Feynman's Last Blackboard: From Bethe Ansatz to Langlands Duality

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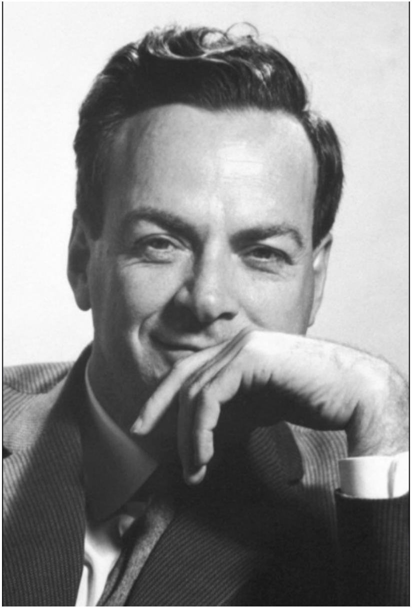
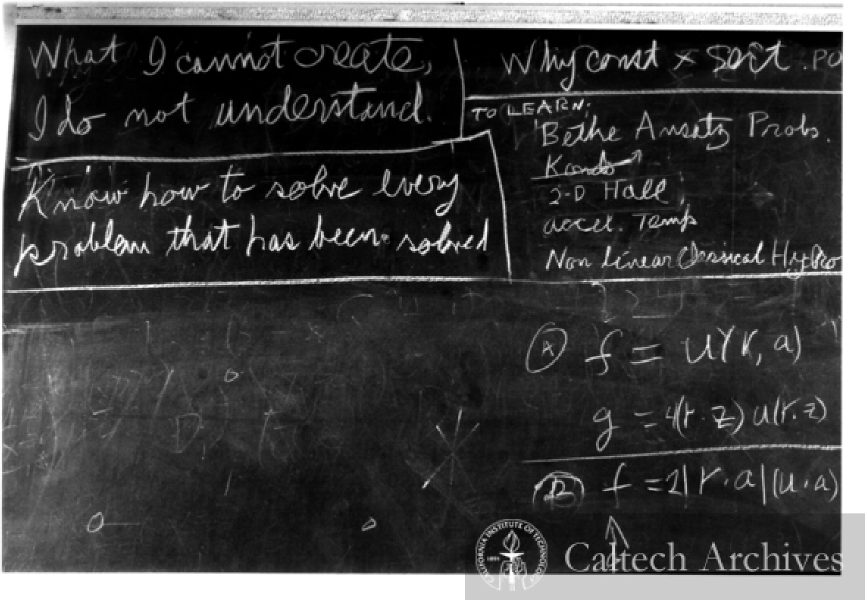
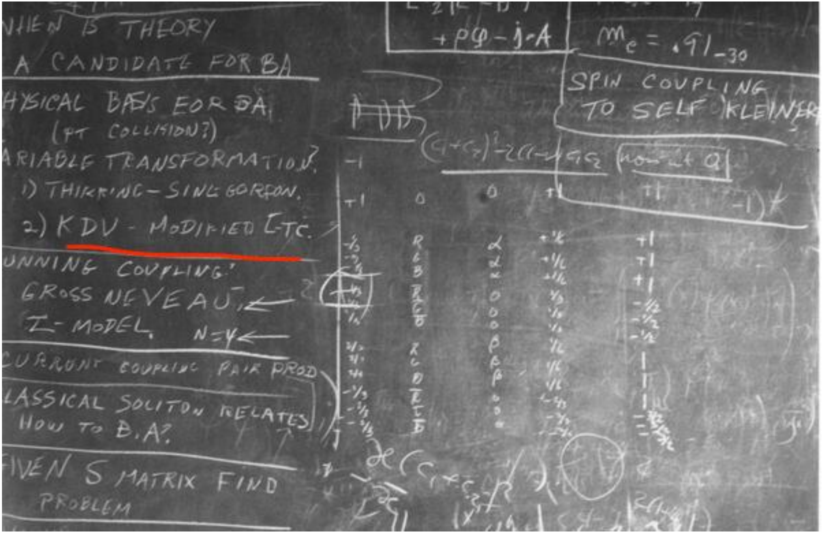
Richard Feynman (1918–1988)



Richard Feynman's Last Blackboard at Caltech




HEN IS THEORY
A CANDIDATE FOR BA
HYSICAL BASIS FOR BA
(PT COLLISION?)
ARIABLE TRANSFORMATION?
1) THIRRING – SINE GORDON.
2) KDV - MODIFIED ETC.
UNNING COUPLING?
GROSS NEVEAU
Σ-MODEL.
N=4
URRENT COUPLING PAIR PROD
LASSICAL SOLITON RELATES
HOW TO B.A?
IVEN S MATRIX FIND
PROBLEM
+ρφ − j·A
$m_e = .91_{-30}$
SPIN COUPLING
TO SELF KLEINER

4/42

Bethe Ansatz

An elegant method for solving quantum-mechanical models, introduced by Hans Bethe in 1931 in the case of Heisenberg's XXX model (1-dim. spin chain with space of states $(\mathbb{C}^2)^{\otimes N}$).

Namely, Bethe proposed an explicit formula for the eigenvectors of the Hamiltonian of the XXX model depending on certain parameters. These are indeed eigenvectors iff a certain system of equations is satisfied – Bethe Ansatz equations (BAE).

This method has subsequently been applied to many other integrable models, both discrete (QM) and continuous (2d QFTs), and proved to be surprisingly successful.

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Why was Feynman so interested in it?

Feynman's notes, Caltech Lunch Seminar, 01/22/1987

(summary of the beginning)

Bethe Ansatz

Many different two-dimensional field theories have been proposed as models to learn from.

Sometimes, surprisingly, they can be solved; for example

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Non-linear Schrödinger
Thirring
sine-Gordon
Gross-Neveu (running coupling constant)
$O(N)$ σ-**model**
Two-dimensional statistical mechanics (Onsager, Baxter)

All solved by the same method: guessing the form of eigenvectors

Bethe Ansatz (1931)

Mystery: When will it work?

Connection to classical solitons [later in the notes: **Quantum KdV**]

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(4) *Fun*

Some recent works linking BA & 4d gauge theory

Lipatov (1993), Faddeev-Korchemsky (1994) QCD $\rightsquigarrow$ XXX model

Minahan-Zarembo (2002), Beisert-Staudacher (2003) N=4 4d SYM (AdS_5/CFT_4) Gromov-Kazakov-Leurent-Volin (2013) QQ-system

Nekrasov-Shatashvili (2009) N=2 4d SYM with Ω-background $\rightsquigarrow$ Yang-Yang functions of integrable systems

Gaiotto-Witten (2011) S-duality in N=4 4d SYM $\rightsquigarrow$ Gaudin model

Costello (2013), Costello-Witten-Yamazaki (2017), Costello-Gaiotto-Yagi (2021) 4d Chern-Simons theory $\rightsquigarrow$ integrable models

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Gaiotto-Lee-Vicedo-Wu (2020) **Kondo problem** & Gaudin model

Gaudin model

Space of states: $(\mathbb{C}^2)^{\otimes N}$, or more generally, $\otimes_{i=1}^N V_{\lambda_i}$

$V_\lambda, \lambda \in \mathbb{Z}_{\geq 0}$ – finite-dim. rep. of $\mathfrak{sl}_2$ of dim. $\lambda + 1$ (spin $\lambda/2$)

Basis of $\mathfrak{sl}_2$: $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad e = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

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Gaudin Hamiltonians (for mutually distinct $z_i \in \mathbb{C}$):

$$H_i = \sum_{j \neq i} \frac{e^{(i)} \otimes f^{(j)} + f^{(i)} \otimes e^{(j)} + \frac{1}{2} h^{(i)} \otimes h^{(j)}}{z_i - z_j}, \quad i = 1, \dots, N$$

(appear on the RHS of the KZ equations).
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Problem: diagonalize them on $\otimes_{i=1}^N V_{\lambda_i}$

More precisely, the decomposition of $\otimes_{i=1}^N V_{\lambda_i}$ under the diagonal $\mathfrak{sl}_2$ action is preserved by the H_i's.

Hence we consider the problem of finding eigenvectors and eigenvalues of the H_i's on the *subspace of highest weight vectors* in $\otimes_{i=1}^N V_{\lambda_i}$ w.r.t. diagonal $\mathfrak{sl}_2$ (i.e. annihilited by the diagonal e) *of weight*

$$\lambda_\infty := \sum_{i=1}^N \lambda_i - 2m$$

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For $m = 0$, this subspace is spanned by

$$|0\rangle = \otimes_{i=1}^N v_{\lambda_i}$$

where v_{λ_i} is the *highest weight vector* in V_{λ_i}.

Bethe Ansatz in Gaudin model

For $w \in \mathbb{C}, w \neq z_i$, let

$$f(w) = \sum_{i=1}^{N} \frac{f^{(i)}}{w - z_i}$$

Define the Bethe vector

$$|w_1, w_2, \ldots, w_m\rangle := f(w_1)f(w_2)\ldots f(w_m)|0\rangle$$

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Lemma. This vector is an **eigenvector of the Gaudin Hamiltonians** iff the following system of Bethe Ansatz equations is satisfied:

$$\sum_{i=1}^{N} \frac{\lambda_i/2}{w_j - z_i} - \sum_{s \neq j} \frac{1}{w_j - w_s} = 0, \qquad j = 1, \ldots, m$$

Eigenvalues of the Gaudin Hamiltonians

$$H_i \,|w_1, w_2, \ldots, w_m\rangle = \mu_i \,|w_1, w_2, \ldots, w_m\rangle$$

Let $\quad v(z) := \sum_{i=1}^{N} \frac{\lambda_i(\lambda_i+2)/4}{(z-z_i)^2} + \sum_{i=1}^{N} \frac{\mu_i}{z-z_i}.$

$$v(z) = u(z)^2 - \partial_z u(z), \qquad u(z) = \sum_{i=1}^{N} \frac{\lambda_i/2}{z-z_i} - \sum_{j=1}^{m} \frac{1}{z-w_j}$$

Miura transformation

$$\partial_z^2 - v(z) = (\partial_z - u(z))(\partial_z + u(z)).$$

PSL_2-opers describe the spectrum

PSL_2-oper (a.k.a. projective connection) is a differential operator

$$\partial_z^2 - v(z) : \quad K^{-1/2} \longrightarrow K^{3/2}$$

(transforms as the stress tensor in CFT)

The joint **spectrum of the Gaudin Hamiltonians:**

PSL_2-opers on $\mathbb{CP}^1$

- with regular singularities at $z_i, i = 1, \ldots, N$, and ∞;
- with leading terms $\lambda_i(\lambda_i + 2)/4, i = 1, \ldots, N$, and $\lambda_\infty(\lambda_\infty + 2)/4$;

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- with **trivial monodromy**

These conditions $\Leftrightarrow$ the PSL_2-oper is the Miura transformation of first-order diff. operator with reg. sing. & residues λ_i at z_i, λ_∞ at z_∞.

Generalization to an arbitrary simple Lie algebra $\mathfrak{g}$

It is easy to construct analogues of the (quadratic) Gaudin Hamiltonians using an invariant bilinear form on $\mathfrak{g}$:

$$H_i = \sum_{j \neq i} \frac{\sum_a J_a^{(i)} J^{a(j)}}{z_i - z_j}, \quad i = 1, \ldots, N$$

Questions:

- Are there higher order commuting Hamiltonians forming a commutative subalgebra $\mathcal{A} \subset U(\mathfrak{g})^{\otimes N}$?
- Is there an explicit formula for the eigenvectors?
- What are the corresponding Bethe Ansatz equations?
- Can we describe the spectrum in terms of geometric objects on $\mathbb{CP}^1$ like opers?

Master Algebra

Feigin-F.-Reshetikhin (1994); F.'s ICMP'94 talk

Let $\widehat{\mathfrak{g}}$ be the affine Kac–Moody algebra associated to $\mathfrak{g}((t))$.

For $k \in \mathbb{C}$, let $\widetilde{U}(\widehat{\mathfrak{g}})_k$ be the completion of $U(\widehat{\mathfrak{g}})$ with level k.

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Example. The coefficients S_n of Sugawara current:

$$S(z) = \frac{1}{2} \sum_a : J^a(z) J_a(z) := \sum_{n \in \mathbb{Z}} S_n z^{-n-2}$$

Commutation relations: $[S_n, J^a_m] = -(k + h^\vee) m J^a_{n+m}$

where $h^\vee$ is the dual Coxeter number ($h^\vee = n$ for $\mathfrak{sl}_n$).

Thus, S_n are central elements of $\widetilde{U}(\widehat{\mathfrak{g}})_k$ when $k = -h^\vee$, **critical level**

The center of the completed enveloping algebra of $\widehat{\mathfrak{g}}$

Let $Z(\widehat{\mathfrak{g}})_k$ be the **center** of $\widetilde{U}(\widehat{\mathfrak{g}})_k$.

Theorem (Feigin-F.)

(1) $Z(\widehat{\mathfrak{g}})_{-h^\vee} \simeq \mathrm{Fun}\,\mathrm{Op}_{{}^L G}(D^\times)$

(2) If $k \neq -h^\vee$, then $Z(\widehat{\mathfrak{g}})_k = \mathbb{C}$.

$\mathrm{Op}_{{}^L G}(D^\times)$ – the space of ${}^L G$-opers on $D^\times$, the punctured formal disc

Here ${}^L G$ – simple Lie group of adjoint type whose Lie algebra

${}^L \mathfrak{g}$ is Langlands dual to $\mathfrak{g}$

Opers

If $\mathfrak{g} = \mathfrak{sl}_2$, then ${}^L G = PSL_2$ and PSL_2-oper on $D^\times$ is a second order differential operator $\partial_z^2 - v(z)$ where $v(z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-2}$.

Therefore, $\text{Fun}\,\text{Op}_{PSL_2}(D^\times)$ is a completion of $\mathbb{C}[v_n]^\sim_{n \in \mathbb{Z}}$.

The isomorphism $Z(V_{-2}(\mathfrak{sl}_2)) \simeq \text{Fun}\,\text{Op}_{PSL_2}(D_x)$

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${}^L G$-opers are, roughly speaking, gauge equivalence classes ${}^L \mathfrak{g}$-valued connections $\partial_z + A(z)$ on $D^\times$

(Drinfeld-Sokolov (1984), Beilinson-Drinfeld (2005))

$\mathrm{Fun}\,\mathrm{Op}_{{}^L G}(D^\times)$ is freely generated by $\ell = \mathrm{rk}(\mathfrak{g})$ series of elements $v_{i,n}, i = 1, \ldots, \ell; n \in \mathbb{Z}$, which under the F-F isomorphism correspond to *higher Sugawaras* in the center $Z(\widehat{\mathfrak{g}})_{-h^\vee}$.

Back to Gaudin model

Using $\widehat{\mathfrak{g}}$-conformal blocks, it is easy to construct a family of homomorphisms $Z(\widehat{\mathfrak{g}})_{-h^\vee} \to U(\mathfrak{g})^{\otimes N}$ depending on $\mathbf{z} = \{z_i, i = 1, \ldots, N\}$, such that

$$S(z) \mapsto \sum_{i=1}^{N} \frac{Cas^{(i)}}{(z - z_i)^2} + \sum_{i=1}^{N} \frac{H_i}{z - z_i}$$

Higher Sugawaras then give rise to *higher Gaudin Hamiltonians*.

The image $\mathcal{A}_{\mathbf{z}}$ is a commutative subalgebra of $U(\mathfrak{g})^{\otimes N}$ and the problem is to diagonalize its action on $\otimes_{i=1}^{N} V_{\lambda_i}$.

F-F-R constructed Bethe vectors and Bethe Ansatz equations using the free field (Wakimoto) realization of $\widehat{\mathfrak{g}}$. However, for $\mathfrak{g} \neq \mathfrak{sl}_2$ they do not always give rise to a basis of eigenvectors.

The spectrum and Langlands duality

Nonetheless, we can use the F-F isomorphism to describe the spectrum of $\mathcal{A}_{\mathbf{z}}$ on $\otimes_{i=1}^N V_{\lambda_i}$ directly without invoking Bethe vectors!

Theorem (Feigin-F.-Rybnikov)
The joint spectrum of the algebra $\mathcal{A}_{\mathbf{z}}$ of generalized Gaudin Hamiltonians on $\otimes_{i=1}^N V_{\lambda_i}$ is in bijection with the set of ${}^L G$-opers on $\mathbb{CP}^1$ with regular singularities at $z_i, i = 1, \ldots, N$, and ∞ with the "leading terms" determined by $\lambda_i, i = 1, \ldots, N$, and λ_∞ and **trivial monodromy**.

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There is also a generalization with irregular singularities, Feigin-F.-Toledano Laredo & Rybnikov (2007).

Langlands Correspondence

In mathematics, Langlands correspondence can be formulated in 3 different domains (in the framework of André Weil's *Rosetta Stone*):

Number Fields *Curves over* $\mathbb{F}_q$ *Curves over* $\mathbb{C}$

Langlands initially formulated his correspondence (in the late 1960s) in the first two domains, aiming to solve difficult questions in Number Theory using tools of Harmonic Analysis.

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Starting in the 1980s, in the works of Deligne, Drinfeld, Laumon and others, similar structures were found in the third domain, giving rise to the *geometric Langlands correspondence*.

However, there was a significant **difference** between the formulations in the first two domains and the third.

In the first two domains, we have the Hilbert space of **functions** on a certain natural discrete set with a measure, attached to a reductive algebraic group G, and a family of commuting **Hecke operators** acting on it. Langlands correspondence describes their joint spectra in terms of homomorpisms of the relevant Galois group to ${}^LG$.

On the other hand, in the geometric Langlands correspondence for a Riemann surface X, we have a category of **sheaves** on the moduli stack of G-bundles on X and **Hecke functors** acting on this category. The geometric Langlands correspondence can be viewed as an equivalence between this category and another category, associated to ${}^LG$.

The prevailing wisdom in the subject was that a function-theoretic formulation was not appropriate, or even possible, for complex curves. **This turned out to be incorrect!**

S-duality

Kapustin-Witten (2006) linked the geometric/categorical Langlands correspondence for a Riemann surface X to the S-duality of (twisted topological) $N = 4$ 4d SYM theories with gauge groups G_c and ${}^L G_c$ on the 4-manifold $\Sigma \times X$.

Specifically, to the equivalence of the corresponding categories of A- and B-branes on the *Hitchin moduli spaces*, which naturally appear after the 2d compactification along X (e.g. Hecke functors become 't Hooft line operators acting on A-branes, etc.). This has inspred a great deal of research in this area.

S-duality has an explanation in terms of string theory (Vafa (1998)): namely, we realize $N = 4$ 4d SYM theories as (orbifolds of) compactifications on dual tori of Type IIA (or IIB) string theories on ALE spaces & applying T-duality twice, for both circles on the torus.

Analytic Langlands Correspondence

Etingof-F.-Kazhdan (2019-2021) proposed a novel **analytic version** of the Langlands correspondence for complex curves (i.e. *function-theoretic* instead of *sheaf-theoretic*), following earlier works by Teschner (2017) and Langlands (2018).

Moreover, the two versions (categorical & analytic) complement each other. We can use each of them to gain new insights about the other.

Analogy: correlation functions in 2D conformal field theory are single-valued bilinear combinations of (multi-valued) conformal and anti-conformal blocks.

Gaiotto-Witten (2021) have given an elegant interpretation of the analytic Langlands correspondence in terms of the S-duality and the *brane quantization* (Gukov-Witten (2008))

A brief summary of E-F-K

For each pointed Riemann surface X and a Lie group G there is a Hilbert space $\mathcal{H}_{X,G}$ of half-densities on Bun_G and a family of commuting operators on it:

- Hecke operators (integral);
- differential operators, holomorphic (Beilinson-Drinfeld) and anti-holomorphic.

$X = \mathbb{CP}^1$ – these differential operators are the generalized Gaudin Hamiltonians (and their complex conjugates)!

Conjecture: The joint spectrum of these commuting operators can be identified with the set of ${}^L G$-opers on X whose monodromy is in the *split real form* ${}^L G(\mathbb{R})$ of ${}^L G(\mathbb{C})$.

This is the analytic Langlands correspondence for **curves over** $\mathbb{C}$.

Analytic Langlands Correspondence for curves over $\mathbb{R}$

Next, we consider the case of an algebraic curve defined over $\mathbb{R}$, rather than over $\mathbb{C}$ (E-F-K, to appear soon).

If our curve is $\mathbb{CP}^1$, this gives us a **unified framework** for a large class of Gaudin models, with tensor products of representations of $\mathfrak{g}$ of different types as the spaces of states. A similar picture appears in higher genera as well.

The Hamiltonians of all of these quantum integrable systems come from the same **master algebra** $Z(\widehat{\mathfrak{g}})_{-h^\vee}$ which (via the F-F isomorphism we have discussed) is isomorphic to the algebra of functions on $\mathrm{Op}_{^LG}(D^\times)$.

This, and the existence of the commuting Hecke operators, implies that the spectrum can be expressed in terms of ${}^L G$-opers on $\mathbb{CP}^1$ (with singularities at our points) whose **monodromy** satisfies certain conditions (depending on the types of these representations).

The simplest case: finite-dimensional representations of $\mathfrak{g}$. Then the spectrum consists of ${}^L G$-opers with **trivial monodromy**.

Other types of representations $\Longrightarrow$ other monodromy conditions.

[A closely related description of the spectra of the Gaudin Hamiltonians has been obtained in some cases by Nekrasov-Rosly-Shatashvili (2011) by other methods.]

This kind of description suggests the following **modern version of Bethe Ansatz:**

It is no longer about finding explicit formulas for eigenvectors of the commuting quantum Hamiltonians but about describing their joint spectrum in terms of *dual classical geometric objects* (e.g. ${}^L G$-opers).

Such a description of the spectrum can be seen as a particular **duality**, which may well be related to a fundamental duality of QFT and/or string theory. Our **challenge** is then to determine what it is.

(Finding a master algebra of commuting quantum Hamiltonians and finding its spectrum can also be helpful.)

For example, in the case of the Gaudin model, mathematically this duality is a special case of the Langlands duality, and it is a manifestation of S-duality of $N = 4$ 4d SYM theories, which can be derived from Type IIA/B string theory.

I will now describe other examples.

Deformations

- We can consider $\widehat{\mathfrak{g}}$ *away from the critical level*. This corresponds to changing the coupling constant of the $N = 4$ 4d (twisted) SYM theory. We still have S-duality but there is no longer a *classical* side; both sides are quantum (**quantum Langlands**).
- We can stay at the critical level but deform $U(\widehat{\mathfrak{g}})$ to the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ (or the Yangian $Y(\mathfrak{g})$). Then Gaudin model gets deformed to a **quantum spin chain** of XXZ (or XXX) type, and on the other side opers become q-opers.
- We can do both deformations (**quantum q-Langlands**).
- We can go from $\mathfrak{g}$ to $\widehat{\mathfrak{g}}$, and hence from $\widehat{\mathfrak{g}}$ to a double loop algebra. As the result, we obtain **affine Gaudin models** and opers become affine opers. (We can also turn on q.)

Quantum Langlands

Recall the **F-F isomorphism:** $Z(\widehat{\mathfrak{g}})_{-h^\vee} \simeq \mathrm{Fun}\,\mathrm{Op}_{{}^LG}(D^\times)$

The RHS is actually the *classical $\mathcal{W}$-algebra* $\mathcal{W}({}^L\mathfrak{g})$ which is the Poisson algebra of functions on the phase space of ${}^L\mathfrak{g}$-**KdV system**.

The center $Z(\widehat{\mathfrak{g}})_{-h^\vee}$ also has a natural Poisson structure, and the F-F isomorphism is in fact an *isomorphism of Poisson algebras*.

Both algebras can be deformed: $\mathcal{W}({}^L\mathfrak{g}) \rightsquigarrow \mathcal{W}_{{}^L\beta}({}^L\mathfrak{g})$, where ${}^L\beta$ is a small parameter.

$Z(\widehat{\mathfrak{g}})_{-h^\vee} \rightsquigarrow \mathcal{W}_\beta(\mathfrak{g})$, where β is large.

Duality (F-F (1991)): $\mathcal{W}_\beta(\mathfrak{g}) \simeq \mathcal{W}_{{}^L\beta}({}^L\mathfrak{g}) \quad$ if $\quad {}^L\beta = \dfrac{1}{n_{\mathfrak{g}}\beta}$

This is connected to both T-duality and S-duality.

F-F isomorphism appears in the limit $\beta \to \infty$.

q-deformation

When we deform $\widehat{\mathfrak{g}}$ to $U_q(\widehat{\mathfrak{g}})$, the Gaudin model gets deformed to the XXZ spin chain for $\mathfrak{g} = \mathfrak{sl}_2$ and its generalizations.

In the simplest case, the space of states becomes $\otimes_{i=1}^N V_{\lambda_i}^q(z_i)$, where $V_{\lambda_i}^q$ is a finite-dimensional (level 0) representation of $U_q(\widehat{\mathfrak{g}})$. The parameters z_i are now the spectral paramers of these representations.

Commuting quantum Hamiltonians are the **transfer-matrices** $T_V(z)$, where $V \in \operatorname{Rep} U_g(\widehat{\mathfrak{g}})$, or more generally, $\operatorname{Rep} U_q(\widehat{\mathfrak{b}}_+)$.

The problem is to diagonalize them on $\otimes_{i=1}^N V_{\lambda_i}^q(z_i)$ (or more general representations of $U_g(\widehat{\mathfrak{g}})$).

For $\mathfrak{sl}_2$

Baxter (1972) Elegant reformulation of Bethe Ansatz:

Let $T(z)$ be the transfer-matrix of the 2-dim. rep. of $U_q(\widehat{\mathfrak{sl}}_2)$, and let $t(z)$ be one of its eigenvalues in $\otimes_{i=1}^N V_{\lambda_i}^q(z_i)$.

Consider the q-difference equation (**Baxter's TQ-relation**):

$$(D^2 - t(z)D + 1)Q(z) = 0, \qquad (D \cdot f)(z) = f(zq^2).$$

Then there is a unique solution $Q(z)$ which is a polynomial (*Baxter polynomial*), up to a universal factor that is the same for all eigenvalues in $\otimes_{i=1}^N V_{\lambda_i}^q(z_i)$. Its roots satisfy the Bethe Ansatz eqs.

Moreover, all solutions of this second order q-difference equation (q-oper!) are then polynomials, up to the same universal factor – this is the q-analogue of the **no monodromy** condition we have encountered in Gaudin model.

The TQ-relation is a q-analogue of the Miura transformation appearing in the formula for the eigenvalues of the Gaudin Hamiltonians.

There exist analogues of the Baxter TQ-relation for a general Lie algebra $\mathfrak{g}$ in terms of the q-characters. (F.-Reshetikhin (1998), F.-Hernandez (2014)):

There are now $\ell = \text{rank}(\mathfrak{g})$ Baxter polynomials $Q_i(z)$, and the eigenvalues of $t_V(z)$ can be written in terms of these $Q_i(z)$.

Beautiful fact: $Q_i(z)$'s are transfer-matrices of special ∞-dim. representations, which actually satisfy a system of relations among themselves, called the **QQ-system**. This system leads to a more concise description of the spectra of quantum Hamiltonians for $U_q(\widehat{\mathfrak{g}})$.

For $\mathfrak{sl}_2$: the q-Wronskian relation of B-L-Z (1996).

For $\mathfrak{sl}_n$: Bazhanov-Frassek-Łukowski-Meneghelli-Staudacher (2011).

For a general simple Lie algebra $\mathfrak{g}$: F.-Hernandez (2016 & to appear).

QQ-system for $\mathfrak{gl}(4|4)$ plays an important role in N=4 4d SYM (and $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence) – *quantum spectral curve* of Gromov-Kazakov-Leurent-Volin (2013).

Moreover, it also appeared in the study of the spectra of **affine opers** that appear on the **dual side** of quantum KdV (Masoero-Raimondo-Valeri (2015)), which we'll discuss shortly.

q-opers

The QQ-system can be described (at least for simply-laced $\mathfrak{g}$) in terms of **Miura** (G,q)**-opers** (F.-Koroteev-Sage-Zeitlin (2020))

Closely related work on *fused flags* by Ekhammar-Shu-Volin (2021)

spectrum of $U_q(\widehat{\mathfrak{g}})$ Hamiltonians	$\leftrightarrow$	$({}^L G, q)$-opers with Miura structure

This is a q-deformation of the Langlands duality we discussed earlier:

spectrum of $\mathfrak{g}$-Gaudin Hamiltonians	$\leftrightarrow$	${}^L G$-opers with Miura structure $\Leftrightarrow$ no monodromy

Quantum q-Langlands and string duality

When we turn on both parameters, q and $k + h^\vee$, we obtain **quantum q-Langlands** duality (Aganagic-F.-Okounkov (2017)):

Origin: Duality in little string theory on an ALE space times a torus, with non-zero string tension (which corresponds to $k + h^\vee \neq 0$).

Affine Gaudin models

We now keep $q = 1$ but replace $\mathfrak{g}$ by $\widehat{\mathfrak{g}}$, so $\widehat{\mathfrak{g}}$ should be replaced by $\widehat{\widehat{\mathfrak{g}}}$.

Then Gaudin model $\rightsquigarrow$ **affine Gaudin model** (Feigin-F. (2007)).

Classical L-operator of the Gaudin model (with irreg. sing. at ∞):

$$L = \sum_{i=1}^{N} \frac{A_i}{z - z_i} + \chi, \qquad A_i \in \mathfrak{g}^*, \qquad \text{with fixed}\, \chi \in \mathfrak{g}^*$$

In the affine Gaudin model:

$$L = \sum_{i=1}^{N} \frac{\partial_t + A_i(t)}{z - z_i} + \chi, \qquad \partial_t + A_i(t) \in \widehat{\mathfrak{g}}_1^*$$

Soliton hierarchies

Simplest case: $N = 0$

$$L = \frac{\partial_t + A(t)}{z} + \chi \qquad \sim \qquad L = \partial_t + A(t) + \chi z$$

General form of an L-**operator an integrable soliton hierarchy**!
z – spectral parameter.

$\chi \in \mathfrak{h}^*$ $\quad \rightsquigarrow \quad$ $\mathfrak{g}$-AKNS hierarchy

$\chi = e_{\alpha_{\max}}$ & Drinfeld-Sokolov reduction $\quad \rightsquigarrow \quad$ $\mathfrak{g}$-KdV hierarchy

$\mathfrak{g} = \mathfrak{sl}_2$: $L = \partial_t - \begin{pmatrix} 0 & v(t) + z \\ 1 & 0 \end{pmatrix} \quad \sim \quad \partial_t^2 - v(t) - z$

Likewise, we obtain an integrable system for any number of singular points in L.

Quantization

For $\mathfrak{sl}_2$: Quantum Hamiltonians: local (Feigin-F. (1992)) and non-local (Bazhanov-Lukyanov-Zamolodchikov (1994))

How to describe their spectra on irreducible reps of Virasoro algebra?

Dorey-Tateo (1998) (special case), B-L-Z (1998, 2003) (in general) related them to spectral determinants of one-dimesional Schrödinger operators of a special kind.

This became known as the **ODE/IM correspondence** that has since been realized in a large class of models.

Important feature of Schrödinger operators: they have regular singularities and *trivial monodromy*!

Feigin-F. (2007) interpreted this as an affine analogue of the **Langlands duality** description of the spectra of the Gaudin model.

Langlands duality for quantum KdV

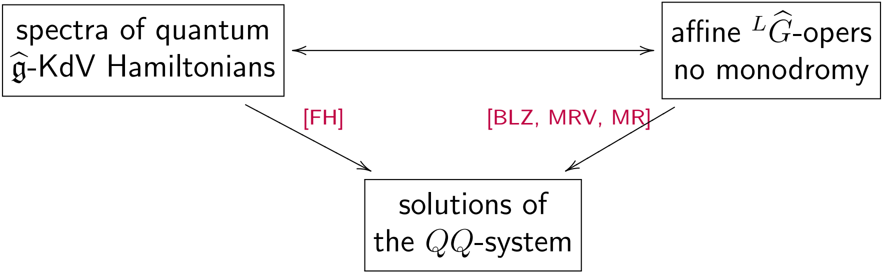
Recall that for the finite Gaudin model:

spectrum of $\mathfrak{g}$-Gaudin Hamiltonians	$\leftrightarrow$	${}^L G$-opers with trivial monodromy

Now, for the affine Gaudin model:

spectrum of $\widehat{\mathfrak{g}}$-Gaudin Hamiltonians	$\leftrightarrow$	affine ${}^L\widehat{G}$-opers with trivial monodromy

Quantum KdV: Link via the QQ-system



[BLZ] Bazhanov-Lukyanov-Zamolodchikov (2003)
[MRV] Masoero-Raimondo-Valeri (2015)
[MR] Masoero-Raimondo (2018)
[FH] F.-Hernandez (2016)

Affine Gaudin models, Kondo Problem & 4d CS theory

Gaiotto-Lee-Vicedo-Wu (2020) Interpretation of the **Kondo problem** in terms of affine Gaudin models.

Kotousov-Lukyanov (2021)

Vicedo (2019) Link between the affine Gaudin models & 4d CS theory.

Costello (2013), Costello-Witten-Yamazaki (2017) 4d Chern-Simons theory $\rightsquigarrow$ quantum integrable models (such as XXX model)

Costello-Gaiotto-Yagi (2021) TQ-relation and QQ-system naturally appear in 4d Chern-Simons theory, where T is a Wilson line operator and Q is a 't Hooft line operator.

Kotousov-Lacroix-Teschner (2022) applications of affine Gaudin models to **nonlinear sigma models**.

Open Problems

- Find the master algebra of affine Gaudin models (it can be viewed as an analogue the center of the enveloping algebra of a double loop algebra at its "critical level")
- Is there a string theory explanation for the affine Gaudin models' Langlands duality?
- Is there a q-deformation of affine Gaudin models, and if so, what is the corresponding Bethe Ansatz?
- Are there applications of "Bethe Ansatz" to *realistic* 4d gauge theories like QCD (Richard Feynman's dream)?