Non-Invertible Symmetry, Duality, and Anomalies

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References and Collaborators

"Non-Invertible Duality Defects in 3+1 Dimensions" [Choi-CC-Hsin-Lam-Shao]

"Non-Invertible Condensation, Duality, and Triality Defects in 3+1 Dimensions"

"Quantum Duality in Electromagnetism and the Fine-Structure Constant" [CC-Ohmori]

"Obstructions to Gapped Phases from Non-Invertible Symmetries" [Apte-CC-Lam]

"Anomalies of (1+1)*D* categorical symmetries" [CC-Zhang]

Key papers by others:

Defects: [Kaidi-Ohmori-Zheng, Niro-Roumpedakis-Sela, Kapustin-Tikhanov, Bhardwaj-Schafer-Nameki]

Anomaly: [Thorngren-Wang, Freed-Moore-Teleman, Kaidi-Ohmori-Zheng, Kaidi-Nardoni-Zafrir-Zheng]

Non-Invertible Symmetry

Symmetry Beyond Groups

Symmetries are characterized by commuting with Hamiltonian

Modern view: topological operator $\mathcal{O}(\Sigma)$ (generalizes current conservation)

Typically, represented by unitary matrices with invertible or group−like fusion:

$$\mathcal{O}_h(\Sigma)\times\mathcal{O}_g(\Sigma) = \mathcal{O}_{hg}(\Sigma)$$

In the last few years, more general algebraic structure realized in many QFT:

$$\mathcal{O}_i(\Sigma)\times\mathcal{O}_j(\Sigma) = \sum_k C_{ij}^k \times\mathcal{O}_k(\Sigma)$$

General symmetry is a topological operator with no inverse: **non-invertible**

Self-Duality and Symmetry

One class of examples arises via duality. Typically duality gives identifications among the possible coupling constants or classical presentations of a theory

In special cases one can have a self-duality: fixed point under identification

The fixed point enjoys extra symmetry, often non-invertible. Simple examples:

- Kramers-Wannier self-duality of the critical 2d Ising model
- Electric-Magnetic self-duality of 4d Maxwell theory at coupling $\frac{e^2}{2\pi} \in \mathbb{Q}$
- $\mathcal{N} = 4,\ SU(N)$ Super Yang-Mills at $\tau = i$. (Non-Abelian EM duality)

Self-Duality in Electromagnetism

[Choi-CC-Hsin-Lam- Shao, Niro-Roumpedakis-Sela, CC-Ohmori]

Higher Symmetry and Duality in EM

Maxwell theory has $U(1)^{(1)} \times U(1)^{(1)}$ 1-form global symmetry that acts on lines

We can couple to two-form background gauge fields $B_e^{(2)}$ and $B_m^{(2)}$

$$S = \frac{1}{2e^2}\int \left(F - B_{\mathrm{e}}^{(2)}\right)^2 + \frac{i}{2\pi}\int F \wedge B_m^{(2)}$$

[Gaiotto-Kapustin Seiberg-Willett]

The symmetry is anomalous with 5d inflow action

$$S = \frac{i}{2\pi}\int B_e^{(2)} \wedge dB_m^{(2)}$$

This anomaly means that we cannot simultaneously gauge $B_e^{(2)}$ and $B_m^{(2)}$

Higher Symmetry and Duality in EM

Something special happens if we restrict to the subgroup:

$$\mathbb{Z}_{N_e}^{(1)} \times \mathbb{Z}_{N_m}^{(1)} \subset U(1)^{(1)} \times U(1)^{(1)}, \qquad \gcd(N_e, N_m) = 1$$

The anomaly vanishes and gauging the one-form symmetry yields:

$$\int \frac{F}{2\pi} = \frac{N_m}{N_e}\, k, \quad k \in \mathbb{Z}$$

Rescalling the gauge field, this is equivalent to an action on the coupling:

gauging: $\mathbb{Z}_{N_e}^{(1)} \times \mathbb{Z}_{N_m}^{(1)}$: $\qquad e \mapsto \left(\frac{N_e}{N_m}\right) e$

Higher Symmetry and Duality in EM

We can combine with the action of S -duality

$$S: \qquad \frac{e^2}{2\pi} \leftrightarrow \frac{2\pi}{e^2} \ , \qquad \frac{F}{2\pi} \leftrightarrow \frac{i * F}{e^2} \ .$$

Composite: $S \circ$ gauging $\mathbb{Z}_{N_e}^{(1)} \times \mathbb{Z}_{N_m}^{(1)}$, has fixed point at rational coupling:

$$\frac{e^2}{2\pi} = \frac{N_m}{N_e}$$

At this coupling, Maxwell theory has new non-invertible symmetry $\mathcal{D}_{N_e,N_m}$

Approximate Symmetry of Nature

IR of Standard Model is Maxwell theory with (presumably irrational) coupling:

$$\frac{e^2}{2\pi} \approx \frac{2}{137.0359908}$$

Truncating to a fixed precision yields approximate symmetries of our world

e.g. taking $N_m = 2,\ N_e = 137$ yields a nearly topological operator, where the energy momentum tensor is nearly continuous across the defect

These symmetries are broken by the coupling to charged matter, i.e. leptons and quarks. They are approximate below the mass scale of these species

Defect Worldvolume Theory

The properties of the defect can be deduced from its worldvolume action

Intuitively given by response to bulk fields A_L and A_R on each side of defect

$$S \sim \left(\frac{i}{2\pi}\right)\frac{N_e}{N_m}\int A_L \wedge dA_R$$

To understand more precisely, unfold using defect gauge fields, a, b

$$S = \frac{i\,N_m}{2\pi}\int a \wedge db - \frac{i\,N_e}{2\pi}\int a \wedge dA_L + \frac{i}{2\pi}\int b \wedge dA_R$$

Defect has $\mathbb{Z}_{N_m}$ topological theory coupling to bulk via one-form symmetry

In leading approx to fine-structure constant: $\mathbb{Z}_2$ gauge theory i.e. toric code

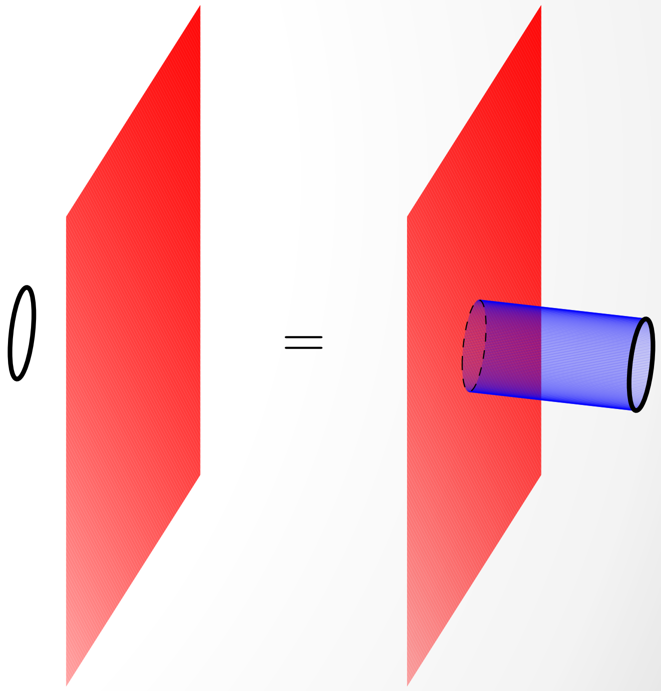
Aspects of Symmetry

$\mathcal{D}_{N_e,N_m}$ acts by dragging topological defect across an operator

On local operators, dictated by electric-magnetic duality $\mathcal{D}(F) = i * F$

On extended operators the action is richer

Sweeping $\mathcal{D}$ past a line operator (dyon) attaches a topological surface, i.e. the charge fractionalizes



$$\mathcal{D}_{N_e,N_m}(q_e, q_m) = \left(\frac{q_m\, N_e}{N_m}, -\frac{q_e\, N_m}{N_e}\right)$$

Order-disorder transformation is ubiquitous for non-invertible symmetry

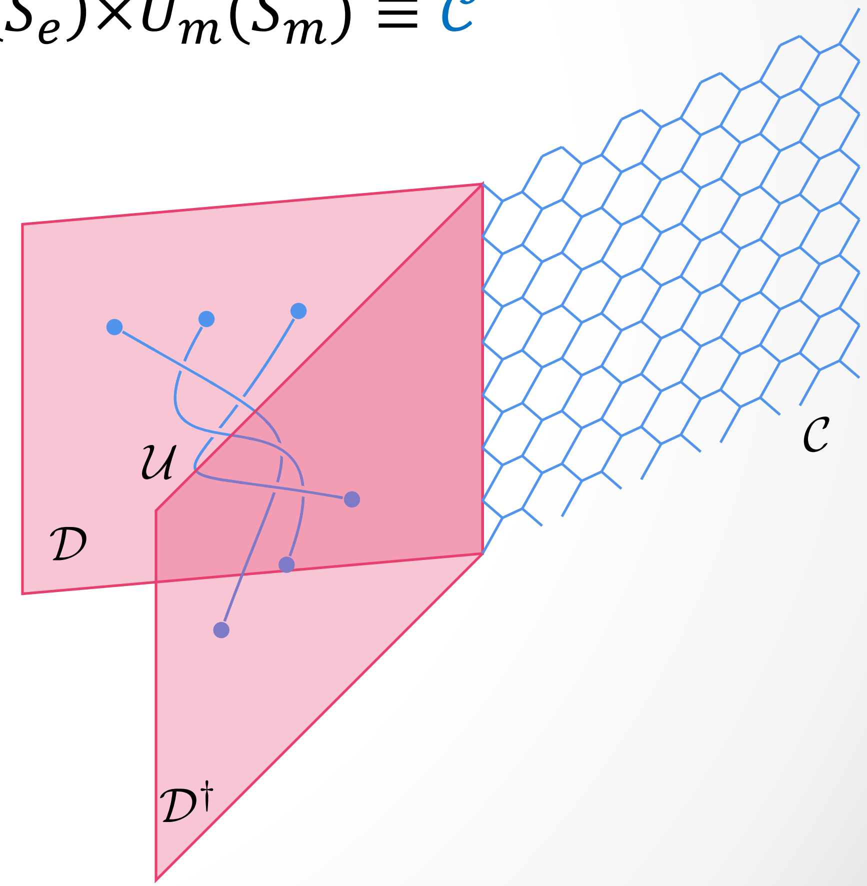
Aspects of Symmetry

The non-invertible nature can be seen in the defect fusion rules

$$\mathcal{D}(\Sigma) \times \mathcal{D}^{+}(\Sigma) = \sum_{S_i \in H_2(\Sigma, \mathbb{Z}_{N_i})} U_e(S_e) \times U_m(S_m) \equiv \mathcal{C}$$

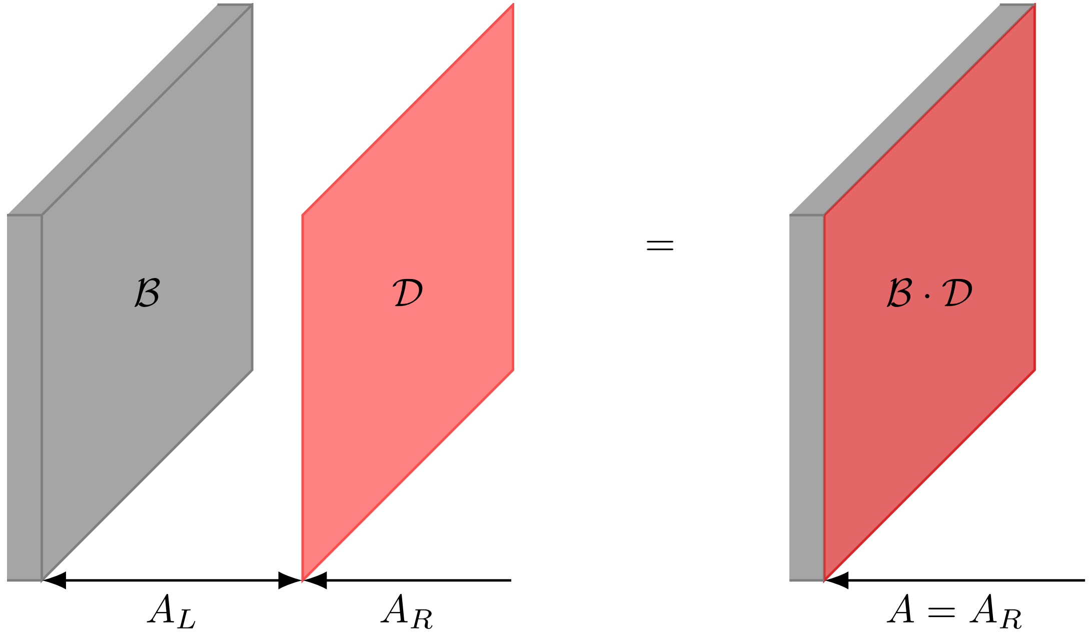
$\mathcal{C}$ is a condensation of the $\mathbb{Z}_{N_e}^{(1)} \times \mathbb{Z}_{N_m}^{(1)}$ one form symmetry defects U

Trivial on local operators but can annihilate extended operators (non-invertible)



Aspects of Symmetry

The defect acts on boundaries. Colliding $\mathcal{D}$ with boundary $\mathcal{B}$ produces $\mathcal{B} \cdot \mathcal{D}$



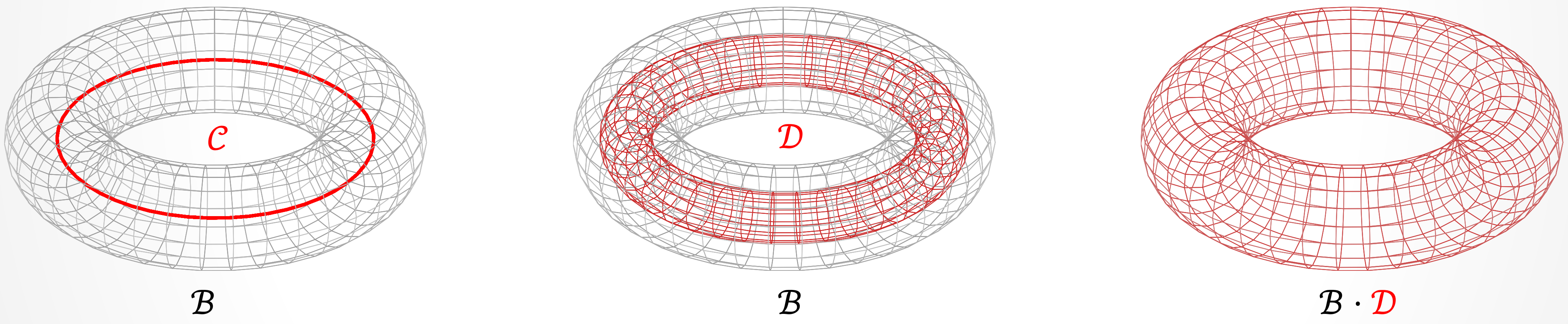
For instance, if $\mathcal{B}$ is an electric conductor (Dirichlet) $F| = 0$

$\mathcal{B} \cdot \mathcal{D}$ magnetic conductor (Neumann) $* F| = 0$ + boundary $\mathbb{Z}_{N_m}$ gauge theory

Hilbert Spaces

$\mathcal{D}$ relates spectra. Non-trivial with topology so energies depend on coupling

Consider a toroidal cavity with boundary $\mathcal{B}$ and instertion of the one-form condensation defect $\mathcal{C}$. Via fusion rules, replace with $\mathcal{B} \cdot \mathcal{D}$ and no insertion



fractional flux on disc ↔ momentum for holonomy of A

Boundary anyon state fixes effective θ-angle seen by the holonomy variable

Dynamical Constraints

[Choi-CC-Hsin-Lam- Shao, Apte-CC-Lam, CC-Zhang]

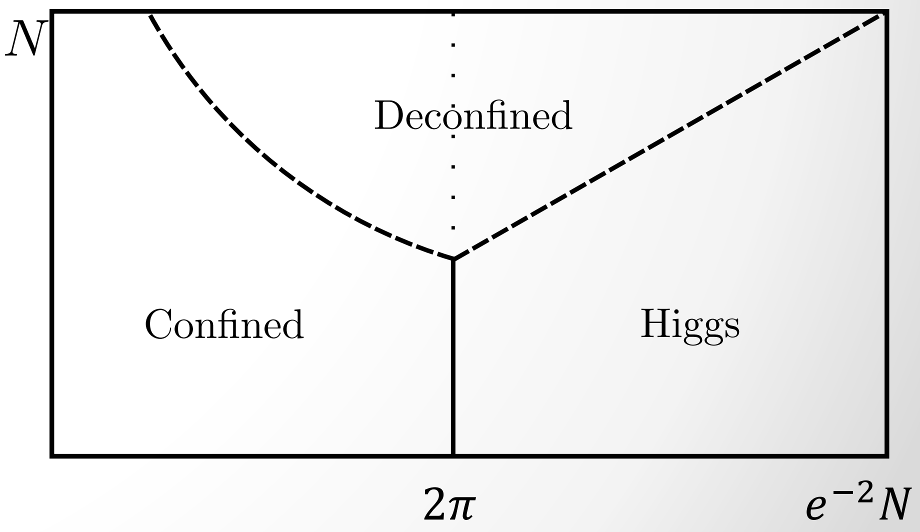
RG Flows with Duality Symmetry

Non-invertible symmetry constrains dynamics in models with $\mathcal{D}$ present in UV
RG flow must preserve $\mathcal{D}$ and any associated invariants, i.e. anomalies

An example is $\mathbb{Z}_N$ lattice gauge theory (Villain formulation). At a special coupling, this theory is invariant under gauging its electric $\mathbb{Z}_N^{(1)}$ symmetry

For small N there is a first order phase transition between a trivial vacuum and $\mathbb{Z}_N$ TQFT.

For large N there is a gapless window and the self-dual theory is a Coulomb phase with $\frac{e^2}{2\pi} = \frac{1}{N}$



RG Flows with Duality Symmetry

From the viewpoint of symmetry we describe this as follows:

- The first order transition is governed by spontaneously breaking of $\mathcal{D}$.

  The Higgs and confining phases differ by gauging the $\mathbb{Z}_N^{(1)}$ symmetry

- The vacuum of the gapless Coulomb phase preserves the symmetry $\mathcal{D}$

  There are massive particle excitations in representations of the duality

  These are magnetic monopoles and charge N electrons with equal mass

Higher Landau Paradigm

More generally want to constrain phases of QFT with non-invertible symmetry

Beyond gapless, and spontaneous symmetry breaking phases, also contemplate a symmetry preserving gapped vacuum i.e. a TQFT

For $\mathcal{D}$ with $N_e = N,$ and $N_m = 1$ there are known constraints:

- Trivially gapped possible if N Pythagorean (-1 is quadratic residue mod N)
- Symmetry preserving gapped phase possible if $N = k^2 \times$ Pythagorean

Allowed N:

SPT	2, 5, 10, 13, 17, 25, 26, 29, 34, 37, 41, 50, 53, 58, ...
TQFT	4, 8, 9, 16, 18, 20, 25, 32, 36, 40, 45, 49, 50, 52, ...

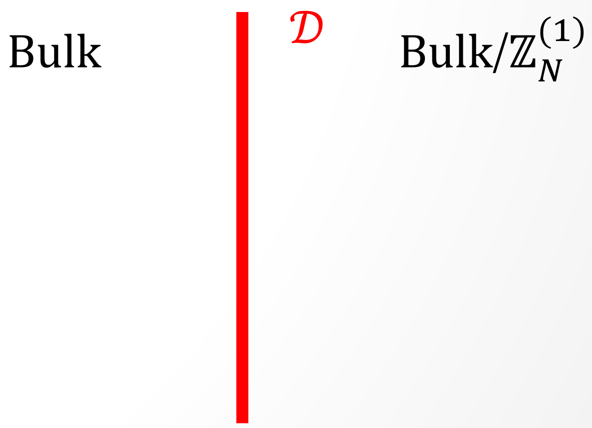
Duality Defects and Invertible Phases

One way to argue for constraints on N is to examine the worldvolume of $\mathcal{D}$

If the bulk is invertible (trivially gapped), $\mathcal{D}$ is a well-defined 3d TQFT

This 3d theory has $\mathbb{Z}_N^{(1)}$ symmetry and is described by a minimal abelian TQFT

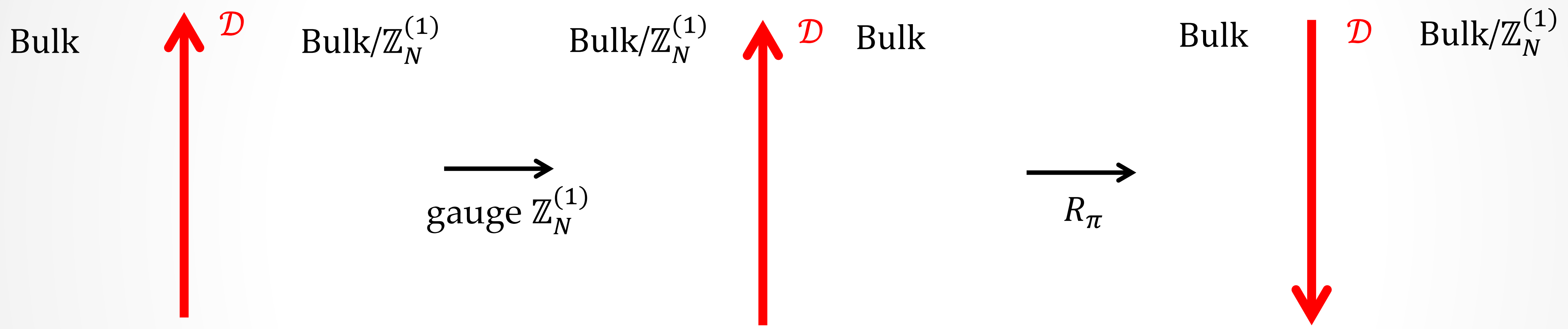
- $\mathbb{Z}_N$ abelian anyon fusion algebra generated by a line ℓ, with $\ell^N = 1$
- spins characterized by $\mathrm{h}(\ell^r) = \frac{p\, r^2}{2N}$



Minimal abelian TQFT $\mathcal{A}^{N,p}$

Duality Defects and Invertible Phases

We can gauge the $\mathbb{Z}_N^{(1)}$ and rotate by π (CPT transformation):



Invariant except that $\mathcal{D}$ has reversed orientation. This means that $\mathcal{D}$ has time reversal symmetry T. With $T^2 = C$ (charge conjugation)

This is only possible if N Pythagorean [Delmastro-Gomis]

Duality Defects and Gapped Phases

Gapped symmetry preserving phases may be similarly constrained

Want a unique local operator so there is a unique local ground state

For example let $N = k^2$, and consider $\mathbb{Z}_k$ topological gauge theory.
The action coupled to a background field $B^{(2)}$ for the $\mathbb{Z}_N^{(1)}$ symmetry is

$$S\left[B^{(2)}\right] = \frac{i\,k}{2\pi}\int a^{(1)} \wedge db^{(2)} + \frac{i\,N}{2\pi}\int b^{(2)} \wedge B^{(2)}$$

And is invariant under gauging $B^{(2)}$. So this theory has $\mathcal{D}$ symmetry

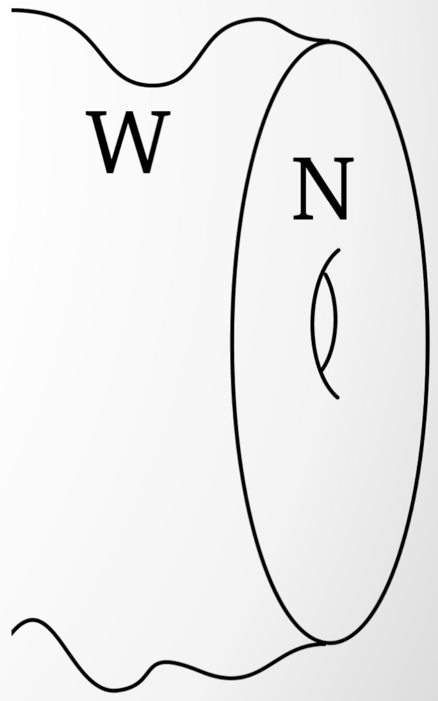
More generally can prove this only possible for $N = k^2 \times$ Pythagorean

Anomalies of Non-Invertible Symmetry

A frontier of progress is to understand anomalies for general symmetry

For standard (invertible) symmetries the paradigm is anomaly inflow:

- d-dimensional spacetime N is boundary of W, dimension d+1
- Background fields for global symmetries extend from N to W
- On W there is a classical (invertible) field theory, this is the anomaly (generalized Chern-Simons theory)
- The anomaly is inert because all dynamics are on N



Anomalies of Non-Invertible Symmetry

For non-invertible symmetries the theory on W upgraded to a non-trivial TQFT

[Kulp-Gaiotto, Gaiotto-Kapustin-Seiberg-Willett, Freed-Moore-Teleman,]

Given a TQFT we then want to study what boundaries are possible

An obstruction to an invertible, trivially gapped boundary, is an anomaly

If spacetime N is 2d, symmetry described by fusion category $\mathcal{A}$ [Thorngren-Wang CC-Zhang]

The topological theory in 3d is the Drinfeld center $Z[\mathcal{A}]$, and anomalies are obstructions to the existence of certain kinds of Lagrangian algebras

Generalization to higher d, involves higher fusion categories and TQFTs

Conclusions

- Non-Invertible symmetries occur in many familiar QFTs
- They imply selection rules and their anomalies constrain RG flows

Future topics:

- Full scope of non-invertible symmetry unclear. Which models have them?
- Can we give a complete characterization of their anomalies?
- Are non-invertible symmetries useful for explaining hierarchies in nature?