The Virasoro Minimal String

Strings 2023

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Based on work in collaboration with Scott Collier, Beatrix Mühlmann and Victor Rodriguez

2d Gravity/Matrix integral duality

Gravity

(2,p) Minimal String

↓ Roughly $p \to \infty$

JT gravity

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Gravity

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Roughly $p \to \infty$

JT gravity

Matrix integral

$$\rho_0(E) = \sinh\left(\frac{p}{2}\,\mathrm{arccosh}(1+E)\right)$$

Brezin, Kazakov '90,
Gross, Migdal '90,
Douglas, Shenker '90, ...

$$\rho_0(E) = \sinh(\sqrt{E})$$

Saad, Shenker, Stanford '19

2d Gravity/Matrix integral duality

Gravity

(2,p) Minimal String

↓ Roughly $p \to \infty$

JT gravity

↑ $b \to 0$

Virasoro Minimal String

⟷

Matrix integral

$$\rho_0(E) = \sinh\left(\frac{p}{2}\,\mathrm{arccosh}(1+E)\right)$$

Brezin, Kazakov '90, Gross, Migdal '90, Douglas, Shenker '90, ...

↓

$$\rho_0(E) = \sinh(\sqrt{E})$$

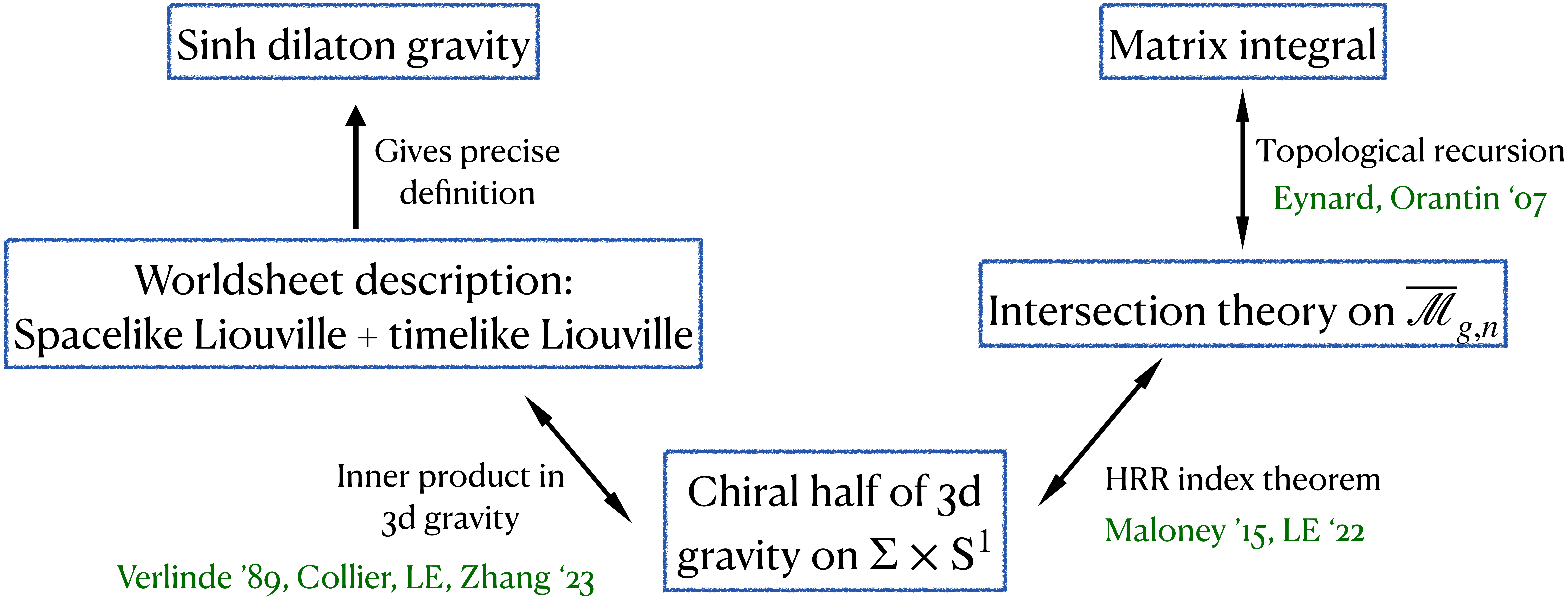
Saad, Shenker, Stanford '19

↑

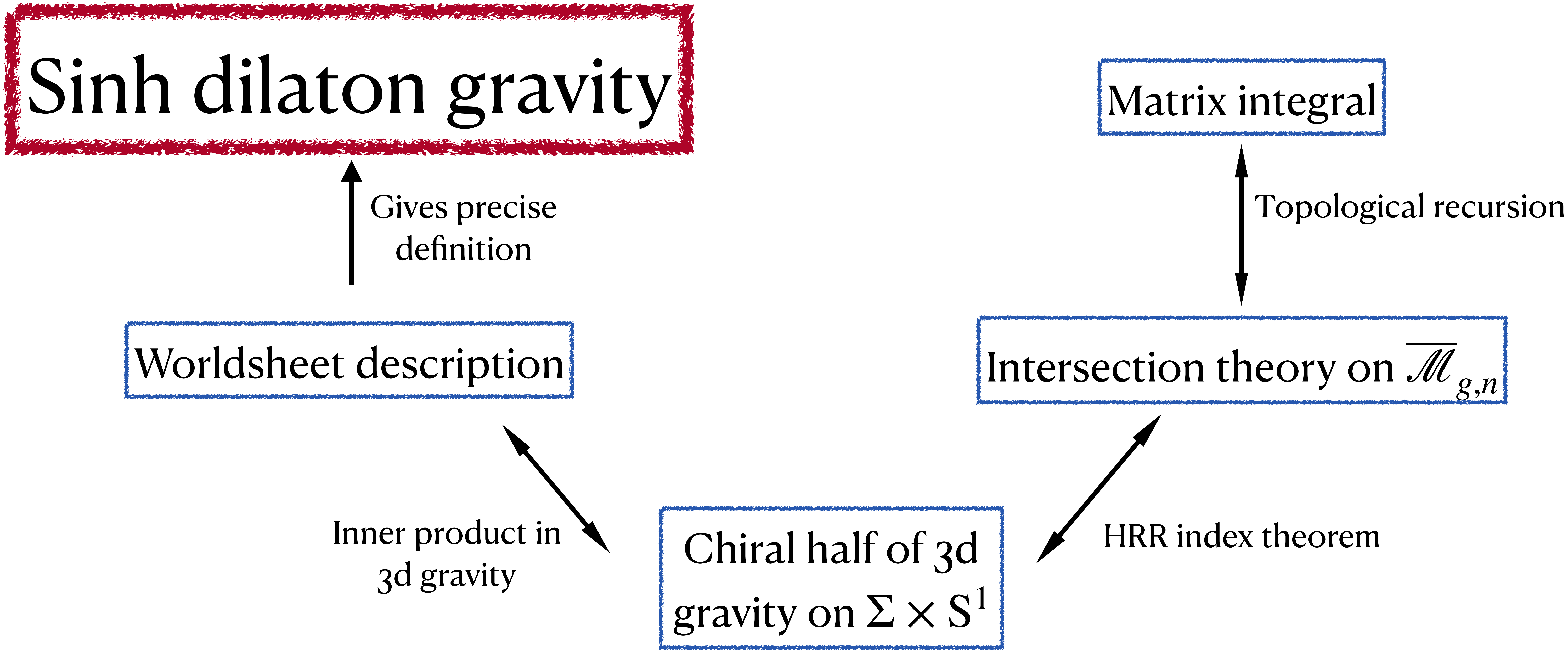
$$\rho_0^{(b)}(E) = \frac{\sinh(b\sqrt{E})\sinh(b^{-1}\sqrt{E})}{\sqrt{E}}$$

Collier, LE, Mühlmann, Rodriguez

Cornerstones



$\Rightarrow$ Gives a complete proof of the correspondence + many additional direct checks


Sinh dilaton gravity
Gives precise definition
Worldsheet description
Inner product in 3d gravity
Chiral half of 3d gravity on $\Sigma \times S^1$
HRR index theorem
Intersection theory on $\overline{\mathscr{M}}_{g,n}$
Topological recursion
Matrix integral

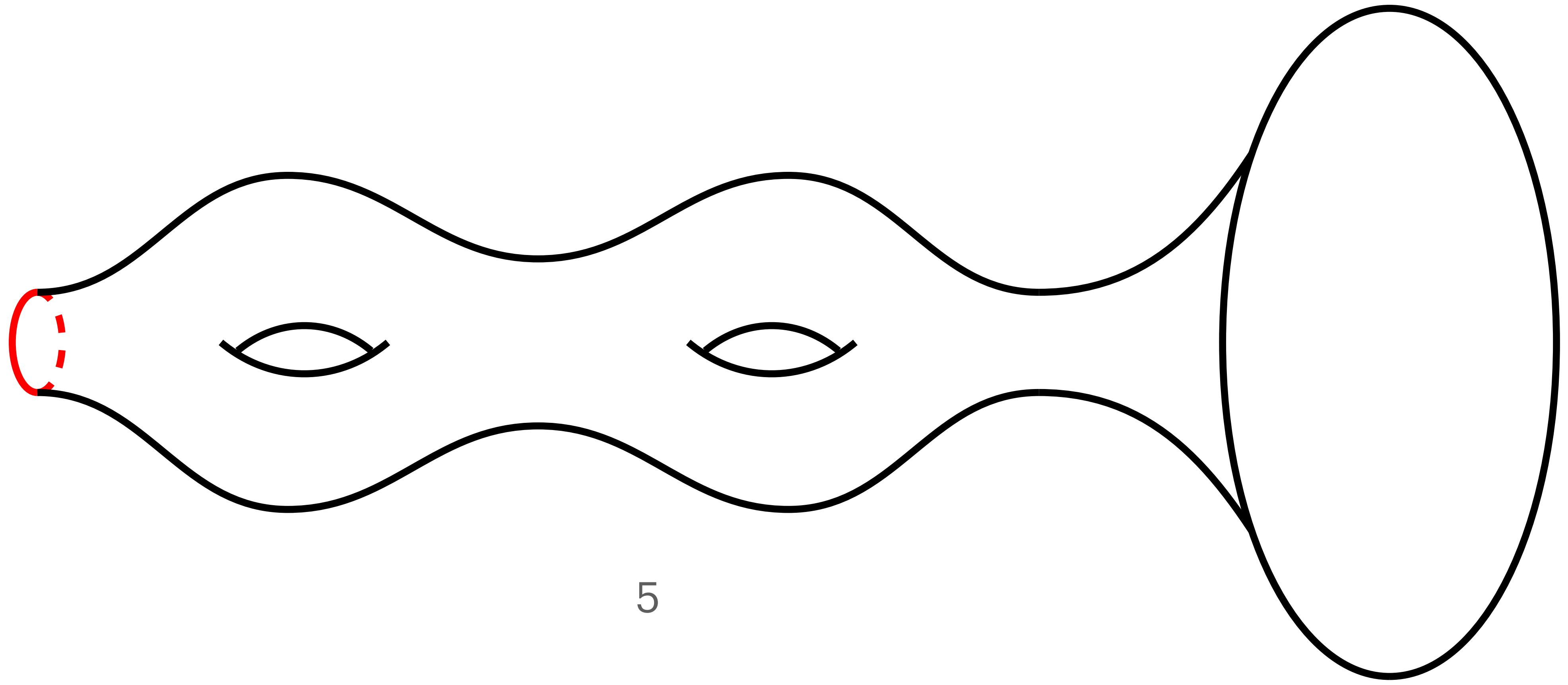
The Virasoro String as a 2d gravity theory

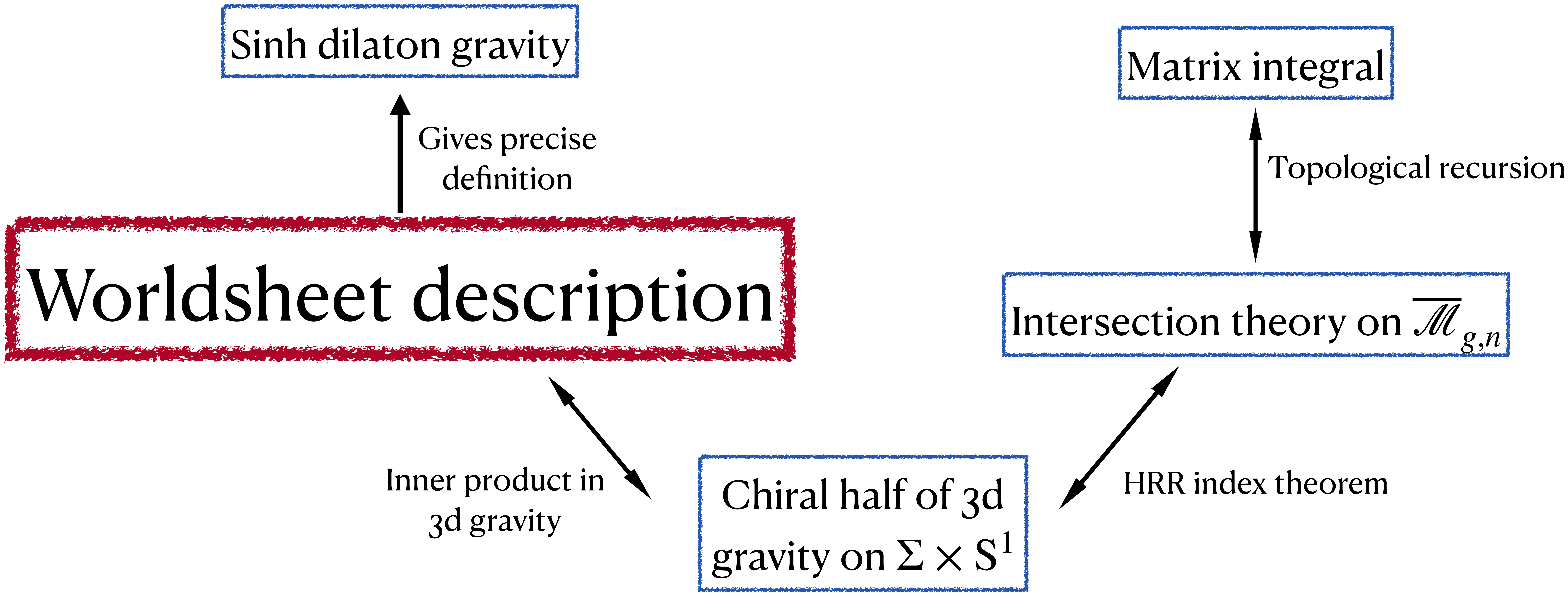
- The simplest way to define the bulk theory is as a dilaton gravity:

$$S = -\frac{1}{2}\int_\Sigma \mathrm{d}^2x\sqrt{g}\,(\Phi\mathscr{R} + W(\Phi)) - \int_{\partial\Sigma}\sqrt{h}\Phi K + \text{Euler term}\ , \qquad W(\Phi) = \frac{\sinh(2\pi b^2\Phi)}{\sin(\pi b^2)}$$

Mertens, Turiaci ’20, Suzuki Takayanagi ’21, Fan, Mertens ‘21

- $b \to 0$: JT-gravity
- Can consider the theory on surfaces with geodesic and/or asymptotic boundaries





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Change of Variables

- Linear combination of the Weyl factor and the dilaton rewrites the action as a sum

$$\text{Spacelike Liouville theory } c = 1 + 6(b + b^{-1})^2$$
$$+$$
$$\text{Timelike Liouville theory } \hat{c} = 1 - 6(b - b^{-1})^2$$

Seiberg, Stanford, Mertens, Turiaci '20

- This defines a critical string theory and can be taken as a more well-defined definition of the theory
- We will **NOT** take the second factor to be a minimal model!

Timelike Liouville theory

- Well-defined, non-unitary, modular invariant and crossing symmetric CFT with $\hat{c} \leq 1$ and spectrum

$$\hat{\Delta} = \frac{\hat{c}-1}{24} + \hat{P}^2 \geq \frac{\hat{c}-1}{24}$$

Zamolodchikov '05, Kostov, Petkova '05,
Harlow, Maltz, Witten '11,
Ribault, Santachiara '15, Ribault, Tsiares WIP

- Correlation functions

$$\left\langle \prod_{j=1}^{n} \hat{V}_{\hat{P}_j}(z_j) \right\rangle_g$$

are analytic in $\hat{P}_i$ and can trivially be analytically continued to imaginary values of $\hat{P}_i$ (no OPE contour deformation necessary)

Bautista, Dabholkar, Erbin '19

The worldsheet theory

- Parametrize

$$c = 1 + 6(b + b^{-1})^2 \,, \qquad \hat{c} = 1 - 6(b - b^{-1})^2$$
$$\Delta_i = \frac{c-1}{24} + P_i^2 \,, \qquad \hat{\Delta}_i = \frac{\hat{c}-1}{24} - P_i^2$$

so that $c + \hat{c} = 26$ and $\Delta_i + \hat{\Delta}_i = 1$.

- Can define the string theory correlators

$$\mathsf{V}^{(b)}_{g,n}(P_1, \ldots, P_n) = \int_{\mathcal{M}_{g,n}} \left\langle \prod_{j=1}^{n} V_{P_j}(z_j) \right\rangle_g \left\langle \prod_{j=1}^{n} \hat{V}_{\hat{P}_j = iP_j}(z_j) \right\rangle_g \times \text{ghosts}$$

``quantum volumes''

⇒ Absolutely convergent integral!

Numerical evaluation

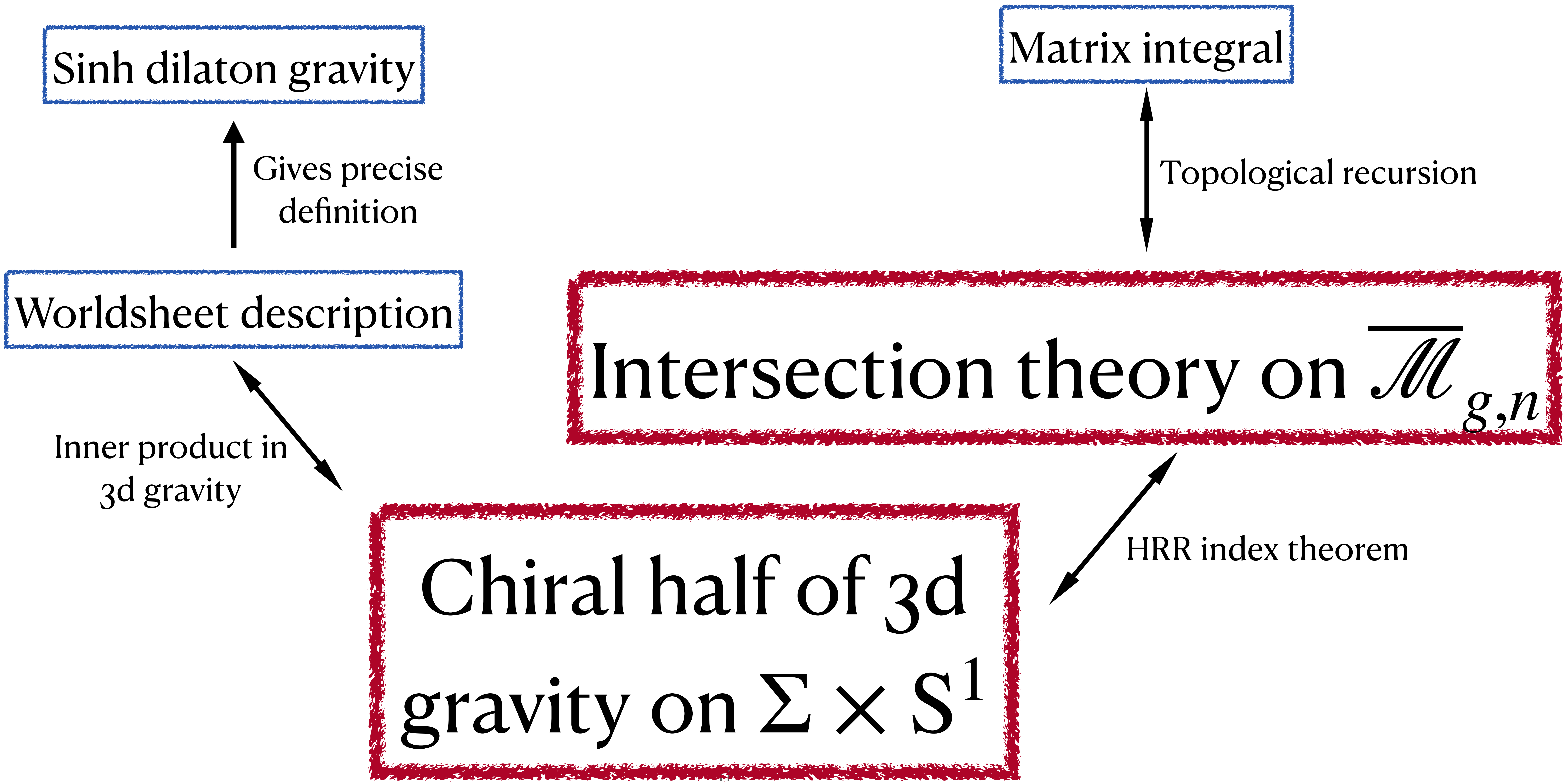
- One can directly put this definition on a computer for low g and n.
- Result (error $< 10^{-3}\,\%$):

$$\mathsf{V}^{(b)}_{0,4}(P_1, P_2, P_3, P_4) = \frac{c-13}{24} + P_1^2 + P_2^2 + P_3^2 + P_4^2$$

$$\mathsf{V}^{(b)}_{1,1}(P_1) = \frac{1}{24}\left(\frac{c-13}{24} + P_1^2\right)$$

See Rodriguez '23 for the case $c = 25$

- Surprisingly simple!
- $b \to 0,\ P_i = \dfrac{\ell_i}{4\pi b}$: Weil-Petersson volumes of moduli space, up to normalization of the measure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Chiral half of 3d gravity on $\Sigma \times S^1$

- The key to understand the dual matrix model is the relation to 3d gravity (more precisely the Virasoro TQFT).
- Inner product on the Hilbert space of half of 3d gravity:

Verlinde '89 Collier, LE, Zhang '23

$$\left\langle \mathcal{F}^{(1)}_{g,n} \,\middle|\, \mathcal{F}^{(2)}_{g,n} \right\rangle = \int_{\mathcal{T}_{g,n}} \overline{\mathcal{F}^{(1)}_{g,n}} \mathcal{F}^{(2)}_{g,n} \left\langle \prod_{j=1}^{n} \hat{V}_{iP_j}(z_j) \right\rangle \times \text{ghosts} = \frac{\prod_j \delta(P_j^{(1)} - P_j^{(2)})}{\prod C_b(P_i^{(1)}, P_j^{(1)}, P_k^{(1)}) \prod \rho_0^{(b)}(P_j^{(1)})}$$

Teichmüller space

DOZZ 3-point function

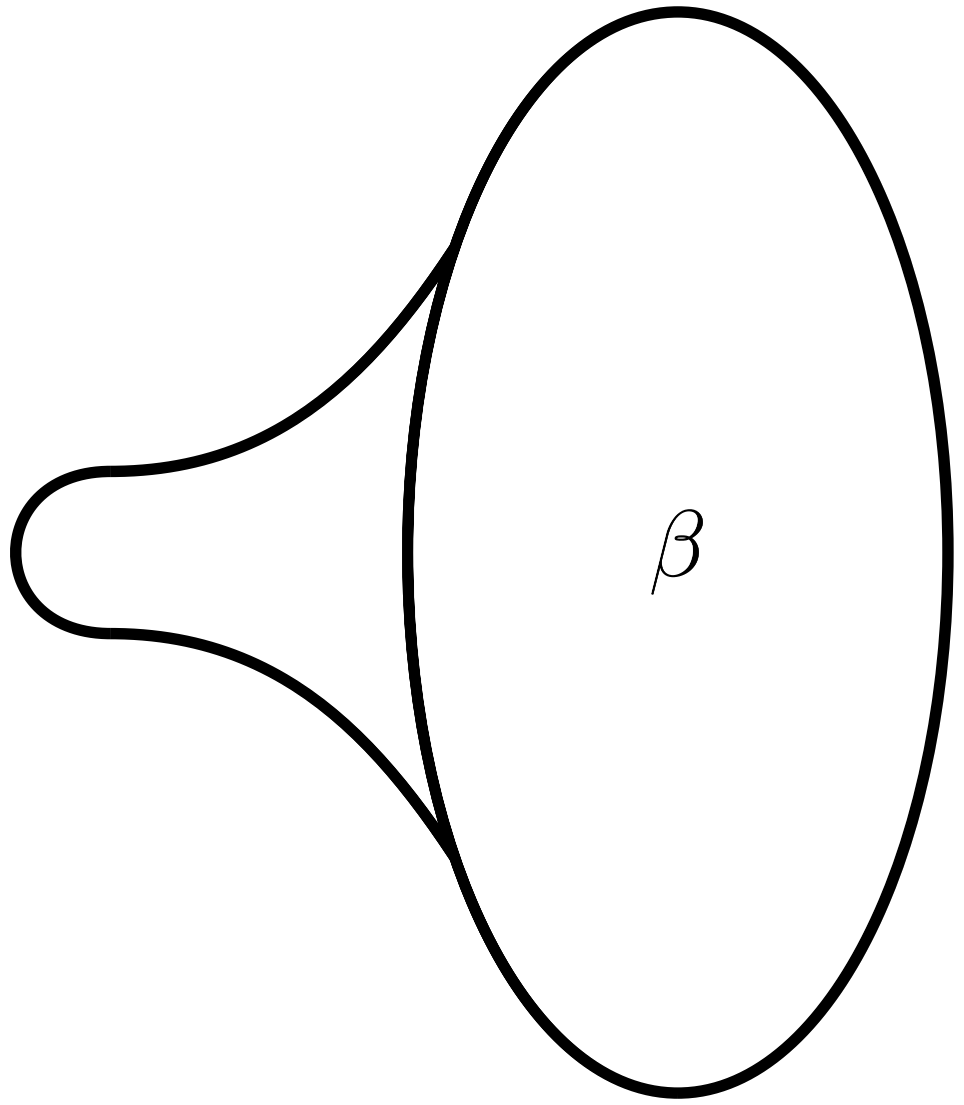
Liouville 2-point function normalization

$$\Longrightarrow \mathsf{V}^{(b)}_{g,n}(P_1, \ldots, P_n) = Z_{\text{chiral 3d gravity}}(\Sigma_{g,n} \times S^1)$$

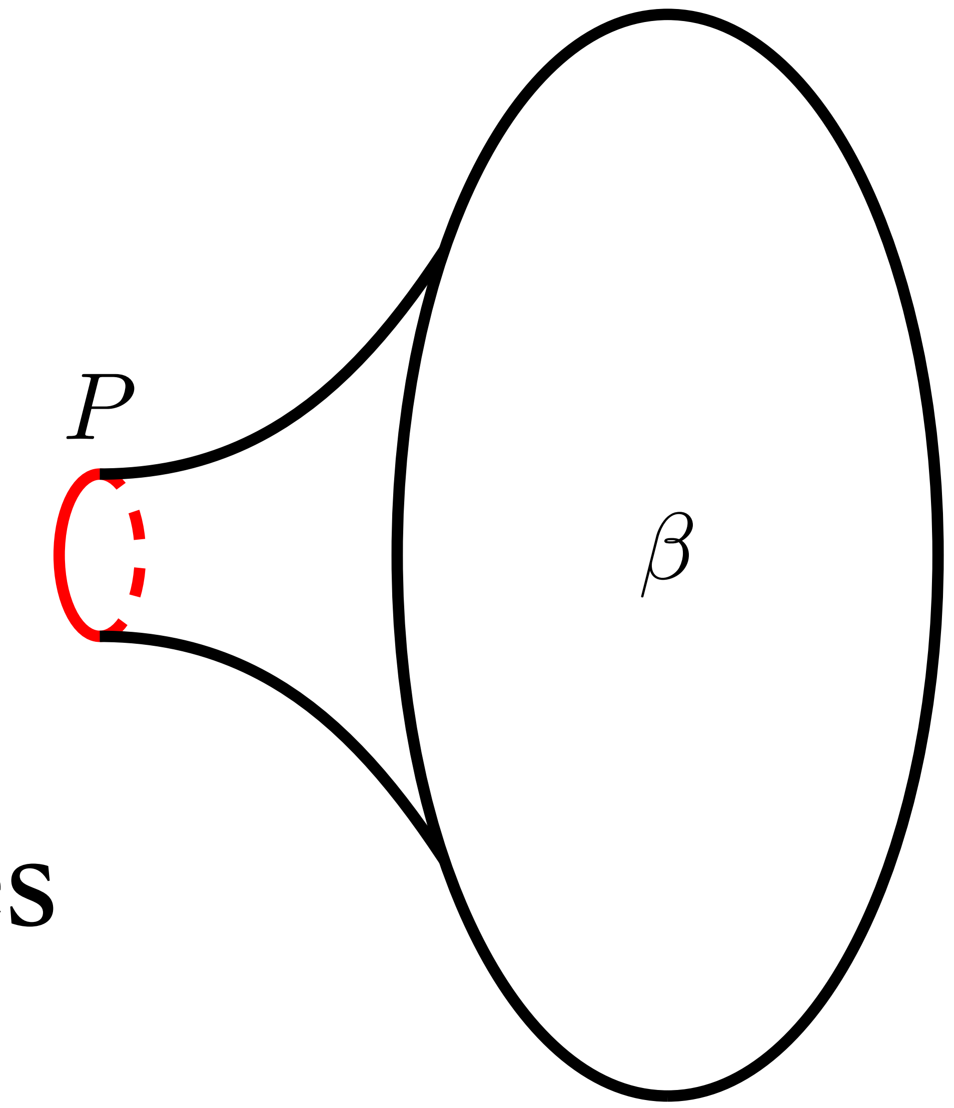
The disk and trumpet

- From 3d gravity:

$$\mathcal{Z}^{(b)}_{\text{disk}}(\beta) = \mathrm{e}^{\frac{\pi^2 c}{6\beta}} \prod_{n=2}^{\infty} \frac{1}{1-\mathrm{e}^{-\frac{4\pi^2 n}{\beta}}} = \frac{1}{\eta\left(\frac{\beta i}{2\pi}\right)} \underbrace{\sqrt{\frac{2\pi}{\beta}} \left(\mathrm{e}^{\frac{\pi^2 (b+b^{-1})^2}{\beta}} - \mathrm{e}^{\frac{\pi^2 (b-b^{-1})^2}{\beta}}\right)}_{Z^{(b)}_{\text{disk}}(\beta)}$$



$$\mathcal{Z}^{(b)}_{\text{trumpet}}(\beta, P) = \mathrm{e}^{-\frac{4\pi^2}{\beta}\left(P^2 - \frac{1}{24}\right)} \prod_{n=1}^{\infty} \frac{1}{1-\mathrm{e}^{-\frac{4\pi^2 n}{\beta}}} = \frac{1}{\eta\left(\frac{\beta i}{2\pi}\right)} \underbrace{\sqrt{\frac{2\pi}{\beta}}\, \mathrm{e}^{-\frac{4\pi^2 P^2}{\beta}}}_{Z^{(b)}_{\text{trumpet}}(\beta, P)}$$



- The dual matrix model only captures the partition function of primaries

Quantization of $\mathscr{M}_{g,n}$

- The Hilbert space of chiral 3d gravity is the quantization of its phase space $\mathscr{M}_{g,n}$
- Partition functions on $\Sigma_{g,n} \times \mathrm{S}^1$ compute the dimension of the 3d gravity Hilbert space and can also be computed from the Hirzebruch-Riemann-Roch index theorem:

$$\mathsf{V}^{(b)}_{g,n}(P_1, \ldots, P_n) = \int_{\overline{\mathscr{M}}_{g,n}} \mathrm{td}(\mathscr{M}_{g,n})\ \mathrm{e}^{\frac{c}{48\pi^2}\omega_{\mathrm{WP}}\left(\ell_i^2 = \frac{96\pi^2}{c}(P_i^2 - \frac{1}{24})\right)}$$

Todd class

Curvature of the (projective) line bundle of conformal blocks

Friedan, Shenker '86

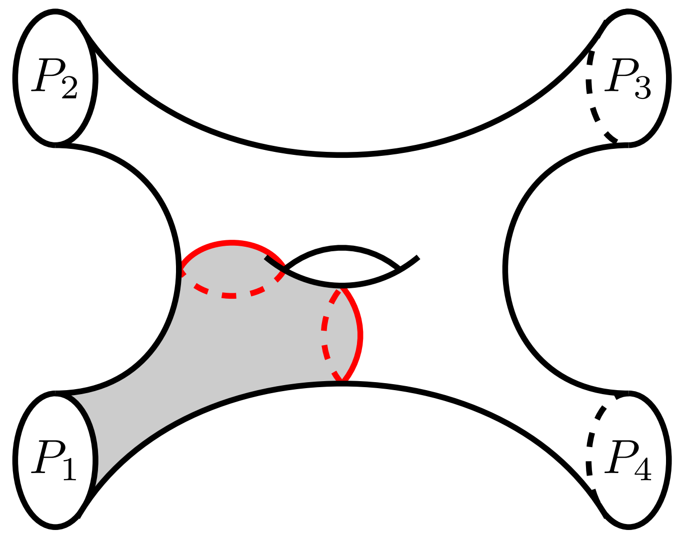
- Reduces the problem to intersection theory on $\overline{\mathscr{M}}_{g,n}$, which is mathematically well-developed

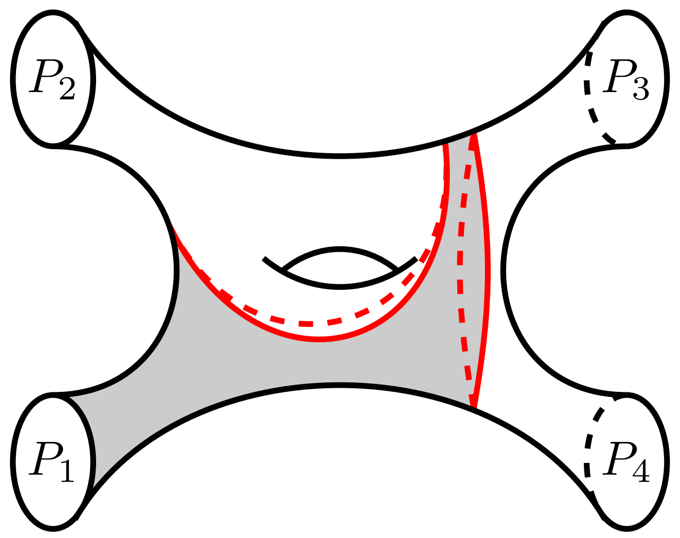
Mumford '83, Witten '90

Deformed Mirzakhani recursion

- Algebraic geometry and topological recursion gives a recursion for $\mathsf{V}^{(b)}_{g,n}(P_1, \ldots, P_n)$:

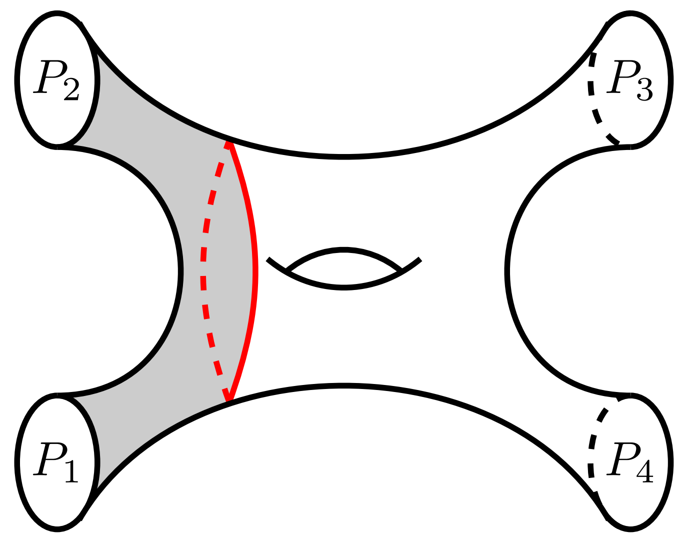
Mirzakhani '06, Eynard, Orantin '07, Eynard '11, LE '22





$$P_1 \mathsf{V}^{(b)}_{g,n}(P_1, \mathbf{P}) = \int_0^\infty (2P\,\mathrm{d}P)\,(2P'\,\mathrm{d}P')\,H(P+P', P_1)\left(\mathsf{V}^{(b)}_{g-1,n+1}(P, P', \mathbf{P}) + \sum_{h=0}^{g} \sum_{I \sqcup J = \{2,\ldots,n\}} \mathsf{V}^{(b)}_{h,|I|+1}(P, \mathbf{P}_I)\mathsf{V}^{(b)}_{g-h,|J|+1}(P', \mathbf{P}_J)\right)$$

$$+ \sum_{i=2}^{n} \int_0^\infty (2P\,\mathrm{d}P)\,\big(H(P, P_1 + P_i) + H(P, P_1 - P_i)\big)\mathsf{V}^{(b)}_{g,n-1}(P, \mathbf{P}\backslash P_i)$$



$$H(x, y) = \frac{y}{2} - \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}t\, \frac{\sin(4\pi t x)\sin(4\pi t y)}{\sinh(2\pi b t)\sinh(2\pi b^{-1} t)}$$

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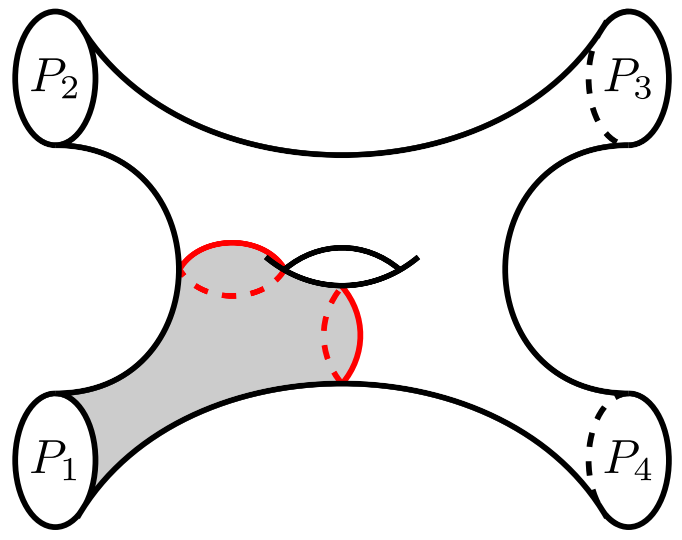

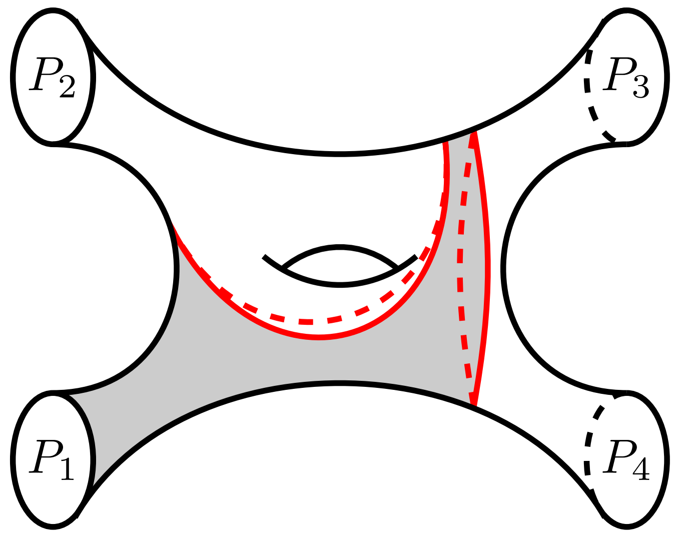

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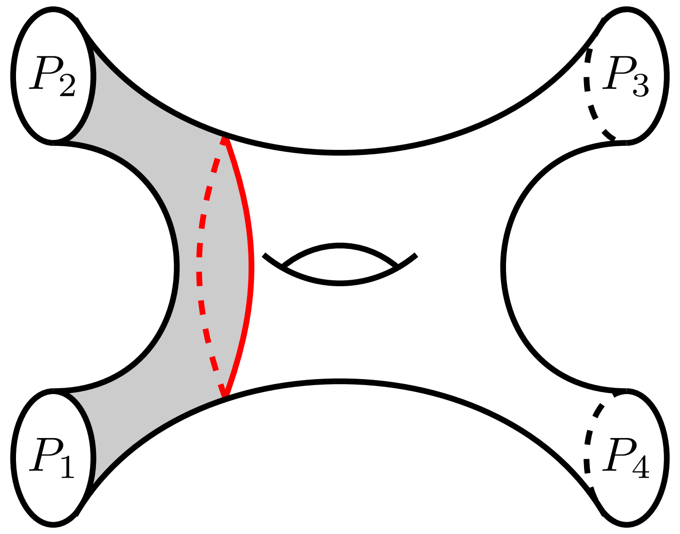

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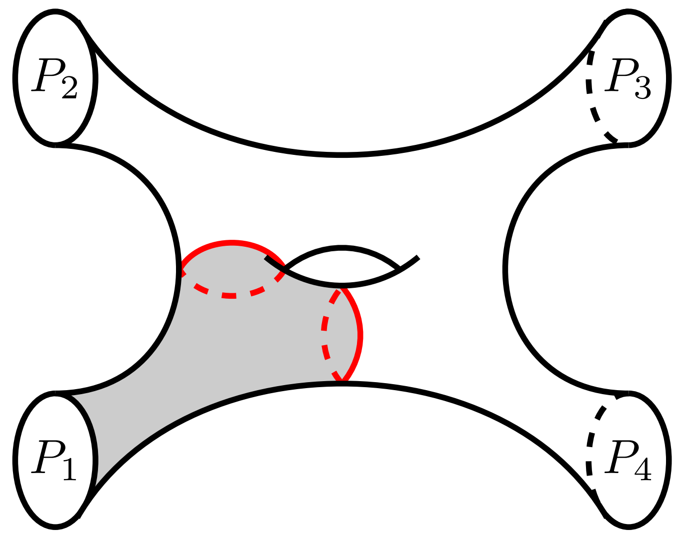

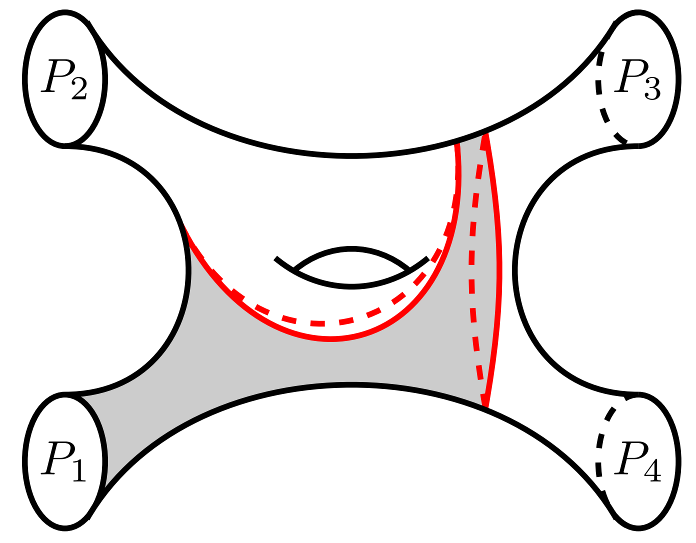

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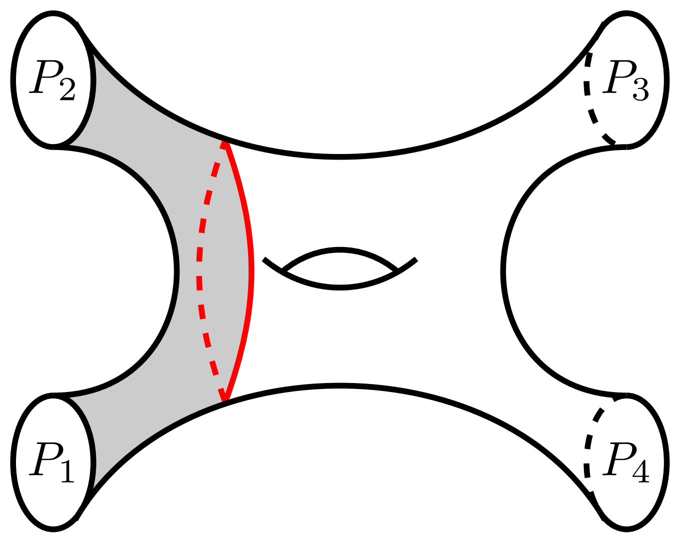

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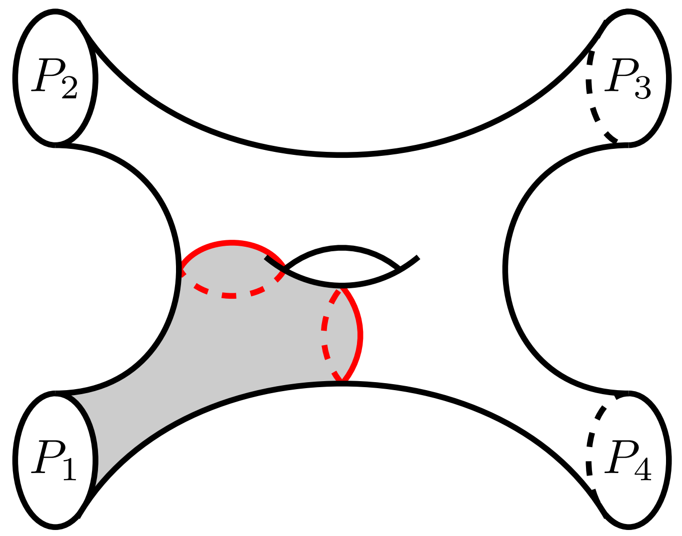

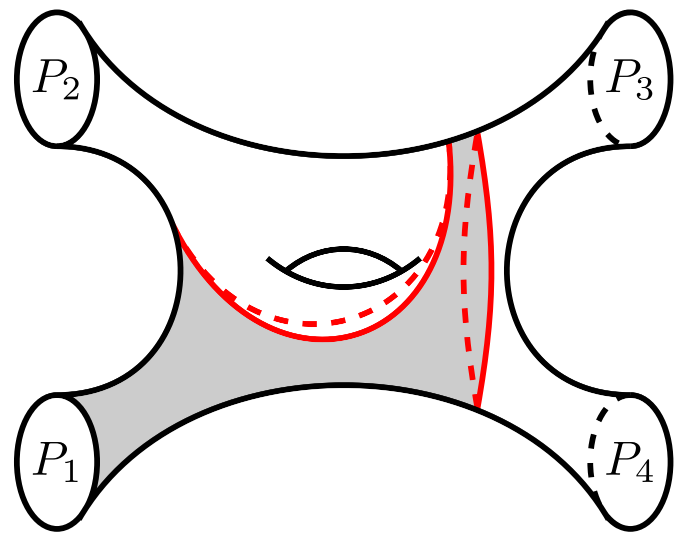

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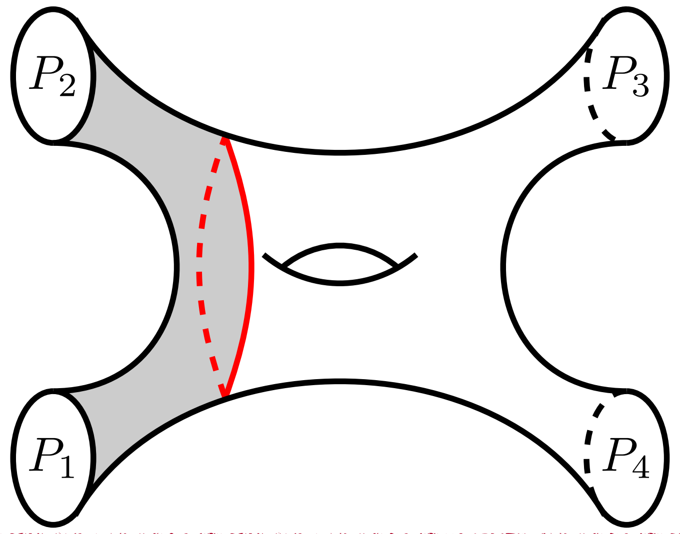

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$\mathsf{V}^{(b)}_{4,2}(P_1, P_2)$

$$-\frac{622\,299\,244\,001\,693\,280\,243\,725\,803}{1\,880\,880\,137\,947\,990\,781\,362\,962\,432\,000}+\frac{219\,335\,796\,233\,528\,858\,329\,441\,507\,c}{575\,779\,634\,065\,711\,463\,682\,539\,520\,000}-\frac{69\,757\,135\,409\,392\,663\,839\,720\,919\,c^2}{366\,405\,221\,678\,180\,022\,343\,434\,240\,000}+\frac{15\,619\,371\,867\,590\,440\,925\,212\,471\,c^3}{284\,981\,839\,083\,028\,906\,267\,115\,520\,000}-\frac{867\,042\,290\,512\,844\,401\,997\,717\,c^4}{85\,494\,551\,724\,908\,671\,880\,134\,656\,000}+$$

$$\frac{543\,227\,165\,257\,573\,020\,923\,711\,c^5}{427\,472\,758\,624\,543\,359\,400\,673\,280\,000}-\frac{2\,246\,260\,904\,409\,640\,116\,479\,c^6}{20\,355\,845\,648\,787\,779\,019\,079\,680\,000}+\frac{2\,845\,436\,456\,582\,265\,558\,709\,c^7}{427\,472\,758\,624\,543\,359\,400\,673\,280\,000}-\frac{46\,834\,973\,498\,118\,846\,053\,c^8}{170\,989\,103\,449\,817\,343\,760\,269\,312\,000}+\frac{1\,254\,323\,810\,769\,084\,659\,c^9}{170\,989\,103\,449\,817\,343\,760\,269\,312\,000}-\frac{11\,824\,960\,042\,824\,149\,c^{10}}{102\,593\,462\,069\,890\,406\,256\,161\,587\,200}+\frac{909\,612\,310\,986\,473\,c^{11}}{1\,128\,528\,082\,768\,794\,468\,817\,777\,459\,200}+$$

$$\left(\frac{142\,325\,184\,871\,938\,013\,073\,185\,159}{110\,207\,820\,582\,890\,084\,845\,486\,080\,000}-\frac{6\,511\,790\,083\,838\,301\,002\,596\,063\,c}{5\,009\,446\,390\,131\,367\,492\,976\,640\,000}+\frac{628\,331\,197\,233\,904\,688\,807\,659\,c^2}{1\,113\,210\,308\,918\,081\,665\,105\,920\,000}-\frac{2\,335\,341\,445\,691\,727\,761\,821\,c^3}{16\,698\,154\,633\,771\,224\,976\,588\,800}+\frac{1\,468\,081\,486\,315\,972\,071\,067\,c^4}{66\,792\,618\,535\,084\,899\,906\,355\,200}-\frac{18\,258\,381\,973\,103\,765\,467\,c^5}{7\,951\,502\,206\,557\,726\,179\,328\,000}+\frac{7\,724\,134\,828\,098\,526\,379\,c^6}{47\,709\,013\,239\,346\,357\,075\,968\,000}-\frac{127\,303\,914\,356\,833\,201\,c^7}{16\,698\,154\,633\,771\,224\,976\,588\,800}+\frac{10\,236\,631\,931\,982\,409\,c^8}{44\,528\,412\,356\,723\,266\,604\,236\,800}-\frac{4\,597\,205\,933\,905\,c^9}{1\,145\,016\,317\,744\,312\,569\,823\,232}+\frac{70\,726\,245\,137\,c^{10}}{2\,290\,032\,635\,488\,625\,139\,646\,464}\right)P_1^2+$$

$$\left(-\frac{581\,612\,149\,034\,552\,326\,800\,083}{417\,453\,865\,844\,280\,624\,414\,720\,000}+\frac{8\,066\,430\,460\,053\,722\,983\,337\,c}{6\,626\,251\,838\,798\,105\,149\,440\,000}-\frac{210\,874\,284\,653\,796\,947\,339\,c^2}{463\,837\,628\,715\,867\,360\,460\,800}+\frac{665\,287\,742\,671\,206\,780\,949\,c^3}{6\,957\,564\,430\,738\,010\,406\,912\,000}-\frac{237\,074\,062\,114\,929\,533\,c^4}{18\,932\,148\,110\,851\,728\,998\,400}+\frac{703\,518\,827\,490\,690\,259\,c^5}{662\,625\,183\,879\,810\,514\,944\,000}-\frac{58\,058\,677\,558\,651\,159\,c^6}{993\,937\,775\,819\,715\,772\,416\,000}+\frac{667\,536\,908\,225\,893\,c^7}{331\,312\,591\,939\,905\,257\,472\,000}-\frac{73\,478\,638\,098\,827\,c^8}{1\,855\,350\,514\,863\,469\,441\,843\,200}+\frac{5\,652\,202\,930\,679\,c^9}{16\,698\,154\,633\,771\,224\,976\,588\,800}\right)P_1^4+$$

$$\left(\frac{5\,995\,026\,910\,338\,472\,651\,499}{9\,276\,752\,574\,317\,347\,209\,216\,000}-\frac{562\,480\,698\,110\,820\,919\,079\,c}{1\,159\,594\,071\,789\,668\,401\,152\,000}+\frac{71\,253\,364\,153\,584\,811\,441\,c^2}{463\,837\,628\,715\,867\,360\,460\,800}-\frac{891\,258\,576\,498\,651\,283\,c^3}{33\,131\,259\,193\,990\,525\,747\,200}+\frac{1\,893\,175\,525\,029\,557\,959\,c^4}{662\,625\,183\,879\,810\,514\,944\,000}-\frac{31\,293\,422\,574\,872\,471\,c^5}{165\,656\,295\,969\,952\,628\,736\,000}+\frac{504\,181\,119\,215\,671\,c^6}{66\,262\,518\,387\,981\,051\,494\,400}-\frac{198\,289\,632\,166\,223\,c^7}{1\,159\,594\,071\,789\,668\,401\,152\,000}+\frac{15\,253\,048\,628\,171\,c^8}{9\,276\,752\,574\,317\,347\,209\,216\,000}\right)P_1^6+$$

$$\left(-\frac{402\,304\,752\,976\,095\,281}{2\,576\,875\,715\,088\,152\,002\,560}+\frac{6\,394\,712\,415\,220\,441\,883\,c}{64\,421\,892\,877\,203\,800\,064\,000}-\frac{80\,218\,809\,299\,861\,297\,c^2}{3\,067\,709\,184\,628\,752\,384\,000}+\frac{4\,878\,849\,517\,588\,783\,c^3}{1\,314\,732\,507\,698\,036\,736\,000}-\frac{80\,764\,061\,136\,863\,c^4}{262\,946\,501\,539\,607\,347\,200}+\frac{45\,584\,271\,483\,677\,c^5}{3\,067\,709\,184\,628\,752\,384\,000}-\frac{3\,587\,095\,147\,649\,c^6}{9\,203\,127\,553\,886\,257\,152\,000}+\frac{275\,930\,395\,973\,c^7}{64\,421\,892\,877\,203\,800\,064\,000}\right)P_1^8+$$

$$\left(\frac{72\,913\,743\,582\,227\,683}{3\,355\,306\,920\,687\,697\,920\,000}-\frac{131\,050\,737\,715\,183\,c}{11\,412\,608\,573\,767\,680\,000}+\frac{391\,389\,581\,499\,617\,c^2}{159\,776\,520\,032\,747\,520\,000}-\frac{1\,297\,731\,398\,717\,c^3}{4\,793\,295\,600\,982\,425\,600}+\frac{104\,732\,700\,497\,c^4}{6\,391\,060\,801\,309\,900\,800}-\frac{8\,245\,100\,357\,c^5}{15\,977\,652\,003\,274\,752\,000}+\frac{634\,238\,489\,c^6}{95\,865\,912\,019\,648\,512\,000}\right)P_1^{10}+\left(-\frac{2\,801\,037\,630\,833}{1\,521\,681\,143\,169\,024\,000}+\frac{8\,383\,530\,282\,047\,c}{10\,651\,768\,002\,183\,168\,000}-\frac{139\,194\,031\,951\,c^2}{1\,065\,176\,800\,218\,316\,800}+\frac{11\,243\,935\,019\,c^3}{1\,065\,176\,800\,218\,316\,800}-\frac{177\,112\,091\,c^4}{426\,070\,720\,087\,326\,720}+\frac{13\,624\,007\,c^5}{2\,130\,353\,600\,436\,633\,600}\right)P_1^{12}+$$

$$\left(\frac{453\,316\,434\,043}{4\,660\,148\,500\,955\,136\,000}-\frac{37\,689\,172\,183\,c}{1\,165\,037\,125\,238\,784\,000}+\frac{3\,047\,305\,747\,c^2}{776\,691\,416\,825\,856\,000}-\frac{1\,372\,033\,c^3}{6\,657\,355\,001\,364\,480}+\frac{105\,541\,c^4}{26\,629\,420\,005\,457\,920}\right)P_1^{14}+\left(-\frac{621\,805\,717}{194\,172\,854\,206\,464\,000}+\frac{10\,064\,501\,c}{12\,944\,856\,947\,097\,600}-\frac{566\,683\,c^2}{9\,246\,326\,390\,784\,000}+\frac{43\,591\,c^3}{27\,738\,979\,172\,352\,000}\right)P_1^{16}+\left(\frac{146\,719}{2\,311\,581\,597\,696\,000}-\frac{242\,983\,c}{24\,271\,606\,775\,808\,000}+\frac{18\,691\,c^2}{48\,543\,213\,551\,616\,000}\right)P_1^{18}+\left(-\frac{299}{433\,421\,549\,568\,000}+\frac{23\,c}{433\,421\,549\,568\,000}\right)P_1^{20}+\frac{P_1^{22}}{317\,842\,469\,683\,200}+$$

$$\left(\frac{142\,325\,184\,871\,938\,013\,073\,185\,159}{110\,207\,820\,582\,890\,084\,845\,486\,080\,000}-\frac{6\,511\,790\,083\,838\,301\,002\,596\,063\,c}{5\,009\,446\,390\,131\,367\,492\,976\,640\,000}+\frac{628\,331\,197\,233\,904\,688\,807\,659\,c^2}{1\,113\,210\,308\,918\,081\,665\,105\,920\,000}-\frac{2\,335\,341\,445\,691\,727\,761\,821\,c^3}{16\,698\,154\,633\,771\,224\,976\,588\,800}+\frac{1\,468\,081\,486\,315\,972\,071\,067\,c^4}{66\,792\,618\,535\,084\,899\,906\,355\,200}-\frac{18\,258\,381\,973\,103\,765\,467\,c^5}{7\,951\,502\,206\,557\,726\,179\,328\,000}+\frac{7\,724\,134\,828\,098\,526\,379\,c^6}{47\,709\,013\,239\,346\,357\,075\,968\,000}-\frac{127\,303\,914\,356\,833\,201\,c^7}{16\,698\,154\,633\,771\,224\,976\,588\,800}+\frac{10\,236\,631\,931\,982\,409\,c^8}{44\,528\,412\,356\,723\,266\,604\,236\,800}-\frac{4\,597\,205\,933\,905\,c^9}{1\,145\,016\,317\,744\,312\,569\,823\,232}+\frac{70\,726\,245\,137\,c^{10}}{2\,290\,032\,635\,488\,625\,139\,646\,464}\right)P_2^2+$$

$$\left(-\frac{15\,087\,180\,119\,394\,943\,486\,433}{3\,339\,630\,926\,754\,244\,995\,317\,760}+\frac{7\,314\,398\,978\,146\,119\,199\,801\,c}{1\,855\,350\,514\,863\,469\,441\,843\,200}-\frac{3\,898\,368\,189\,561\,632\,995\,c^2}{2\,650\,500\,753\,519\,242\,059\,776}+\frac{430\,115\,432\,033\,782\,399\,687\,c^3}{1\,391\,512\,886\,147\,602\,081\,382\,400}-\frac{765\,872\,990\,143\,746\,979\,c^4}{18\,932\,148\,110\,851\,728\,998\,400}+\frac{90\,866\,752\,394\,731\,253\,c^5}{26\,505\,007\,355\,192\,420\,597\,760}-\frac{7\,496\,421\,897\,226\,457\,c^6}{39\,757\,511\,032\,788\,630\,896\,640}+\frac{430\,867\,228\,492\,903\,c^7}{66\,262\,518\,387\,981\,051\,494\,400}-\frac{237\,114\,331\,042\,693\,c^8}{1\,855\,350\,514\,863\,469\,441\,843\,200}+\frac{18\,239\,563\,926\,361\,c^9}{16\,698\,154\,633\,771\,224\,976\,588\,800}\right)P_1^2P_2^2+$$

$$\left(\frac{1\,515\,661\,271\,084\,217\,251}{345\,117\,283\,270\,734\,643\,200}-\frac{1\,988\,679\,658\,390\,064\,887\,c}{603\,955\,245\,723\,785\,625\,600}+\frac{6\,292\,436\,729\,173\,816\,967\,c^2}{6\,039\,552\,457\,237\,856\,256\,000}-\frac{1\,404\,531\,609\,941\,707\,c^3}{7\,703\,510\,787\,293\,184\,000}+\frac{238\,554\,011\,575\,379\,c^4}{12\,325\,617\,259\,669\,094\,400}-\frac{78\,835\,826\,259\,047\,c^5}{61\,628\,086\,298\,345\,472\,000}+\frac{1\,269\,873\,622\,489\,c^6}{24\,651\,234\,519\,338\,188\,800}-\frac{35\,669\,807\,719\,c^7}{30\,814\,043\,149\,172\,736\,000}+\frac{2\,743\,831\,363\,c^8}{246\,512\,345\,193\,381\,888\,000}\right)P_1^4P_2^2+$$

$$\left(-\frac{4\,169\,146\,749\,797\,786\,549}{2\,300\,781\,888\,471\,564\,288\,000}+\frac{18\,536\,611\,594\,604\,751\,257\,c}{16\,105\,473\,219\,300\,950\,016\,000}-\frac{46\,470\,247\,800\,495\,727\,c^2}{153\,385\,459\,231\,437\,619\,200}+\frac{98\,862\,405\,451\,603\,123\,c^3}{2\,300\,781\,888\,471\,564\,288\,000}-\frac{233\,699\,221\,180\,997\,c^4}{65\,736\,625\,384\,901\,836\,800}+\frac{131\,869\,526\,330\,063\,c^5}{766\,927\,296\,157\,188\,096\,000}-\frac{10\,375\,789\,877\,531\,c^6}{2\,300\,781\,888\,471\,564\,288\,000}+\frac{798\,137\,682\,887\,c^7}{16\,105\,473\,219\,300\,950\,016\,000}\right)P_1^6P_2^2+$$

$$\left(\frac{102\,940\,208\,269\,921\,663}{268\,424\,553\,655\,015\,833\,600}-\frac{1\,293\,887\,108\,538\,973\,c}{6\,391\,060\,801\,309\,900\,800}+\frac{551\,643\,040\,451\,477\,c^2}{12\,782\,121\,602\,619\,801\,600}-\frac{9\,140\,867\,270\,821\,c^3}{1\,917\,318\,240\,392\,970\,240}+\frac{737\,480\,670\,553\,c^4}{2\,556\,424\,320\,523\,960\,320}-\frac{58\,049\,937\,517\,c^5}{6\,391\,060\,801\,309\,900\,800}+\frac{4\,465\,379\,809\,c^6}{38\,346\,364\,807\,859\,404\,800}\right)P_1^8P_2^2+\left(-\frac{61\,010\,611\,938\,263}{1\,331\,471\,000\,272\,896\,000}+\frac{26\,064\,213\,896\,191\,c}{1\,331\,471\,000\,272\,896\,000}-\frac{61\,784\,873\,657\,c^2}{19\,021\,014\,289\,612\,800}+\frac{34\,923\,517\,931\,c^3}{133\,147\,100\,027\,289\,600}-\frac{550\,013\,659\,c^4}{53\,258\,840\,010\,915\,840}+\frac{42\,308\,743\,c^5}{266\,294\,200\,054\,579\,200}\right)P_1^{10}P_2^2+$$

$$\left(\frac{431\,257\,711\,547}{133\,147\,100\,027\,289\,600}-\frac{35\,829\,108\,887\,c}{33\,286\,775\,006\,822\,400}+\frac{2\,895\,612\,371\,c^2}{22\,191\,183\,337\,881\,600}-\frac{228\,105\,839\,c^3}{33\,286\,775\,006\,822\,400}+\frac{17\,546\,603\,c^4}{133\,147\,100\,027\,289\,600}\right)P_1^{12}P_2^2+\left(-\frac{831\,734\,761}{6\,067\,901\,693\,952\,000}+\frac{13\,454\,873\,c}{404\,526\,779\,596\,800}-\frac{5\,301\,673\,c^2}{2\,022\,633\,897\,984\,000}+\frac{407\,821\,c^3}{6\,067\,901\,693\,952\,000}\right)P_1^{14}P_2^2+\left(\frac{1\,304\,249}{385\,263\,599\,616\,000}-\frac{719\,771\,c}{1\,348\,422\,598\,656\,000}+\frac{55\,367\,c^2}{2\,696\,845\,197\,312\,000}\right)P_1^{16}P_2^2+\left(-\frac{1937}{43\,342\,154\,956\,800}+\frac{149\,c}{43\,342\,154\,956\,800}\right)P_1^{18}P_2^2+$$

$$\frac{P_1^{20}P_2^2}{4\,127\,824\,281\,600}+\left(-\frac{581\,612\,149\,034\,552\,326\,800\,083}{417\,453\,865\,844\,280\,624\,414\,720\,000}+\frac{8\,066\,430\,460\,053\,722\,983\,337\,c}{6\,626\,251\,838\,798\,105\,149\,440\,000}-\frac{210\,874\,284\,653\,796\,947\,339\,c^2}{463\,837\,628\,715\,867\,360\,460\,800}+\frac{665\,287\,742\,671\,206\,780\,949\,c^3}{6\,957\,564\,430\,738\,010\,406\,912\,000}-\frac{237\,074\,062\,114\,929\,533\,c^4}{18\,932\,148\,110\,851\,728\,998\,400}+\frac{703\,518\,827\,490\,690\,259\,c^5}{662\,625\,183\,879\,810\,514\,944\,000}-\frac{58\,058\,677\,558\,651\,159\,c^6}{993\,937\,775\,819\,715\,772\,416\,000}+\frac{667\,536\,908\,225\,893\,c^7}{331\,312\,591\,939\,905\,257\,472\,000}-\frac{73\,478\,638\,098\,827\,c^8}{1\,855\,350\,514\,863\,469\,441\,843\,200}+\frac{5\,652\,202\,930\,679\,c^9}{16\,698\,154\,633\,771\,224\,976\,588\,800}\right)P_2^4+$$

$$\left(\frac{1\,515\,661\,271\,084\,217\,251}{345\,117\,283\,270\,734\,643\,200}-\frac{1\,988\,679\,658\,390\,064\,887\,c}{603\,955\,245\,723\,785\,625\,600}+\frac{6\,292\,436\,729\,173\,816\,967\,c^2}{6\,039\,552\,457\,237\,856\,256\,000}-\frac{1\,404\,531\,609\,941\,707\,c^3}{7\,703\,510\,787\,293\,184\,000}+\frac{238\,554\,011\,575\,379\,c^4}{12\,325\,617\,259\,669\,094\,400}-\frac{78\,835\,826\,259\,047\,c^5}{61\,628\,086\,298\,345\,472\,000}+\frac{1\,269\,873\,622\,489\,c^6}{24\,651\,234\,519\,338\,188\,800}-\frac{35\,669\,807\,719\,c^7}{30\,814\,043\,149\,172\,736\,000}+\frac{2\,743\,831\,363\,c^8}{246\,512\,345\,193\,381\,888\,000}\right)P_1^2P_2^4+$$

$$\left(-\frac{81\,265\,498\,358\,836\,753}{21\,303\,536\,004\,366\,336\,000}+\frac{51\,621\,685\,541\,452\,819\,c}{21\,303\,536\,004\,366\,336\,000}-\frac{4\,793\,399\,235\,599\,c^2}{7\,514\,474\,781\,081\,600}+\frac{275\,350\,341\,074\,561\,c^3}{3\,043\,362\,286\,338\,048\,000}-\frac{4\,556\,431\,615\,489\,c^4}{608\,672\,457\,267\,609\,600}+\frac{1\,632\,453\,349\,c^5}{4\,508\,684\,868\,648\,960}-\frac{28\,900\,433\,497\,c^6}{3\,043\,362\,286\,338\,048\,000}+\frac{2\,223\,110\,269\,c^7}{21\,303\,536\,004\,366\,336\,000}\right)P_1^4P_2^4+$$

$$\left(\frac{116\,221\,823\,108\,630\,887}{83\,882\,673\,017\,192\,448\,000}-\frac{208\,717\,834\,135\,603\,c}{285\,315\,214\,344\,192\,000}+\frac{622\,964\,103\,254\,429\,c^2}{3\,994\,413\,000\,818\,688\,000}-\frac{10\,323\,400\,176\,349\,c^3}{599\,161\,950\,122\,803\,200}+\frac{118\,989\,019\,687\,c^4}{114\,126\,085\,737\,676\,800}-\frac{65\,563\,911\,829\,c^5}{1\,997\,206\,500\,409\,344\,000}+\frac{5\,043\,377\,833\,c^6}{11\,983\,239\,002\,456\,064\,000}\right)P_1^6P_2^4+\left(-\frac{96\,278\,168\,222\,743}{380\,420\,285\,792\,256\,000}+\frac{287\,944\,172\,374\,457\,c}{2\,662\,942\,000\,545\,792\,000}-\frac{4\,778\,295\,195\,817\,c^2}{266\,294\,200\,054\,579\,200}+\frac{55\,122\,700\,459\,c^3}{38\,042\,028\,579\,225\,600}-\frac{6\,077\,042\,621\,c^4}{106\,517\,680\,021\,831\,680}+\frac{467\,464\,817\,c^5}{532\,588\,400\,109\,158\,400}\right)P_1^8P_2^4+$$

$$\left(\frac{15\,656\,858\,317}{616\,421\,759\,385\,600}-\frac{6\,504\,539\,509\,c}{770\,527\,199\,232\,000}+\frac{225\,304\,429\,c^2}{220\,150\,628\,352\,000}-\frac{2\,760\,979\,c^3}{51\,368\,479\,948\,800}+\frac{212\,383\,c^4}{205\,473\,919\,795\,200}\right)P_1^{10}P_2^4+\left(-\frac{10\,047\,898\,549}{6\,934\,744\,793\,088\,000}+\frac{23\,222\,147\,c}{66\,045\,188\,505\,600}-\frac{64\,054\,237\,c^2}{2\,311\,581\,597\,696\,000}+\frac{4\,927\,249\,c^3}{6\,934\,744\,793\,088\,000}\right)P_1^{12}P_2^4+\left(\frac{894\,433}{19\,263\,179\,980\,800}-\frac{2\,468\,063\,c}{337\,105\,649\,664\,000}+\frac{189\,851\,c^2}{674\,211\,299\,328\,000}\right)P_1^{14}P_2^4+\left(-\frac{689}{891\,813\,888\,000}+\frac{53\,c}{891\,813\,888\,000}\right)P_1^{16}P_2^4+$$

$$\frac{149\,P_1^{18}P_2^4}{28\,894\,769\,971\,200}+\left(\frac{5\,995\,026\,910\,338\,472\,651\,499}{9\,276\,752\,574\,317\,347\,209\,216\,000}-\frac{562\,480\,698\,110\,820\,919\,079\,c}{1\,159\,594\,071\,789\,668\,401\,152\,000}+\frac{71\,253\,364\,153\,584\,811\,441\,c^2}{463\,837\,628\,715\,867\,360\,460\,800}-\frac{891\,258\,576\,498\,651\,283\,c^3}{33\,131\,259\,193\,990\,525\,747\,200}+\frac{1\,893\,175\,525\,029\,557\,959\,c^4}{662\,625\,183\,879\,810\,514\,944\,000}-\frac{31\,293\,422\,574\,872\,471\,c^5}{165\,656\,295\,969\,952\,628\,736\,000}+\frac{504\,181\,119\,215\,671\,c^6}{66\,262\,518\,387\,981\,051\,494\,400}-\frac{198\,289\,632\,166\,223\,c^7}{1\,159\,594\,071\,789\,668\,401\,152\,000}+\frac{15\,253\,048\,628\,171\,c^8}{9\,276\,752\,574\,317\,347\,209\,216\,000}\right)P_2^6+$$

$$\left(-\frac{4\,169\,146\,749\,797\,786\,549}{2\,300\,781\,888\,471\,564\,288\,000}+\frac{18\,536\,611\,594\,604\,751\,257\,c}{16\,105\,473\,219\,300\,950\,016\,000}-\frac{46\,470\,247\,800\,495\,727\,c^2}{153\,385\,459\,231\,437\,619\,200}+\frac{98\,862\,405\,451\,603\,123\,c^3}{2\,300\,781\,888\,471\,564\,288\,000}-\frac{233\,699\,221\,180\,997\,c^4}{65\,736\,625\,384\,901\,836\,800}+\frac{131\,869\,526\,330\,063\,c^5}{766\,927\,296\,157\,188\,096\,000}-\frac{10\,375\,789\,877\,531\,c^6}{2\,300\,781\,888\,471\,564\,288\,000}+\frac{798\,137\,682\,887\,c^7}{16\,105\,473\,219\,300\,950\,016\,000}\right)P_1^2P_2^6+$$

$$\left(\frac{116\,221\,823\,108\,630\,887}{83\,882\,673\,017\,192\,448\,000}-\frac{208\,717\,834\,135\,603\,c}{285\,315\,214\,344\,192\,000}+\frac{622\,964\,103\,254\,429\,c^2}{3\,994\,413\,000\,818\,688\,000}-\frac{10\,323\,400\,176\,349\,c^3}{599\,161\,950\,122\,803\,200}+\frac{118\,989\,019\,687\,c^4}{114\,126\,085\,737\,676\,800}-\frac{65\,563\,911\,829\,c^5}{1\,997\,206\,500\,409\,344\,000}+\frac{5\,043\,377\,833\,c^6}{11\,983\,239\,002\,456\,064\,000}\right)P_1^4P_2^6+\left(-\frac{82\,581\,629\,006\,669}{190\,210\,142\,896\,128\,000}+\frac{11\,761\,688\,186\,071\,c}{63\,403\,380\,965\,376\,000}-\frac{65\,062\,718\,669\,c^2}{2\,113\,446\,032\,179\,200}+\frac{47\,287\,090\,409\,c^3}{19\,021\,014\,289\,612\,800}-\frac{248\,250\,899\,c^4}{2\,536\,135\,238\,615\,040}+\frac{19\,096\,223\,c^5}{12\,680\,676\,193\,075\,200}\right)P_1^6P_2^6+$$

$$\left(\frac{632\,984\,073\,619}{9\,510\,507\,144\,806\,400}-\frac{52\,595\,154\,703\,c}{2\,377\,626\,786\,201\,600}+\frac{850\,184\,687\,c^2}{317\,016\,904\,826\,880}-\frac{334\,884\,199\,c^3}{2\,377\,626\,786\,201\,600}+\frac{25\,760\,323\,c^4}{9\,510\,507\,144\,806\,400}\right)P_1^8P_2^6+\left(-\frac{1\,334\,014\,123}{247\,669\,456\,896\,000}+\frac{21\,582\,923\,c}{16\,511\,297\,126\,400}-\frac{8\,504\,899\,c^2}{82\,556\,485\,632\,000}+\frac{654\,223\,c^3}{247\,669\,456\,896\,000}\right)P_1^{10}P_2^6+\left(\frac{6\,395\,581}{27\,518\,828\,544\,000}-\frac{504\,257\,c}{13\,759\,414\,272\,000}+\frac{38\,789\,c^2}{27\,518\,828\,544\,000}\right)P_1^{12}P_2^6+$$

$$\left(-\frac{211\,159}{42\,138\,206\,208\,000}+\frac{16\,243\,c}{42\,138\,206\,208\,000}\right)P_1^{14}P_2^6+\frac{947\,P_1^{16}P_2^6}{22\,473\,709\,977\,600}+\left(-\frac{402\,304\,752\,976\,095\,281}{2\,576\,875\,715\,088\,152\,002\,560}+\frac{6\,394\,712\,415\,220\,441\,883\,c}{64\,421\,892\,877\,203\,800\,064\,000}-\frac{80\,218\,809\,299\,861\,297\,c^2}{3\,067\,709\,184\,628\,752\,384\,000}+\frac{4\,878\,849\,517\,588\,783\,c^3}{1\,314\,732\,507\,698\,036\,736\,000}-\frac{80\,764\,061\,136\,863\,c^4}{262\,946\,501\,539\,607\,347\,200}+\frac{45\,584\,271\,483\,677\,c^5}{3\,067\,709\,184\,628\,752\,384\,000}-\frac{3\,587\,095\,147\,649\,c^6}{9\,203\,127\,553\,886\,257\,152\,000}+\frac{275\,930\,395\,973\,c^7}{64\,421\,892\,877\,203\,800\,064\,000}\right)P_2^8+$$

$$\left(\frac{102\,940\,208\,269\,921\,663}{268\,424\,553\,655\,015\,833\,600}-\frac{1\,293\,887\,108\,538\,973\,c}{6\,391\,060\,801\,309\,900\,800}+\frac{551\,643\,040\,451\,477\,c^2}{12\,782\,121\,602\,619\,801\,600}-\frac{9\,140\,867\,270\,821\,c^3}{1\,917\,318\,240\,392\,970\,240}+\frac{737\,480\,670\,553\,c^4}{2\,556\,424\,320\,523\,960\,320}-\frac{58\,049\,937\,517\,c^5}{6\,391\,060\,801\,309\,900\,800}+\frac{4\,465\,379\,809\,c^6}{38\,346\,364\,807\,859\,404\,800}\right)P_1^2P_2^8+\left(-\frac{96\,278\,168\,222\,743}{380\,420\,285\,792\,256\,000}+\frac{287\,944\,172\,374\,457\,c}{2\,662\,942\,000\,545\,792\,000}-\frac{4\,778\,295\,195\,817\,c^2}{266\,294\,200\,054\,579\,200}+\frac{55\,122\,700\,459\,c^3}{38\,042\,028\,579\,225\,600}-\frac{6\,077\,042\,621\,c^4}{106\,517\,680\,021\,831\,680}+\frac{467\,464\,817\,c^5}{532\,588\,400\,109\,158\,400}\right)P_1^4P_2^8+$$

$$\left(\frac{632\,984\,073\,619}{9\,510\,507\,144\,806\,400}-\frac{52\,595\,154\,703\,c}{2\,377\,626\,786\,201\,600}+\frac{850\,184\,687\,c^2}{317\,016\,904\,826\,880}-\frac{334\,884\,199\,c^3}{2\,377\,626\,786\,201\,600}+\frac{25\,760\,323\,c^4}{9\,510\,507\,144\,806\,400}\right)P_1^6P_2^8+\left(-\frac{81\,463\,343}{9\,906\,778\,275\,840}+\frac{6\,589\,499\,c}{3\,302\,259\,425\,280}-\frac{519\,311\,c^2}{3\,302\,259\,425\,280}+\frac{39\,947\,c^3}{9\,906\,778\,275\,840}\right)P_1^8P_2^8+\left(\frac{6\,923\,243}{13\,759\,414\,272\,000}-\frac{545\,831\,c}{6\,879\,707\,136\,000}+\frac{41\,987\,c^2}{13\,759\,414\,272\,000}\right)P_1^{10}P_2^8+\left(-\frac{14\,053}{963\,158\,999\,040}+\frac{1081\,c}{963\,158\,999\,040}\right)P_1^{12}P_2^8+\frac{1781\,P_1^{14}P_2^8}{11\,236\,854\,988\,800}+$$

$$\left(\frac{72\,913\,743\,582\,227\,683}{3\,355\,306\,920\,687\,697\,920\,000}-\frac{131\,050\,737\,715\,183\,c}{11\,412\,608\,573\,767\,680\,000}+\frac{391\,389\,581\,499\,617\,c^2}{159\,776\,520\,032\,747\,520\,000}-\frac{1\,297\,731\,398\,717\,c^3}{4\,793\,295\,600\,982\,425\,600}+\frac{104\,732\,700\,497\,c^4}{6\,391\,060\,801\,309\,900\,800}-\frac{8\,245\,100\,357\,c^5}{15\,977\,652\,003\,274\,752\,000}+\frac{634\,238\,489\,c^6}{95\,865\,912\,019\,648\,512\,000}\right)P_2^{10}+\left(-\frac{61\,010\,611\,938\,263}{1\,331\,471\,000\,272\,896\,000}+\frac{26\,064\,213\,896\,191\,c}{1\,331\,471\,000\,272\,896\,000}-\frac{61\,784\,873\,657\,c^2}{19\,021\,014\,289\,612\,800}+\frac{34\,923\,517\,931\,c^3}{133\,147\,100\,027\,289\,600}-\frac{550\,013\,659\,c^4}{53\,258\,840\,010\,915\,840}+\frac{42\,308\,743\,c^5}{266\,294\,200\,054\,579\,200}\right)P_1^2P_2^{10}+$$

$$\left(\frac{15\,656\,858\,317}{616\,421\,759\,385\,600}-\frac{6\,504\,539\,509\,c}{770\,527\,199\,232\,000}+\frac{225\,304\,429\,c^2}{220\,150\,628\,352\,000}-\frac{2\,760\,979\,c^3}{51\,368\,479\,948\,800}+\frac{212\,383\,c^4}{205\,473\,919\,795\,200}\right)P_1^4P_2^{10}+\left(-\frac{1\,334\,014\,123}{247\,669\,456\,896\,000}+\frac{21\,582\,923\,c}{16\,511\,297\,126\,400}-\frac{8\,504\,899\,c^2}{82\,556\,485\,632\,000}+\frac{654\,223\,c^3}{247\,669\,456\,896\,000}\right)P_1^6P_2^{10}+\left(\frac{6\,923\,243}{13\,759\,414\,272\,000}-\frac{545\,831\,c}{6\,879\,707\,136\,000}+\frac{41\,987\,c^2}{13\,759\,414\,272\,000}\right)P_1^8P_2^{10}+\left(-\frac{6929}{334\,430\,208\,000}+\frac{533\,c}{334\,430\,208\,000}\right)P_1^{10}P_2^{10}+$$

$$\frac{487\,P_1^{12}P_2^{10}}{1\,605\,264\,998\,400}+\left(-\frac{2\,801\,037\,630\,833}{1\,521\,681\,143\,169\,024\,000}+\frac{8\,383\,530\,282\,047\,c}{10\,651\,768\,002\,183\,168\,000}-\frac{139\,194\,031\,951\,c^2}{1\,065\,176\,800\,218\,316\,800}+\frac{11\,243\,935\,019\,c^3}{1\,065\,176\,800\,218\,316\,800}-\frac{177\,112\,091\,c^4}{426\,070\,720\,087\,326\,720}+\frac{13\,624\,007\,c^5}{2\,130\,353\,600\,436\,633\,600}\right)P_2^{12}+\left(\frac{431\,257\,711\,547}{133\,147\,100\,027\,289\,600}-\frac{35\,829\,108\,887\,c}{33\,286\,775\,006\,822\,400}+\frac{2\,895\,612\,371\,c^2}{22\,191\,183\,337\,881\,600}-\frac{228\,105\,839\,c^3}{33\,286\,775\,006\,822\,400}+\frac{17\,546\,603\,c^4}{133\,147\,100\,027\,289\,600}\right)P_1^2P_2^{12}+$$

$$\left(-\frac{10\,047\,898\,549}{6\,934\,744\,793\,088\,000}+\frac{23\,222\,147\,c}{66\,045\,188\,505\,600}-\frac{64\,054\,237\,c^2}{2\,311\,581\,597\,696\,000}+\frac{4\,927\,249\,c^3}{6\,934\,744\,793\,088\,000}\right)P_1^4P_2^{12}+\left(\frac{6\,395\,581}{27\,518\,828\,544\,000}-\frac{504\,257\,c}{13\,759\,414\,272\,000}+\frac{38\,789\,c^2}{27\,518\,828\,544\,000}\right)P_1^6P_2^{12}+\left(-\frac{14\,053}{963\,158\,999\,040}+\frac{1081\,c}{963\,158\,999\,040}\right)P_1^8P_2^{12}+\frac{487\,P_1^{10}P_2^{12}}{1\,605\,264\,998\,400}+\left(\frac{453\,316\,434\,043}{4\,660\,148\,500\,955\,136\,000}-\frac{37\,689\,172\,183\,c}{1\,165\,037\,125\,238\,784\,000}+\frac{3\,047\,305\,747\,c^2}{776\,691\,416\,825\,856\,000}-\frac{1\,372\,033\,c^3}{6\,657\,355\,001\,364\,480}+\frac{105\,541\,c^4}{26\,629\,420\,005\,457\,920}\right)P_2^{14}+$$

$$\left(-\frac{831\,734\,761}{6\,067\,901\,693\,952\,000}+\frac{13\,454\,873\,c}{404\,526\,779\,596\,800}-\frac{5\,301\,673\,c^2}{2\,022\,633\,897\,984\,000}+\frac{407\,821\,c^3}{6\,067\,901\,693\,952\,000}\right)P_1^2P_2^{14}+\left(\frac{894\,433}{19\,263\,179\,980\,800}-\frac{2\,468\,063\,c}{337\,105\,649\,664\,000}+\frac{189\,851\,c^2}{674\,211\,299\,328\,000}\right)P_1^4P_2^{14}+\left(-\frac{211\,159}{42\,138\,206\,208\,000}+\frac{16\,243\,c}{42\,138\,206\,208\,000}\right)P_1^6P_2^{14}+\frac{1781\,P_1^8P_2^{14}}{11\,236\,854\,988\,800}+\left(-\frac{621\,805\,717}{194\,172\,854\,206\,464\,000}+\frac{10\,064\,501\,c}{12\,944\,856\,947\,097\,600}-\frac{566\,683\,c^2}{9\,246\,326\,390\,784\,000}+\frac{43\,591\,c^3}{27\,738\,979\,172\,352\,000}\right)P_2^{16}+$$

$$\left(\frac{1\,304\,249}{385\,263\,599\,616\,000}-\frac{719\,771\,c}{1\,348\,422\,598\,656\,000}+\frac{55\,367\,c^2}{2\,696\,845\,197\,312\,000}\right)P_1^2P_2^{16}+\left(-\frac{689}{891\,813\,888\,000}+\frac{53\,c}{891\,813\,888\,000}\right)P_1^4P_2^{16}+\frac{947\,P_1^6P_2^{16}}{22\,473\,709\,977\,600}+\left(\frac{146\,719}{2\,311\,581\,597\,696\,000}-\frac{242\,983\,c}{24\,271\,606\,775\,808\,000}+\frac{18\,691\,c^2}{48\,543\,213\,551\,616\,000}\right)P_2^{18}+\left(-\frac{1937}{43\,342\,154\,956\,800}+\frac{149\,c}{43\,342\,154\,956\,800}\right)P_1^2P_2^{18}+\frac{149\,P_1^4P_2^{18}}{28\,894\,769\,971\,200}+\left(-\frac{299}{433\,421\,549\,568\,000}+\frac{23\,c}{433\,421\,549\,568\,000}\right)P_2^{20}+\frac{P_1^2P_2^{20}}{4\,127\,824\,281\,600}+\frac{P_2^{22}}{317\,842\,469\,683\,200}$$

Properties of the quantum volumes

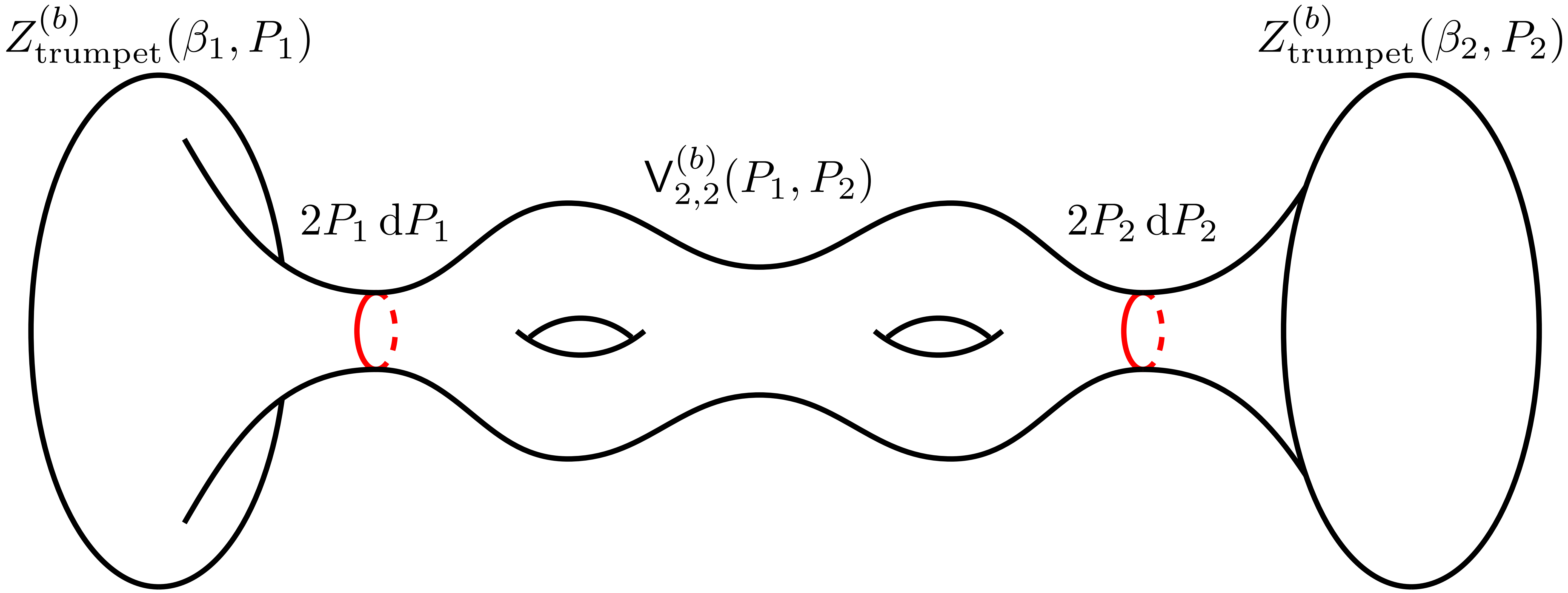
- This gives an effective algorithm to compute all $\mathsf{V}^{(b)}_{g,n}(P_1, \ldots, P_n)$.
- They are polynomial in $\mathbb{Q}[c, P_1^2, \ldots, P_n^2]$ of degree $3g - 3 + n$.
- They are invariant under switching the role of spacelike and timelike Liouville theory:

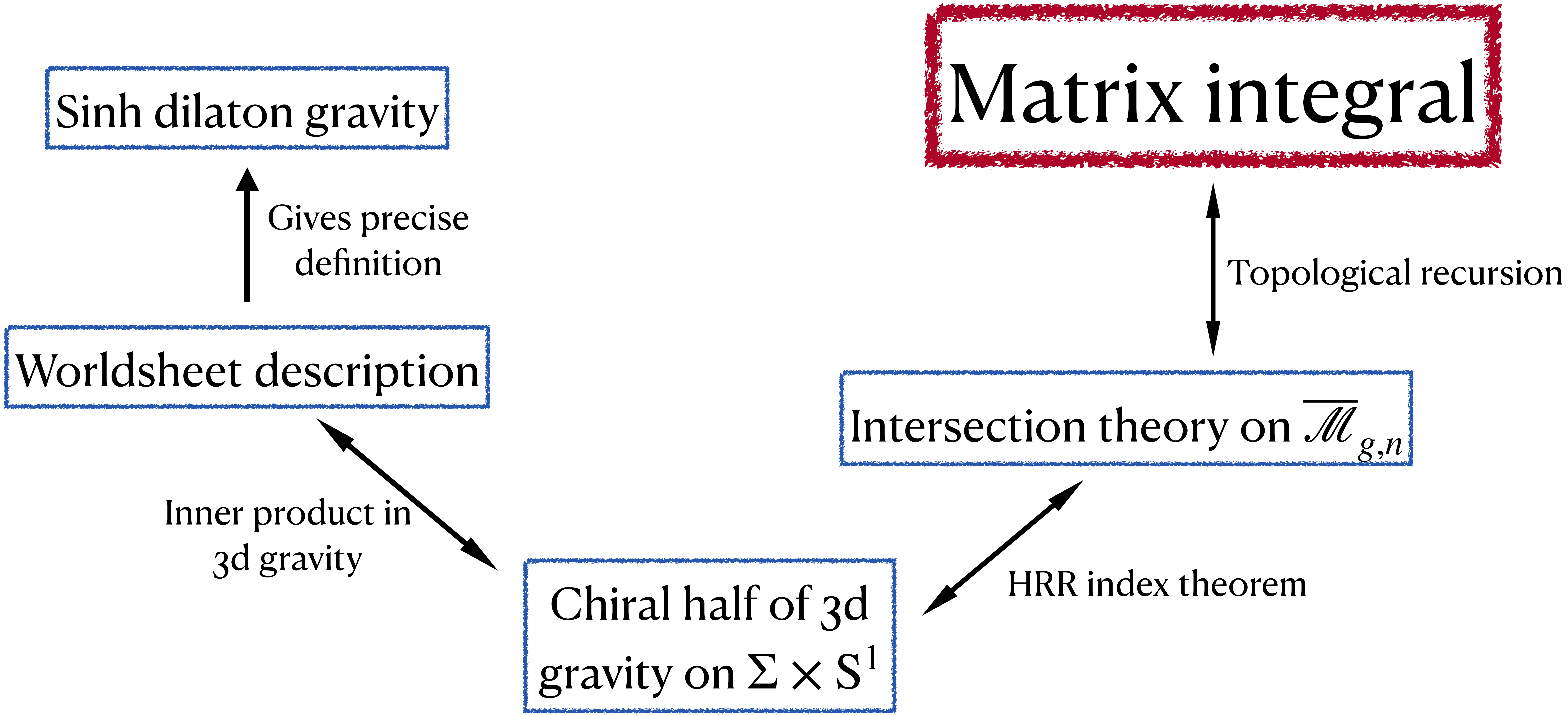
$$\mathsf{V}^{(ib)}_{g,n}(iP_1, \ldots, iP_n) = (-1)^{3g-3+n}\mathsf{V}^{(b)}_{g,n}(P_1, \ldots, P_n)$$

Adding asymptotic boundaries

- Asymptotic boundaries can be added by gluing trumpets:

$$Z^{(b)}_{g,n}(\beta_1, \ldots, \beta_n) = \int_0^\infty \prod_{j=1}^{n} \left(2P_j\, \mathrm{d}P_j\, Z^{(b)}_{\text{trumpet}}(\beta_j, P_j)\right) \mathsf{V}^{(b)}_{g,n}(P_1, \ldots, P_n)$$





Sinh dilaton gravity
Gives precise definition
Worldsheet description
Inner product in 3d gravity
Chiral half of 3d gravity on $\Sigma \times S^1$
HRR index theorem
Intersection theory on $\overline{\mathcal{M}}_{g,n}$
Topological recursion
Matrix integral

The density of states

- The deformed Mirzakhani recursion is equivalent to the loop equations of a doubly-scaled matrix model
- Its leading density of states is the universal Cardy density in a CFT_2

$$\rho_0^{(b)}(E)\,\mathrm{d}E = 4\sqrt{2}\ \sinh(2\pi b P)\sinh(2\pi b^{-1} P)\,\mathrm{d}P\ , \qquad E = P^2$$

- This motivated us to call the corresponding bulk theory the Virasoro minimal string
- JT-limit $b \to 0$: $\rho_0^{(b)}(E) \to \sinh(2\pi\sqrt{E})$ after rescaling the energy

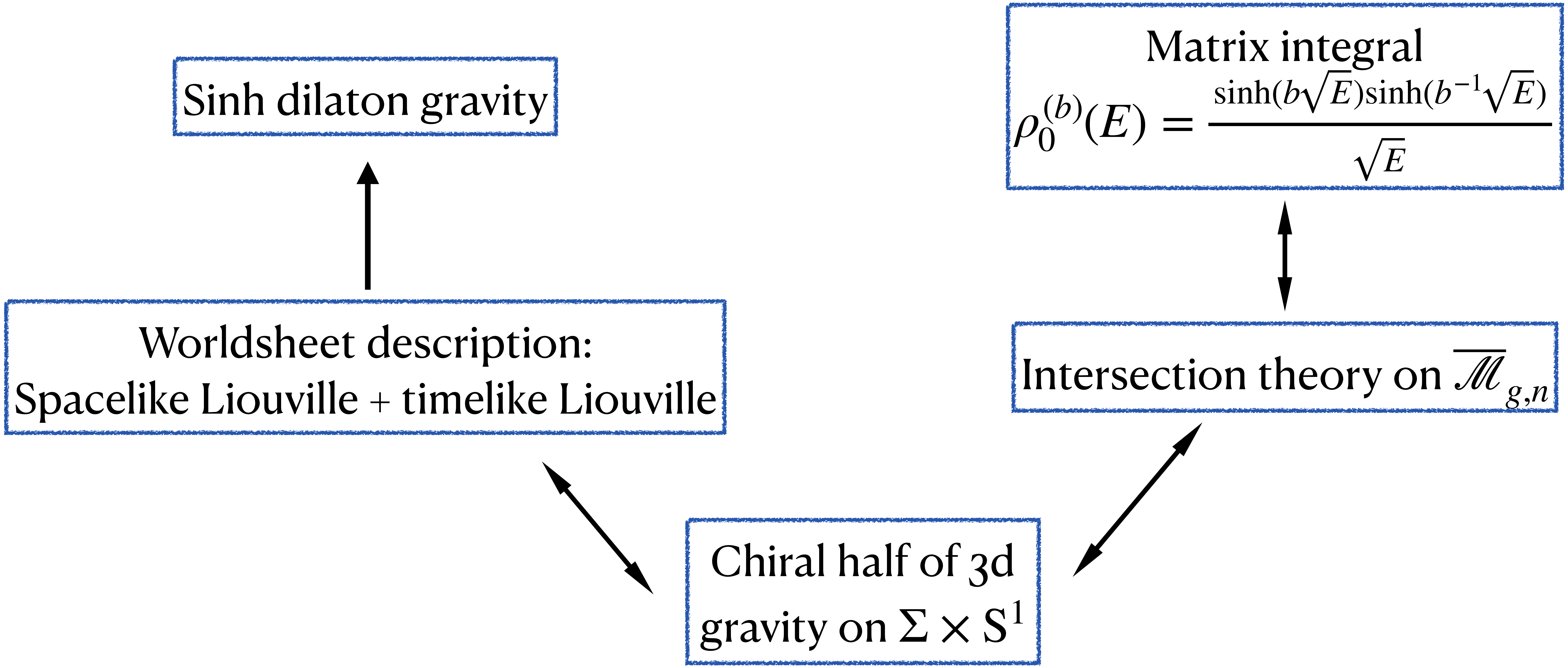
The full matrix integral

- Knowing the leading density of states $\rho_0^{(b)}(P)\,\mathrm{d}P$ determines the perturbative part of the double-scaled matrix integral completely.
- Our arguments **derive** this correspondence directly
- Much simpler than the $(2,p)$ minimal string:
  - Analytic in b and P_i
  - No contact terms

Non-perturbative effects

- The matrix integral also makes non-perturbative predictions
- Correspond to ZZ-instanton-like contributions from the worldsheet
- For $b = 1$, the theory is non-perturbatively stable
- For $b \neq 1$:
  - change the integration contour of the matrix integral or — Saad, Shenker, Stanford '19
  - Deform the model at low energies — Johnson '19
- Predicts large-g behavior of $\mathsf{V}^{(b)}_{g,0}$ for $b \neq 1$

$$\mathsf{V}^{(b)}_{g,0} \sim \frac{1}{2\sqrt{2}\pi^{\frac{5}{2}}} \left(4\sqrt{2} b \sin(\pi b^2)\right)^{2-2g} (1 - b^4)^{2g-\frac{5}{2}} \Gamma\left(2g - \frac{5}{2}\right)$$


Sinh dilaton gravity
Worldsheet description:
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Matrix integral
$\rho_0^{(b)}(E) = \frac{\sinh(b\sqrt{E})\sinh(b^{-1}\sqrt{E})}{\sqrt{E}}$

Thank you!