Six- and eleven-dimensional theories via superspace torsion and Poisson brackets

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This talk is an overview of some joint work (past, present, and future) with numerous collaborators.

Here is an incomplete list:

Ilka Brunner, Martin Cederwall, Kevin Costello, Tudor Dimofte, Richard Eager, Chris Elliott, Owen Gwilliam, Fabian Hahner, John Huerta, Simon Jonsson, Simone Noja, Jakob Palmkvist, Natalie Paquette, Surya Raghavendran, Johannes Walcher, Brian R. Williams.

I owe thanks to all of them, and to the many other people I've been fortunate enough to talk with and learn from (some of whom are in the audience).

I'd like to give an overview of the program that has emerged out of these collaborations: its tools and aims; its successes so far; and its future directions.

To sum up this program, and to serve as a frame for the lecture, here is a list of slogans:

1 – Superspace geometry is like almost-complex geometry.
2 – Theories are like their twists.
3 – Symmetries and deformations belong together.
4 – Higher-dimensional current algebras are worth thinking about.

Going in order, I will try and convey what I mean by each of these four cryptic phrases, and (in particular) what I mean by the word "like."

At the end of the day, I'd like to arrive at the following result, which is an illustrative special case:

1 – From the geometry of $\mathscr{N} = (2,0)$ superspace, one obtains a six-dimensional current algebra closely related to the conformal supergravity multiplet.

2 – The corresponding chiral algebra consists of Virasoro (or W_2) currents. The holomorphic twist gives the currents of the exceptional super Lie algebra $E(3|6)$.

3 – It governs both superconformal symmetry and moduli.

4 – Working perturbatively at the holomorphic level, the reduction of this object to five dimensions agrees with $\mathfrak{sl}(2)$ maximal super Yang–Mills theory.

Very concrete progress towards the physics of fivebranes!

1 – Superspace geometry

As we all know, superspace is characterized by the algebra of functions on it. I have the usual commuting spacetime coordinates x^μ, but also fermionic coordinates θ^a.

Usually, the coordinates θ^a parameterize one or more odd copies of spin bundles on the underlying spacetime. (Here, a is a collective index.)

Importantly, though, a superspace X is equipped with one additional geometric structure: a *non-integrable odd distribution of maximal dimension*, encoding local supersymmetry transformations.

This idea goes back, at least, to Yuri Manin.

On flat space, this amounts to specifying the *supercovariant derivatives*, given by the usual formulas

$$Q_a = \frac{\partial}{\partial\theta^a} + \gamma^\mu_{ab}\theta^b\frac{\partial}{\partial x^\mu}, \qquad D_a = \frac{\partial}{\partial\theta^a} - \gamma^\mu_{ab}\theta^b\frac{\partial}{\partial x^\mu},$$

when acting on the left and on the right respectively.

In the literature, one finds the statement that "flat superspace has torsion:" Since they do not commute, the covariant derivatives cannot be a coordinate basis in any coordinate system on superspace.

An almost-complex structure is defined by precisely the same kind of data: a distribution

$$\overline{T} \subset T_{\mathbb{C}}$$

in the (complexified) tangent bundle. The structure is integrable precisely when its “torsion” vanishes.

So any structure in almost-complex geometry that I can define using that data has an analogue in superspace.

A very similar perspective was taken by Berkovits and Howe. See also the work of Tanaka.

If I want to study a complex manifold, I shouldn't think about the *smooth* functions. Its structure is better captured by the algebra of *holomorphic* functions.

How can I recover that algebra from the geometric data we have to work with?

Here is one recipe:

- Consider the differential forms $\Omega^\bullet(X)$.
- Give them a "weight grading": $|dz| = -1$, $|d\bar{z}| = 0$.
- The differential decomposes into weighted pieces:

$$d = \bar{\partial} + \partial.$$

- The weight-zero piece of the differential is $\bar{\partial}$.
- In weight $-k$, I recover the $(k, \bullet)$-forms.

This doesn't quite work when the complex structure is not integrable. However, there *is* a good generalization in the almost complex case, due to Cirici and Wilson.

Their insight is simple to describe. In the almost complex case, the de Rham differential decomposes as

$$d = \gamma + \overline{\partial} + \partial + \overline{\gamma},$$

where γ is the Nijenhuis tensor.

While $\overline{\partial}$ is no longer a square-zero differential, γ is. And $\overline{\partial}$ squares to zero up to γ-exact terms. So we should think of the Dolbeault operator, not as a square-zero differential on the graded algebra $\Omega^\bullet$, but on the *differential graded algebra* $(\Omega^\bullet, \gamma)$.

Based on our analogy, we should try and understand superspace by following this recipe and constructing appropriate analogues of “holomorphic objects” on it.

In particular:

- What is the algebra of “holomorphic” functions?
- What are “holomorphic” vector fields?

The latter object will govern symmetries and deformations, in similar fashion to Kodaira–Spencer theory.

So now we can just watch what happens...

From this point forward, I will often leave the scare quotes implicit.

The de Rham forms on flat superspace are generated by coordinates x^μ, θ^a and one-forms $dx^\mu, d\theta^a$. But we need to work with respect to the left-invariant frame

$$\lambda^a = d\theta^a, \quad v^\mu = dx^\mu + \theta^a \gamma^\mu_{ab} d\theta^b.$$

In this basis, the de Rham differential becomes

$$d = \lambda^a \lambda^b \gamma^\mu_{ab} \frac{d}{dv^\mu} + \lambda^a \left(\frac{d}{d\theta^a} - \gamma^\mu_{ab} \theta^b \frac{d}{dx^\mu} \right) + v^\mu \frac{d}{dx^\mu}.$$

λ has weight 0, and v weight -1. Thus, in our analogy,

$$\gamma = \lambda^a \lambda^b \gamma^\mu_{ab} \frac{d}{dv^\mu}, \quad \bar{\partial} = \lambda^a \left(\frac{d}{d\theta^a} - \gamma^\mu_{ab} \theta^b \frac{d}{dx^\mu} \right), \quad \partial = v^\mu \frac{d}{dx^\mu}.$$

$\bar{\partial}$ squares to zero up to the "pure spinor constraint" $\lambda^a \gamma^\mu_{ab} \lambda^b = 0$, which is imposed by the differential γ.

Berkovits, Cederwall, Howe, Nilsson, ...

The multiplet $A^\bullet$ of “holomorphic” functions:

Superspace	*Structure sheaf*	*On-shell?*	$\dim_{\mathbb{C}}$	*CY?*
3d $\mathscr{N}=1$	vector		1	
4d $\mathscr{N}=1$	vector		2	
6d $\mathscr{N}=(1,0)$	vector		3	
6d $\mathscr{N}=(2,0)$	abelian tensor	$\checkmark^*$	1	
$P^1 \times P^2$	T	$\checkmark^*$	1	
10d $\mathscr{N}=(1,0)$	vector	$\checkmark$	5	$\checkmark^\dagger$
10d $\mathscr{N}=(2,0)$	supergravity (IIB)	$\checkmark^*$	1	
$\wedge^2(T)\oplus T^*$	min. BCOV	$\checkmark^*$	1	
11d $\mathscr{N}=1$	supergravity	$\checkmark$	2	$\checkmark$
$\mathrm{Gr}(2,5)$	$E(5\|10)$	$\checkmark$	2	$\checkmark$

The star refers to *presymplectic* on-shell (BV) theories, which include self-dual fields. The dagger refers to a subtlety that will not be important here (“Gorenstein, but not maximally Cohen–Macaulay.”)

Here, the “complex dimension” is the largest k such that the sheaf of “holomorphic k-forms” is nontrivial, and a “Calabi–Yau structure” is a trivialization of the sheaf of holomorphic top forms, making $A^\bullet$ a Calabi–Yau k-algebra.

Why those particular non-standard examples?

The construction outlined applies to *any* choice of an appropriate superspace—indeed, the only input is the structure constants γ^μ_{ab} of the supertranslation algebra. But nothing says these have to be of any standard form.

It’s useful to set things up in this generality. As I will now explain, the nonstandard examples above govern the twists of $\mathscr{N} = (2,0)$ supersymmetry, eleven-dimensional supersymmetry, and type IIB, respectively.

2 – Theories and their twists

The key point can be put as follows: *On flat superspace, twisting is the odd version of dimensional reduction*. In a precise sense, the theory and its twists have the same structure.

After all, dimensional reduction takes invariants of a vector field generating an even translation. Twisting takes invariants of a supercharge; in a superfield formalism, this is just a vector field generating an odd translation.

From this perspective, it's not surprising that superfield formulations or actions do not change dramatically under twisting. (When I dimensionally reduce, I write *the same* action functional, restricted to a smaller space of fields.)

One rigorous formulation of this point is the following:

Theorem (IAS–Williams)

The construction outlined above commutes with twisting. In particular, the twist of $A^\bullet$ by a square-zero supercharge Q is obtained by applying $A^\bullet$ to the algebra of residual supertranslations in the desired twist.

As a corollary, the holomorphic twist of the type IIB supergravity multiplet is (free) minimal BCOV theory, as conjectured by Costello–Li.

Because of the computational efficiency of the formalism, checking this—which would be deeply unwieldy in components—becomes a task you can do in five minutes. And it's not just about free theories...

This statement is true up to taking potentials for various field strengths in BCOV.

- *Baulieu:* Holomorphically twisted ten-dimensional super Yang–Mills theory is holomorphic Chern–Simons theory on $\mathbb{C}^5$.
- *Berkovits; Schwarz; Witten:* Ten-dimensional super Yang–Mills theory is holomorphic Chern–Simons theory on superspace.
- *Costello:* Maximally twisted eleven-dimensional supergravity should be Poisson–Chern–Simons theory on $\mathbb{C}^2 \times \mathbb{R}^7$.
- *Raghavendran–IAS–Williams:* Holomorphically twisted eleven-dimensional supergravity on $\mathbb{C}^5 \times \mathbb{R}$ should be the exceptional Lie superalgebra $E(5|10)$.
- *Cederwall; Hahner–IAS: Untwisted* eleven-dimensional supergravity is Poisson–Chern–Simons theory on superspace. So is $E(5|10)$.

While it's great to interpret well-known theories in terms of superspace geometry, I want to emphasize that these tools are useful *beyond* an interpretational level. Sometimes new structures become apparent. And sometimes one knows *more* about a twisted theory than its untwisted parent...

As an example of the first kind, notice that this formulation of eleven-dimensional supergravity gives the fields a *2-shifted Poisson algebra* structure. This is roughly the same as an E_3 structure—which is precisely what one would expect if it arose from a first-quantized description via a *three-dimensional* theory.

(This is analogous to the 1-shifted bracket of Lian–Zuckerman and Getzler in string theory, or the BV algebra structure on polyvector fields in the B-model.)

Beem–Ben-Zvi–Bullimore–Dimofte–Neitzke, Elliott–Williams, ...

As an example of the second kind: $\mathscr{N} = (2,0)$ theories in six dimensions admit two twists. It is expected that the nonminimal twist of the $\mathfrak{sl}(n)$ theory is the W_n algebra.

Beem, Rastelli, and van Rees showed that twisting $\mathscr{N} = (2,0)$ theories by a particular superconformal element of type $Q + S$ produces chiral algebras, and conjectured that these are W-currents.

They write: *"the structure of the computable correlators may hold some clues about the right language with which to describe* $(2,0)$ *SCFTs more generally."*

By work of Oh–Yagi and Jeong, one expects a relation to the nonminimal twist; further strong evidence is given by Costello's description in the context of twisted holography in the omega background.

Alday–Gaiotto–Tachikawa, Gaiotto, ...

3 – Superspace symmetries and deformations

We know that the object that controls symmetries and deformations of a complex manifold is the sheaf of holomorphic vector fields. (This is normal Kodaira–Spencer theory: $H^1(T)$ corresponds to Beltrami differentials.)

To understand deformations of superspace, we should think about "holomorphic" vector fields—in other words, derivations of $A^\bullet$. Again, this object can be computed straightforwardly.

Either amazingly or unsurprisingly, this produces the *conformal supergravity multiplet*, in any dimension and with any amount of supersymmetry.

Innumerable angles: Ogievetsky–Sokatchev, Wess–Zumino, Schwarz, Cremmer–Ferrara, Santi–Spiro, Figueroa-o'Farrill–Santi, d'Auria–Fré, generalized geometry…

For untwisted $\mathscr{N} = (2,0)$ supersymmetry, here is the result:

$$\begin{array}{llllll} 0: & \mathrm{Vect} & \Pi S_+(R) & \Omega^0(\mathrm{ad}_R) & & \\ & \downarrow & \downarrow & \downarrow & & \\ 1: & \mathrm{Met}_0 & \Pi\,\mathrm{RS}(R) & \Omega^1(\mathrm{ad}_R) \oplus \Omega^3_+(\mathbf{5}) & \Pi S_-(\mathbf{16}) & \Omega^0(\mathbf{14}). \end{array}$$

And here it is for holomorphic $\mathscr{N} = (2,0)$ supersymmetry:

$$0: \quad \mathrm{Vect}_{\mathrm{hol}} \quad \Omega^1_{\mathrm{hol}} \otimes \Pi R' \quad \Omega^0_{\mathrm{hol}}(\mathrm{ad}_{R'}).$$

The first of these, $\mathscr{L}_{(2,0)}$, is a derived version of the conformal supergravity multiplet of Bergshoeff, Sezgin, and van Proeyen. The second is the exceptional simple Lie superalgebra $E(3|6)$ constructed by Kac.

4 – Higher current algebras

We just constructed a family of local L_∞ algebras that encode the symmetries and deformations of superspace. These naturally couple to any superconformal theory.

Costello and Gwilliam worked out a powerful generalization of Noether's theorem for any local L_∞ algebra $\mathscr{L}$. They construct a P_0 factorization algebra of *currents*, which I will call $\mathrm{Cur}(\mathscr{L})$. To a field theory T with a symmetry by $\mathscr{L}$, they then associate a map of P_0 factorization algebras

$$\mathrm{Cur}(\mathscr{L}) \to \mathrm{Obs}(T).$$

(It is worth remarking that this works, in particular, for any infinitesimal higher-form symmetry.) There is a quantum version as well, but I will work semiclassically.

One should think about these constructions as analogous to typical computations in two-dimensional field theory, where current algebras are ubiquitous. While they are not defined by least-action principles, they have perfectly well-defined OPE structures.

The difference is analogous to that between a standard (symplectic) phase space and a Poisson manifold. Both have algebras of functions that admit deformation quantizations.

When we ask for a Hamiltonian group action on M, we ask for a moment map $\mathfrak{g} \to \mathbb{C}^\infty(M)$. This extends to a map from polynomials in $\mathfrak{g}$ to functions on M, which I can think of as a Poisson map from M to $\mathfrak{g}^\vee$.

So I can think of $\mathfrak{g}^\vee$ itself as a degenerate ("non-Lagrangian") phase space. . . .

Moyal, Weyl, Kontsevich; Kirillov, Kostant, Souriau; . . .

At this point, I hope it is clear that *we have constructed a semiclassical object with $\mathscr{N} = (2,0)$ superconformal symmetry, which is not described by a least-action problem.*

This object is just

$$\mathrm{Cur}(\mathscr{L}_{(2,0)}):$$

the current algebra of the moduli problem of deformations of $(2,0)$ superspace.

By analogy: "W_2 is the current algebra of the moduli problem of deformations of a complex curve."

Thanks to the general results above, many consistency checks are *automatic*. For example, the associated chiral algebra in the nonminimal twist is W_2, or Virasoro currents—without doing any computations at all.

Similarly, the holomorphic version, describing the dynamics of 1/16 BPS operators, is $\mathrm{Cur}(E(3|6))$.

But the connection to 5d super Yang–Mills theory is *not* automatic. I will check this at the holomorphic level by matching the dimensional reduction of $E(3|6)$ currents to the perturbative holomorphic twist of the $\mathfrak{sl}(2)$ theory.

There is a subtlety related to the central extension; I will comment on this later.

After dimensionally reducing, we obtain a local L_∞ algebra of the form

$$
\mathscr{L}^{\text{red}}_{(2,0)} : \qquad
\begin{array}{ccc}
\mathrm{Vect}_{\text{hol}} & \Omega^1_{\text{hol}} \otimes \Pi R' & \Omega^0_{\text{hol}}(\mathrm{ad}_{R'}). \\
\Omega^0_{\text{hol}} \cdot \partial_w & (\Omega^0_{\text{hol}} \cdot dw) \otimes \Pi R' & \\
\hline
(X^\mu, x) & (\psi^a_\mu, \xi^a) & \rho^{ab}
\end{array}
$$

Inside of $\mathrm{Cur}(\mathscr{L}^{\text{red}}_{(2,0)})$, these generators are shifted down by one. But we need to take *compactly supported* sections, meaning that the nontrivial generators in cohomology are overall in degree +2.

Perturbative holomorphic super Yang–Mills theory with $\mathfrak{g} = \mathfrak{su}(2)$ is also described (in BV) by a local L_∞ algebra of the form

$$\mathscr{E}: \qquad \begin{array}{ccc} \Omega^0_{\mathrm{hol}} \otimes \mathfrak{g} & \Omega^0_{\mathrm{hol}} \otimes \Pi R' \otimes K^{1/2} & \Omega^3_{\mathrm{hol}} \otimes \mathfrak{g}. \\ \hline \alpha & \phi^a & \beta \end{array}$$

Overall, α and β are observables of odd parity, and the ϕ^a are of even parity. Each observable has degree $+1$. $\mathfrak{su}(2)$ has only one quadratic Casimir invariant, so the gauge-invariant observables (at lowest order in holomorphic derivatives) are generated by quadratic expressions in β, ϕ^a, and the holomorphic derivatives of α.

The map can be written down very explicitly:

$$\rho^{ab} \mapsto \mathrm{tr}\left(\phi^a \phi^b\right),$$
$$\xi^a \mapsto \mathrm{tr}\left(\phi^a \beta\right), \qquad \psi^a_\mu \mapsto \mathrm{tr}\left(\phi^a \partial_\mu \alpha\right),$$
$$X^\mu \mapsto \mathrm{tr}\left(\beta \vee \partial_\mu \alpha\right), \qquad x \mapsto \mathrm{tr}(\partial_1 \alpha \partial_2 \alpha).$$

At a hand-waving level, and ignoring the central extension, one can already see how the relevant components of the Poisson brackets match up (recalling that α is conjugate to β and ϕ_1 to ϕ_2).

More properly, one might frame this as a map from $\mathscr{L}^{\mathrm{red}}_{(2,0)}$ to (shifted) local functionals in holomorphic Yang–Mills.

At an intuitive level: compare to Drinfeld–Sokolov, Feigin–Frenkel...

To fully work out the structure of this algebra, one needs to understand possible central extensions of $\mathscr{L}_{(2,0)}$. One expects a unique local central extension, closely connected to the superconformal invariant studied in the literature. (In progress with Williams.)

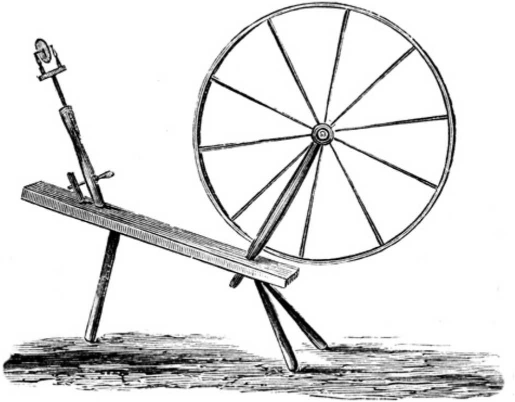
We expect to construct higher-rank examples as higher analogues of W_n algebras. W_∞ is understood: at the semiclassical level, it is eleven-dimensional supergravity! The superspace description of higher-spin generators is also understood. (To appear with Hahner, Raghavendran, and Williams.)

Building on intuitions from twisted holography, we expect to recover analogues of the Gelfand–Dickey Poisson bracket in this higher setting. (In progress with Raghavendran and Williams.)

Butter–Novak–Tartaglino-Mazzucchelli; Raghavendran–Williams; IAS–Williams

Many open questions and future directions. Here are a few:

- There should be a higher analogue of the Miura transform which constructs a map from higher W_n currents to the observables of n abelian tensor multiplets; useful for tensor branch physics? (I thank Procházka and Rapčák for conversations about this.)
- Can I learn more about first-quantized applications of pure spinor techniques from this perspective? (Ongoing work with Huerta.)
- Is there an E_3 algebra whose factorization homology on the two-sphere reproduces the 2-shifted Poisson structure of eleven-dimensional supergravity?
- What is an oper on Spec $A^\bullet$?



Thanks for your attention!

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