# A tensor model for approximate conformal field theories

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Appearing soon with Alex Belin, Jan de Boer, Pranjal Nayak, Julian Sonner

- Introduction
- Approximate CFTs
- Random tensor model
- 3d gravity

# Ensembles for pseudo-random microdata

- Features of high energy microstates of chaotic quantum systems are pseudo-random.
- One can encode their statistical properties in an ensemble, such that typical elements preserve EFT observables.
- For quantum mechanics, this gives a random matrix model for the matrix elements of simple operators between the high energy microstates, for example $H_{ij}, \mathcal{O}_{ij}$ .
- For CFTs, there are also the three point functions of high dimension operators, so it is a random tensor model, $C_{ijk}$ .

# Holographic matrix ensembles

- The Saad Shenker Stanford matrix model for Jackiw-Teitelboim gravity reinterprets a minimal string theory matrix model as the Wigner ensemble for the holographically dual Hamiltonian. The 't Hooft expansion of $< \mathrm{Tr}(e^{-\beta_1 H})\ \mathrm{Tr}(e^{-\beta_2 H}).. >$ gives the sum over 2d topologies, as dense Feynman diagrams fill out the surfaces in a double scaled limit.

- The lack of euclidean factorization is puzzling from the perspective of exact 2d holography.

- How does it work in higher dimensional versions of this?

[Mertens Turiaci; Collier Maloney Maxfield Tsiares; Cotler Jensen; Belin de Boer Liska; Belin de Boer Nayak Sonner; Anous Belin de Boer Liska; Chandra Collier Hartman Maloney ...]

# CFTs

- The microdata of a CFT consists of the dilation operator acting on the Hilbert space graded by spin, and the structure constants, C. These give a matrix and a tensor.

- The operators best characterized by an ensemble are in the dense part of the spectrum. For simplicity, could consider a large c 2d CFT, with no operators below the black hole threshold beside the identity Virasoro module.

- If there were light operators, these would define matrices $C_{LHH}$

- At small c, the discussion applies to the high dimension region of the spectrum.

# OPE statistics from 3d gravity

- The Cardy formula gives the asymptotics of the spectral density

$$\rho \approx \exp\left[2\pi(\frac{c}{6}(h-\frac{c}{24}))\right]\exp\left[2\pi(\frac{c}{6}(\bar{h}-\frac{c}{24}))\right]$$

- The leading order variance of the structure constants is given by $|C_{ijk}|^2 \approx C_0(h_i,h_j,h_k)C_0(\bar{h}_i,\bar{h}_j,\bar{h}_k)$ , where $C_0$ is the Liouville three point function given by the DOZZ formula, normalized by the spectral density.

- The associated Gaussian ensemble reproduces certain handlebody and euclidean wormhole topologies in 3d gravity.

[Chandra Collier Hartman Maloney]

# Higher d has more constraints

- An important point is that unlike QM, where any Hamiltonian is allowed, CFTs have many consistency constraints. 4 point crossing encodes locality, the vanishing of operator commutators outside the lightcone. In 2d, there is also torus 1 point function modular invariance.

- Therefore, we do not expect there to exist ensembles over exact CFT data, in contrast to QM Hamiltonians. No disordered exact duals are possible for higher dimensional gravity.

- We will define a notion of approximate CFT, which in an appropriate limit would localize to a true CFT.

# Which gravity EFTs have UV completions?

- In AdS, this turns into a problem of classifying CFTs: is there an exact CFT whose light operators behave like a given EFT's correlation functions in AdS?

- Around the AdS vacuum, perturbative solutions to the bootstrap exist for any EFT. There are some sharp "naturalness" bounds on the Wilson coefficients from dispersion relations.

  [Heemskerk Penedones Polchinski Sully; Caron-Huot Mazac Rastelli Simmons-Duffin]

- For black hole backgrounds, corresponding to thermal correlators, there are consistent ensembles for any EFT.

  [DLJ Kolchmeyer Mukhametzhanov Sonner]

- Major constraints must come from mutually locality of black hole microstate operators among themselves.

# Approximate CFTs

- Consists of a full collection of CFT data, subject to the conditions that crossing is obeyed up to a tolerance T, subject to bounds on
  - Total number of insertions
  - Dimension of external operators
  - Genus
  - Avoidance of extreme Lorentzian kinematics (Regge regime)

- These allow $e^S$ moduli of deformation away from a true CFT.

# From EFT to microstate

- One can show that this implies approximate crossing for almost all operators, including heavy operators, above the previous cutoff. This is done by performing an appropriate inversion transform on a higher genus or higher point function to pick out contributions of (bands of) heavy operators in some channel.

- It follows from Moore-Seiberg that it is equivalent to having 4 point crossing and torus 1 point modular invariance approximately satisfied for almost all operators.

- The CFT data purely for light operators can be taken to be fixed, or approximately fixed.

# Variance of crossing

- Although the expectation value of the crossing equation vanishes in the gaussian ensemble, individual instantiations strongly violate it and are not approximate CFTs. This is captured by a large variance, which must be cancelled by higher moments.

From genus 2

$$G_{1122}(x)|_{\mathrm{s}} = \sum_k |C_{12k}|^2 |\mathcal{F}_{1221}(O_k|x)|^2, \quad G_{1122}(x)|_{\mathrm{t}} = \sum_{k'} C_{11k'} C_{22k'} |\mathcal{F}_{1122}(O_{k'}|1-x)|^2$$

$$\overline{s-t} = 0, \text{ but } \overline{(s-t)^2} = \overline{s^2} + \overline{t^2} - 2\overline{st} \neq 0$$

- The s t cross term vanishes in the Gaussian ensemble. Therefore there must be a quartic moment to cancel the variance.

$$\overline{C_{ijk}C_{iml}C_{njl}C_{nmk}}\Big|_c = \left\{ \begin{matrix} \mathcal{O}_k & \mathcal{O}_j & \mathcal{O}_i \\ \mathcal{O}_l & \mathcal{O}_m & \mathcal{O}_n \end{matrix} \right\} = \left| \mathbf{F}_{kl} \left[ \begin{matrix} n & j \\ m & i \end{matrix} \right] C_0(h_i,h_j,h_k) C_0(h_k,h_n,h_m) \rho_0(h_l)^{-1} \right|^2$$

# “Local” version of crossing

- The Ponsot-Teschner crossing kernel F is exactly the object which transforms s-channel to t-channel blocks.

$$\mathcal{F}_{ijmn}\left(O_k|x\right) = \int \mathrm{d}[O_l]\, \mathbf{F}_{kl}\left[\begin{array}{cc} n & j \\ m & i \end{array}\right] \mathcal{F}_{imjn}\left(O_l|1-x\right)$$

- Expand the crossing equation in s-channel principal series

$$\sum_q \left( C_{i_1 i_2 q} C_{i_3 i_4 q} \delta^{(2)}\left(P_s - P_q\right) - C_{i_1 i_4 q} C_{i_2 i_3 q} \left| \mathbf{F}_{P_q P_s} \left[\begin{array}{cc} P_3 & P_4 \\ P_2 & P_1 \end{array}\right] \right|^2 \right) = 0\,,$$

- For Virasoro, the principal series are the above threshold physical weights. For other operators (including Id), the δ involves contour manipulation. In approximate CFTs, the restriction on kinematics implies the δ is smeared.

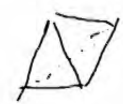
# Tensor model

- Instead of specifying moments, write an explicit ensemble:

$$\mathcal{Z} = \int D[L_0, \bar{L}_0]\, D[C]\, e^{-a\, V[L_0, \bar{L}_0, C]}$$

- The maximum ignorance ensemble consistent with crossing has V given by the sum of squares of the constraints.

$$V_4 = 2 \sum_{i_1 \cdots i_4}{}' \sum_{p,q} \left( \frac{C_{i_1 i_2 p} C_{i_3 i_4 p} C_{i_1 i_2 q} C_{i_3 i_4 q}}{|\rho_0(p) C_0(12p) C_0(34p)|^2} \delta^{(2)}\left(P_p - P_q\right) - \frac{C_{i_1 i_2 p} C_{i_3 i_4 p} C_{i_1 i_4 q} C_{i_2 i_3 q}}{|C_0(12p) C_0(34p) C_0(23q) C_0(14q)|^2} \left\{ \begin{array}{ccc} \mathcal{O}_q & \mathcal{O}_4 & \mathcal{O}_1 \\ \mathcal{O}_p & \mathcal{O}_2 & \mathcal{O}_3 \end{array} \right\} \right)$$



- p=Id, 1=2, 3=4 gives the propagator.

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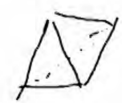

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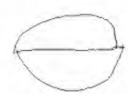


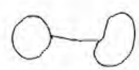
# Scaling limit

- The random tensor model is defined by truncating to a finite number of primaries, and then taking a triple scaled limit, in which the rank and overall coefficient ***a*** are taken to infinity, while the width of the smeared delta functions is taken to zero.
- In principle, in the strict limit, crossing is imposed exactly and one either finds a specific exact CFT or no solutions. What is interesting is to study the theory in the ‘t Hooft matrix + triple line tensor Feynman expansion. The expansion parameter is $e^{-c}$
- The ultimate fate of the strict limit is a “doubly non-perturbative” question - sum over all topologies.

# Modular invariance

- Similarly, modular invariance of torus 1 point functions can be written in terms of the modular inversion kernel, S, and we take its square as part of the potential.

$$V_{\mathrm{S},\neq\mathbf{1}} = 2\sum_{i,j,k}{}' \left|\rho_0(P_i)C_0(P_i,P_i,P_j)\right|^2 C_{iij}C_{kkj}\left(\delta^{(2)}\left(P_i-P_k\right)-\left|\mathbf{S}\left[\mathcal{O}_j\right]_{P_iP_k}\right|^2\right)$$



- This is another quadratic term. With the Id inserted, one gets

$$V_{\mathrm{S},\mathbf{1}} = 64\left(\sum_{i,j}{}' \left|\sinh(2\pi bP_i)\sinh\left(2\pi\frac{P_i}{b}\right)\right|^2\delta^{(2)}(P_i-P_j)-\sum_{i}{}' \left|\sinh(2\pi bP_i)\sinh\left(2\pi\frac{P_i}{b}\right)\right|^2-8\sum_{i,j}{}' \left|\sinh(2\pi bP_i)\sinh\left(2\pi\frac{P_i}{b}\right)\cos(4\pi P_iP_j)\right|^2\right)$$

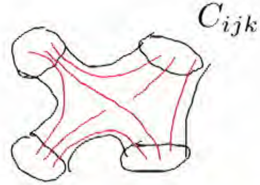
- T invariance imposes integrality of spin

$$V_T = \mathrm{Tr}\sin^2(\pi(L_0-\bar{L}_0))$$

# 3d gravity

- Conjecture: in the case with no fixed primaries aside from Id (thus a gap to the black hole threshold), then in $e^{-c}$ expansion, the triple scaled limit of the tensor model is exactly pure 3d gravity, including the sum over all hyperbolic 3-manifolds!

- Building block of the tensor part is the 4 boundary wormhole associated to the 6j vertex. The index loops are filled in with 't Hooft diagrams of the matrices – leading disk corresponds to BTZ filling of solid torus.




[Collier Eberhardt Zhang]

# 2 and 4 point resummed

- However, the bare propagator of the tensor model has a factor of $-\frac{1}{a}$. This is similar to imposing $\langle x^2 \rangle = v$ by an integral

$$\int dx\ e^{-a(x^2/v-1)^2} \sim \int dx\ e^{-a(-2x^2/v+x^4/v^2)}$$

- Need to resum infinitely many diagrams. One can check Schwinger-Dyson equations

- Very similar to simplicial 3d gravity of Regge, … Boulatov also involving 6j symbol vertices. Here matrix part also plays important role. But there is a sum over many diagrams for each 3-manifold. Integrability seems to imply that they are all proportional! So also like Turaev-Viro, Collier-Eberhardt-Zhang.

# Relation to M5 branes

- An intriguing fact is that the reduction of the 2 M5 brane theory on $S^3$ in the context of $\mathcal{N}=2$ sphere partition functions is exactly the integration cycle for SL(2,C) Chern-Simons theory that gives TTQFT on the 3-manifold.

  [Dimofte Gaiotto Gukov; Cordova DLJ; Mikhaylov]

- There is probably a connection of the tensor model to the M5 brane similar to the matrix model/minimal string.

- A distinction again is that here the sum over all of the complicated 3-manifold topologies ends up imposing exact crossing, rigidifying the model.

# Summary

- Defined a notion of approximate CFT, which is CFT data that obeys crossing up to small corrections away from extreme kinematics.

- One can define ensembles of such data, which must be strongly non-Gaussian.

- Leads to a tensor model that is completely determined by conformal symmetries – maximum ignorance ensemble.

- Its expansion is related to 3d gravity, connecting simplicial gravity and VTQFT.