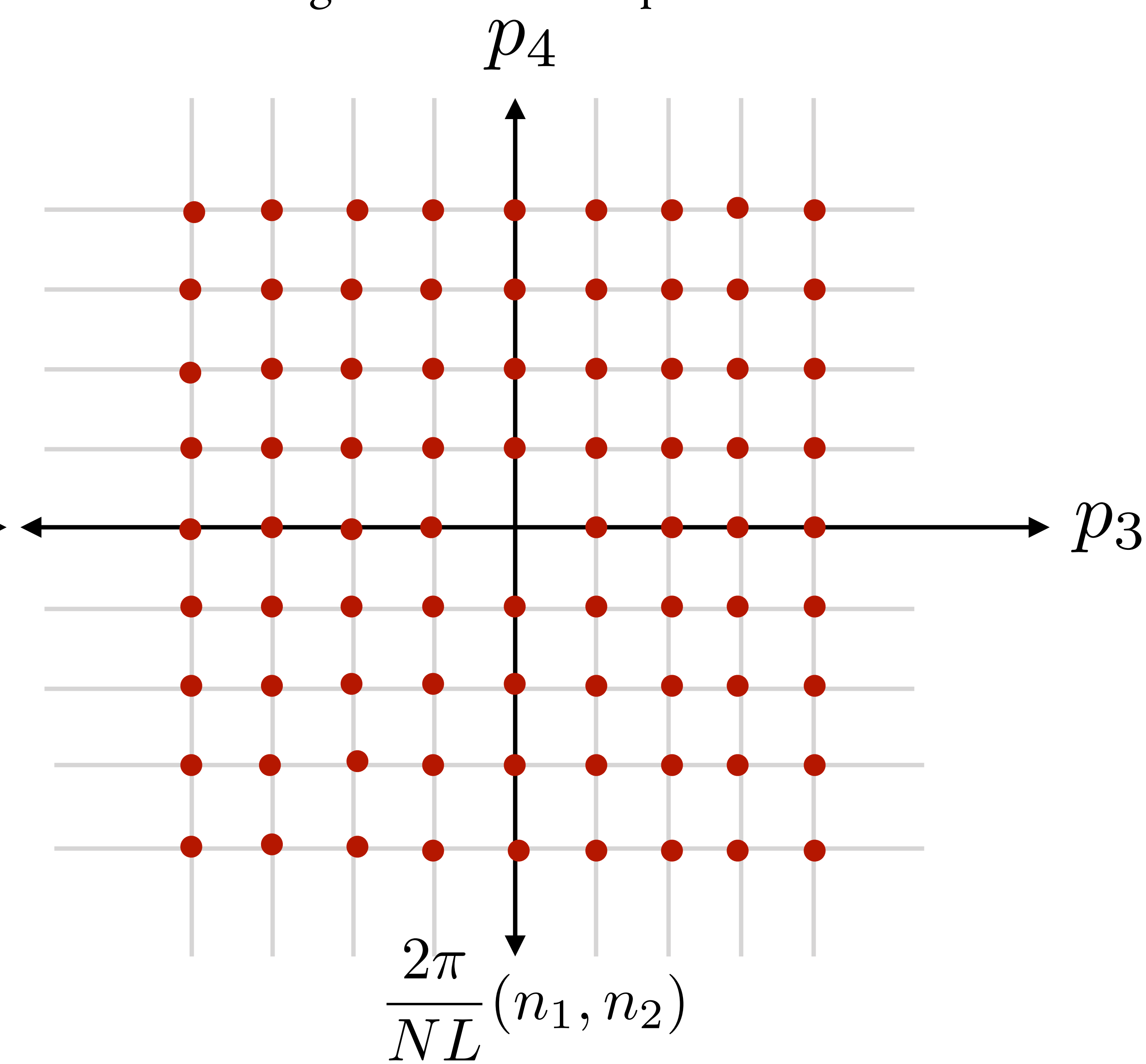
Perturbative spectrum on R2 x T2

without flux



with flux

No gapless modes in 2d!
Forms a continuum in the N ➡∞!
Large-N volume independence.



Wilson loops $\Longrightarrow$ Perimeter law in perturbation theory. (TQFT in pert. th.)
How about non-perturbatively? Is this TQFT destabilized?

Fractional instantons $\Longrightarrow$ Center vortices

$$Q_{\rm top} = \frac{1}{8\pi^2}\int_{T^4} {\rm tr}\left((\tilde{F}-B)^2\right) \in -\frac{1}{N}\frac{\varepsilon_{\mu\nu\rho\sigma}n_{\mu\nu}n_{\rho\sigma}}{8} + \mathbb{Z}.$$

('t Hooft, van Baal,...)

$$\mathrm{Re}(S_{\rm YM}) \geq \frac{8\pi^2}{g^2}|Q_{\rm top}| = \frac{8\pi^2}{Ng^2},$$

Assuming solution that satisfies BPS-bound exists.

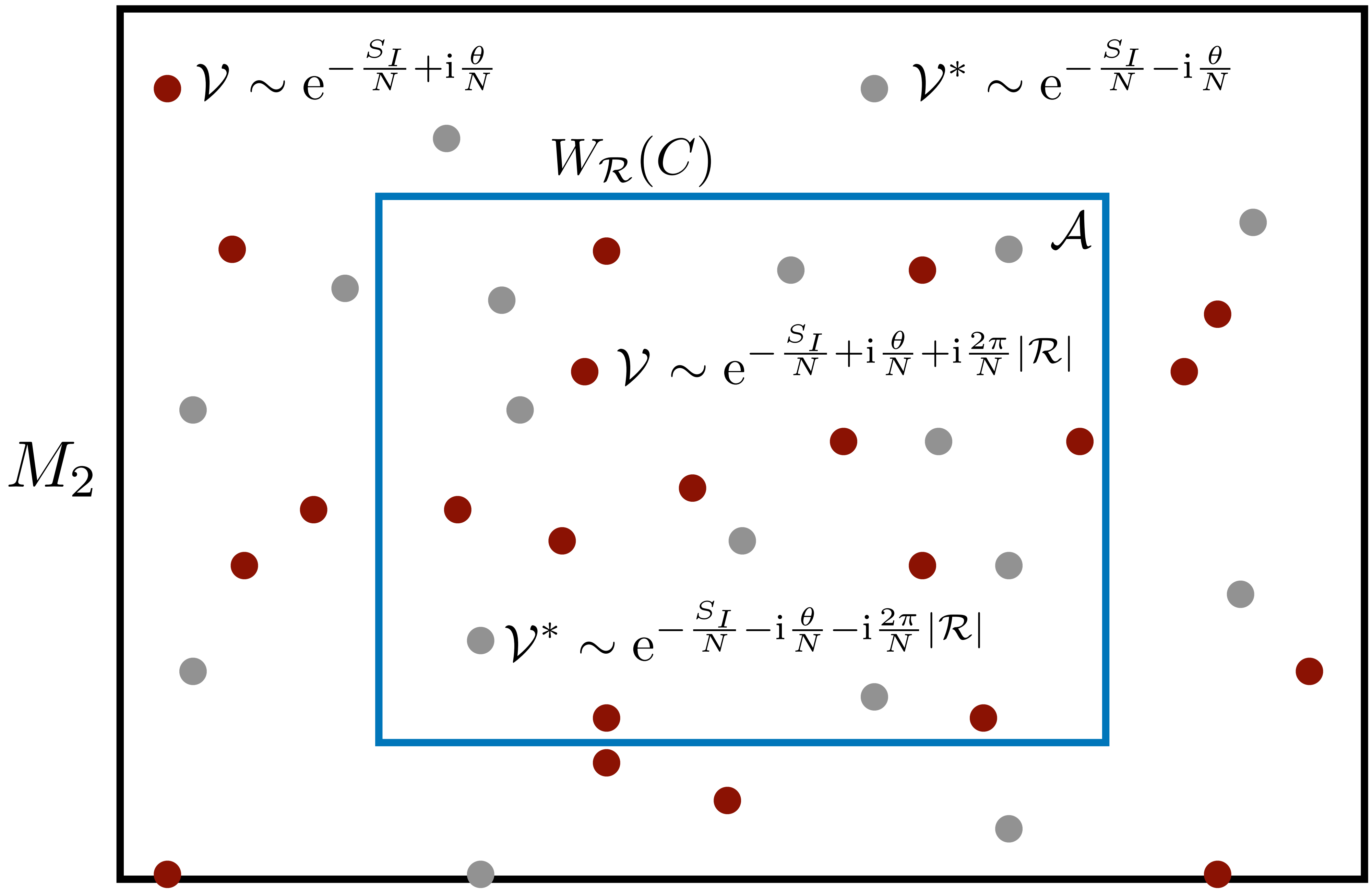
There is compelling evidence from lattice that there is. (Gonzalez-Arroyo, Montero, Garcia-Perez 1990s). Furthermore,

$$W_{\mathcal{R}}(C) = \exp(2\pi i|\mathcal{R}|/N)$$

when a vortex is inside the loop. Non-trivial mutual statistics between the Wilson loop and the vortex!

Proliferation of vortices and semi-classics

Semi-classical description here is very similar to charge-N abelian Higgs model in 2d.



First, ignore the Wilson loop. Let us just look to partition function.

Proliferation of vortices and semi-classics

$$Z(\theta) = \sum_{n,\overline{n}\geq 0} \frac{V^{n+\overline{n}}}{n!\overline{n}!} K^{n+\overline{n}} \mathrm{e}^{-(n+\overline{n})S_I/N} \mathrm{e}^{\mathrm{i}(n-\overline{n})\theta/N} \delta_{n-\overline{n}\in N\mathbb{Z}},$$

Integer top. charge

$$Z(\theta) = \sum_{n,\overline{n}\geq 0} \frac{V^{n+\overline{n}}}{n!\overline{n}!} K^{n+\overline{n}} \mathrm{e}^{-(n+\overline{n})S_I/N} \mathrm{e}^{\mathrm{i}(n-\overline{n})\theta/N} \sum_{k=0}^{N-1} \mathrm{e}^{-\frac{2\pi \mathrm{i}\, k}{N}(n-\overline{n})}$$

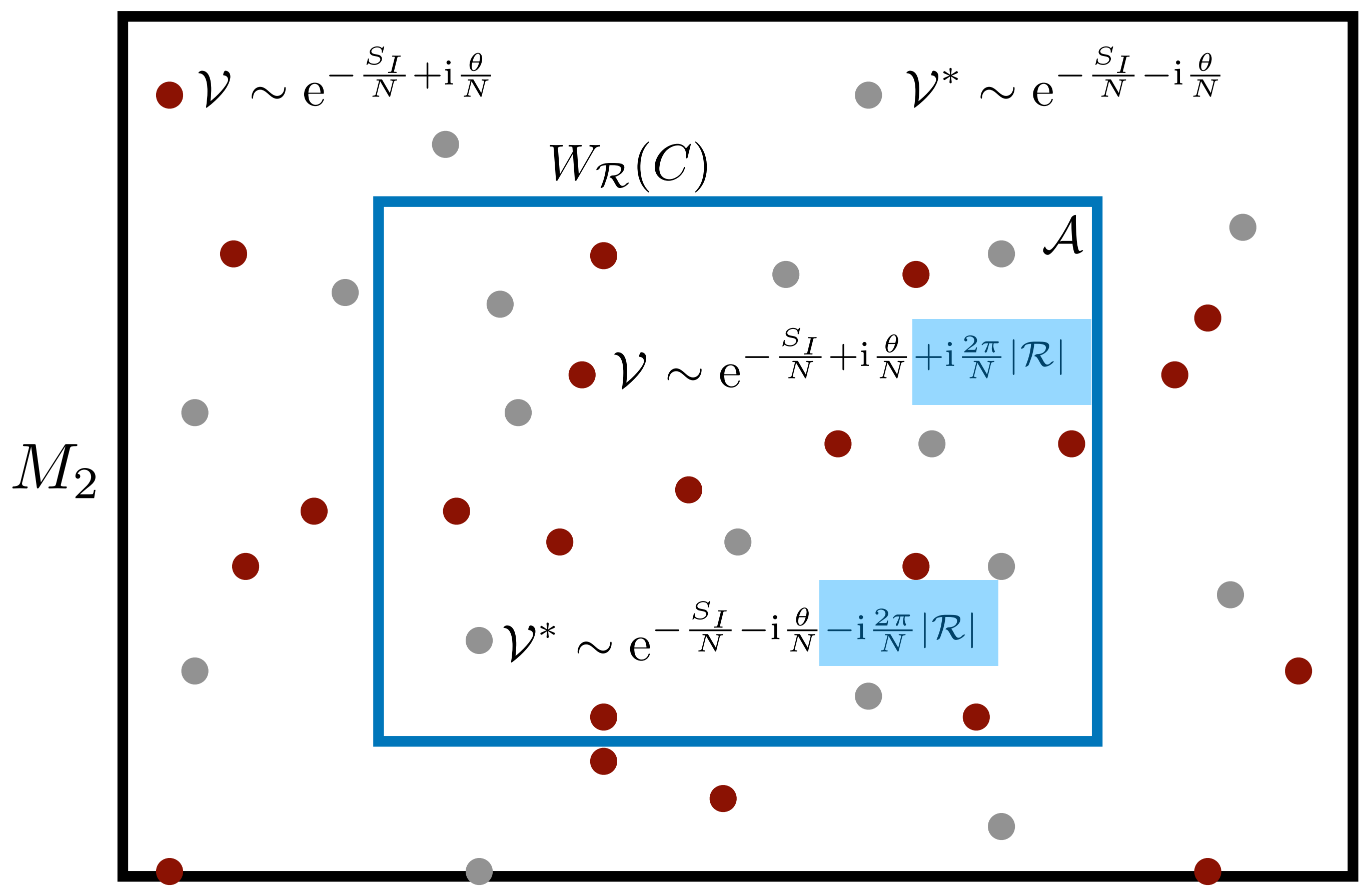
$$= \sum_{k=0}^{N-1} \exp\left[VK\mathrm{e}^{-S_{\mathrm{I}}/N+\mathrm{i}(\theta-2\pi k)/N} + VK\mathrm{e}^{-S_{\mathrm{I}}/N-\mathrm{i}(\theta-2\pi k)/N} \right]$$

$$= \sum_{k=0}^{N-1} \exp\left[-V\left(-2K\mathrm{e}^{-S_I/N} \cos\left(\frac{\theta-2\pi k}{N}\right)\right)\right],$$

Multi-branched vacuum structure

$$E_k(\theta) = -\Lambda^2 (\Lambda L_s)^{5/3} \cos\left(\frac{\theta-2\pi k}{N}\right).$$

$$\langle W_{\mathcal{R}}(C)\rangle = \frac{1}{Z(\theta)} \sum_{n_1,n_2,\overline{n}_1,\overline{n}_2} \frac{\mathcal{A}^{n_1+\overline{n}_1}(V-\mathcal{A})^{n_2+\overline{n}_2}}{n_1!\,n_2!\,\overline{n}_1!\,\overline{n}_2!} \left(K\,\mathrm{e}^{-S_I/N}\right)^{n_1+n_2+\overline{n}_1+\overline{n_2}}$$
$$\times\, \mathrm{e}^{\mathrm{i}(n_1+n_2-\overline{n}_1-\overline{n}_2)\theta/N}\, \mathrm{e}^{2\pi\mathrm{i}(n_1-\overline{n}_1)|\mathcal{R}|/N}\, \delta_{n_1+n_2-\overline{n}_1-\overline{n}_2\in N\mathbb{Z}}$$
$$= \frac{1}{Z(\theta)} \sum_{k=0}^{N-1} \mathrm{e}^{-VE_k(\theta)} \exp\Big(-\mathcal{A}\big(E_k(\theta+2\pi|\mathcal{R}|)-E_k(\theta)\big)\Big).$$



Extra phases: Due to non-trivial Mutual statistics of Wilson loops with center vortex.

Only the ones inside the loop acquire these phases.

$$T_{\mathcal{R}}(\theta) = E_0(\theta+2\pi|\mathcal{R}|) - E_0(\theta)$$
$$\sim \Lambda^2(\Lambda L_s)^{5/3}\left(\cos\frac{\theta}{N} - \cos\frac{\theta+2\pi|\mathcal{R}|}{N}\right).$$

In the V →∞ limit, we obtain Finite string tension.

Why is this compactification special?

Mixed anomaly: $\mathbb{Z}_N^{[1]}$ and CP at $\theta = \pi$.

$$Z_{\theta+2\pi}[B_{\rm 4d}] = \exp\left(\frac{\mathrm{i}N}{4\pi}\int_{M_2\times T^2} B_{\rm 4d}\wedge B_{\rm 4d}\right) Z_\theta[B_{\rm 4d}].$$

Gaiotto, Kapustin, Komargodski, Seiberg, 2017

$$CP : Z_{\theta=\pi}[B_{\rm 4d}] \to Z_{\theta=-\pi}[B_{\rm 4d}]$$
$$= \exp\left(-\frac{\mathrm{i}N}{4\pi}\int_{M_2\times T^2} B_{\rm 4d}\wedge B_{\rm 4d}\right) Z_{\theta=\pi}[B_{\rm 4d}].$$

Vacuum at **θ**= **π** can not be trivial.

Compactification and mixed anomaly

$$B_{4\mathrm{d}} = B_{2\mathrm{d}} + A_3 \wedge \frac{\mathrm{d}x_3}{L_s} + A_4 \wedge \frac{\mathrm{d}x_4}{L_s} + \frac{2\pi n_{34}}{N} \frac{\mathrm{d}x_3 \wedge \mathrm{d}x_4}{L_s^2}.$$

- $B_{2\mathrm{d}}$: 2-form gauge field for $\mathbb{Z}_N^{[1]}$, which couples to $W_{\mathcal{R}}(C)$ inside M_2.
- A_3: 1-form gauge field for one of $\mathbb{Z}_N^{[0]}$, which couples to P_3.
- A_4: 1-form gauge field for another $\mathbb{Z}_N^{[0]}$, which couples to P_4.

$$Z_{\theta+2\pi}^{(n_{34})}[B_{2\mathrm{d}}, A_3, A_4] = \exp\left(\mathrm{i}\, n_{34} \int_{M_2} B_{2\mathrm{d}} - \frac{\mathrm{i}N}{2\pi} \int_{M_2} A_3 \wedge A_4\right) Z_{\theta}^{(n_{34})}[B_{2\mathrm{d}}, A_3, A_4].$$

Mixed anomaly between 1-form and CP survives when n(34) is non-zero.
θ= **π** can not be trivial. Consistent with semi-classical description.

Can we do similar analysis in QCD with fundamental quarks?

According to 't Hooft (81), the answer is no. With the introduction of fundamental matter, we loose $\mathbb{Z}_N$ one-form symmetry and it is not possible to impose 't Hooft's tbc consistently.

't Hooft (81): "Note that these classes disappear if a field in the fundamental representation of SU(N) is added to the system (these fields would make unacceptable jumps at the boundary)."

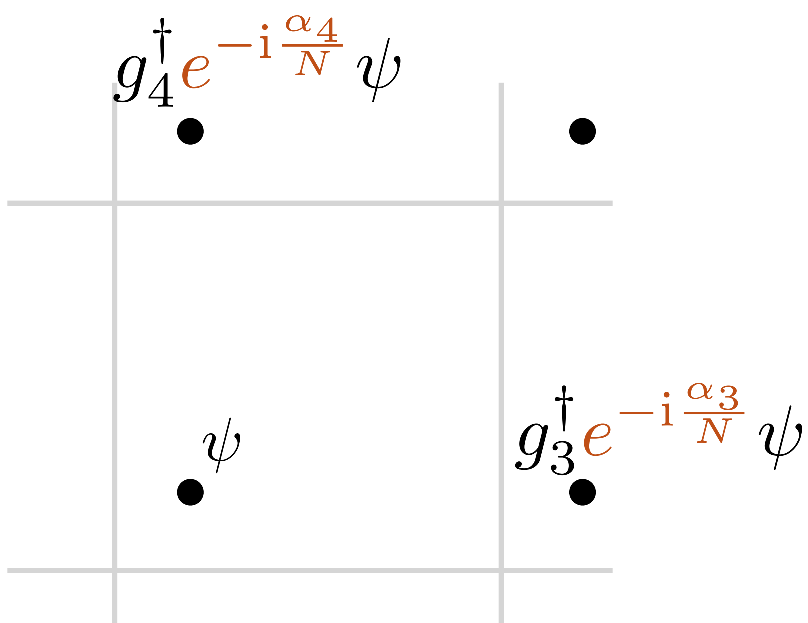
This actually turns out to be too fast, and there is a way around it.

’t Hooft flux in the presence of fundamental quarks

Obstacle: One cannot naively introduce ’t Hooft flux in the presence of fundamental matter field. (No center symmetry).

Way around: Turn on a $U(1)_B$ baryon magnetic flux background. Since $U(1)_B = U(1)_q / \mathbb{Z}_N$, ’t Hooft flux can still be inserted through the common center group of SU(N) and $U(1)_B$

Misumi, Sakai, Tanizaki 2017



$$\psi(x_3 + L, x_4) = e^{-i\alpha_3(x_4)/N} g_3(x_4)^\dagger \psi(x_3, x_4),$$
$$\psi(x_3, x_4 + L) = e^{-i\alpha_4(x_4)/N} g_4(x_3)^\dagger \psi(x_3, x_4)$$

The perturbative massless spectrum in baryon number background ⟹ Solve Dirac eq. with t.b.c. ⟹ Nf 2d massless fermions

⟹ Zero modes: Jacobi Theta function, quasi-periodic.

2d N_f-flavor massless Dirac fermions can be mapped to 2d level-1 $U(N_f)$ WZW

$$S = \frac{1}{8\pi}\int_{M_2} \mathrm{tr_f}(\mathrm{d}\tilde{U}\wedge\star\mathrm{d}\tilde{U}^\dagger) + \frac{1}{12\pi}\int_{M_3}\mathrm{tr_f}[(\tilde{U}^\dagger\mathrm{d}\tilde{U})^3],$$

From Non-abelian Bosonization, Witten

Almost chiral Lagrangian.... Wait a bit more. Center-vortex generate

$$\Delta S \sim -\frac{1}{L^2}\mathrm{e}^{-S_\mathrm{I}/N}\left(\mathrm{e}^{\mathrm{i}\theta/N}(\det\tilde{U})^{1/N} + \mathrm{e}^{-\mathrm{i}\theta/N}(\det\tilde{U}^\dagger)^{1/N}\right).$$

which lifts one of the gapless degrees of freedom. So, IR is SU(Nf)1 level-1 WZW model with central charge Nf-1.

In the large-N limit, the center-vortex term takes the form

$$\Delta S \sim \frac{\Lambda^2 (\Lambda L)^{\frac{5N-2N_f}{3N}}}{N^2} \left(\mathrm{i} \ln \left(\det(\tilde{U}) \right) - \theta \right)^2$$

Which gives η' mass, consistent with the Witten-Veneziano formula.

If one actually assumes that adiabatic continuity holds, this construction is a derivation of the chiral Lagrangian. Let us show this.

What happens at large $T_2 \times R_2$? Chiral Lagrangian perspective

If you compactify chiral largrangian on large $T_2 \times R_2$ and consider physics at length scales larger than T_2 size, you land on 2d Principle Chiral Model.

PCM is asymptotically free and gapped in 2d. This does not look anything like what we obtained at small $T_2 \times R_2$.

But since we are considering theory in $U(1)_B$ background, we must couple it baryon (Skyrmion) current, and this changes the story.

What happens at large T2 x R2? Chiral Lagrangian perspective

U(1)$_B$ background must be coupled to baryon (Skyrmion) current:

$$J_{\mathrm{B}} = \frac{1}{24\pi^2}\mathrm{tr_f}[(U^\dagger \mathrm{d}U)^3]$$

$$\int_{M_4} A_{\mathrm{B}} \wedge J_{\mathrm{B}} = \int_{M_5} \mathrm{d}A_{\mathrm{B}} \wedge J_{\mathrm{B}}.$$

$$\int_{M_3 \times T^2} \mathrm{d}A_{\mathrm{B}} \wedge J_B = \frac{1}{12\pi}\int_{M_3} \mathrm{tr_f}[(U^\dagger \mathrm{d}U)^3] =: \Gamma_{\mathrm{WZW}}[U]$$

Level-1 WZW. Perfect match between microscopic QCD analysis and macroscopic chiral Lagrangian analysis.

Outlook

- All of the known non-trivial strong coupling (confinement, chiral S.B., multi-branch structure etc.) phenomena can be continuously connected to weak coupling!

- All of the above are NP phenomena, controlled by $\exp[-8\pi^2/(g^2N)]$ effects that can take place both at weak and strong coupling.

- Two genuinely different confinement mechanisms in two reliable semi-classical regimes. Monopole-instantons or magnetic bions on $R_3 \times S_1$ (not discussed in this talk) vs. center vortex on $R_2 \times T_2$. A quite interesting puzzle!

- Since everything matches to strong coupling expectations, it is impossible not to speculate that the semi-classical basis (fractional instanton saddles and critical points at infinity) may actually be a complete basis in the sense of resurgence. This is a quite intriguing possibility.