Irreversibility, QNEC, and defects

Gonzalo Torroba

Centro Atómico Bariloche, Argentina

Collaboration with Horacio Casini, Ignacio Salazar-Landea

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Irreversibility of the renormalization group is a key property of nonperturbative QFT.
Goal: find renormalization group charges that partially characterize conformal field theories, and that decrease under the RG.

- Two directions where irreversibility has been developed:

1) Irreversibility theorems in unitary relativistic QFT have been established in d=2, 3 and 4 space-time dimensions (C, F, A)
2) Irreversibility for defect RG flows, embedded in CFTs. Started with the g-theorem.

- Framework for generalizing this:

- D-dimensional CFT coupled to d-dimensional planar static defect
- relevant interactions on defect trigger RG flow between UV and IR fixed points. Bulk remains conformal
- QFTs without defect: special case D=d

Overview of irreversibility inequalities

So far, several proofs for different (D, d). Those based on properties of energy-momentum tensor are:

$d \diagdown D$	2	3	4	5	$\cdots$
1	reflection positivity for stress tensor	reflection positivity for stress tensor	reflection positivity for stress tensor	reflection positivity for stress tensor	reflection positivity for stress tensor
2	reflection positivity for stress tensor	reflection positivity of dilaton	reflection positivity of dilaton	reflection positivity of dilaton	reflection positivity of dilaton
3		No proof	No proof	No proof	No proof
4			unitarity dilaton scattering	unitarity dilaton scattering	unitarity dilaton scattering

D=d=2: [Zamolodchikov]'s original C-thm; D=d=4: [Komargodski, Schwimmer] (dilaton)
Later on: d=2, any D [Jensen, OBannon], d=4, any D: [Wang]. Dilaton methods
d=1, D=2: [Friedan, Konechny].
And recently: d=1, any D: [Cuomo, Komargodski, Raviv-Moshe]. Generalized to d=2, any D by [Sachar, Sinha, Smolkin].

In parallel, irreversibility theorems have also been obtained using methods from quantum information theory:

$d \setminus D$	2	3	4	5	...
1	positivity of relative entropy	positivity of relative entropy	positivity of relative entropy	positivity of relative entropy	positivity of relative entropy
2	SSA or positivity of relative entropy	SSA + QNEC or positivity of relative entropy	SSA + QNEC or positivity of relative entropy	SSA + QNEC or positivity of relative entropy	SSA + QNEC or positivity of relative entropy
3		SSA	SSA + QNEC	SSA + QNEC	SSA + QNEC
4			SSA	SSA + QNEC	SSA + QNEC

D=d=2 entropic C-thm by [Casini, Huerta];
extended by [Casini, Huerta] to D=d=3 (not available via correlators/dilaton)
D=d=4 by [Casini, Teste, GT]. And unifies d=2,3,4 (Markov prop)
d=1, any D [Casini, Salazar, GT]; d=2, D=3 [Casini, Salazar, GT]

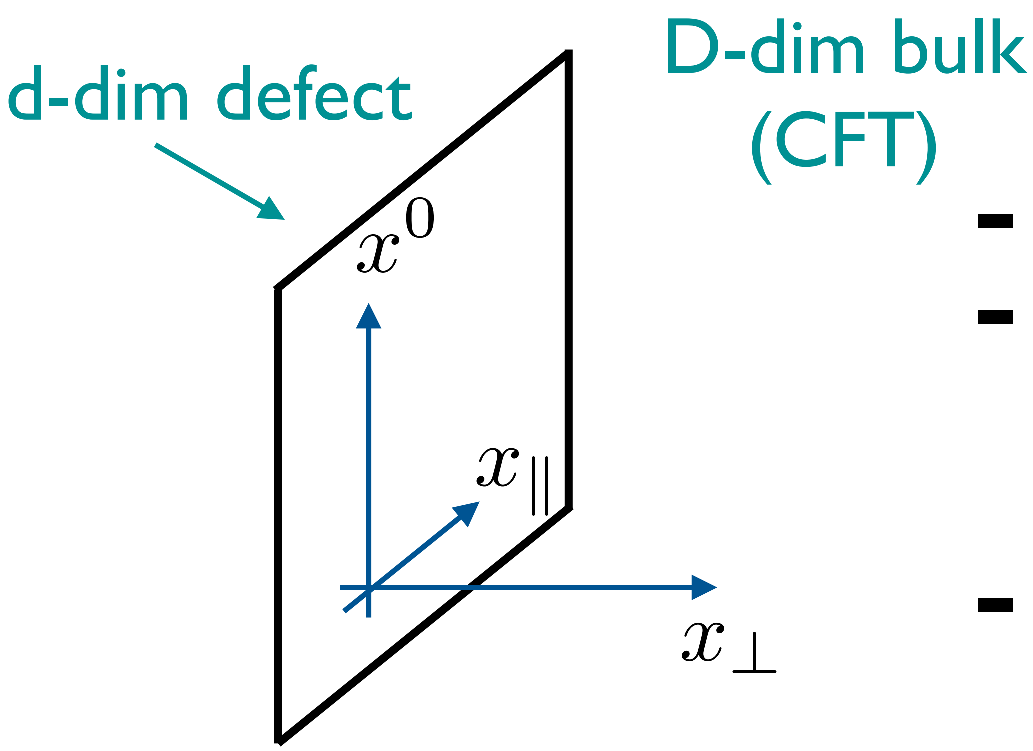
… This covers almost 40 years of developments! But a simple and general understanding is still lacking.

In this talk: we prove an inequality for second derivatives of the Relative Entropy. This establishes the remaining irreversibility thms, and unifies all known theorems for (D, d). [Casini, Salazar, GT, 2023]

A. Key ingredients

QFT setup

D-dimensional CFT with a d-dimensional planar defect



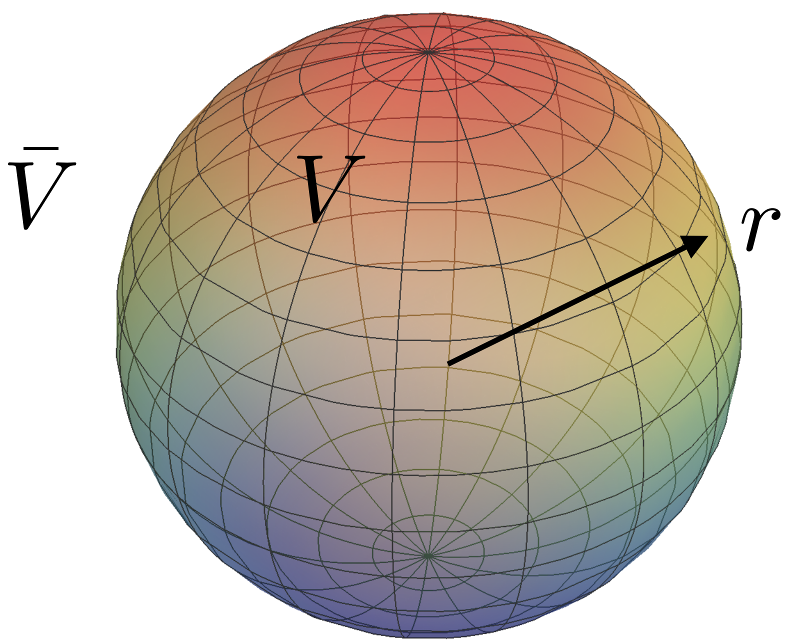
- the defect is conformal in the UV
- turn on relevant deformations on the defect

$$S = S_{UV,CFT} + \int d^d x \, g \mathcal{O}$$

- triggers an RG flow, which ends on a different IR conformal defect
- the bulk does not flow

Particular case: d=D, no bulk. Gives QFT RG flow without defects

Entanglement entropy



density matrix $\rho_r = \mathrm{tr}_{\bar{V}} |0\rangle\langle 0|$

von Neumann entropy $S(r) = -\mathrm{tr}_V(\rho_r \log \rho_r)$

Can probe RG flow by

$$mr \ll 1 \ (\mathrm{UV}) \ \Rightarrow \ mr \gg 1 \ (\mathrm{IR})$$

- Structure near a fixed point, first without defects (D=d):

$$S(r) = S_{local} + S_{non-local}$$

$$S_{local}(r) = \mu_{d-2} r^{d-2} + \mu_{d-4} r^{d-4} + \ldots$$ UV divergent

$$S_{non-local}(r) = \begin{cases} (-1)^{\frac{d}{2}-1} 4A \log \frac{r}{\epsilon} & d \ \text{even} \\ (-1)^{\frac{d-1}{2}} F & d \ \text{odd} \end{cases}$$ universal

RG irreversibility: decrease of A or F; proved for $d = 2, 3, 4$

The universal terms can also be isolated from the partition function on the sphere (free energy in de Sitter), or from the Weyl anomaly if d even.

- For d-dim conformal defect in a D-dim CFT, there are additional terms:

$$S(r) = \mu_{D-2} r^{D-2} + \mu_{D-4} r^{D-4} + \ldots + \tilde{\mu}_{d-2} r^{d-2} + \tilde{\mu}_{d-4} r^{d-4} + \ldots$$
$$+ \begin{cases} (-1)^{\frac{D-2}{2}} 4A \log \frac{r}{\epsilon} & D \text{ even} \\ (-1)^{\frac{D-1}{2}} F & D \text{ odd} \end{cases} + \begin{cases} (-1)^{\frac{d-2}{2}} 4\tilde{A} \log \frac{r}{\epsilon} & d \text{ even} \\ (-1)^{\frac{d-1}{2}} \tilde{F} & d \text{ odd} \end{cases}$$

For defect RG flows, quantities with tildes flow. One can try to prove irreversibility theorems for the universal $\tilde{F}, \tilde{A}$, but in general this does not work (exception: D=d+1).

[Jensen, O'Bannon]
[Kobayashi, Nishioka, Sato, Watanabe]

One reason: the defect contributes nonzero energy. The universal term in the EE no longer coincides with the one in the free energy. We should look at the quantum-information analog of free energy.

Relative entropy

For two density matrices σ and ρ, the relative entropy is

$$S_{rel}(\rho|\sigma) = \mathrm{tr}(\rho\log\rho - \rho\log\sigma)$$

Introducing the modular Hamiltonian $H = -\log\sigma$

$$S_{rel}(\rho|\sigma) = \mathrm{tr}(\rho\log\rho - \rho\log\sigma + \sigma\log\sigma - \sigma\log\sigma)$$

$$= \langle H\rangle_\rho - \langle H\rangle_\sigma - (S_\rho - S_\sigma)$$

$$= \Delta\langle H\rangle - \Delta S$$ ⟵ difference of "free energies"

For irreversibility in QFT, we will compare two density matrices:

$\sigma =$ vacuum density matrix for UV fixed point

$\rho =$ vacuum density matrix for QFT w/relevant deformations

The modular Hamiltonian for a CFT on a sphere is

$$H_\sigma = \int_\Sigma d^{D-1}x\, \eta^\mu \xi^\nu\, T_{\mu\nu}$$

-conf. transf. of Rindler Hamiltonian
- valid also with conformal defects

η^μ : unit normal to Cauchy surface Σ

$$\xi^\nu = \frac{\pi}{R}(R^2 - (x^0)^2 - \vec{x}^2, -2x^0 x^i)$$ Killing vector, R: radius of sphere

Note: $\langle H \rangle_\rho$ depends on choice of Cauchy surface. Reason: states evolve with different action.

- for space-like Σ, $\langle H \rangle_\rho \sim R^D$ dominates relative entropy
- for null Σ, $\langle H \rangle_\rho \sim R^{D-2}$ comparable to EE. We choose this limit.

It contributes a universal term, proportional to energy of defect. Then

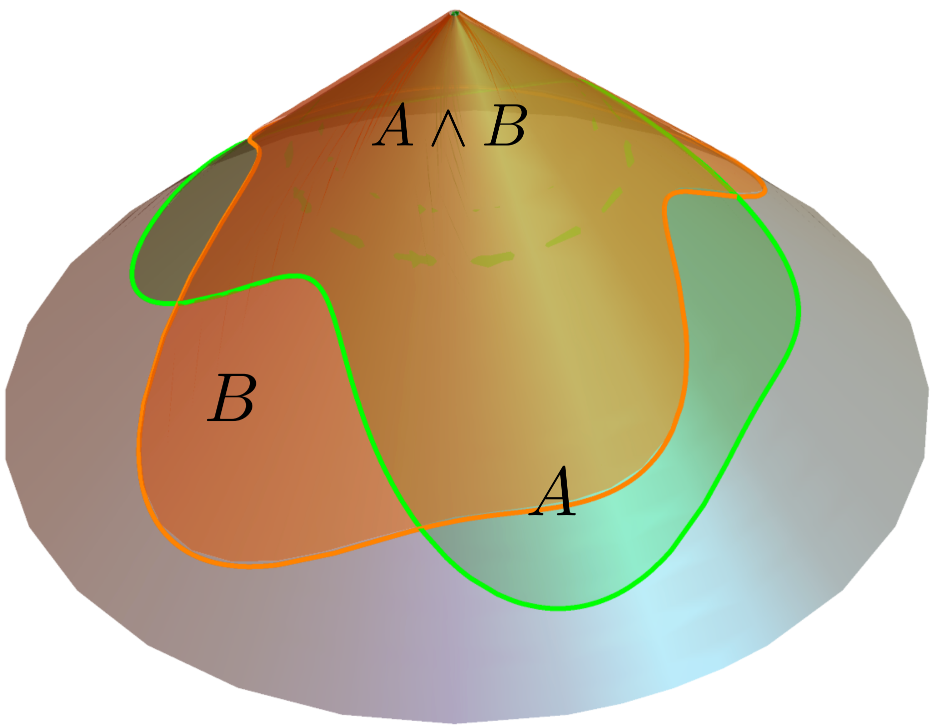
$$-\lim_{R\to\infty} S_{\rm rel}(R) = \Delta\mu'_{d-2}\, R^{d-2} + \Delta\mu'_{d-4}\, R^{d-4} + \ldots + \begin{cases} (-)^{\frac{d-2}{2}} 4\, \Delta A' \log(R/\epsilon) & d \text{ even} \\ (-)^{\frac{d-1}{2}} \Delta F' & d \text{ odd} \end{cases}$$

Strong subadditivity and Markov property

Key property of unitary quantum mechanics: strong subadditivity of the EE [Lieb, Ruskai, 1973]

$$S(A)+S(B)\geq S(A\cup B)+S(A\cap B)$$

For a CFT and for regions with boundary in null cone, Markov property:
[Casini, Testé, GT, 2017]



$$S(A\cup B)+S(A\cap B)=S(A)+S(B)$$

$$\Leftrightarrow \quad \log\rho_{A\cup B}=\log\rho_A+\log\rho_B-\log\rho_{A\cap B}$$

✓This is called a quantum Markov state
✓mod Hamiltonian local on null surfaces
✓Tracing out a subsystem becomes a reversible process

Combining SSA w/Markov, we obtain strong superadditivity

$$S_{\rm rel}(A)+S_{\rm rel}(B)\leq S_{\rm rel}(A\cup B)+S_{\rm rel}(A\cap B)$$

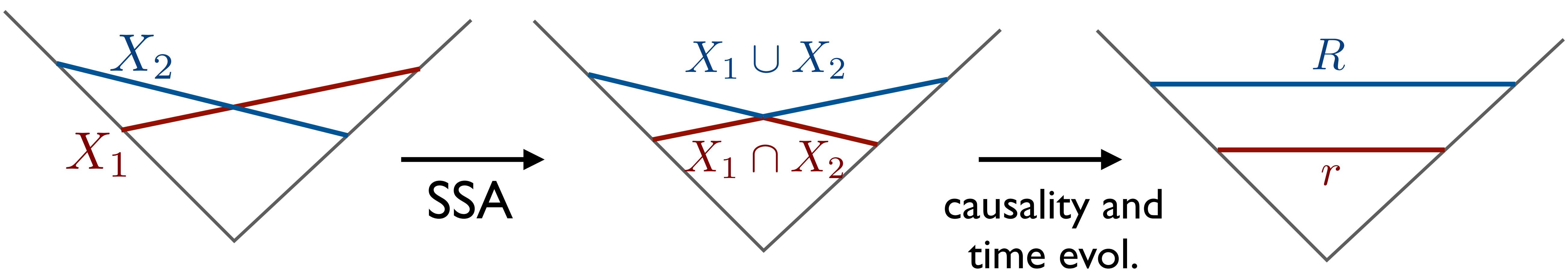
B. Proof of irreversibility inequality

We will explain the proof for a d=2 defect, in D-dim CFT.
General case: [Casini, Salazar, GT, 2023] and talk at IFQ next week

The entropic C-theorem

[Casini, Huerta, '04, '12]

Warm-up without defect, ie D=d=2



$$\Delta S(X_1) + \Delta S(X_2) \geq \Delta S(X_1 \cup X_2) + \Delta S(X_1 \cap X_2) = \Delta S(R) + \Delta S(r)$$

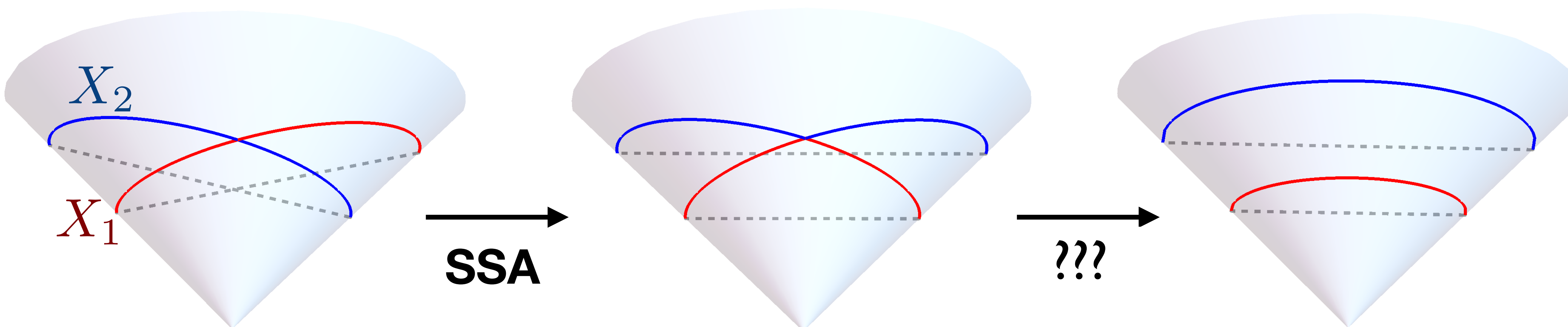
$2\Delta S(\sqrt{rR}) \geq \Delta S(R) + \Delta S(r)$. When $r \to R \Rightarrow (R\,\Delta S'(R))' \leq 0$

implies the decrease of the central charge $C_{IR} < C_{UV}$

d=2 defect

Let's try the same for a d=2 defect, in a D-dim CFT bulk.

The problem is that now the spheres extend into the bulk, and their causal union and intersection no longer give causal diamonds of spheres.



➡New ingredient: use QNEC in the bulk (conformal), to bound the entropy of union and intersection by entropies of diamonds (spheres)

As we will see, this requires the relative entropy:

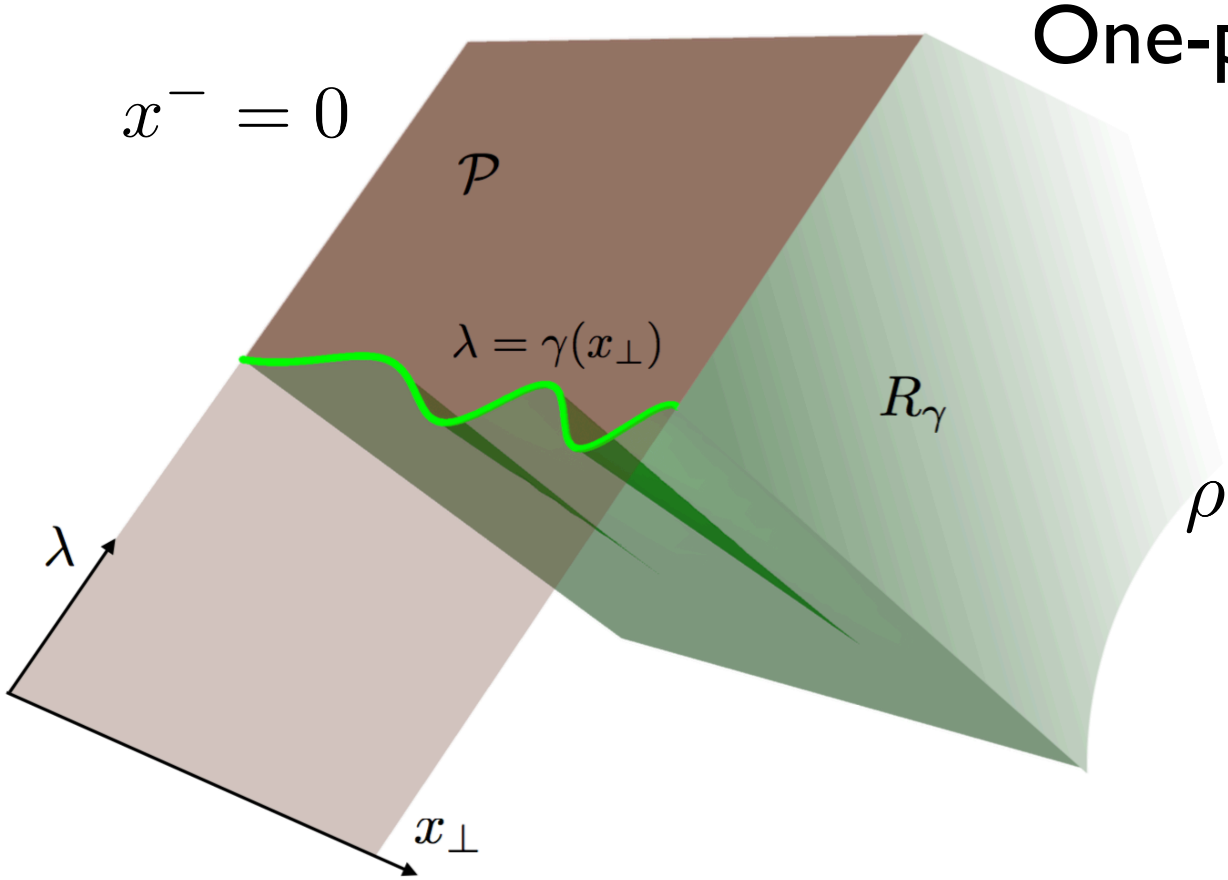
$$S_{\rm rel}(X_1) + S_{\rm rel}(X_2) \le S_{\rm rel}(X_1 \cup X_2) + S_{\rm rel}(X_1 \cap X_2) \le S_{\rm rel}(R) + S_{\rm rel}(r)$$

Then we would get defect irreversibility, $(R S'_{\rm rel}(R))' \ge 0$

QNEC

[Bousso, Fisher, Leichenauer, Wall]
[Balakrishnan, Faulkner, Khandker, Wang]
[Ceyhan, Faulkner]

Originated from quantum focusing conjecture, but is valid in general QFT. Simplest setup: deformations on null plane



One-parameter deformation of null boundary:

$$x^+ = \gamma_s(x_\perp) = \gamma(x_\perp) + a(x_\perp)s$$
$$(a(x_\perp) \geq 0)$$

Relative entropy between σ (vacuum) and ρ (excited state) is convex under null defs

$$\frac{d^2 S_{\rm rel}(\gamma_s)}{ds^2} \geq 0$$

QNEC is also valid on light-cone if we have a CFT. It is valid in the CFT bulk of the theory with defect.

$$\Rightarrow [S_{\rm rel}(R) - S_{\rm rel}(X_1 \cup X_2)] - [S_{\rm rel}(X_1 \cap X_2) - S_{\rm rel}(r)] \geq 0$$

and this establishes the irrev. inequality for d=2 and all D.

C. Conclusions

The result for general (d,D) is $RS''_{\rm rel}(R) - (d-3)S'_{\rm rel}(R) \geq 0$

[Casini, Salazar, GT, 2023]

- The result is independent of D; depends only on defect dim. d
- The theory on defect is nonlocal (it interacts w/bulk). It's remarkable that we get the same inequality as with no bulk (d=D)
- Relative entropy appears, required by QNEC.
- Implies irreversibility of defect RG flows for d=2,3,4 and all D.

Future directions:

1) Irreversibility for d>4? Requires new tools.
2) Theories with less symmetries? Interesting for condensed matter
3) Relation to non-entropic results?