# Regge sum rules and stringy amplitudes

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Strings 2023

based on ongoing work with Kelian Häring

[Talk by G. Remmen]

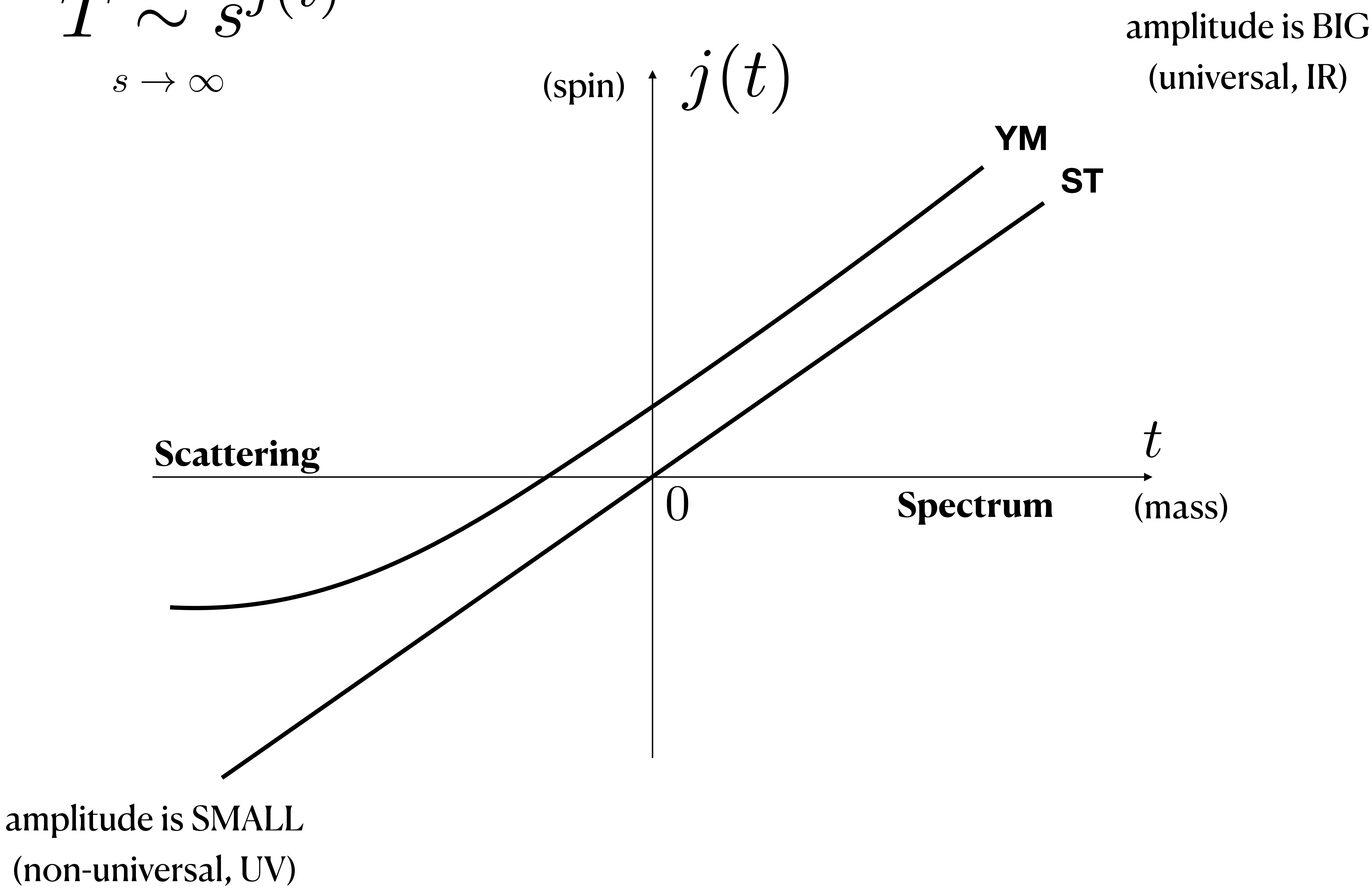
One of the characteristic features of stringy scattering is the Regge behavior at high energies

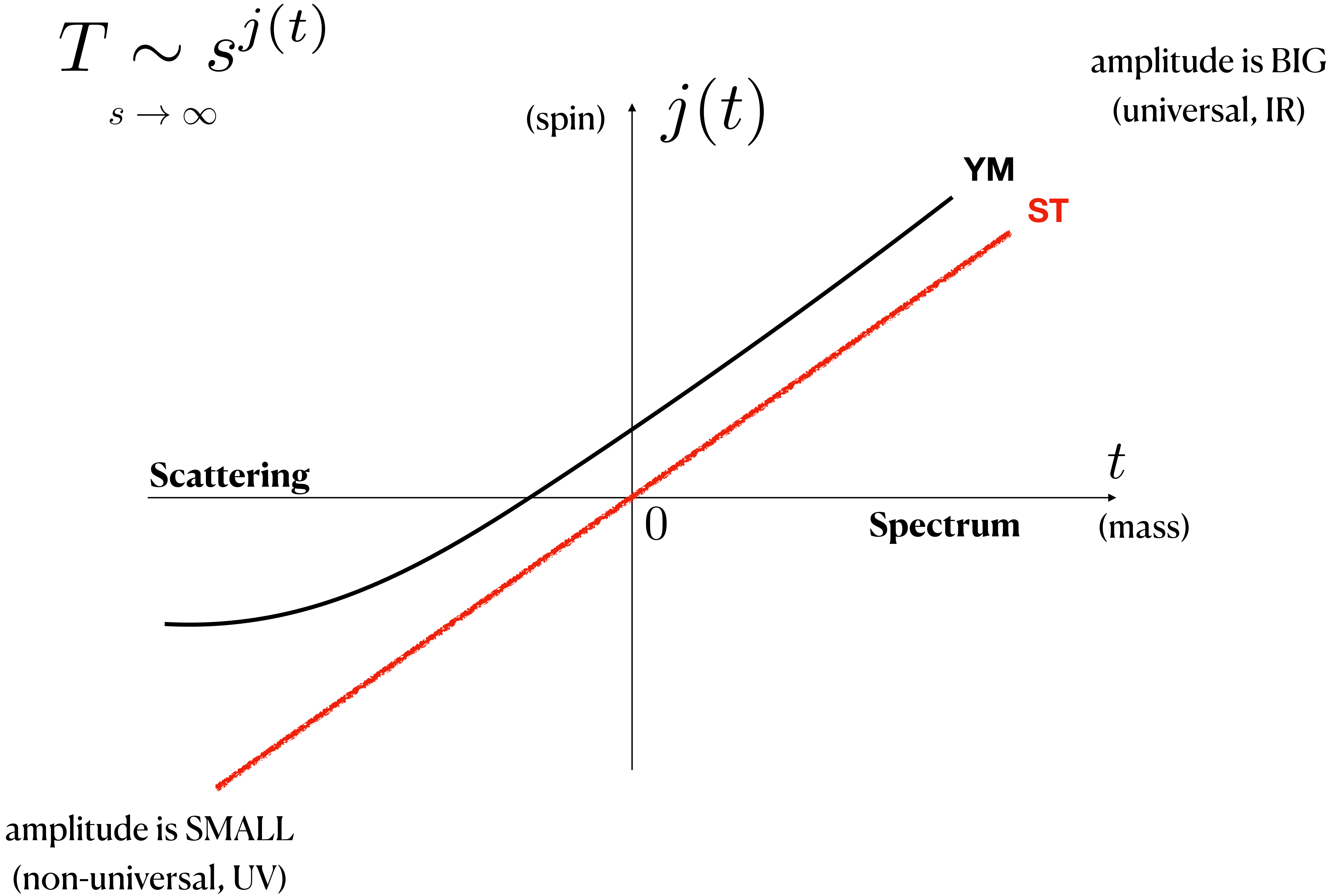
**(tree level)**

$$T(s,t) \sim s^{j(t)} \qquad s \to \infty \quad t \text{ fixed}$$

| | |
|---|---|
| $j(t)$ | **leading Regge trajectory** |

$$T \sim s^{j(t)}$$
$$s \to \infty$$





$T \sim s^{j(t)}$
$s \to \infty$
(spin)
$j(t)$
amplitude is BIG
(universal, IR)
YM
ST
Scattering
$t$
0
Spectrum
(mass)
amplitude is SMALL
(non-universal, UV)

$$S = \frac{1}{M_P^2} \int d^4x \sqrt{-g} \Big( R + \beta_{R^3} R^3 + \beta_{R^4} C^2 + \tilde{\beta}_{R^4} \tilde{C}^2 + ... \Big)$$

Causality requires massive higher spin particles.

$$\beta_i \sim \frac{1}{m_{\rm HS}^{\#}}$$

**mass of higher spin particles**

[Camanho, Edelstein, Maldacena, AZ, Arkani-Hamed, T-C Huang, Y-t Huang, Chiang, Rodina, Weng, Tolley, Wang, Zhou, Caron-Huot, van Duong, Sinha, Zahed, Mazac, Rastelli, Simmons-Duffin, Y-Z Li, Bellazzini, Riembau, Riva, Carrillo Gonzales, de Rham, Jaitly, Pozsgay, Tokareva, Henriksson, McPeak, Russo, Vichi…]

[Rastelli (review), Strings 2022]

[Caron-Huot, Mizera (discussion) Strings 2023]

Sharper bounds from causality+unitarity+crossing.

Wilson coefficients control the low-energy expansion of the amplitude $T(s,t) = ... + \sum_{k,j} a_{k,j} s^{k-j} t^j$

What happens to the bounds if in addition we put in the leading Regge trajectory?

**(How does the UV softness constrain the IR?)**

[D. Gross: “Go multiparticle!”]

[D. Gross: “Don’t follow your elders”]

# Plan:

0. Regge sum rules

I. Open strings

II. High energy, fixed angle scattering

III. Closed strings

0. Regge sum rules

I. Open strings

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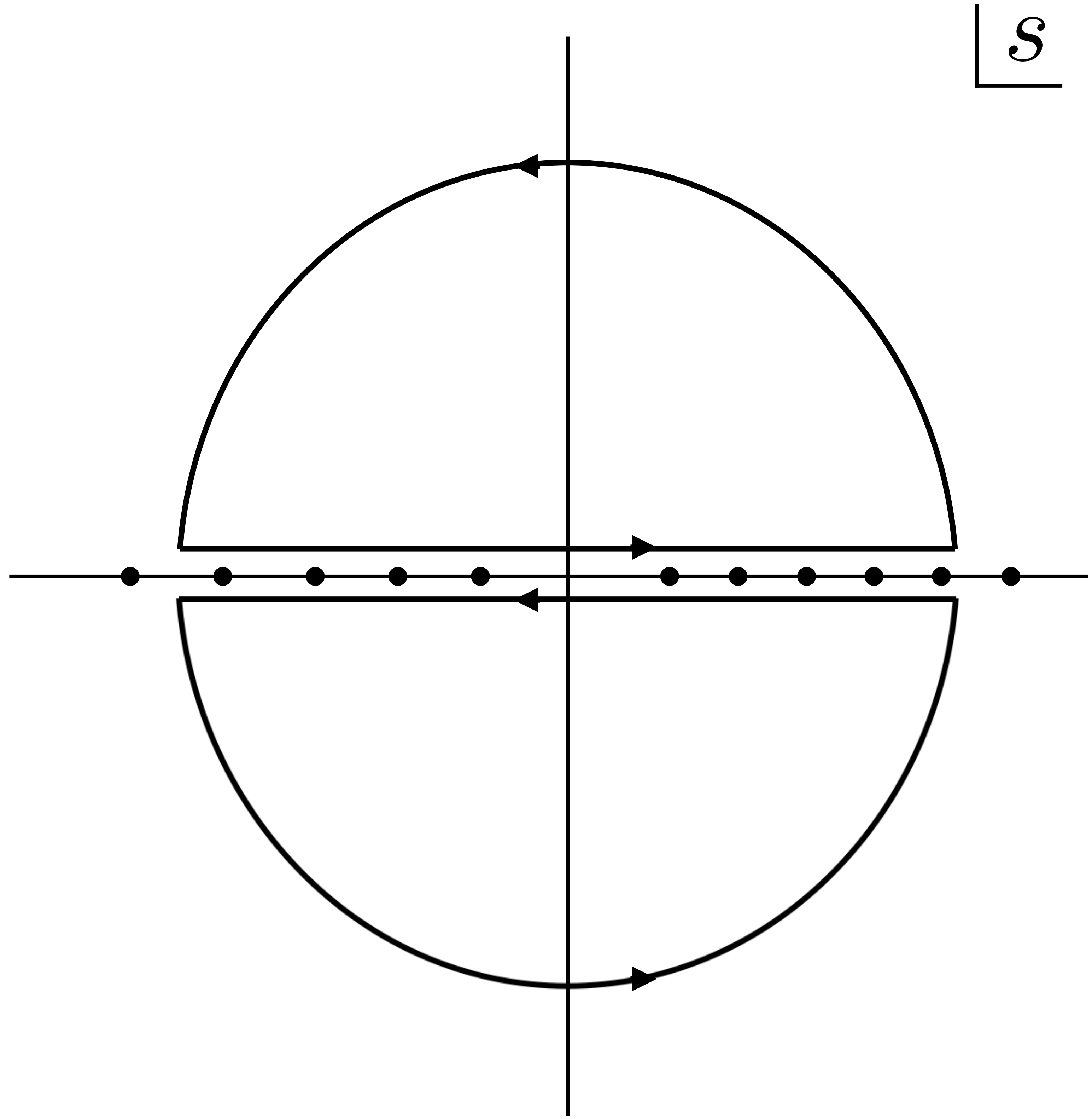
III. Closed strings

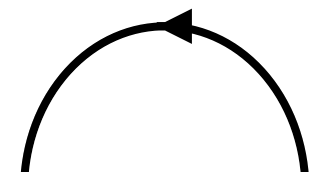



$= 0$

$$\oint_{\mathcal{C}} ds' (s')^n T(s', t) = 0$$

- Regge sum rules (RSR): $t < 0$

$$\oint_{\mathcal{C}_\infty} ds'(s')^n T(s',t) = 0$$



$$\sim \int \frac{ds}{s} s^{j(t)+1+n}$$

$$\int_0^\infty ds'(s')^n T_s(s',t) = 0, \quad j(t) < -1-n$$

**(scattering is soft)**

**discontinuity in s**

- Finite energy sum rules (FESR): $t > 0$

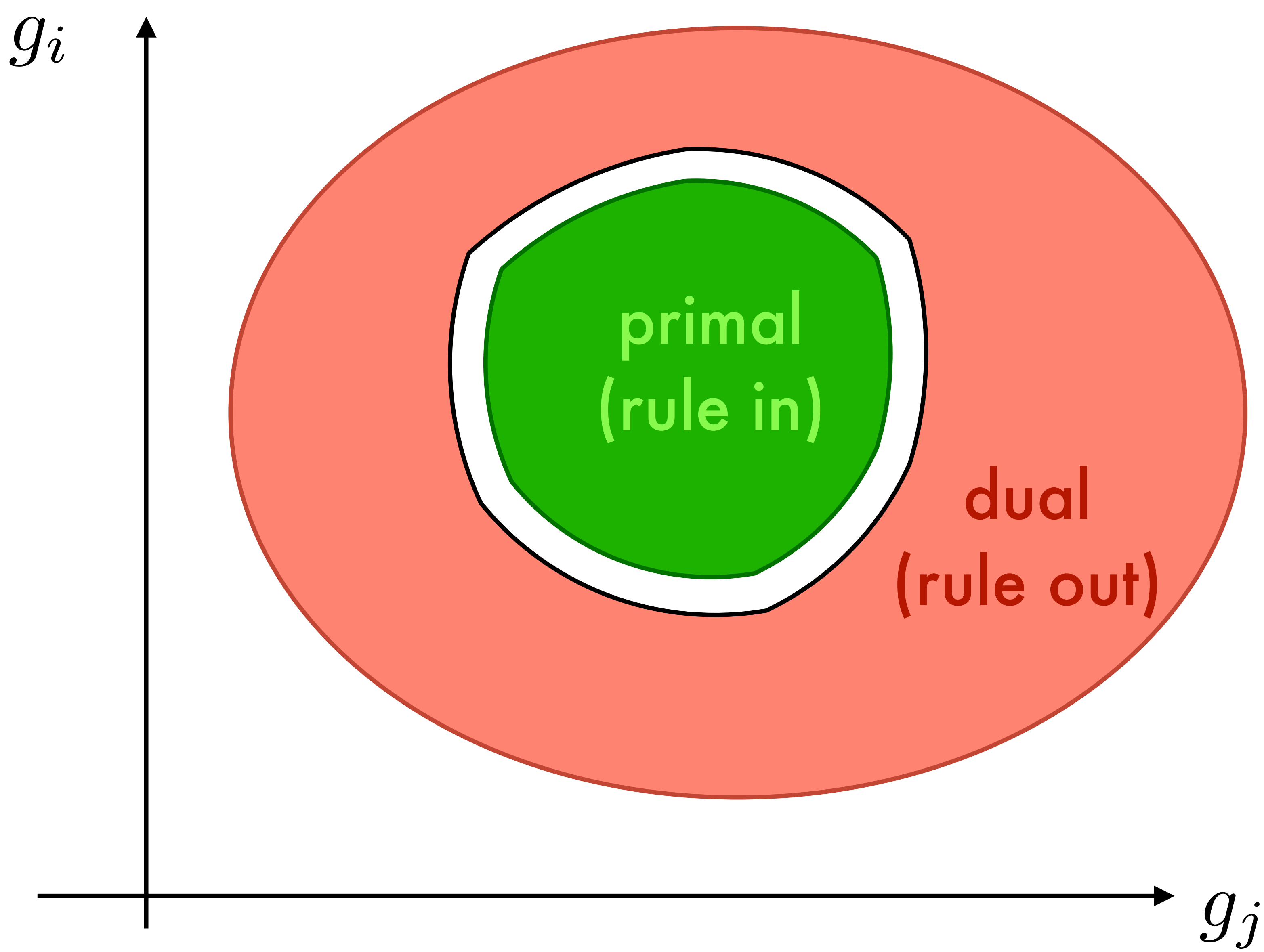
$$\int_0^s ds' T_s(s',t) \sim s^{j(t)+1}$$

**(no accumulation point in the spectrum+HS particles)**

Basic idea: Explore the allowed region by explicitly constructing amplitudes with desired properties.

$$\mathrm{RSR}: \quad \langle (-1)^J c^J \rangle = 0 \qquad c > 1$$

$$\langle \ldots \rangle \equiv \int dm^2 \rho(m^2) \ldots, \quad \rho(m^2) \geq 0.$$



To proceed we consider an ansatz for $T(s, t)$ with manifest:

[Veneziano, Mandelstam, Khuri, Matsuda, Altarelli, Rubinstein, Virasoro, Sivers, Yellin, Gross, …]
[Cheung, Remmen, Geiser, Lindwasser, Arkani-Hamed, Huang, Huang, …]

1. Meromorphy $$T(s,t) \sim \frac{1}{s-m^2}$$

2. Crossing symmetry $$T(s,t) = T(t,s)$$

3. Polynomial residues $$-\mathrm{Res}_{s=m^2} T(s, \frac{s}{2}(z-1)) = \sum_J c_J P_J(z)$$

4. FESR+RSR

5*. All particles have $$m^2 = n, \quad n \in \mathbb{Z}$$

We will then need to impose (numerically):

6. Unitarity $$c_J \geq 0$$

0. Regge sum rules

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For the open string case the ansatz is

$$T(s,t) = \sum_{i=0}^{\infty} \sum_{k=0}^{i} b_{ik} \frac{\Gamma(i-s)\Gamma(i-t)}{\Gamma(i+k-s-t)}$$

- All terms are linearly independent.
- For t>0, FESR is automatic: $T_s(s,t) \sim s^t$
- For t<0, RSR is easy: $\int_0^\infty ds'(s')^n T_s(s',t) = 0$
- Constraints on $b_{ik}$ come from unitarity:

$$\mathrm{Res}_{s=n} T(s,t) = \sum_{J=0}^{n} c_J P_J(1+\frac{2t}{n}), \quad c_J \geq 0.$$

- The ansatz is complete (see back-up slide).

**Comment 1**: The truncated ansatz with $i_{\max} \geq 1$

$$T(s,t) = \sum_{i=0}^{i_{\max}} \sum_{k=0}^{i} b_{ik} \frac{\Gamma(i-s)\Gamma(i-t)}{\Gamma(i+k-s-t)}$$

satisfies RSR and violates unitarity (unless Veneziano).

**Comment 2**: The Cheung-Remmen ansatz

$$T(s,t) = \sum_{i=0}^{\infty} \frac{1}{i!} \frac{r}{r+i} \frac{\Gamma(i-s)\Gamma(i-t)}{\Gamma(i-s-t)}, \quad r \geq -1/2.$$

violates RSR and satisfies unitarity.

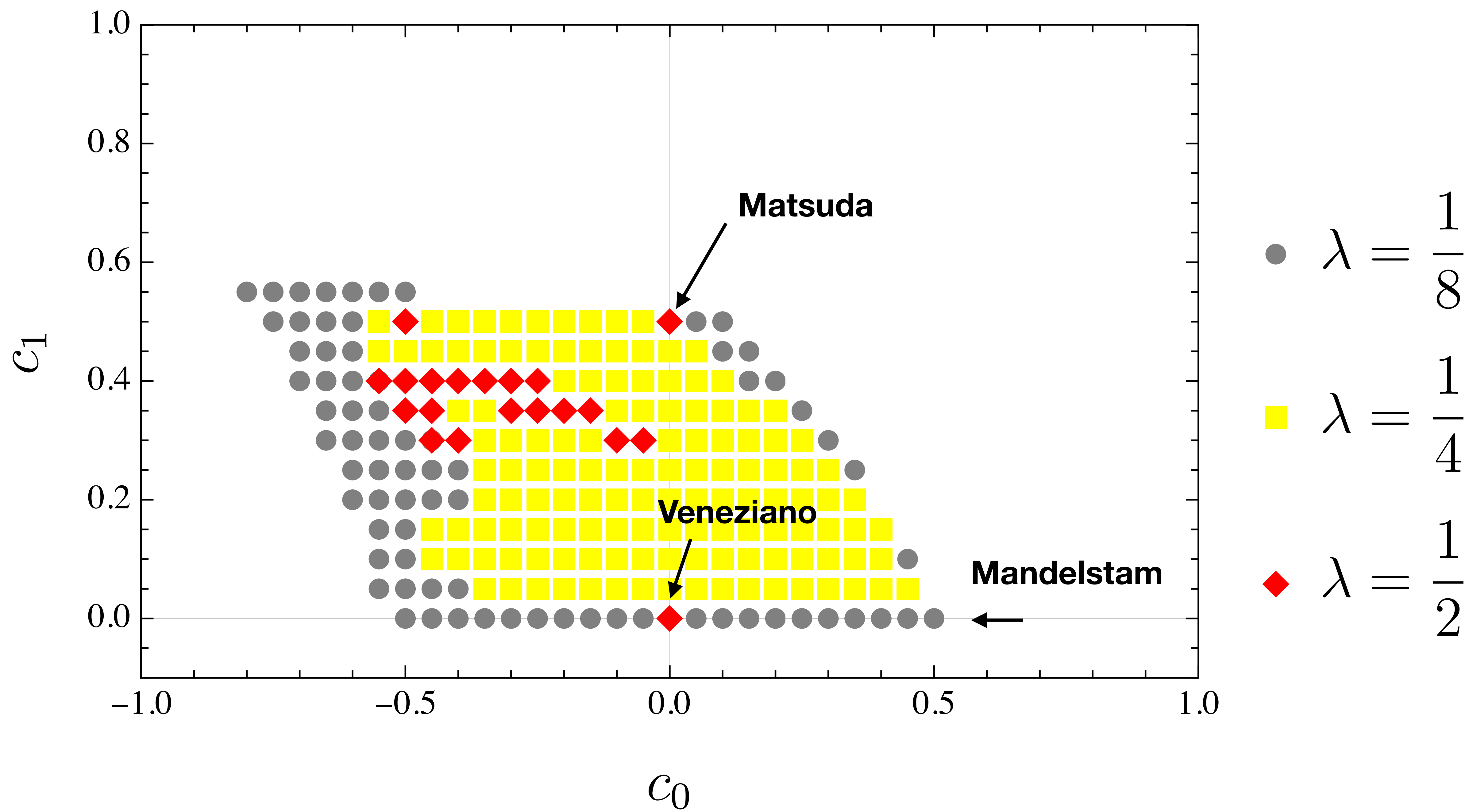
fixed J singularities

$$T_s(s,t) \sim s^t \qquad\qquad T(s,t) \sim f(t)s^t + \frac{r}{(1+t)s} + \dots$$

**Comment 3**: There exist amplitudes different from the Veneziano amplitude that satisfy RSR and satisfy unitarity

$$T_{c_0,c_1,\lambda}(s,t) = \int_0^1 dz\, z^{-s-1}(1-z)^{-t-1}(1-4\lambda(1-z)z)^{c_0+c_1(s+t)}, \quad 0 \leq \lambda < 1/2$$

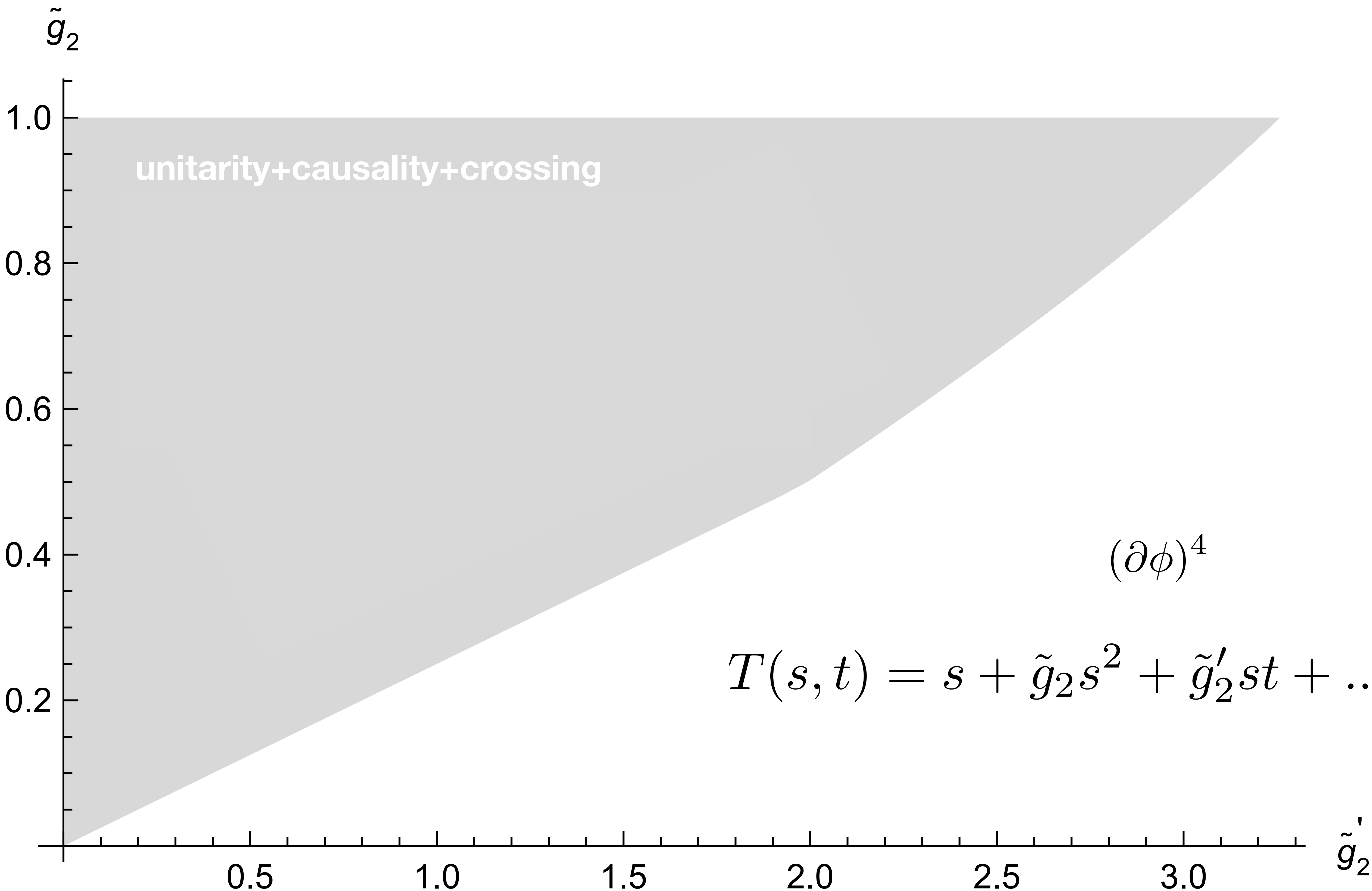
# Unitarity (numerically, $n_{\max} = 0,..,50,...,450$)




We observe that unitarity implies $c_1 \geq 0$.

# Bounds on Wilson coefficients

[Albert, Rastelli '22]

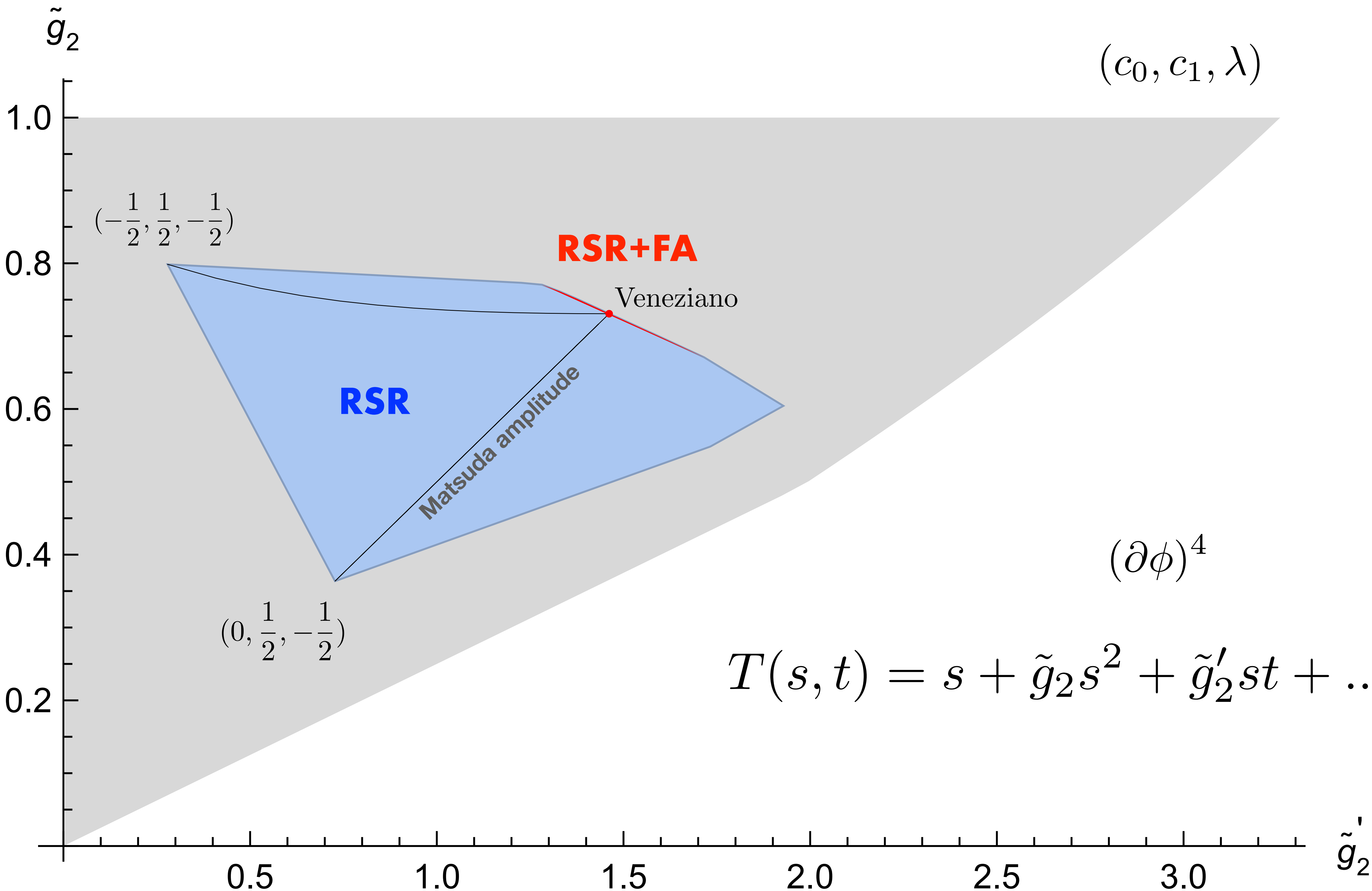



$(\partial\phi)^4$

$$T(s,t) = s + \tilde{g}_2 s^2 + \tilde{g}_2' st + \ldots$$

# Bounds on Wilson coefficients

[Albert, Rastelli '22]




$$T(s,t) = s + \tilde{g}_2 s^2 + \tilde{g}_2' st + \ldots$$

0. Regge sum rules

I. Open strings

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$$s \to \infty$$

$$s/t \text{ fixed}$$

Consider the semi-classical limit $s, t \gg 1$. Here we explicitly get

$$\log T(s,t) = (s+t)\log(s+t) - s\log s - t\log t$$

$$+c_1\left(t\log\frac{1}{2}\left(1-\tilde{\lambda}\frac{s-t}{s+t}+\sqrt{1-\tilde{\lambda}}\sqrt{1-\tilde{\lambda}\frac{(s-t)^2}{(s+t)^2}}\right)+\{s\leftrightarrow t\}\right)$$

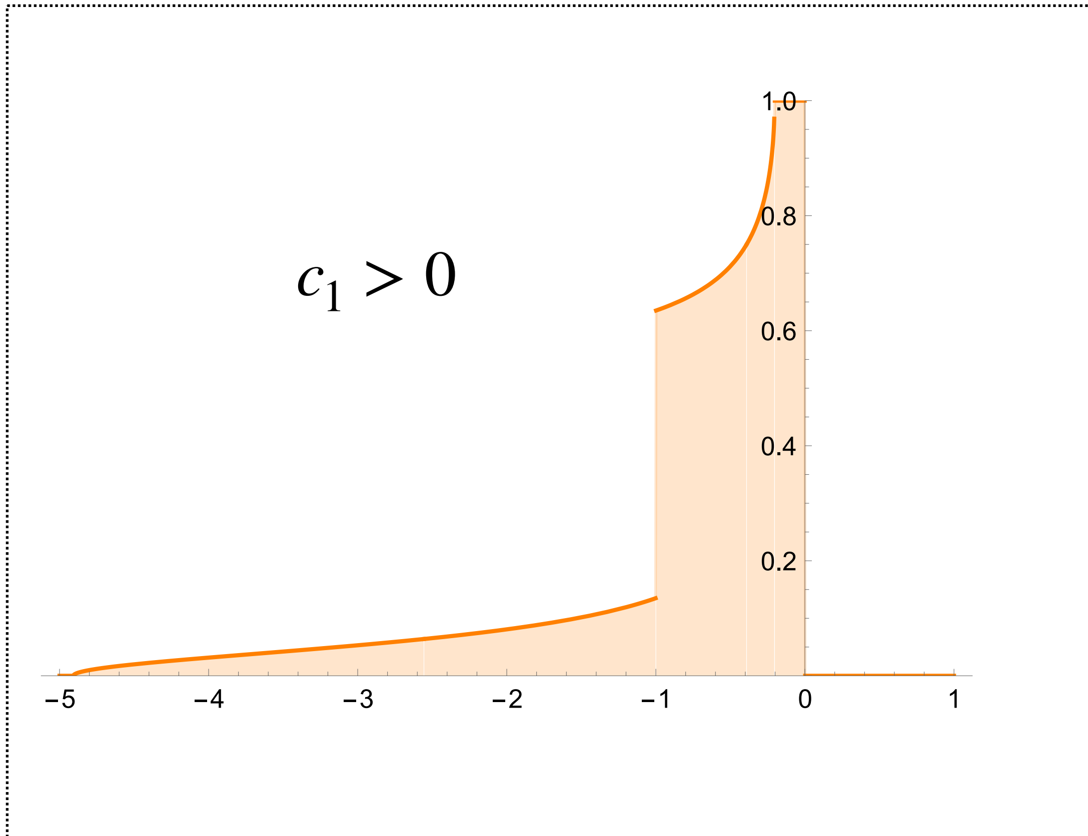
$$0 \le \tilde{\lambda} = 4\lambda(1-\lambda) < 1$$

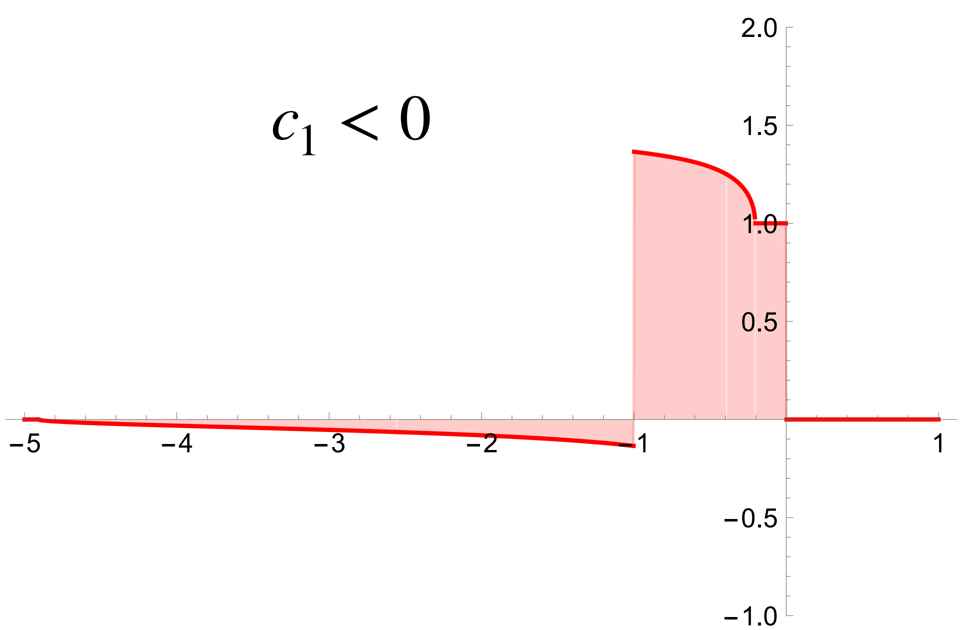
In this regime the amplitude is controlled by the zeros of positive sums of Legendre polynomials

$$\log T(s,t) \sim \log\sum_J^s c_J(s) P_J(1+\frac{2t}{s})$$

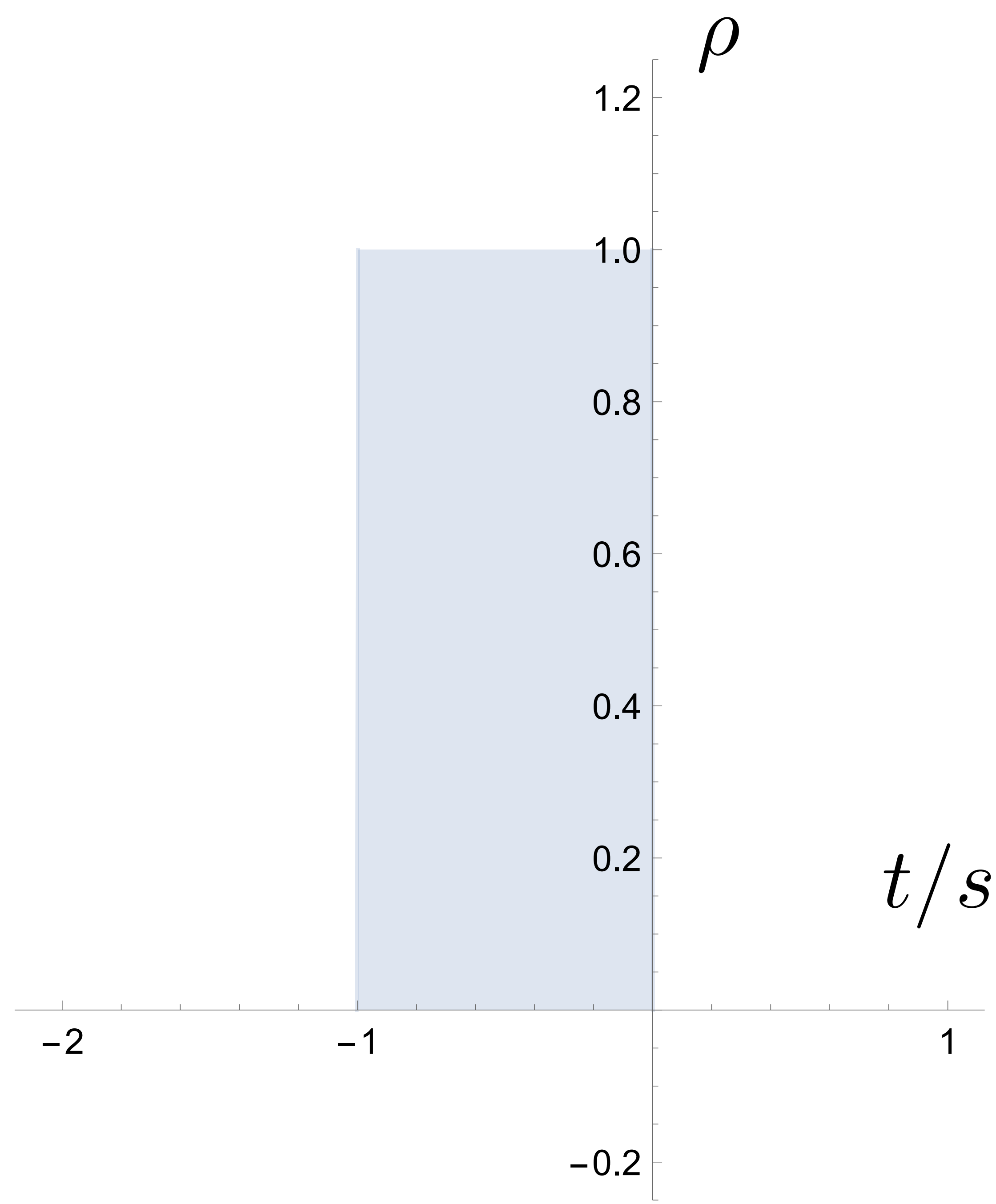
[Caron-Huot, Komargodski, Sever, AZ ’16]

$$\log T(s,t) = s\int_{-\infty}^{0} dz\rho(z)\log\left(1-\frac{t}{sz}\right), \quad \rho(z)\geq 0$$




$\Rightarrow c_1 \geq 0$




[Caron-Huot, Komargodski, Sever, AZ '16]

Maybe a simpler bootstrap target for $s, t \gg 1$ is [M. Correia, AZ]

$$\log T(s,t) \leq (s+t)\log(s+t) - s\log s - t\log t$$

we proved it assuming the zeros are real and are supported along the negative axis.

<u>Problem in 2d electrostatics:</u> Find the bound on the electric field generated by the distribution of positive charges

$$\int_{-\infty}^{0} dz \rho(z) = 1, \quad \rho(z) \geq 0 \ .$$

that satisfies crossing

**(dual problem)**

$$\int_{-\infty}^{0} dz \rho(z) \Big( \beta \log(1 - \frac{1}{\beta z}) + \log(1 - \frac{\beta}{z}) \Big) = 0, \quad \beta > 0 \ .$$

**Veneziano behavior is extremal**

Given an upper bound on scattering for complex angles, one can use analyticity to derive a lower bound on scattering for real angles

$$\text{Regge} \leq (\text{Fixed real angle}) \times (\text{Fixed complex angle})$$

[Hadamard three-circle theorem]

$$\Rightarrow \quad (\text{Fixed real angle}) \geq \frac{\text{Regge}}{(\text{Fixed complex angle})}$$

[Cerulus, Martin '64]
[Tourkine, AZ '23]
[Buoninfante, Tokuda, Yamaguchi '23]

Stringy Cerulus-Martin bound:

$$\max_{|z|\leq z_0} |T(s,z)| \gtrsim e^{-s \log \frac{1+\sqrt{z_0^2-1}}{z_0}}$$

0. Regge sum rules

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Consider the MHV amplitude for gravitons

$$T_{--++}(s,t,u) = (\langle 12\rangle[34])^4 f(s|t,u) \stackrel{CMF}{=} s^4 f(s|t,u)$$

$$f(s|t,u) = -\frac{\Gamma(-s)\Gamma(-t)\Gamma(u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)}$$
$$+ \sum_{c_s,c_t,d_s,d_t} \alpha_{c_s,c_t,d_s,d_t} \frac{\Gamma(c_s - s)\Gamma(c_t - t)\Gamma(c_t - u)}{\Gamma(d_s + s)\Gamma(d_t + t)\Gamma(d_t + u)}$$

The Regge limit is

$$f(s|t,u) \sim s^{-2+2t} \qquad s \to \infty$$

$$f(s|t,u) \sim t^{2+2s} \qquad t \to \infty$$

Unitarity has to be checked!

# Observation: A truncated ansatz

$$f_{N_{\max}}(s|t,u) = -\frac{\Gamma(-s)\Gamma(-t)\Gamma(u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)} + \sum_{c_s,c_t=0}^{N_{\max}} \sum_{d_s,d_t=1}^{2N_{\max}} \alpha_{c_s,c_t,d_s,d_t} \frac{\Gamma(c_s-s)\Gamma(c_t-t)\Gamma(c_t-u)}{\Gamma(d_s+s)\Gamma(d_t+t)\Gamma(d_t+u)}$$

is consistent with unitarity.
It automatically satisfies crossing, FESR, and RSR.

The number of independent parameters in the ansatz is

$$3N_{\max}^2 + N_{\max} - 2$$

We can therefore explore bounds on Wilson coefficients numerically (as a function of $N_{\max}$).

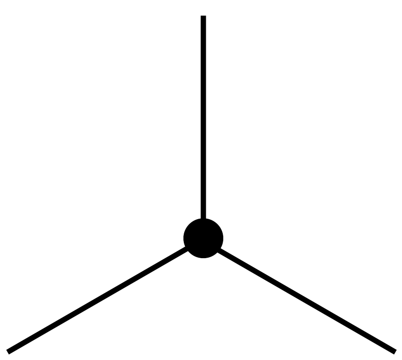
[Paulos, Penedones, Toledo, van Rees, Vieira '17]

$$N_{\max} = 20: \quad 1218 \ \ \text{terms}$$

The low-energy expansion of the amplitude takes the form

$$f(s|t,u) = \frac{1}{stu} + |\beta_{R^3}|^2 \frac{tu}{s} - |\beta_\phi|^2 \frac{1}{s} + \sum_{k\geq j\geq 0} a_{k,j} s^{k-j} t^j$$

First, we look at the correction to the graviton three-point function



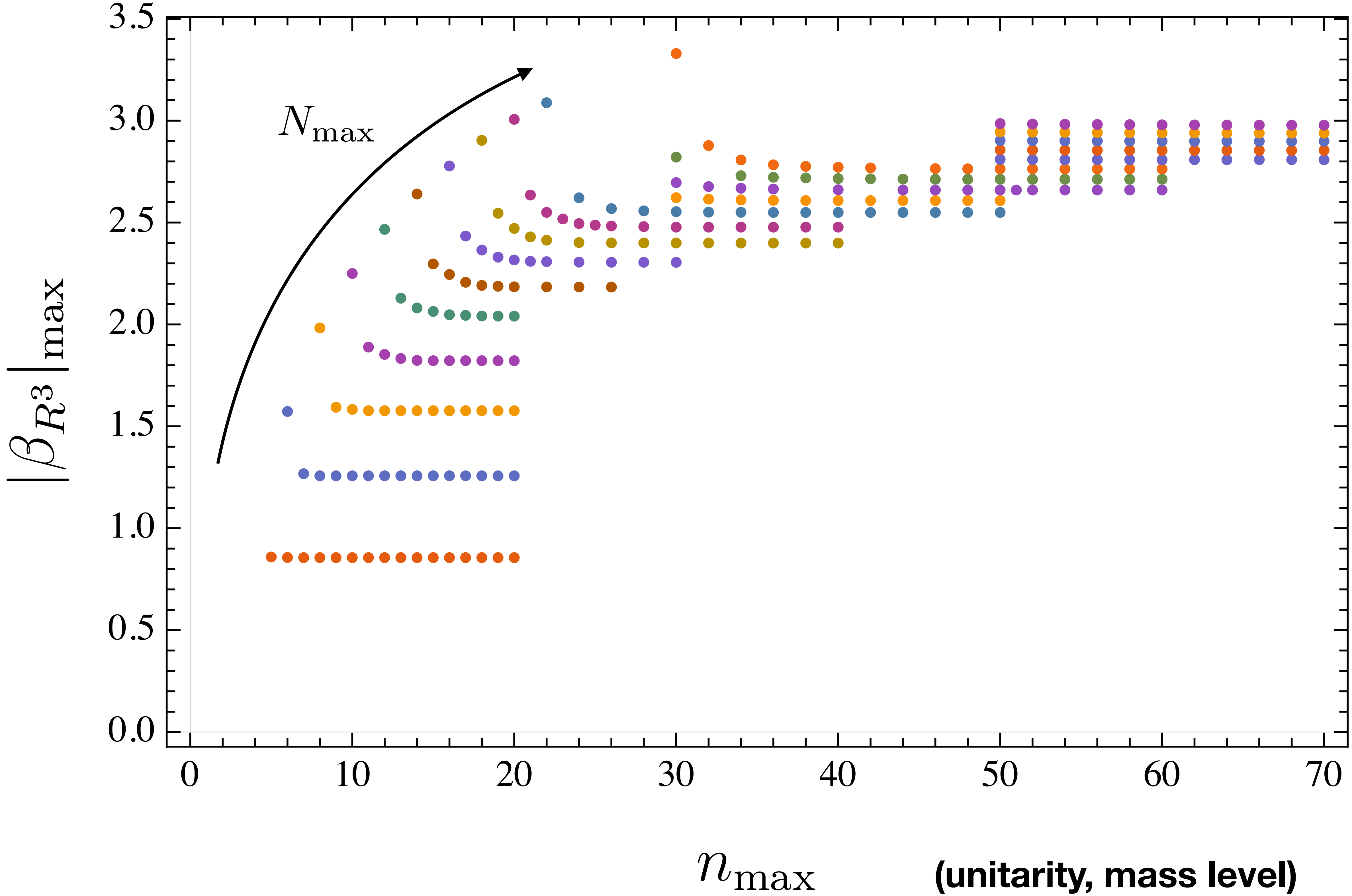
$$|\beta_{R^3}| \lesssim \frac{\log m_{\rm HS} L_{\rm IR}}{m_{\rm HS}^4}$$

[Camanho, Edelstein, Maldacena, AZ '14]

$$|\beta_{R^3}| \leq \# \frac{\log m_{\rm HS} R_{\rm AdS}}{m_{\rm HS}^4}$$

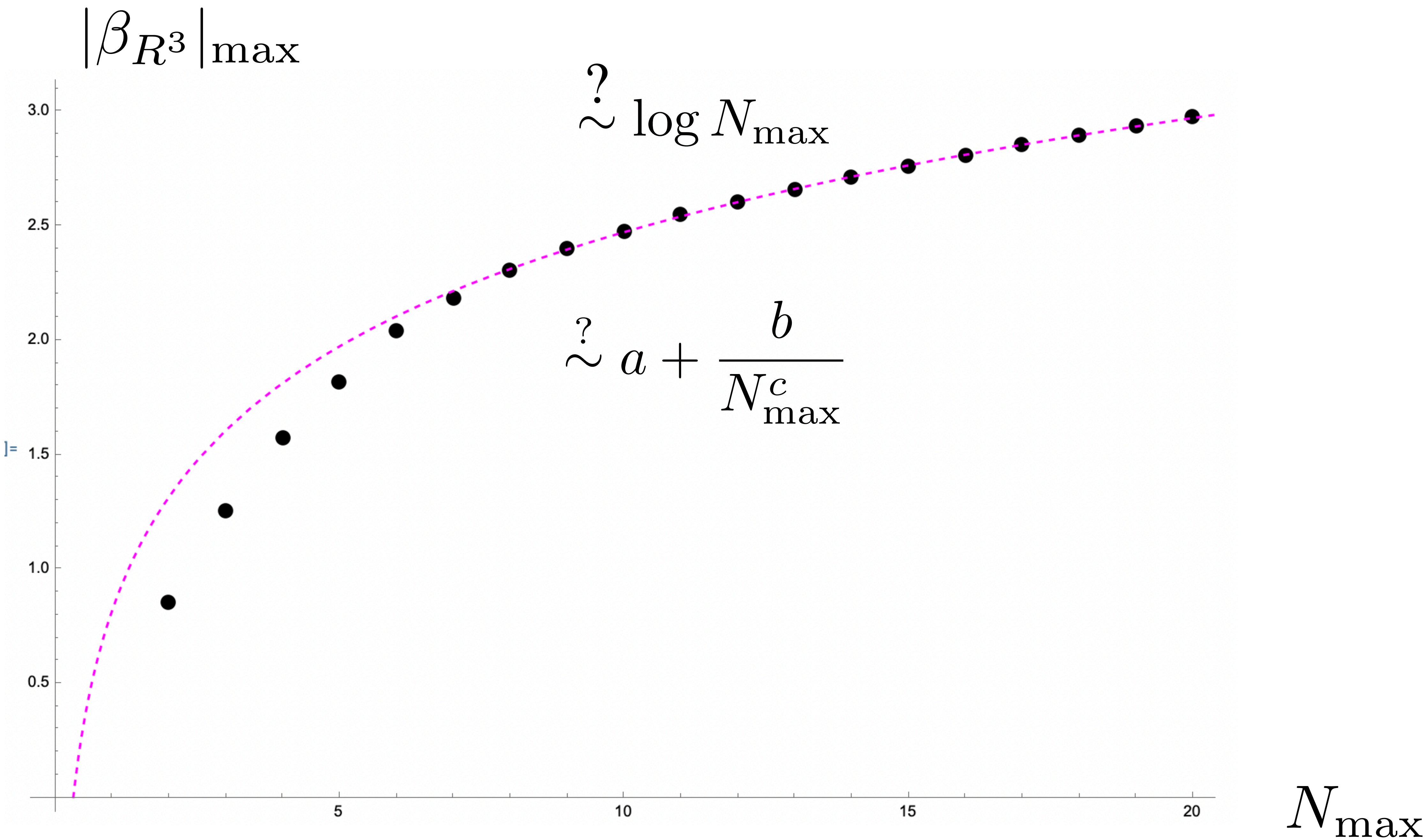
[Caron-Huot, Mazac, Rastelli, Simmons-Duffin '21]

# Bound on three-point coupling (causality+RSR)




**(unitarity, mass level)**

# Bound on three-point coupling (causality+RSR)

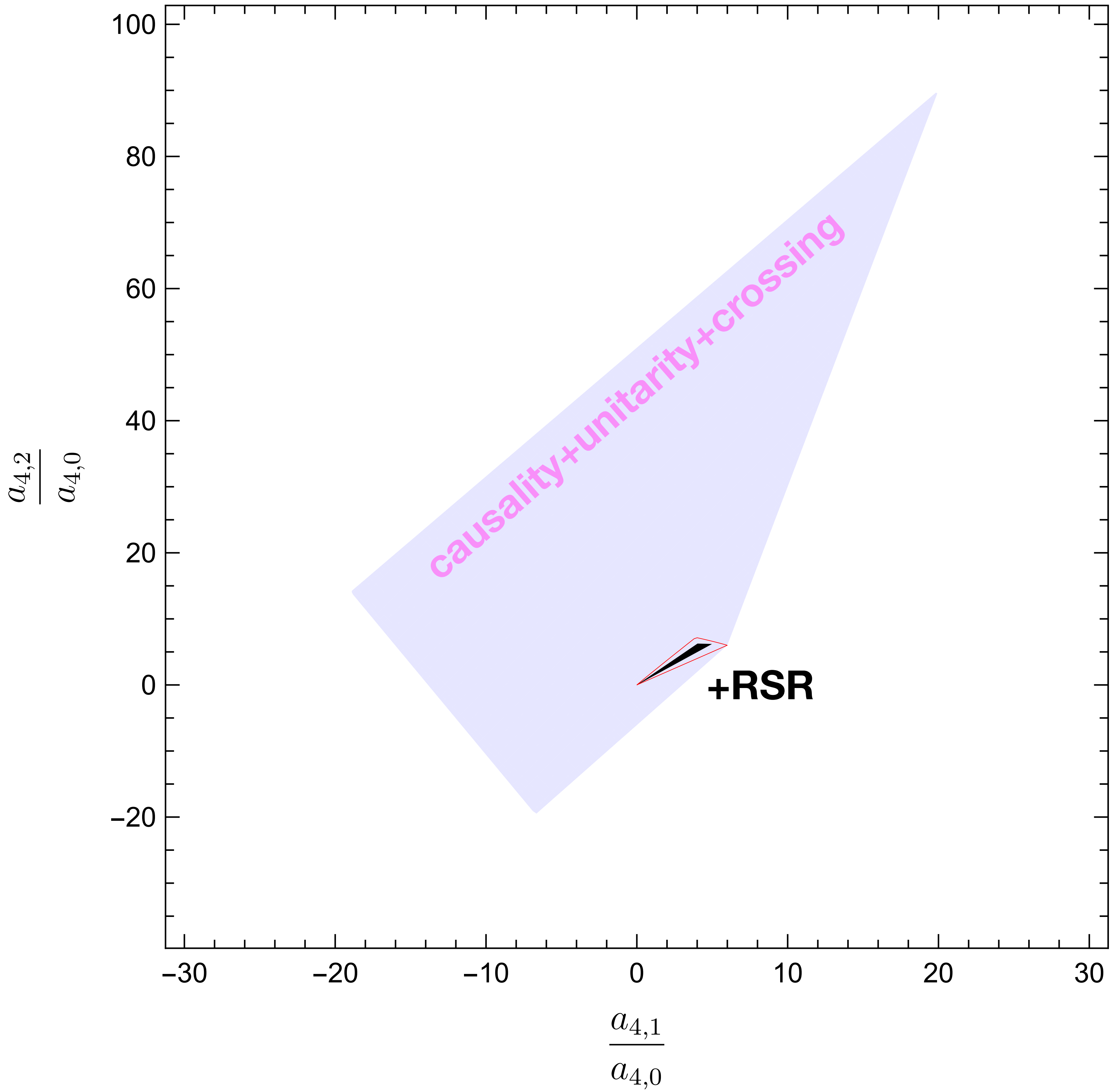



We loose the bound: $|\beta_{R^3}| \lesssim \log N_{\max}$

The extremal solution is non-unitary in d>4.

[Caron-Huot, Li, Parra-Martinez, Simmons-Duffin '22]

# Bound on Wilson coefficients




[Arkani-Hamed, Huang, Huang ’20]
[Bern, Kosmopolous, AZ ’21]

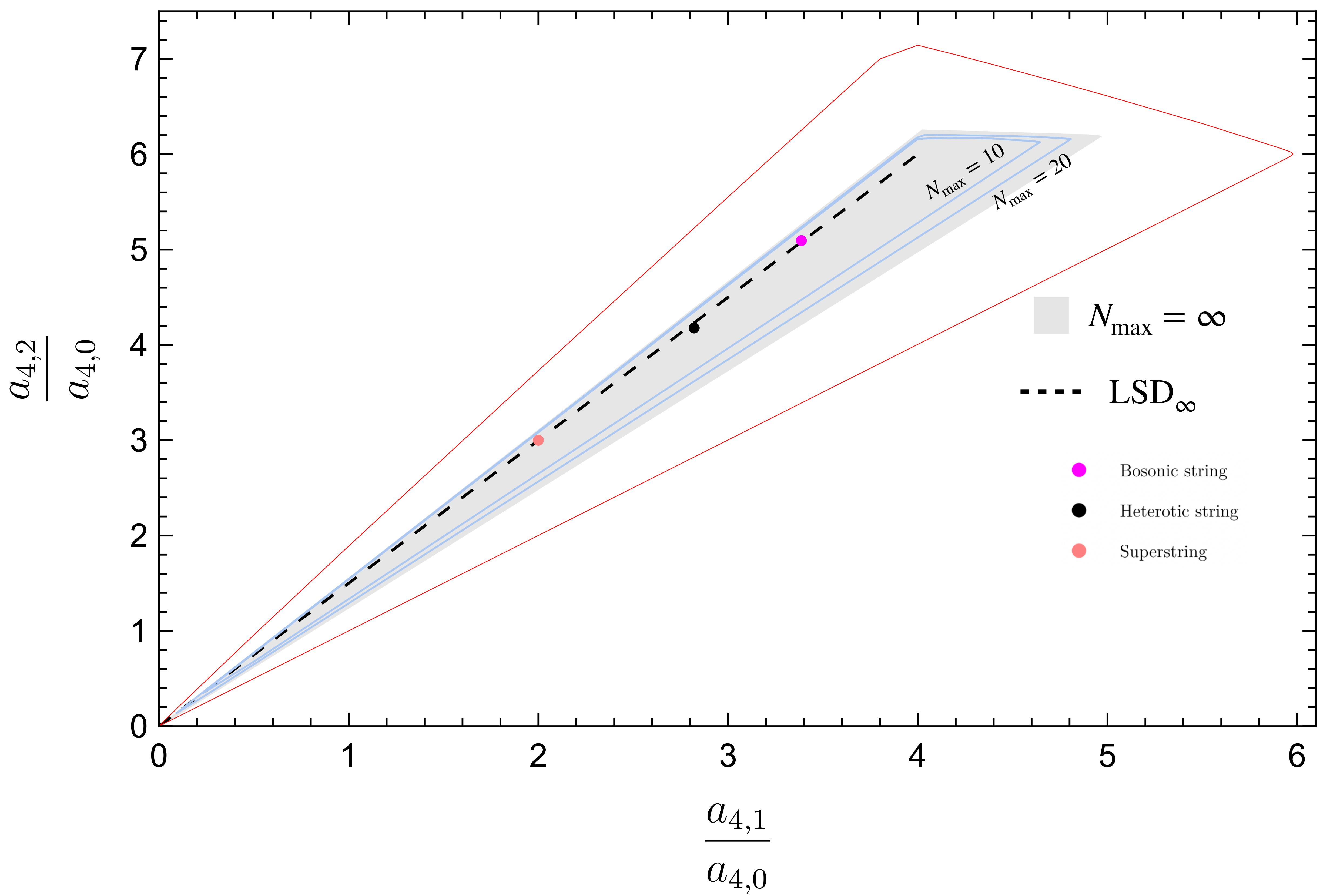
**+extra parametric inhomogeneous constraint (“the pancake effect”)**

$$\frac{a_{4,0}}{a_{0,0}} \geq 10^{-2}$$

[Caron-Huot, Li, Parra-Martinez, Simmons-Duffin ’22]

[Chiang, Y-t Huang, Li, Rodina, Wang ’22]

# Bound on Wilson coefficients



## How does the UV softness constrain the IR?
## Take away:

1. The Veneziano amplitude admits unitary deformations with linear Regge trajectory (soft in the UV).

2. The high energy, fixed angle behavior of such amplitudes is not unique (both for real and complex angles).

3. We probed allowed regions on Wilson coefficients for amplitudes that satisfy RSR and observed that they got smaller.

4. Our results are incomplete (equidistant spectrum+tree-level). A playground for future studies.

# Future directions:

- More general $j(t)$ (discretized AdS) [Veneziano, Yankielowicz, Onofri '17]
- Apply to scattering of gauge bosons [Bachu, Hillman '22] [Arkani-Hamed, Strings 2016]
- Relax the assumption on the equidistant spectrum (is it equivalent to FESR?)
- Multi-particle unitarity (or 1-loop) [Gross '69] [Mizera, Eberhardt '23]
- Physical origin of the deformations
- Higher d
- Similar sum rules hold in QG (is there a good ansatz in that case?) [Guerrieri, Murali, Penedones, Vieira '22]
- FESR+RSR+FA $\overset{?}{\to}$ LSD

**thank you!**

# Back-up slides

Is this ansatz complete (given the assumptions)?

[Khuri '69]

$$T(s,t) \simeq \frac{1}{\Gamma(1+t)\sin \pi t}\Big(a_0(t)(-s)^t + a_1(t)(-s)^{t-1} + \ldots\Big) + \ldots \,, \quad s \to \infty$$

**fixed J singularities**

Analyticity implies that $a_i(t)$ are entire functions.
Let's try to reproduce the leading one using the ansatz

$$a_0(t) = \sum_{i=0}^{\infty} b_{i0} \frac{\Gamma(i-t)}{\Gamma(-t)}$$

This is an expansion in terms of Newton polynomials with interpolating points being integers $t_i = i$. It is convergent given

$$h(\theta, a_0) = \lim_{r\to\infty} \sup \frac{1}{r} \log |a_0(re^{i\theta})|$$

[Buck '48]

$$h(\theta, a_0) < \cos\theta \log(2\cos\theta) + \theta\cos\theta, \quad |\theta| < \pi/2$$

The latter condition is related to unitarity in the semi-classical limit $s, t \gg 1$.

[Caron-Huot, Komargodski, Sever, AZ '16]

$$T_{c_0,c_1,\lambda}(s,t) = \frac{\Gamma(-s)\Gamma(-t)}{\Gamma(-s-t)}\ {}_3F_2\left(-s,-t,-c_0-c_1(s+t);-\frac{s+t}{2},\frac{1-s-t}{2};\lambda\right)$$

- For $\lambda = 0$ it reduces to Veneziano. [Veneziano '68]

- $c_1 = 0,\ c_0 \neq 0,\ \lambda = \frac{1}{4}$. [Mandelstam '68]

**(have not checked unitarity)**

- $c_1 = \frac{1}{2},\ c_0 = 0,\ \lambda$. [Matsuda '69]

$$T_{0,1/2,\lambda}(s,t) = \frac{\Gamma(-s)\Gamma(-t)}{\Gamma(-s-t)}\ {}_2F_1\left(-s,-t,\frac{1-s-t}{2},\lambda\right)$$

- It satisfies FESR+RSR

- For $c_1 \neq 0$ the fixed angle regime is different from the Gross-Mende formula

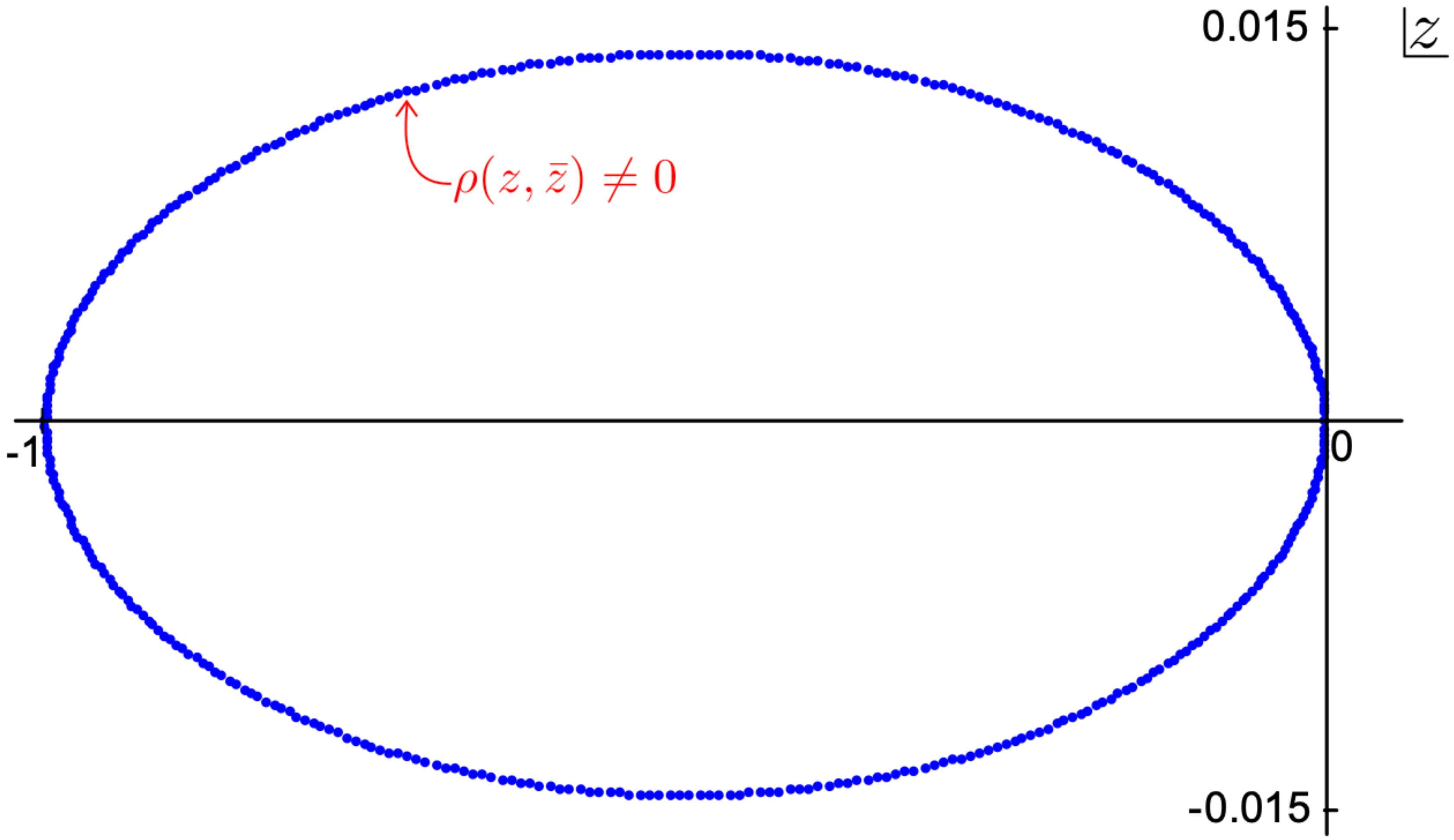
[Caron-Huot, Komargodski, Sever, AZ '16]

To do that let us study unitarity+crossing in the semi-classical limit $s, t \gg 1$. The problem becomes

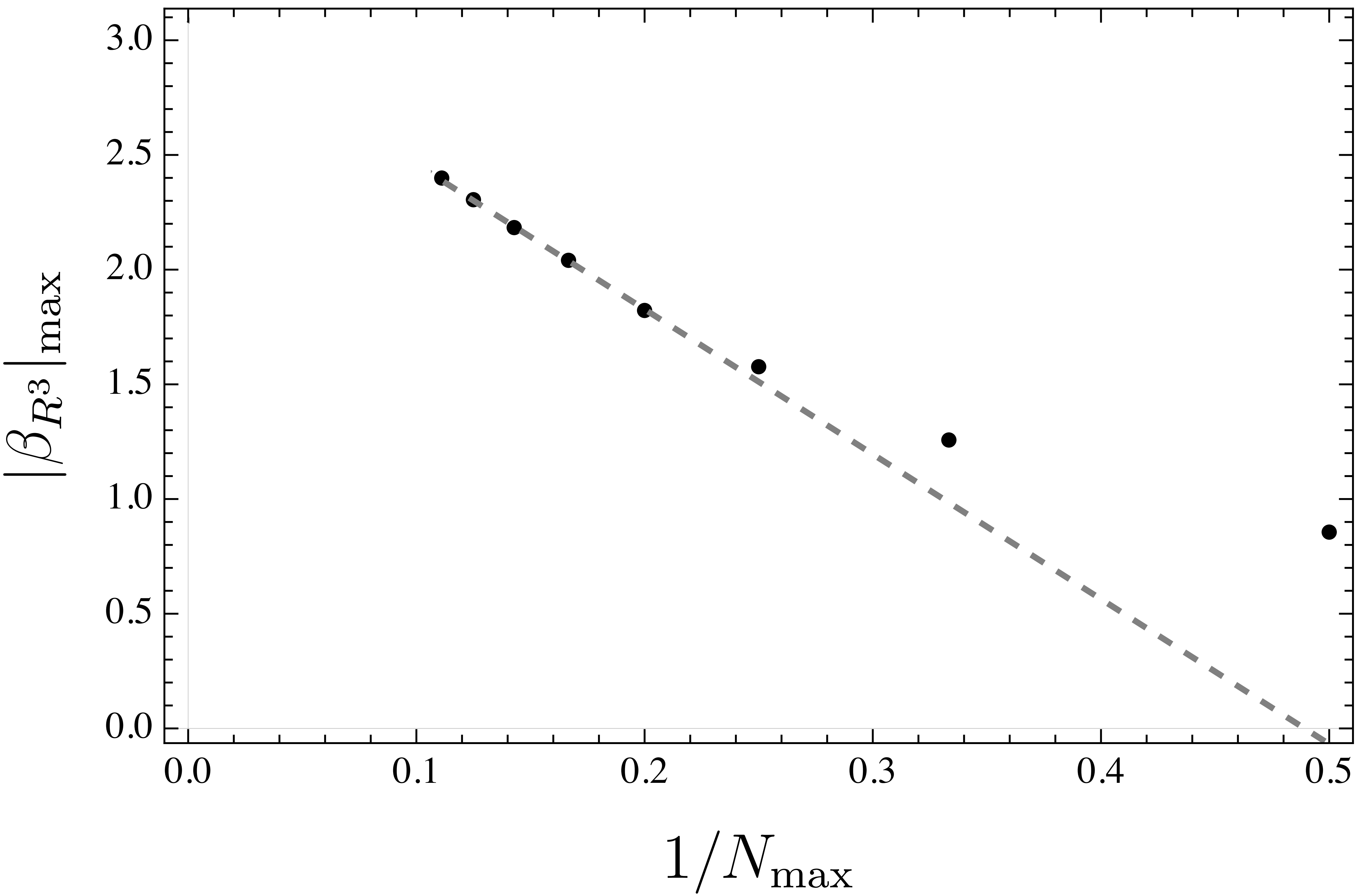
$$\sum_{J}^{J(s)} c_J P_J(1+\frac{2t}{s}) = \sum_{J}^{J(t)} c_J P_J(1+\frac{2s}{t})$$

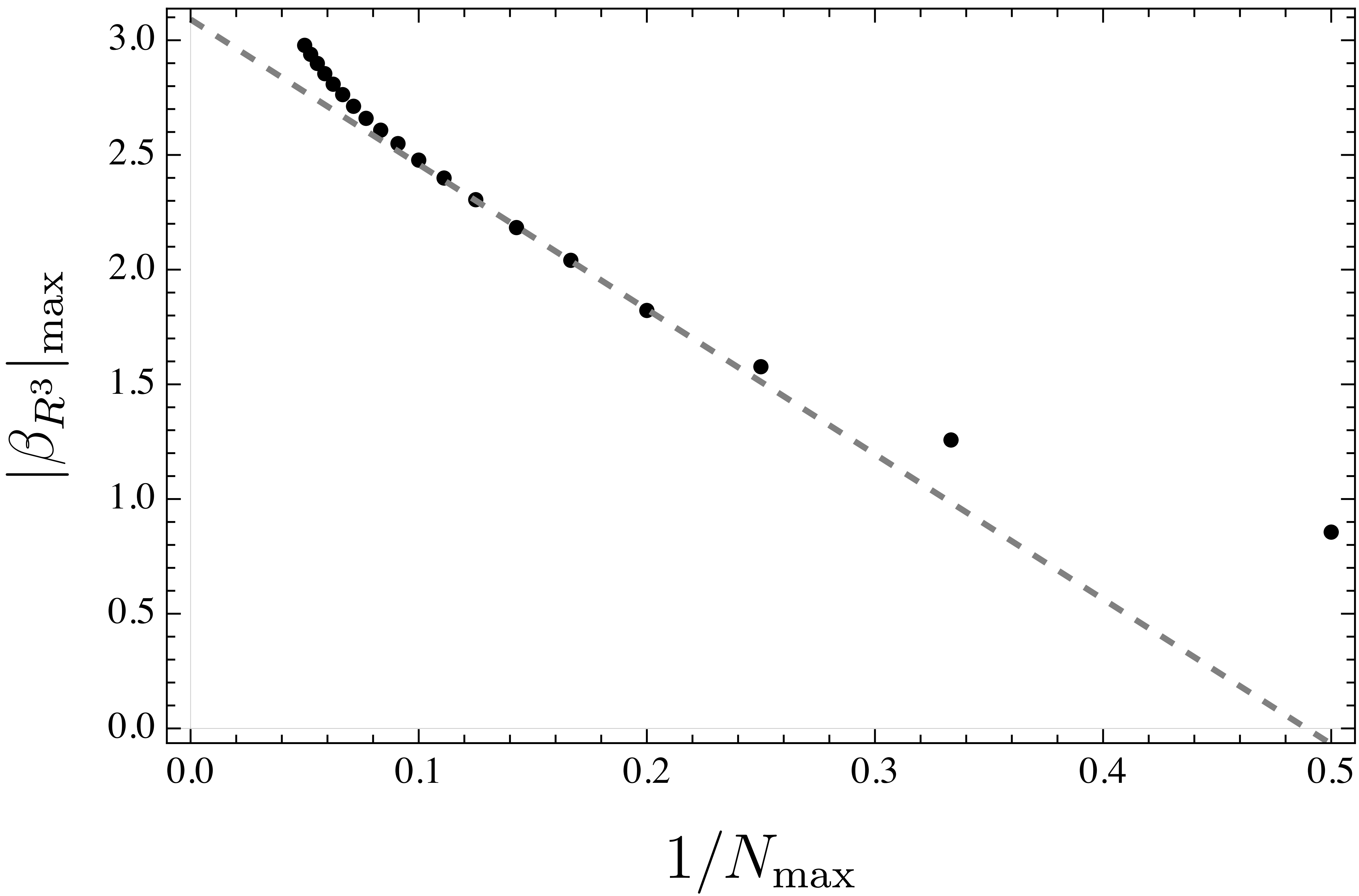
Assuming that unitarity + Regge limit forces the distribution of zeros of Legendre polynomials $P_J(1+2z)$ to be an ellipse localized between $z=-1$ and $z=0$, one can show that the amplitude is unique

$$\log T(s,t) = (s+t)\log(s+t) - s\log s - t\log t$$



0.015
$z$
$\rho(z, \bar{z}) \neq 0$
-1
0
-0.015


3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.0
0.1
0.2
0.3
0.4
0.5
$|\beta_{R^3}|_{\max}$
$1/N_{\max}$



3.0
2.5
2.0
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1.0
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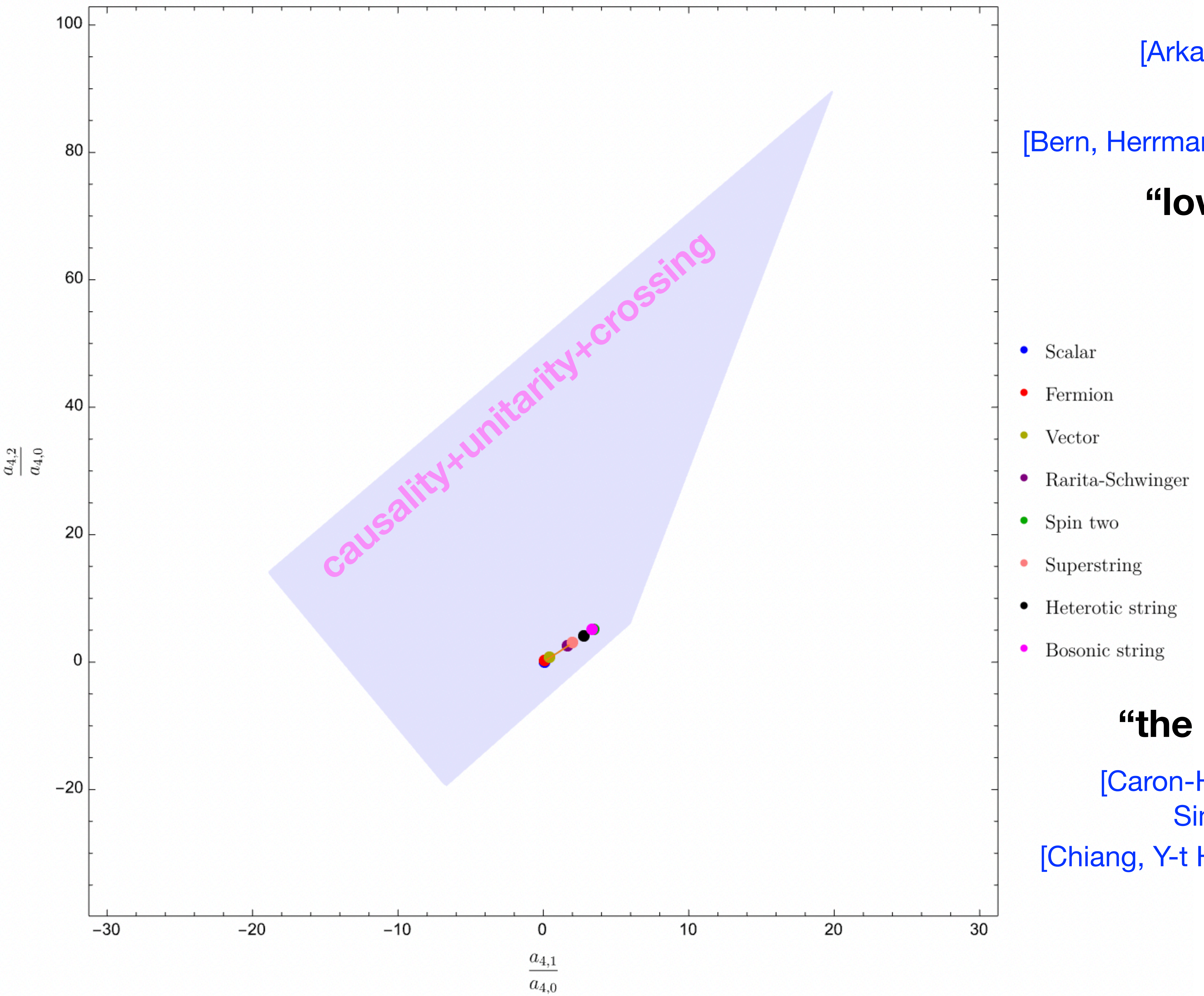
0.0
$|\beta_{R^3}|_{\max}$
0.0
0.1
0.2
0.3
0.4
0.5
$1/N_{\max}$




[Arkani-Hamed, Huang, Huang ’20]
[Bern, Kosmopolous, AZ ’21]

[Bern, Herrmann, Kosmopolous, Roiban ’22]

**“low spin dominance”**

**“the pancake effect”**

[Caron-Huot, Li, Parra-Martinez, Simmons-Duffin ’22]

[Chiang, Y-t Huang, Li, Rodina, Wang ’22]

parameteric knowledge of an extra Wilson coefficient can dramatically affect the bound