Bespoke Dual Resonance

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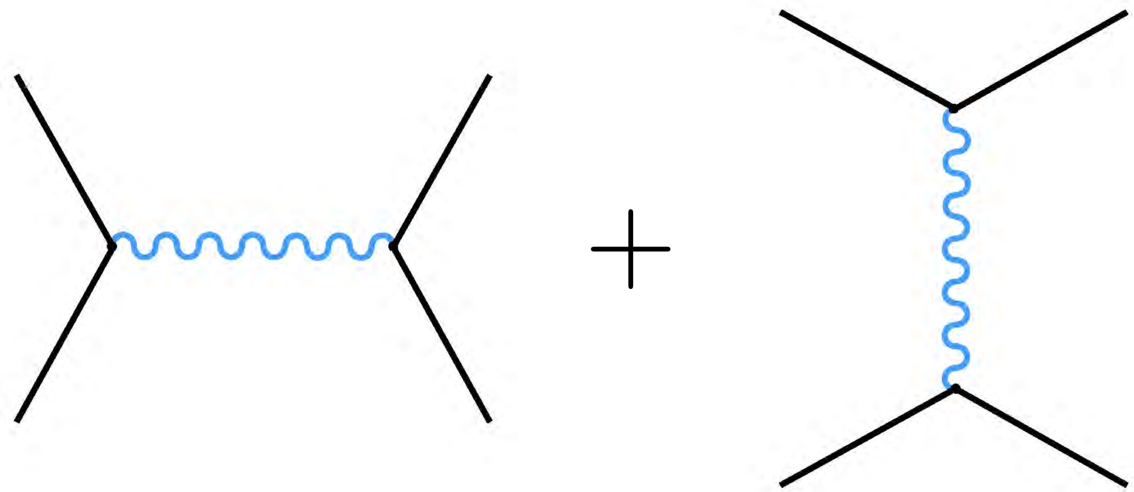
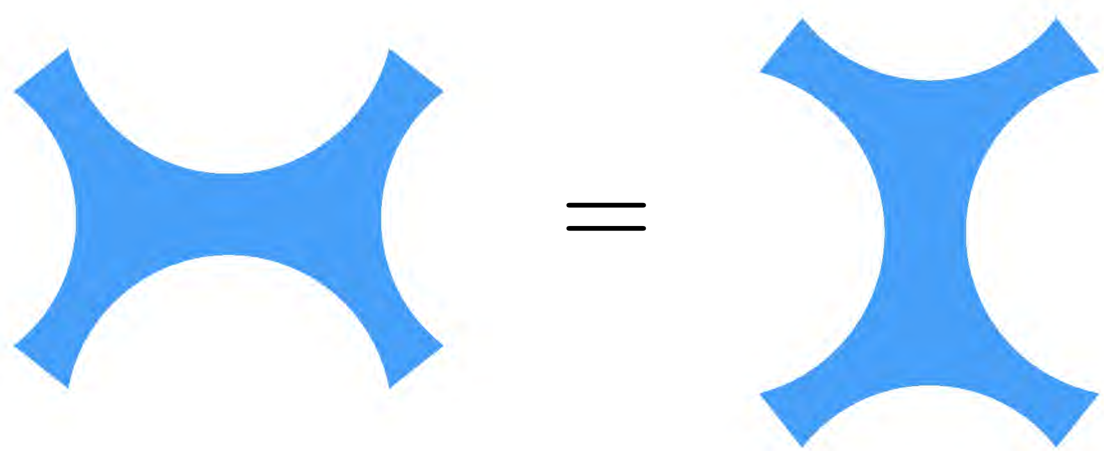
Dual resonance

- In a QFT, we build amplitudes as the *sum* of channels of different topologies:



s-channel (annihilation) $\quad$ t-channel (scattering)

- However, in string theory, the two topologies are indistinguishable, due to the worldsheet:



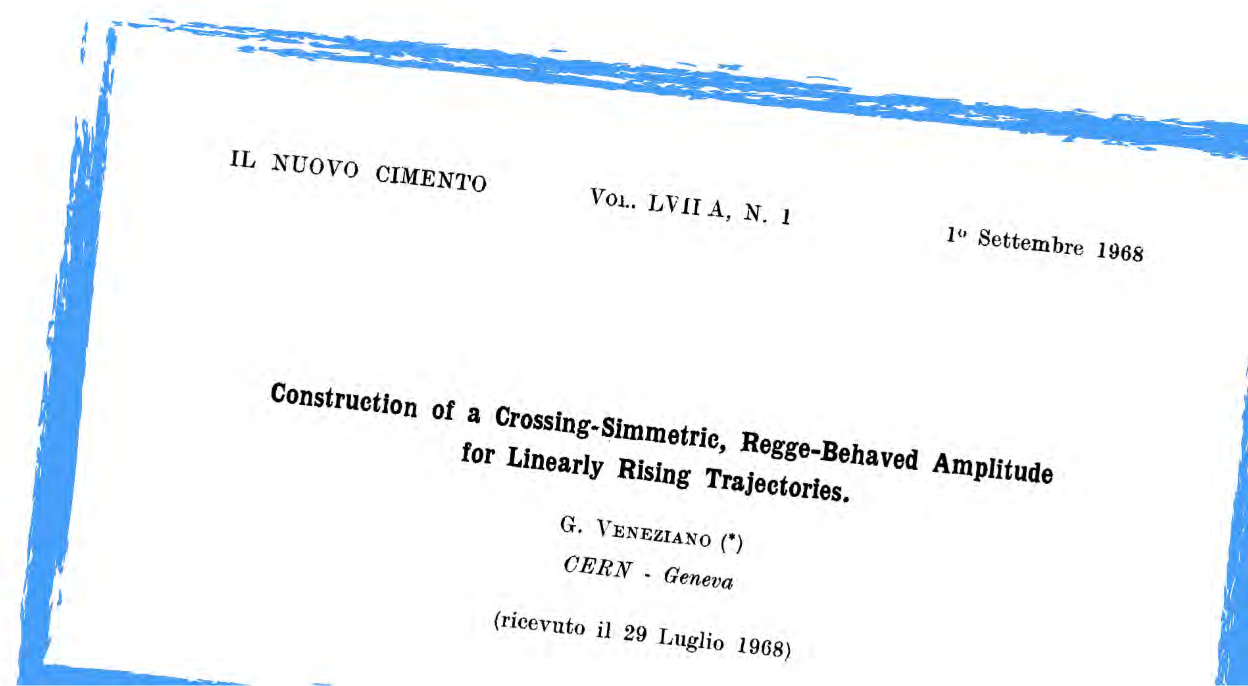
Dual resonance

String amplitudes

- What do string amplitudes do?
    - Ultraviolet-complete low-energy physics by taming Planck-scale pathologies in amplitudes.
    - Accomplish this by adding a tower of massive higher-spin degrees of freedom. (Cannot add just one higher-spin state without making the problem worse. e.g., CEMZ [1407.5597])
- So string theory answers the question of how to build an amplitude exchanging higher-spin modes consistently at high energies:

Veneziano amplitude: (1968)

$$A_V(s,t) = \frac{\Gamma(-s)\Gamma(-t)}{\Gamma(-s-t)}$$


IL NUOVO CIMENTO VOL. LVII A, N. 1 1° Settembre 1968

Construction of a Crossing-Simmetric, Regge-Behaved Amplitude for Linearly Rising Trajectories.

G. VENEZIANO (*)
CERN - Geneva

(ricevuto il 29 Luglio 1968)

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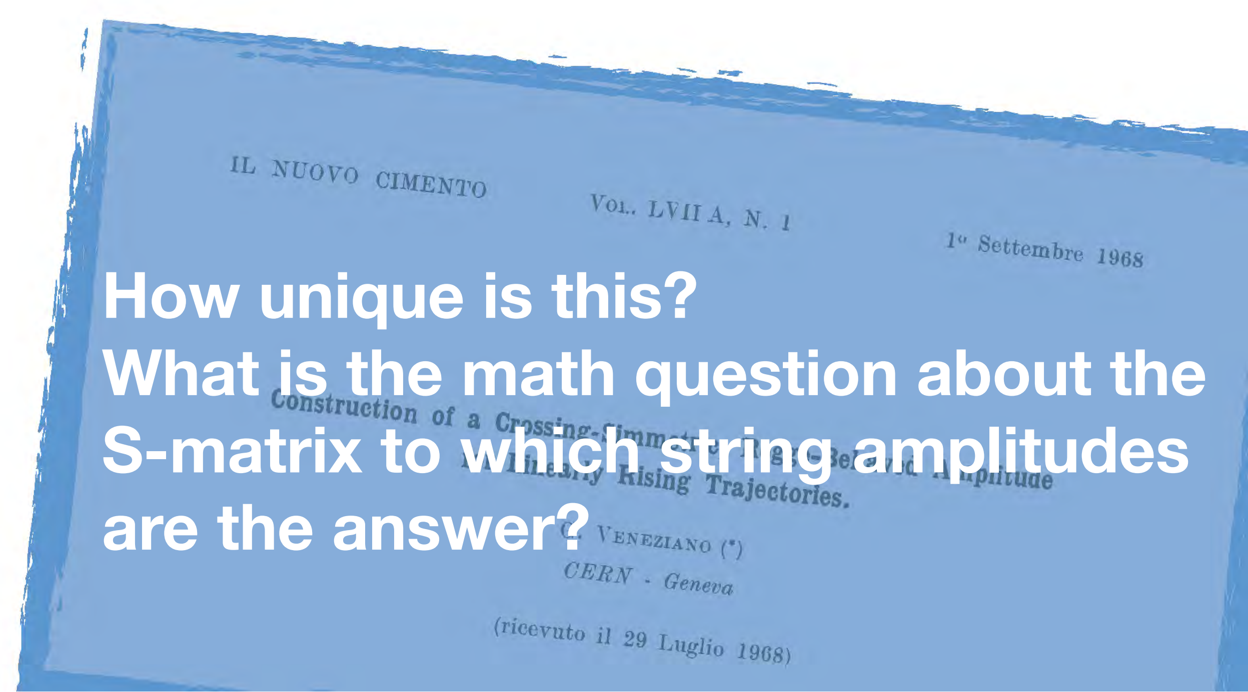

How unique is this?
What is the math question about the S-matrix to which string amplitudes are the answer?

String amplitudes

What makes (tree-level, planar) string amplitudes unique?

- Dual resonance?
- Towers of higher-spin states?
- Tame UV behavior?
- Straightforward generalization to n-point amplitudes?
- Worldsheet integral representation?

What properties of Veneziano amplitudes enable these miracles?

- Regge spectrum with $m_n^2 \propto n$?

String amplitudes

What makes (tree-level, planar) string amplitudes unique?

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In this talk, we will construct *non-string* amplitudes that satisfy all the desired properties marked with ✗, even while exhibiting *custom non-Regge spectra*.

The question of what uniquely fixes string amplitudes remains open, and we now have a plethora of other amplitudes to understand: Are they somehow part of string theory? What physics do they describe?

Dual resonance and UV finiteness

- Dual resonance is deeply tied to the asymptotic scaling of the amplitude in the Regge limit (large s, fixed t) Cheung, **GR** [2302.12263]

- For an amplitude satisfying crossing $A(s,t) = A(t,s)$, with tree-level poles at $s, t = \mu(n)$, then as long as the residue at $s = \infty$ is well defined,

$$\frac{1}{2\pi i} \oint_{s=\infty} \mathrm{d}s \frac{A(s,t)}{s} = A_\infty(t)$$

  we have dual resonance:

$$A(s,t) = \frac{1}{2\pi i} \oint_{s'=s} \frac{ds'}{s'-s} A(s',t)$$

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Polynomial residues:
finite # of states on each resonance

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Need infinite number of poles
to resum the propagator

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- For Veneziano amplitude, $A_V(s \to \infty, t) \sim s^t \implies A_\infty(t) = 0$ for $t < 0$, so

$$A_V(s,t) = \frac{\Gamma(-s)\Gamma(-t)}{\Gamma(-s-t)} = \sum_{n=0}^{\infty} \frac{R_V(n,t)}{n - s}$$

$$R_V(n,t) = \frac{1}{n!} \frac{\Gamma(t+n+1)}{\Gamma(t+1)} = \frac{1}{n!} \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} t^k$$

Dual Resonance for Any Spectrum

Spectral curve

- Define a function $f(\mu, \nu)$ whose zero locus will fix the spectrum of the theory:

$$\underbrace{f(\mu}_{\text{kinematic argument}}, \underbrace{\nu)}_{\text{level argument}} = 0$$

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- We will pick $f(\mu,\nu) = P(\nu) - \mu Q(\nu)$ for some polynomials P, Q so the spectrum is a rational polynomial:

$$\mu(n) = \frac{P(n)}{Q(n)}$$

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- If we choose P, Q to be monic, with P of degree h and Q of degree $h-1$, then $\mu(n)$ is asymptotically Regge, as required on general grounds.

Caron-Huot, Komargodski, Sever, Zhiboedov [1607.04253]

$$P(\nu) = \sum_{k=0}^{h} p_k \nu^{h-k} \qquad Q(\nu) = \sum_{k=1}^{h} q_k \nu^{h-k}$$

$$p_0 = q_1 = 1$$

- For sufficiently large h, we can fit any finite number of specified masses in the spectrum.

Galois meets Veneziano

- Write f as a product over its roots: $f(\mu, \nu) = \prod_{\alpha} (\nu - \nu_{\alpha}(\mu))$
- When $s, t = \mu(n)$, there exists some $\nu_{\alpha} \in \{\nu\}$ that equals n.

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- When $s,t = \mu(n)$, there exists some $\nu_\alpha \in \{\nu\}$ that equals n.
- We define our amplitude by the *Galois sum* over the Veneziano amplitude, sending $s,t \to \nu_\alpha(s), \nu_\beta(t)$:

$$\boxed{A(s,t) = \sum_{\alpha,\beta} A_V(\nu_\alpha(s), \nu_\beta(t))}$$

- Sum is over the Galois group of the roots of f
- Simple poles at $s,t = \mu(n)$

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- Sum is over the Galois group of the roots of f
- Simple poles at $s,t = \mu(n)$
- We can write our amplitude in a remarkable *d*log form as a kinematic transformation of the Mandelstam variables:

$$\begin{aligned} A(s,t) &= \frac{1}{2\pi i}\oint \sum_\alpha \frac{d\sigma}{\sigma - \nu_\alpha(s)} \frac{1}{2\pi i}\oint \sum_\beta \frac{d\tau}{\tau - \nu_\beta(t)} A_V(\sigma,\tau) \\ &= \oint \frac{\mathrm{dlog}\, f(s,\sigma)}{2\pi i} \oint \frac{\mathrm{dlog}\, f(t,\tau)}{2\pi i} A_V(\sigma,\tau) \end{aligned}$$

Asymptotics and control theory

- In the Regge limit of $s \to \infty$ at fixed t, string amplitudes scale exponentially,

$$A_V(s,t) \sim s^t$$

- How do the roots $\nu_\alpha(s)$ behave in this limit?
    - One (call it ν_0) asymptotes to s: $\lim_{s\to\infty} \nu_0(s)/s = 1$
    - The other $h-1$ limit to the s-independent roots of $Q(\nu)$

- The Regge limit of our bespoke amplitude therefore goes like:

$$A_\infty(t) \sim \lim_{s\to\infty} \sum_\beta \left(s^{\nu_\beta(t)} + \sum_{\alpha\neq 0} \nu_\alpha(s)^{\nu_\beta(t)} \right)$$

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- Dual resonance demands well defined $A_\infty(t)$, which requires:

$\mathrm{Re}(\nu_\beta(t)) < 0$ for all β, over our chosen range of t

$\implies$ Control theory: Hurwitz stability, Kharitonov's theorem, etc.

$$A_\infty(t) = \sum_{\alpha\neq 0} \sum_\beta A_V(\lim_{s\to\infty} \nu_\alpha(s), \nu_\beta(t))$$

Dual resonance and Newton's identities

- Given f satisfying the control theory conditions, $A(s,t)$ has a dual resonant representation,

$$A(s,t) = A_\infty(t) + \sum_{n=0}^{\infty} \frac{R(n,t)}{\mu(n) - s}$$

- The branch cuts in s cancel in the Galois sum over the propagators:

$$\sum_\alpha \frac{1}{n - \nu_\alpha(s)} = \frac{\partial_n f(s,n)}{f(s,n)}$$

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- The residues $R(n,t)$ are polynomials in t. All of the branch cuts in the $\nu_\beta(t)$ have precisely cancelled. This is a consequence of Galois theory, and can be formally proved using the fundamental theorem of symmetric polynomials.

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- The residues $R(n,t)$ are polynomials in t. All of the branch cuts in the $\nu_\beta(t)$ have precisely cancelled. We can calculate them directly by computing the power sums $d_k(t) = \sum_\alpha \nu_\alpha(t)^k$ using Newton's identities:

$$d_k(t) = (-1)^k \begin{vmatrix} p_1 - tq_1 & 1 & 0 & \cdots & 0 \\ 2(p_2 - tq_2) & p_1 - tq_1 & 1 & \cdots & 0 \\ 3(p_3 - tq_3) & p_2 - tq_2 & p_1 - tq_1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ k(p_k - tq_k) & p_{k-1} - tq_{k-1} & p_{k-2} - tq_{k-2} & \cdots & p_1 - tq_1 \end{vmatrix}$$

$$R(n,t) = \frac{\mu'(n)}{n!} \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} d_k(t)$$

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- Given f satisfying the control theory conditions, $A(s,t)$ has a dual resonant representation,

$$A(s,t) = A_\infty(t) + \sum_{n=0}^{\infty} \frac{R(n,t)}{\mu(n)-s}$$

- Even more directly, we can make use of the *d*log form of the amplitude:

$$R(n,t) = \mu'(n) \sum_\beta R_V(n, \nu_\beta(t)) = \frac{\mu'(n)}{2\pi i} \oint \mathrm{dlog}(f(t,\tau)) R_V(n,\tau)$$

- Deforming the contour to $\tau = \infty$, we can explicitly calculate the residue coefficients:

$$R(n,t) = \sum_{k=0}^{n} b_k(n) t^k = \sum_{\ell=0}^{n} a_{n,\ell} G_\ell^{(D)}(\cos\theta)$$

$$b_k(n) = \frac{(-1)^{n-k}\mu'(n)}{k!(n-k)!} \lim_{\tau\to\infty} \left[\tau^{n-k+1} \partial_\tau^{n-k} \left(R_V(n,\tau)\, \partial_\tau \partial_t^k \log f(t,\tau)\big|_{t=0} \right)\right]$$

Simplest nonlinear model

- A particularly nice choice of polynomials is the following:

$$\begin{aligned} P(n) &= n^2 + \delta(n+1) \\ Q(n) &= n+1 \end{aligned} \quad \Longrightarrow \quad \mu(n) = \frac{n^2}{n+1} + \delta$$

- Residue at infinity gives a quartic contact term:

$$A_\infty(t) = \frac{1}{2\pi i} \oint_{s=\infty} \frac{\mathrm{d}s}{s} A(s,t) = 1$$

- Dual resonant amplitude:

$$A(s,t) = 1 + \sum_{n=0}^{\infty} \frac{R(n,t)}{\mu(n) - s}$$

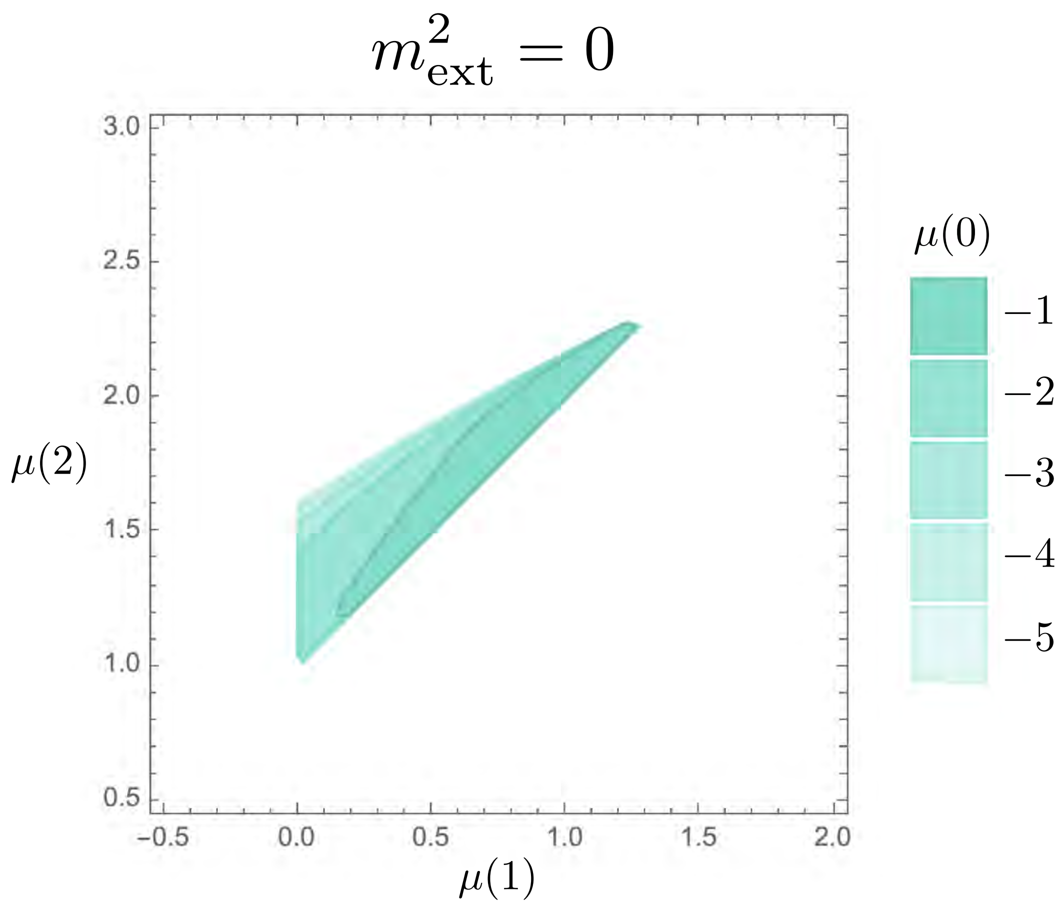
- Satisfies partial wave unitarity for all $\delta \in [-0.5, -0.354]$, setting $m_{\text{ext}} = 0$

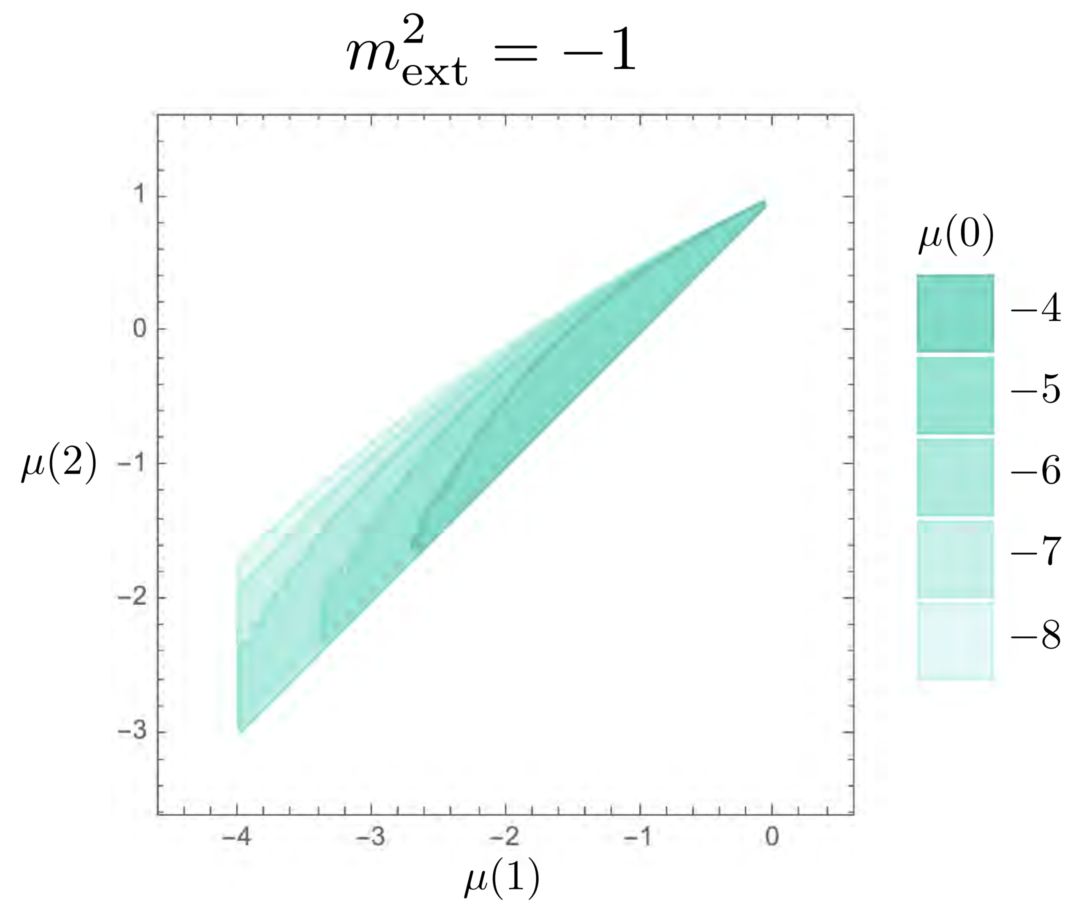
Simplest nonlinear model

- General $h = 2$ model:

$$\mu(n) = \frac{n^2 + p_1 n + p_2}{n + q_2}$$

- Parameter space satisfying dual resonance and partial wave unitarity:





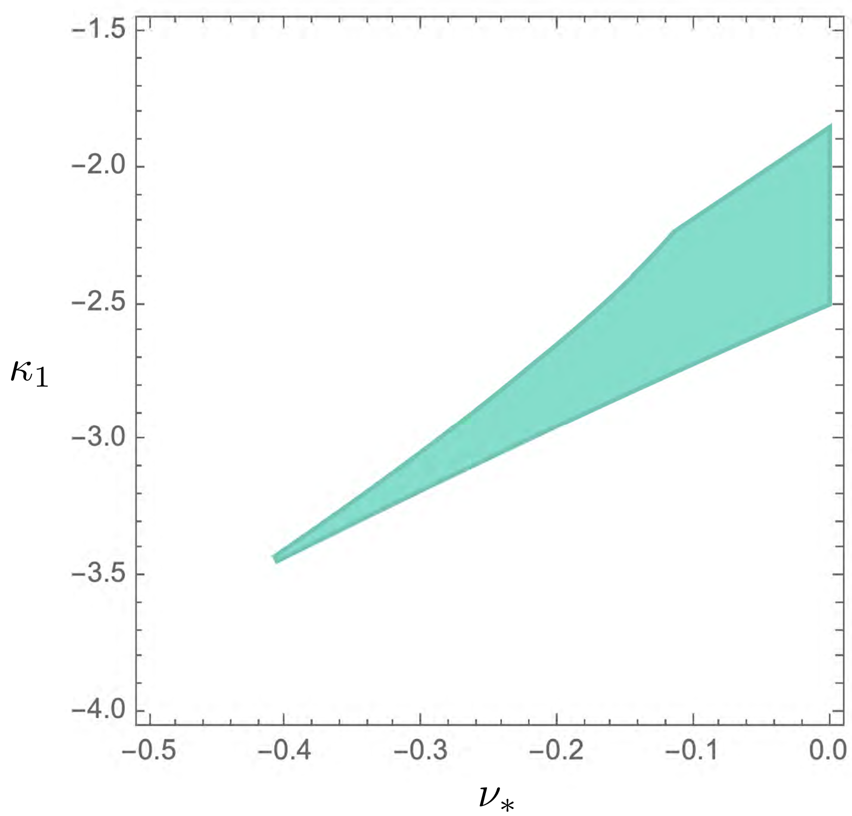
Post-Regge expansion

- Expand in series around the asymptotic spectrum:

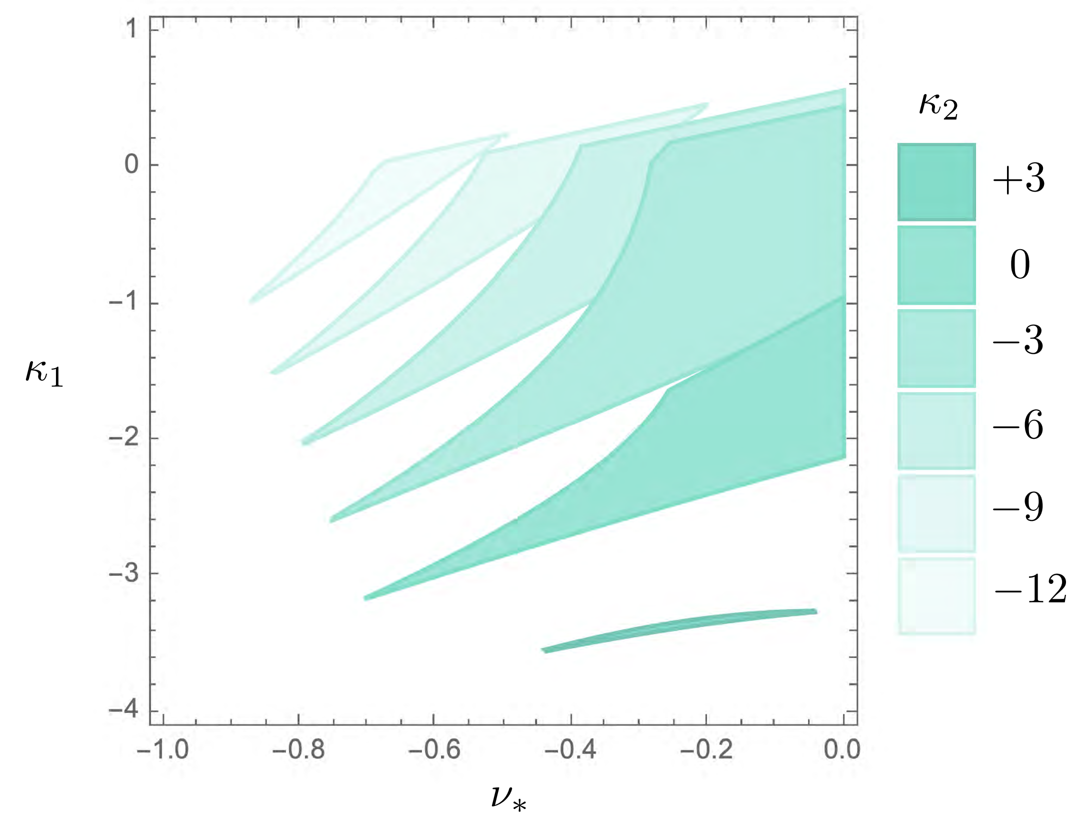
$$\mu(n) = (n - \nu_*) + \kappa_1 + \frac{\kappa_2}{n - \nu_*} + \cdots + \frac{\kappa_h}{(n - \nu_*)^{h-1}}$$

- Fixing $m^2_{\text{ext}} = 0$ and $a_{0,0} = a_{1,1} = 0$, unitarity and dual resonance require $\nu_* \in [-1.229, 0)$ in the $h = 2$ case.

- For $h = 3$:



$h = 4$:



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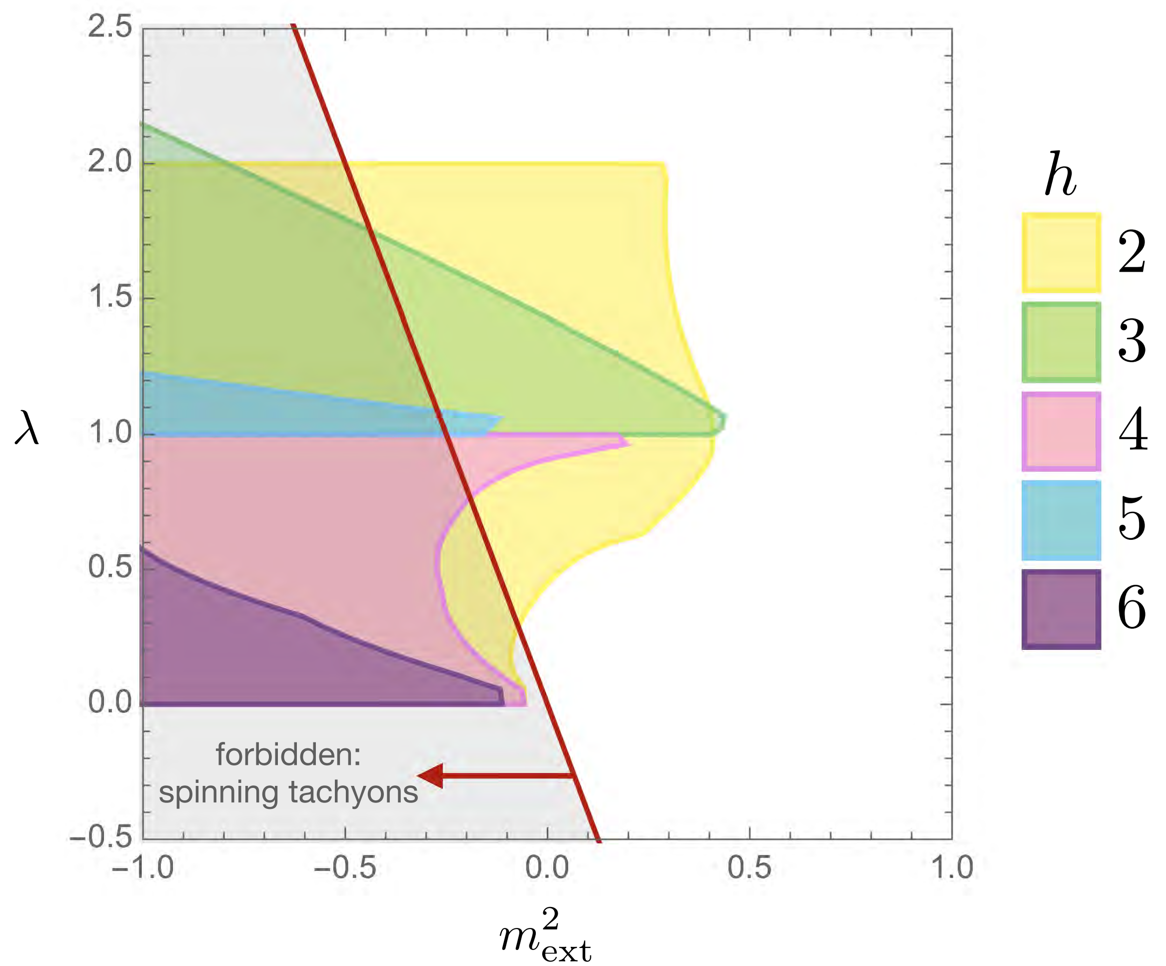
- Fixing $a_{0,0} = a_{1,1} = 0$ and $\mu(n) = \lambda n + \mu(0)$ for $0 \leq n \leq h-1$, the parameter space becomes:



Higher-point generalization

- There is a natural generalization of our construction to the scattering of an arbitrary number of particles.

- Write the planar basis of Mandelstam invariants as $\{s_I\}$
(e.g., four-point basis is $\{s_I\} = \{s_{12}, s_{23}\}$)

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- There is a natural generalization of our construction to the scattering of an arbitrary number of particles.
- Write the planar basis of Mandelstam invariants as $\{s_I\}$ (e.g., four-point basis is $\{s_I\} = \{s_{12}, s_{23}\}$)
- Take the higher-point string amplitude $A_V(\{s_I\})$ and remap each planar invariant:

$$
\begin{aligned}
A(\{s_I\}) &= \left(\prod_I \sum_{\alpha_I}\right) A_V(\{\nu_{\alpha_I}(s_I)\}) \\
&= \left(\prod_I \oint \frac{\mathrm{d}\log f(s_I, \sigma_I)}{2\pi i}\right) A_V(\{\nu_{\alpha_I}(\sigma_I)\})
\end{aligned}
$$

Worldsheet representation

- Worldsheet integral form of the Veneziano amplitude:

$$A_V(s_{12}, s_{23}) = \int_0^1 dx\, x^{-s_{12}-1}(1-x)^{-s_{23}-1}$$

- The structure of the integrand allows the Galois sum to factorize, giving a worldsheet integral representation of our bespoke amplitudes:

$$A(s_{12}, s_{23}) = \int_0^1 dx \sum_{\alpha_{12}} x^{-\nu_{\alpha_{12}}(s_{12})-1} \sum_{\alpha_{23}} (1-x)^{-\nu_{\alpha_{23}}(s_{23})-1}$$

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- Defining the special function $\rho(x, s) = \sum_\alpha x^{-\nu_\alpha(s)}$, this generalizes straightforwardly to higher-point scattering:
    - Write $s_{ij} = \sum_I c_{ijI} s_I$ in planar basis
    - Take Koba-Nielsen integral form of the higher-point amplitude and send $\prod_{i<j}(x_i - x_j)^{-s_{ij}} \longrightarrow \prod_I \rho\left(\prod_{i<j}(x_i - x_j)^{c_{ijI}}, s_I\right)$

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- For example, five-point string amplitude:

$$A_V(s_{12}, s_{23}, s_{34}, s_{45}, s_{51}) = \int_0^1 \int_0^1 dx\, dy\, \frac{x^{-s_{12}}\left(\frac{1-x}{1-xy}\right)^{-s_{23}} y^{-s_{45}}\left(\frac{1-y}{1-xy}\right)^{-s_{34}}(1-xy)^{-s_{51}}}{x(1-x)y(1-y)}$$

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- For example, five-point **bespoke** amplitude:

$$A(s_{12}, s_{23}, s_{34}, s_{45}, s_{51}) = \int_0^1 \int_0^1 dx\, dy\, \frac{\rho(x, s_{12})\rho\left(\frac{1-x}{1-xy}, s_{23}\right)\rho(y, s_{45})\rho\left(\frac{1-y}{1-xy}, s_{34}\right)\rho(1-xy, s_{51})}{x(1-x)y(1-y)}$$

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- One can directly check that factorization on the $n=0$ state holds as for strings:

$$\lim_{s_* \to 0} s_*\, A_V(\{s_I\}) = \sum_{L \perp R} A_V(\{s_{I_L}\}) A_V(\{s_{I_R}\})$$

Worldsheet representation

- Worldsheet integral form of the Veneziano amplitude:

$$A_V(s_{12}, s_{23}) = \int_0^1 dx\, x^{-s_{12}-1}(1-x)^{-s_{23}-1}$$

- The structure of the integrand allows the Galois sum to factorize, giving a worldsheet integral representation of our bespoke amplitudes:

$$A(s_{12}, s_{23}) = \int_0^1 dx \sum_{\alpha_{12}} x^{-\nu_{\alpha_{12}}(s_{12})-1} \sum_{\alpha_{23}} (1-x)^{-\nu_{\alpha_{23}}(s_{23})-1}$$

- Defining the special function $\rho(x,s) = \sum_\alpha x^{-\nu_\alpha(s)}$, this generalizes straightforwardly to higher-point scattering:
    - Write $s_{ij} = \sum_I c_{ijI} s_I$ in planar basis
    - Take Koba-Nielsen integral form of the higher-point amplitude and send $\prod_{i<j}(x_i - x_j)^{-s_{ij}} \longrightarrow \prod_I \rho\left(\prod_{i<j}(x_i-x_j)^{c_{ijI}}, s_I\right)$

- One can directly check that factorization on the $n=0$ state holds as for strings:

$$\lim_{s_*\to\mu(0)} (s_* - \mu(0))A(\{s_I\}) = \mu'(0) \sum_{L\perp R} h^{(|L|-1)} A(\{\nu_{\alpha_{I_L}}(s_{I_L})\}) h^{(|R|-1)} A(\{\nu_{\alpha_{I_R}}(s_{I_R})\})$$

Stringy Dynamics from an Amplitudes Bootstrap

Bootstrapping string theory?

- So far, we've built a remarkable class of "bespoke dual resonant" amplitudes with arbitrary spectra $m_n^2 = \mu(n)$ by using the Veneziano amplitude as input and kinematically transmuting it using our Galois sum.

- But how can one **derive** the structure of Veneziano from a bootstrap?

- Let's assume a linear spectrum $m_n^2 = n$ and see if the dynamics of string theory can be derived using minimal assumptions:

  i) Crossing Symmetry
  ii) Polynomial Residues
  iii) High-Energy Boundedness

Integer spectrum bootstrap

- As we have shown, dual resonance and a well-defined pole at infinity are equivalent, so we start with a dual resonant form of the amplitude, with arbitrary residues.

- A priori, this is a two-variable problem:

$$A(s,t) = \sum_{n=0}^{\infty} \frac{R_n(t)}{n-s} = \sum_{n=0}^{\infty} \frac{R_n(s)}{n-t} = A(t,s)$$

- Turn into a single-variable problem by choosing special kinematics,

$$t = s - k, \qquad k \in \mathbb{N}$$

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Crossing becomes:

$$A(s,s-k) = A(s-k,s) \implies \sum_{n=0}^{\infty} \frac{R_n(s-k)}{n-s} = \sum_{n=0}^{\infty} \frac{R_n(s)}{n+k-s}$$

$$\implies \sum_{n=k}^{\infty} \frac{R_n(s-k) - R_{n-k}(s)}{n-s} = -\sum_{n=0}^{k-1} \frac{R_n(s-k)}{n-s}$$

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Integer spectrum bootstrap

$$\sum_{n=k}^{\infty} \frac{R_n(s-k) - R_{n-k}(s)}{n-s} = -\sum_{n=0}^{k-1} \frac{R_n(s-k)}{n-s}$$

finite number of terms,
no poles at $s = n \leq k$

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Integer spectrum bootstrap

$$\sum_{n=k}^{\infty} \frac{R_n(s-k) - R_{n-k}(s)}{n-s} = -\sum_{n=0}^{k-1} \frac{R_n(s-k)}{n-s}$$

demanding
no poles at $s = n \leq k$
yields

$$\boxed{R_n(n-k) = R_{n-k}(n), \qquad 1 \leq k \leq n}$$

- Strictly speaking, neither necessary nor sufficient for crossing. We will take the residue constraint above as motivation and see what we find. All subsequent examples will indeed satisfy this constraint and converge.

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Integer spectrum bootstrap

We have n conditions

$$R_n(n-k) = R_{n-k}(n), \qquad 1 \leq k \leq n$$

on the $n+1$ free parameters in the residue ansatz:

$$R_n(t) = \sum_{m=0}^{n} \lambda_{n,m} t^m$$

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Defining $\lambda_m \equiv \lambda_{m,m}$ and for brevity writing $x! \equiv \Gamma(x+1)$ for $x \in \mathbb{C}$, we find the general solution:

$$R_n(t) = \sum_{m=0}^{n} \frac{\lambda_m}{m!} \frac{t!}{(t-m)!} \frac{n!}{(n-m)!}$$

Remarkably, we numerically find that choosing λ_m such that the s-channel representation of $A(s,t)$ converges always yields a crossing-symmetric amplitude: **an infinite-parameter family of dual resonant amplitudes with linear spectra.**

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Veneziano amplitude

- Let us choose $\lambda_m = \dfrac{1}{m!}$
- The Vandermonde identity then implies $R_n(t) = \dfrac{(t+n)!}{t!n!}$
- The amplitude is thus:

$$\begin{aligned}
A(s,t) &= \sum_{n=0}^{\infty} \frac{1}{n-s} \binom{t+n}{n} \\
&= -\frac{1}{s} \sum_{n=0}^{\infty} \frac{(-s)_n (1+t)_n}{(1-s)_n} \frac{1}{n!} \\
&= -\frac{1}{s}\, {}_2F_1 \left[\begin{matrix} -s,\ 1+t \\ 1-s \end{matrix} ;1 \right] \\
&= \boxed{\frac{\Gamma(-s)\Gamma(-t)}{\Gamma(-s-t)}}
\end{aligned}$$

Veneziano amplitude

Hypergeometric amplitude

- Let us choose $\lambda_m = \dfrac{r!}{(m+r)!}, \qquad r \in \mathbb{R}$

$$\implies R_n(t) = \frac{(t+n+r)!r!}{(t+r)!(n+r)!}$$

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Hypergeometric amplitude

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- From the definition of the generalized hypergeometric function,

$$ {}_mF_n \left[\begin{matrix} a_1, \ldots, a_m \\ b_1, \ldots, b_n \end{matrix} ; z \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_m)_k}{(b_1)_k \cdots (b_n)_k} \frac{z^k}{k!}$$

the amplitude becomes

$$A(s,t) = \sum_{n=0}^{\infty} \frac{R_n(t)}{n-s} = -\frac{1}{s}\, {}_3F_2 \left[\begin{matrix} 1,\ -s,\ 1+t+r \\ 1-s,\ 1+r \end{matrix} ; 1 \right]$$

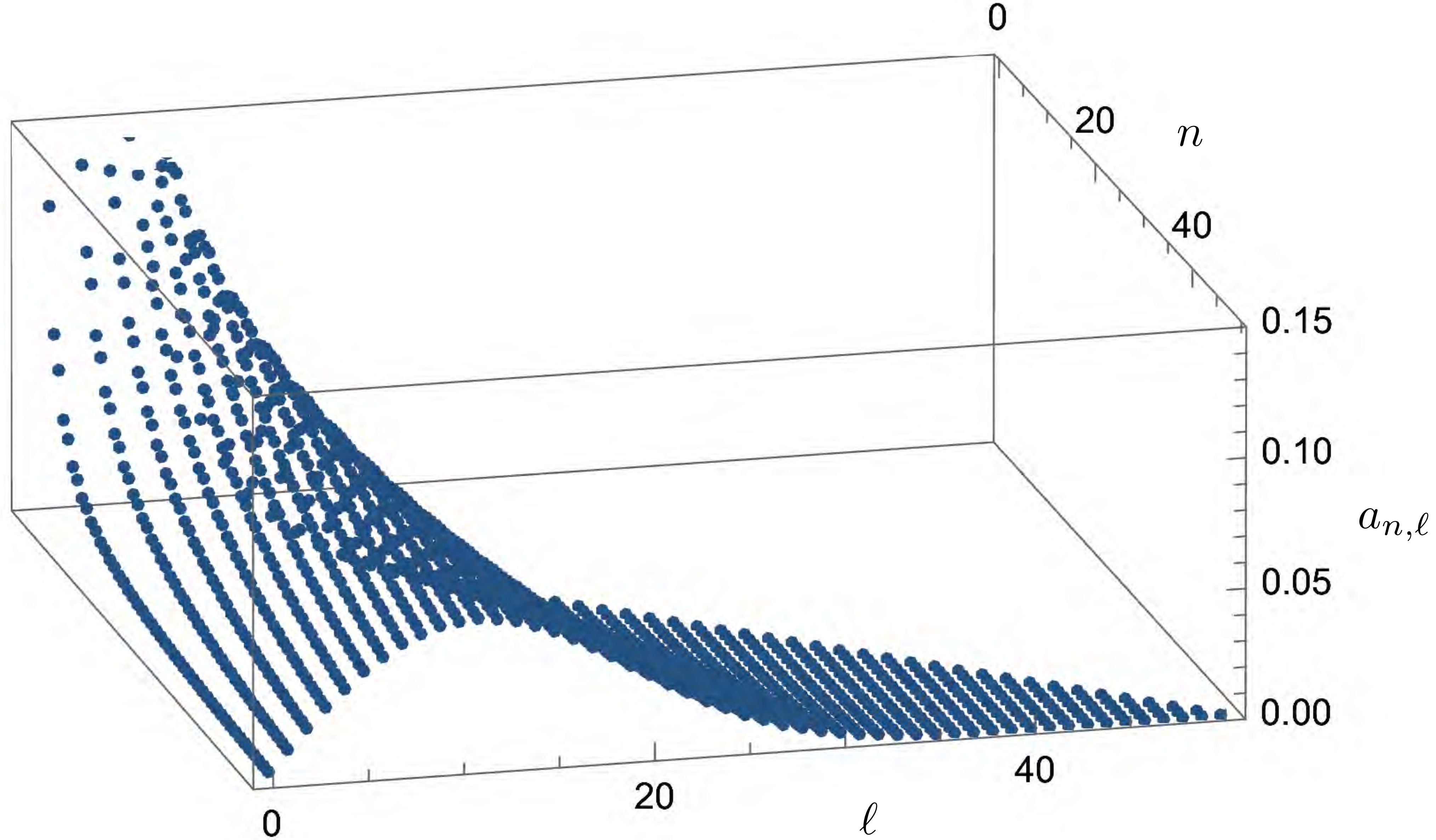
- Using a Thomae transformation,

$$\boxed{A(s,t) = \frac{\Gamma(-s)\Gamma(-t)}{\Gamma(-s-t)}\, {}_3F_2 \left[\begin{matrix} -s,\ -t,\ r \\ -s-t,\ 1+r \end{matrix} ; 1 \right]}$$

New hypergeometric amplitude

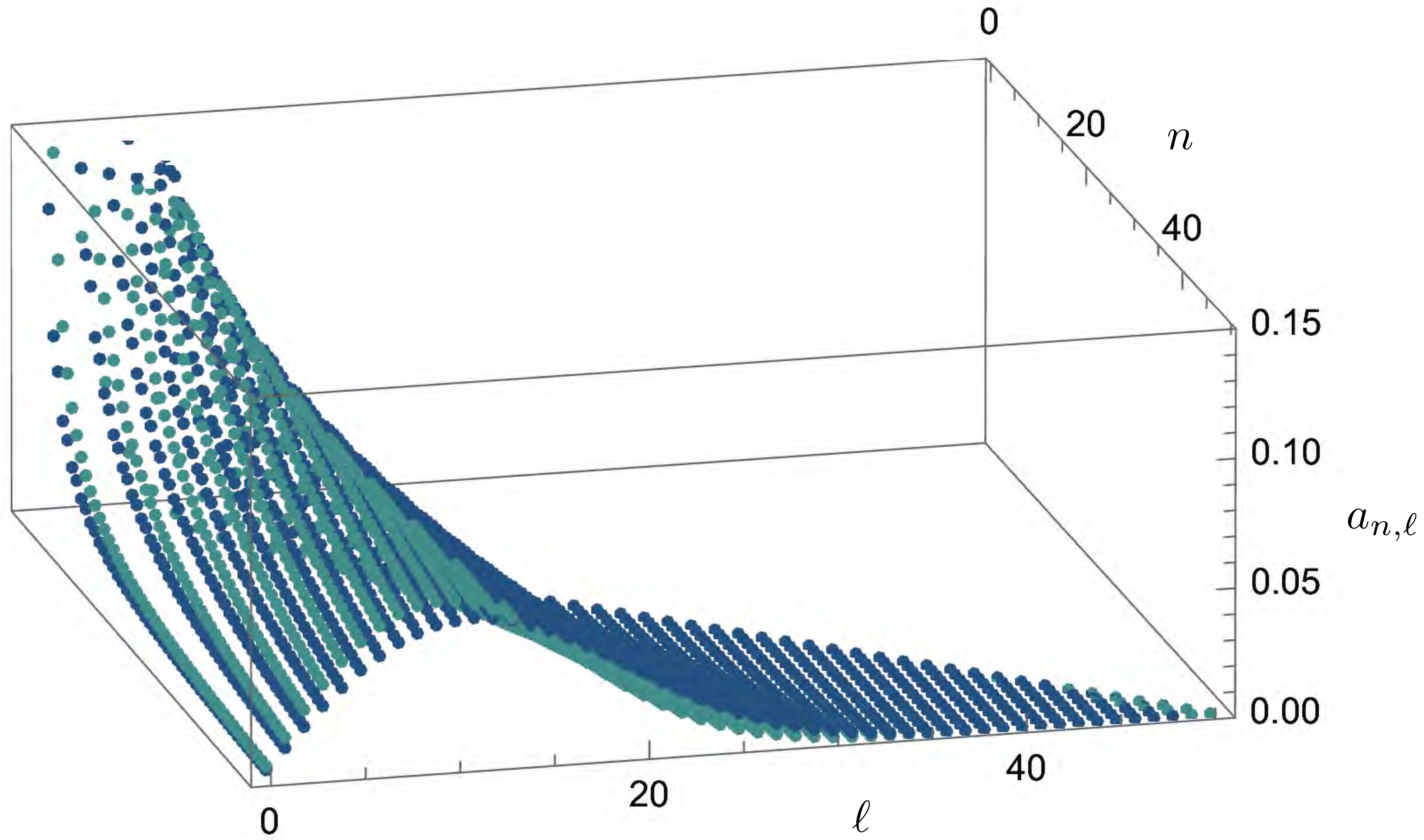
Cheung, **GR** [2302.12263]

Unitarity



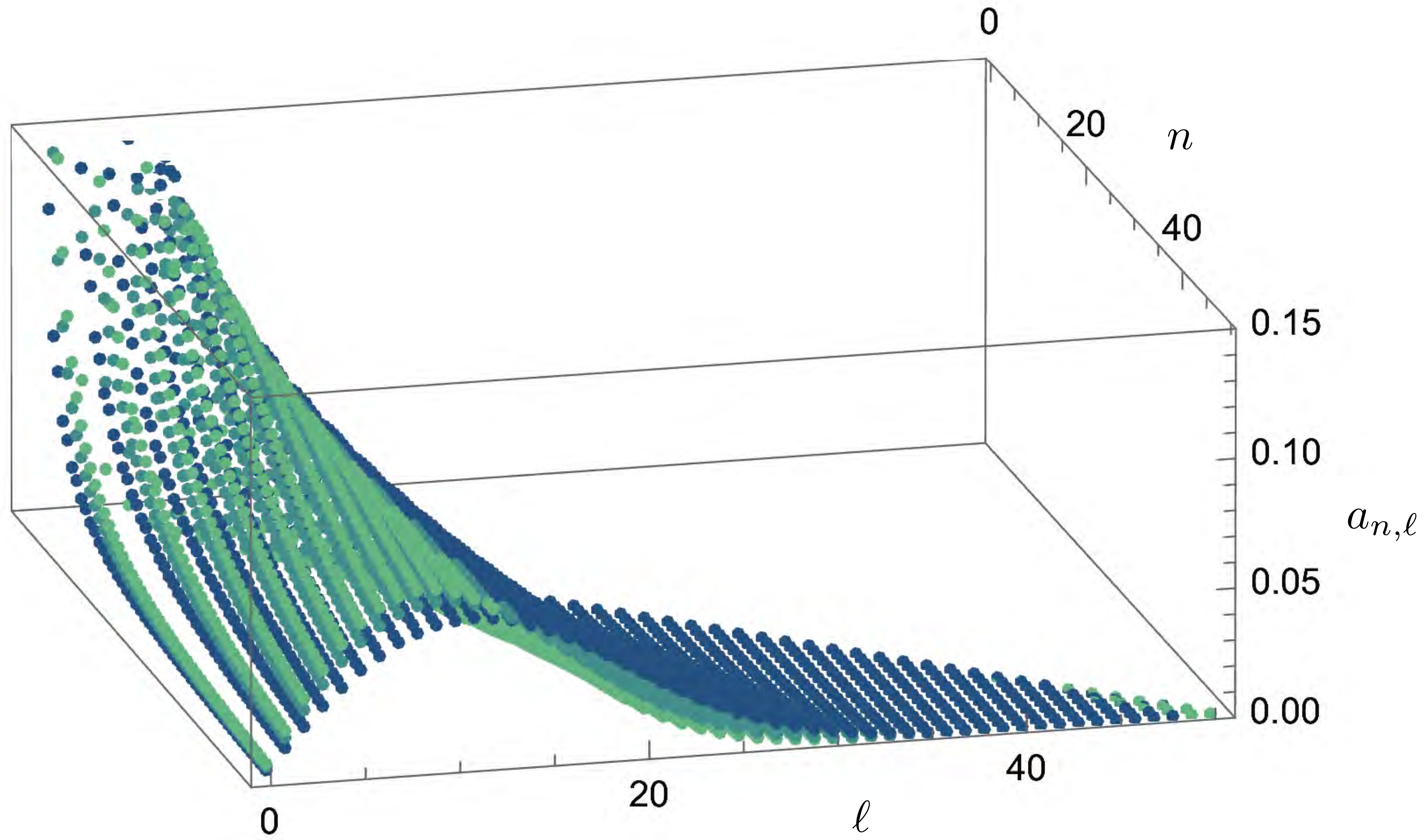
$$r = 0$$

Unitarity



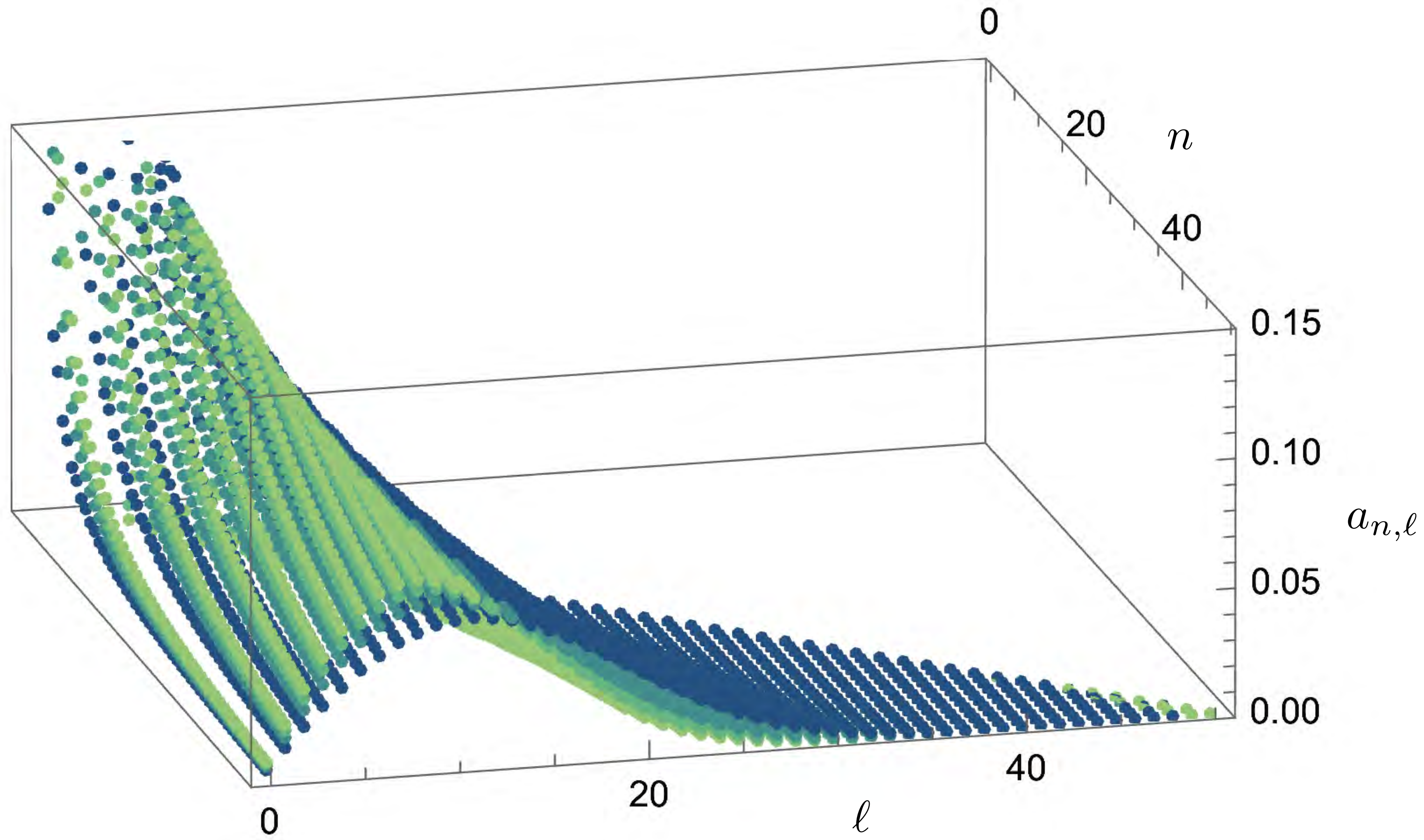
$r = 1$

Unitarity



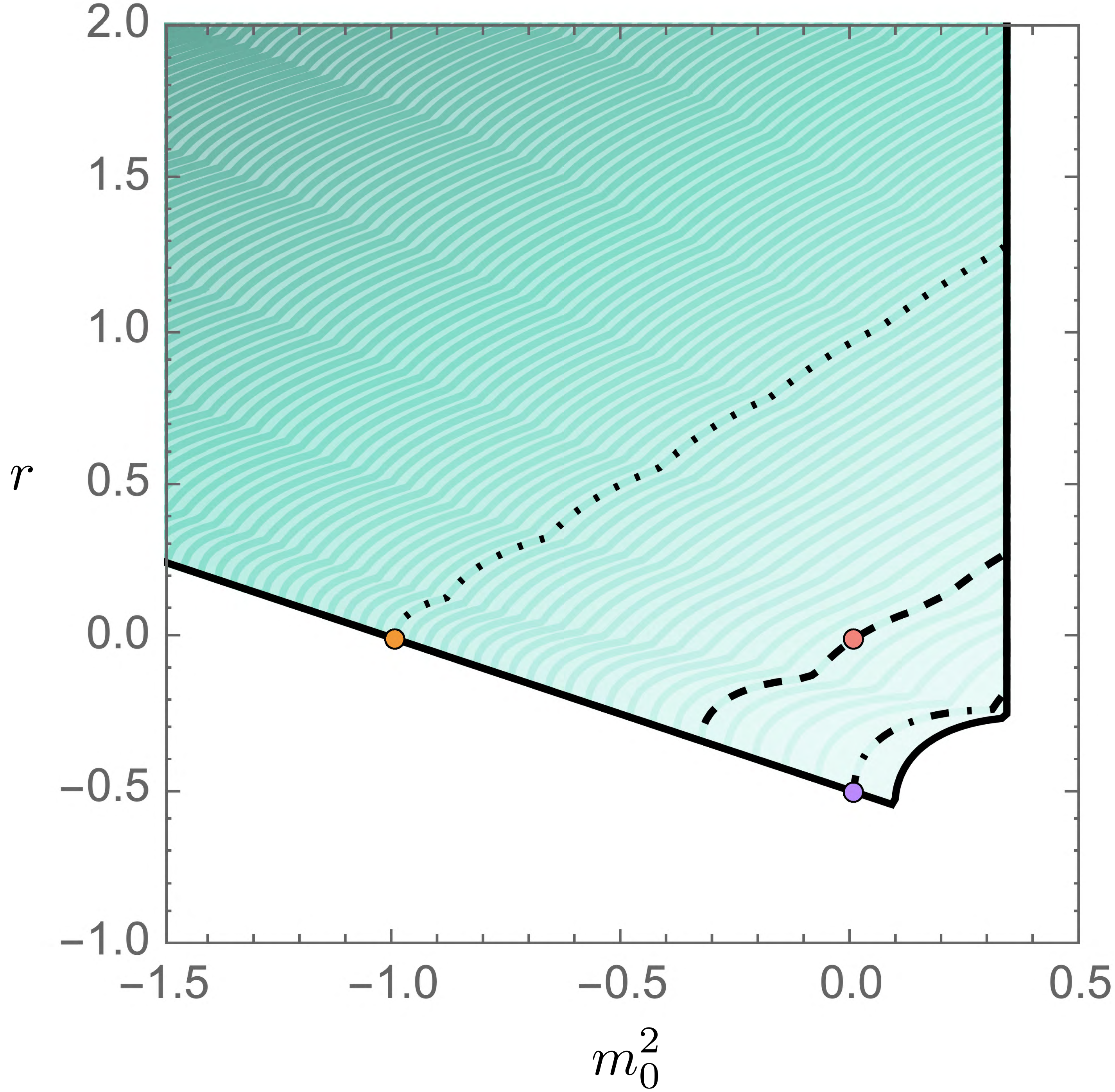
$\boldsymbol{r = 2}$

Unitarity



$r = 3$

Unitarity



Hard scattering

- In the high-energy, fixed-angle limit,

$$|s|,\ |t| \to \infty, \qquad t/s \text{ fixed}$$

  the hypergeometric amplitude exhibits the scaling:

$$A(s,t) \sim e^{B(s,t)} + \frac{r}{st} + \cdots\,, \qquad B(s,t) = (s+t)\log(s+t) - s\log s - t\log t + \cdots$$

- In the physical region, $\cos\theta = 1 + \frac{2t}{s} \in [-1,1]$, one has $B < 0$, so the amplitude falls off as a power law $\sim r/st$, unless $r = 0$, where the exponential decay of the string amplitude obtains.

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- In the unphysical $t > 0$ region, $B > 0$ and we find the universal scaling predicted by Caron-Huot, Komargodski, Sever, Zhiboedov [1607.04253]:

$$\log A \sim (s+t)\log(s+t) - s\log s - t\log t$$

A worldsheet interpretation?

Remarkably, the hypergeometric amplitude has an integral representation,

$$A(s,t) = r \int_0^1 \int_0^1 \mathrm{d}x \, \mathrm{d}y \, \frac{x^{-s-1} y^{r-1} (1-xy)^t}{(1-x)^{t+1}}$$

reminiscent the Koba-Nielsen form for the Veneziano amplitude,

4-point: $$A^{(4)}_{\mathrm{Ven}} = \int_0^1 \mathrm{d}x \, \frac{x^{-s-1}}{(1-x)^{t+1}}$$

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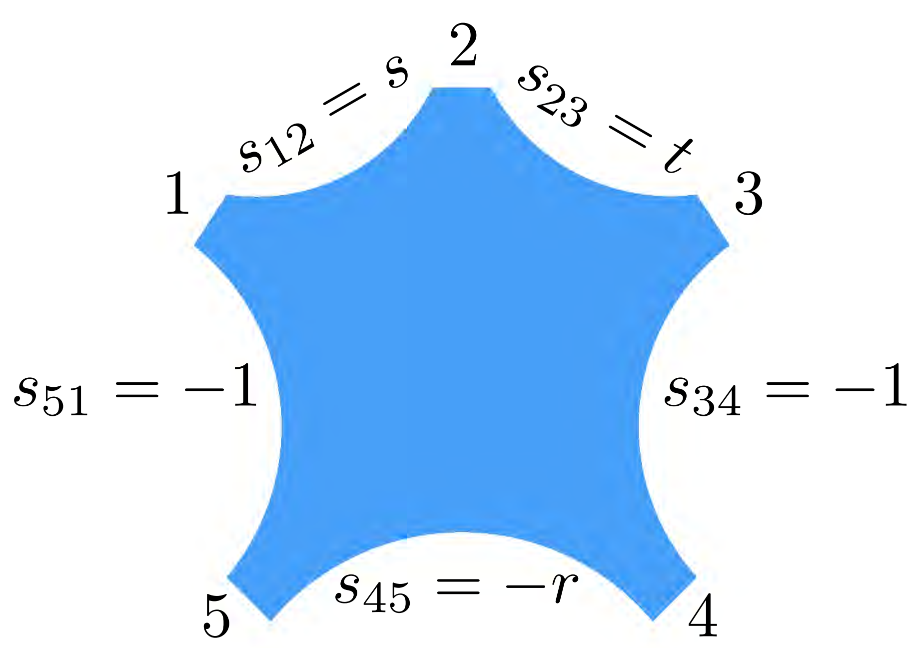
reminiscent the Koba-Nielsen form for the Veneziano amplitude,

5-point: $$A^{(5)}_{\text{Ven}} = \int_0^1\int_0^1 \mathrm{d}x\,\mathrm{d}y\,\frac{x^{-s_{12}-1}y^{-s_{45}-1}(1-xy)^{s_{23}+s_{34}-s_{51}}}{(1-x)^{s_{23}+1}(1-y)^{s_{34}+1}}$$



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Coon amplitudes

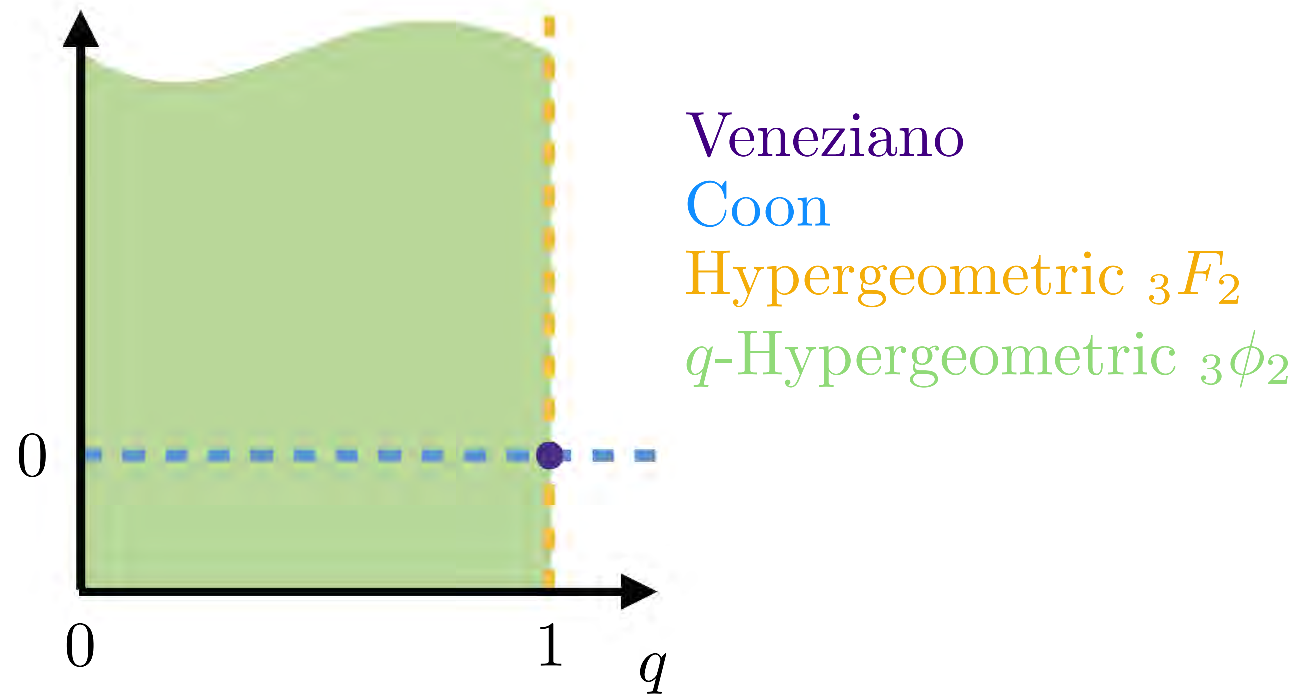
- Historically, string amplitudes *predate* the realization that the theory was about strings at all. Exploring amplitudes can lead to new physics, as we've seen from this talk.
- Also satisfying our physical constraints is the q-deformed generalization of Veneziano discovered by Coon (1969), unfortunately forgotten for decades:

Recent surge of interest:

- unitarity
  Figueroa, Tourkine [2201.12331]; Bhardwaj et al. [2212.00764]; Jepsen [2303.02149]
- string amplitudes with similar properties
  Maldacena, **GR** [2207.06426]
- construction and generalization
  Cheung, **GR** [2210.12163, 2302.12263]; Geiser, Lindwasser [2207.08855, 2210.14920]

q-hypergeometric amplitude

- We can generalize this construction to the q-deformed integers $[n]_q = \frac{1-q^n}{1-q}$, obtaining a family of amplitudes that subsumes the Veneziano, Coon, and hypergeometric amplitudes:



$$A(\sigma,\tau) = \sum_{n=0}^{\infty} \frac{q^{\tau(\sigma-n)} R_n([\tau]_q)}{[n-\sigma]_q} = q^{\sigma\tau} \frac{\Gamma_q(-\sigma)\Gamma_q(-\tau)}{\Gamma_q(-\sigma-\tau)} \, {}_3\phi_2 \left[\begin{matrix} q^{-\sigma}, q^{-\tau}, q^{r} \\ q^{-\sigma-\tau}, q^{1+r} \end{matrix} ; q; q \right]$$

New q-hypergeometric amplitude

Cheung, **GR** [2302.12263]

Discussion

Conclusions

- We have constructed new infinite-parameter families of amplitudes obeying:
    - Meromorphicity
    - Crossing symmetry
    - Polynomial residues
    - Partial wave unitarity
    - UV boundedness
    - Dual resonance

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    - Worldsheet-like representation
    - n-point generalization

- Bootstrap construction with Regge spectrum: hypergeometric amplitudes

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- Galois sum construction: bespoke spectra
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  - n-point generalization

- Bootstrap construction with Regge spectrum: hypergeometric amplitudes

To prove that string theory is the unique theory of quantum gravity, we must explore whether it can be bootstrapped from first principles.

Alternative structures that we find along the way can help us understand the mechanism by which strings become inevitable and give us insights into new structures within string theory itself.

Questions