Bootstrapping mesons at large N

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2312.15013 + wip w/ J. Henriksson, L. Rastelli, A. Vichi

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Large *N* QCD

We consider 4-dimensional QCD with N_f massless quarks in ‘t Hooft’s large N limit. At leading order in $1/N$, the theory decomposes into three decoupled sectors of stable asymptotic states:

- Mesons
  $q\bar{q}$
- Baryons
  $\epsilon_{ij\ldots k} q^i q^j \ldots q^k$
- Glueballs
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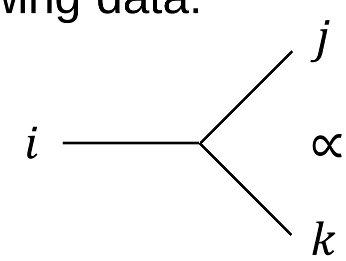
We will focus on the meson sector. The theory then consists of an infinite collection of weakly coupled mesons. We parametrize it with the following data:

$$\left\{\left(m_i^2, J_i\right), \lambda_{ijk}, \lambda_{ijkl}, \ldots\right\}$$

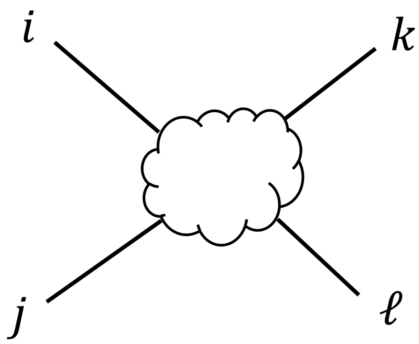


$$\propto \lambda_{ijk} \sim \frac{1}{\sqrt{N}}$$

We will proceed to place bounds on these data.

Bootstrap philosophy

These data parametrize a space of putative "large N confining gauge theories". We will carve out allowed regions in it by requiring consistency of meson S-matrices.



S-matrix Bootstrap

- Unitarity
- Regge boundedness
- Crossing symmetry
- …

[Martin 1969, Pham & Truong 1985, Ananthanarayan et al.1995, …]

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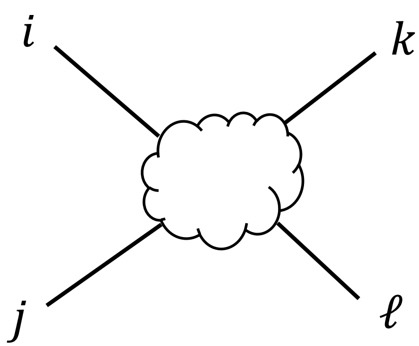

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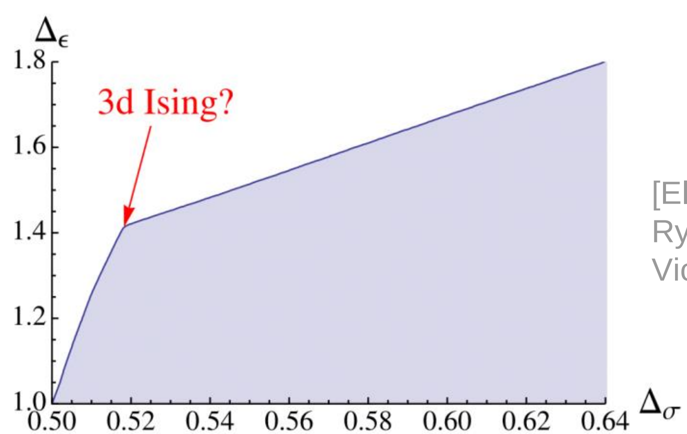
The goal is to choose the right set of assumptions that will corner large N QCD, much like the Ising model in the CFT bootstrap.



[El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin, Vichi 2012]

Pion scattering

We start by scattering the lowest mesons in the spectrum, the pions π^a, which are Goldstone bosons for

$$U(N_f)_L \times U(N_f)_R \to U(N_f)_{\mathrm{diag}}.$$

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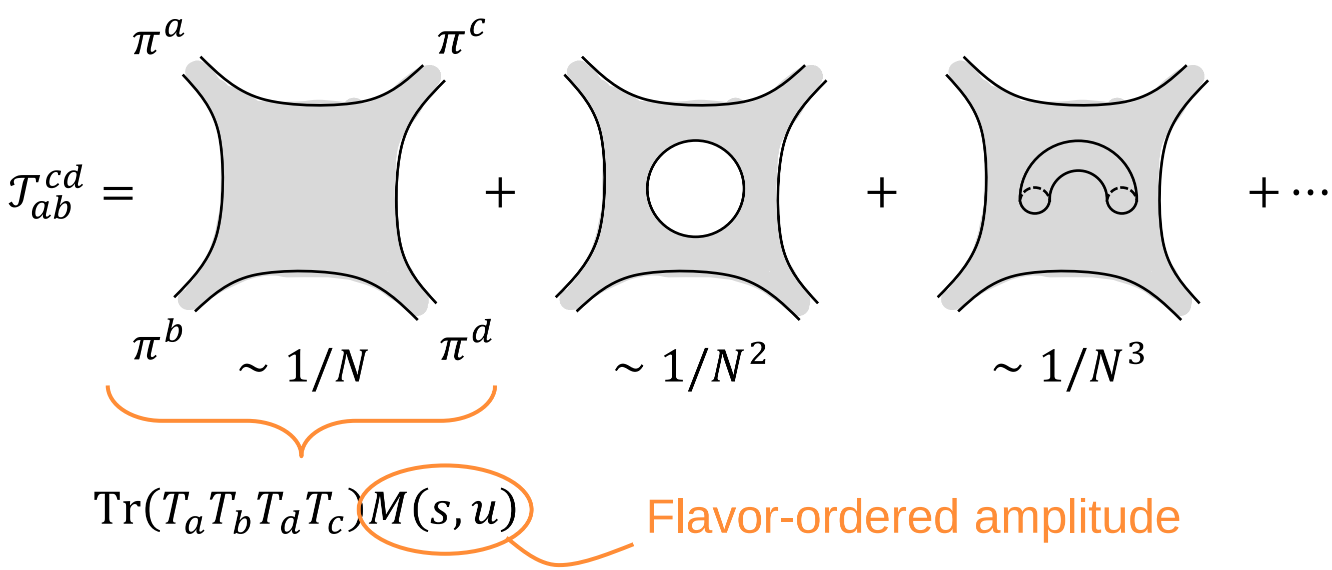
In the large N limit, diagrammatic contributions arrange in a topological expansion. At leading order, only disk diagrams contribute to the $2 \to 2$ pion amplitude.



Properties of *M*(*s*,*u*)

- **Crossing symmetry:** Invariance under the exchange of external pions implies

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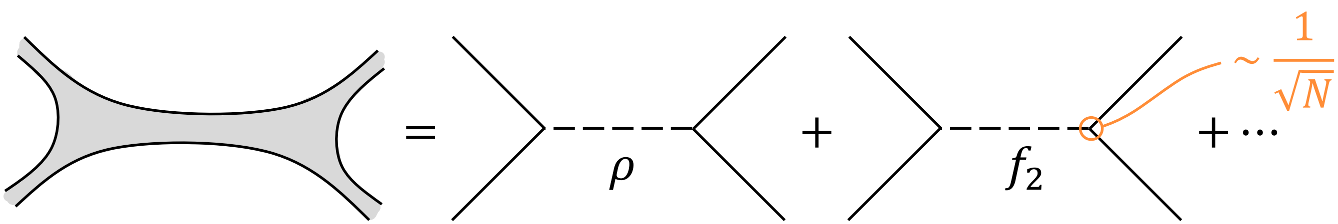
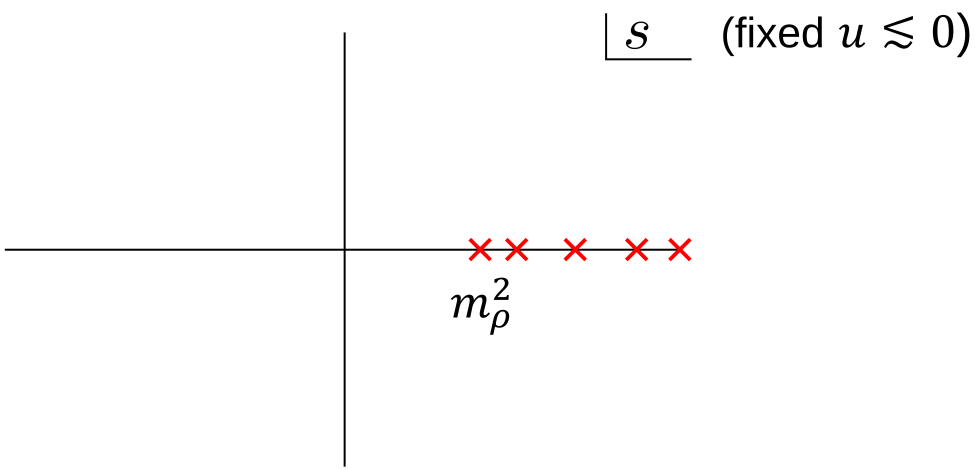
- **Analytic structure:** $M(s,u)$ is meromorphic with poles from tree-level exchanges of physical mesons.





Properties of *M(s,u)*

- **Unitarity:** The residues of $M(s,u)$ admit a partial wave expansion with positive coefficients,

$$\operatorname{Im} M(s,u) = \sum_{i \,\in\, \text{spect.}} \underbrace{\lambda_{\pi\pi i}^2}_{\geq 0} m_i^2 \pi \delta(s - m_i^2) P_{J_i}\left(1 + \frac{2u}{s}\right).$$

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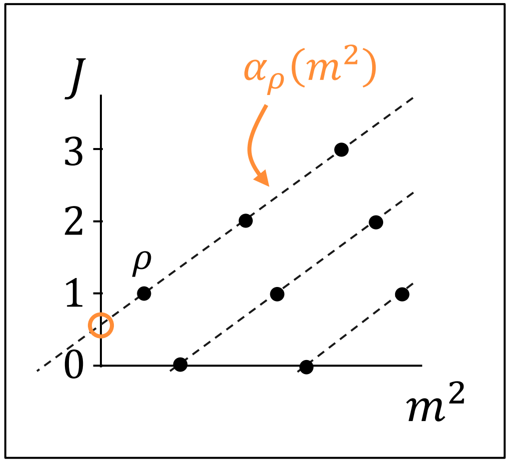
- **Regge behavior:** In the Regge limit of $|s| \to \infty$ with fixed $u \lesssim 0$, the growth of the amplitude is controlled by the intercept of the leading Regge trajectory,

$$M(s,u) \sim s^{\alpha(0)}\,.$$



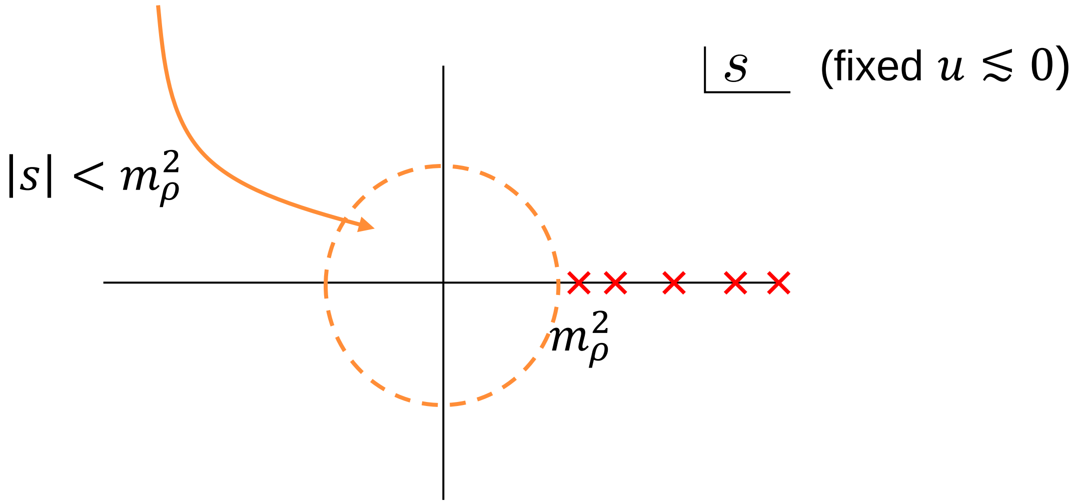
This is the trajectory of the rho, so

$$\lim_{|s|\to\infty} \frac{M(s,u)}{s} = 0\,.$$

Properties of *M(s,u)*

- **Low-energy expansion:** For energies below the first pole m_ρ^2, $M(s,u)$ admits a polynomial expansion,

$$M_{\text{low}}(s,u) \approx g_{1,0}(s+u) + g_{2,0}(s^2+u^2) + 2g_{2,1}su + \cdots.$$



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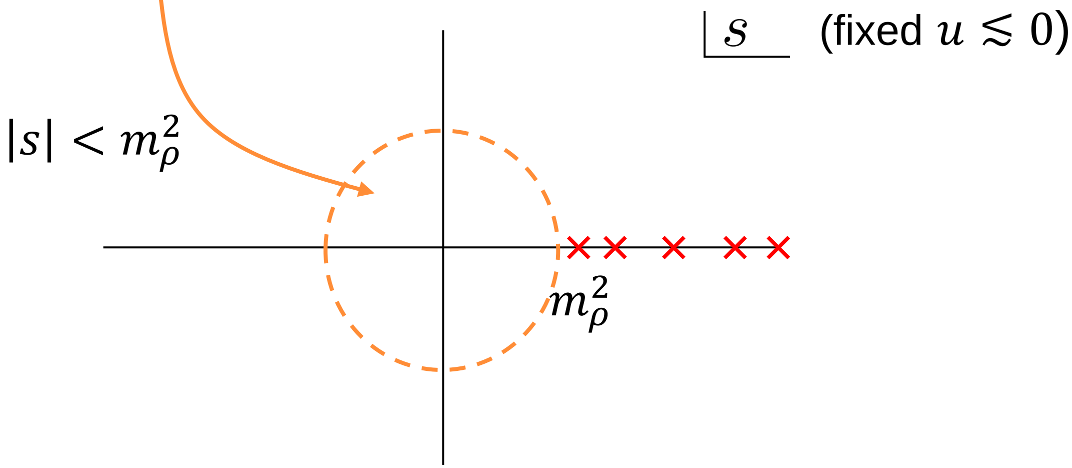
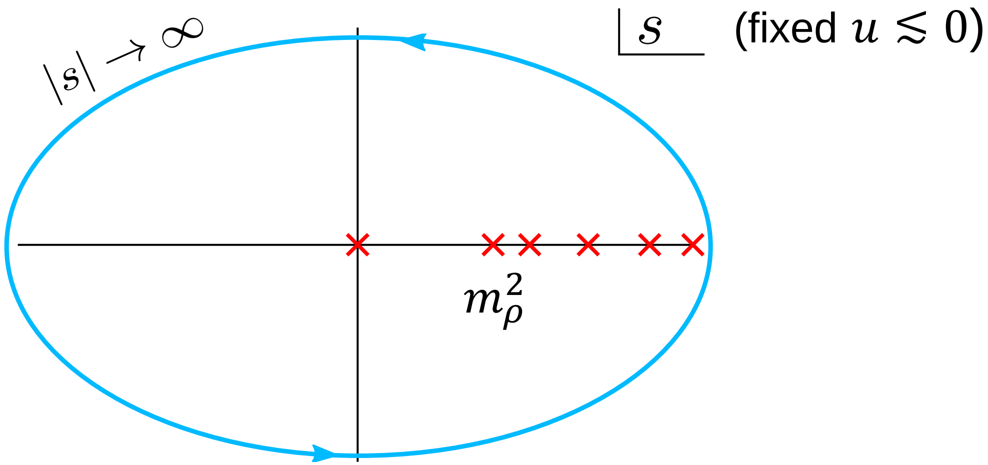


These coefficients are in one-to-one correspondence with Wilson coefficients of the chiral Lagrangian,

$$\mathcal{L}_{\text{Ch}} = -\frac{f_\pi^2}{4}\text{Tr}\left(\partial_\mu U^\dagger \partial^\mu U\right) + \kappa_1 \text{Tr}\left(\left(\partial_\mu U^\dagger \partial^\mu U\right)^2\right) + \kappa_2 \text{Tr}\left(\partial_\mu U^\dagger \partial_\nu U \partial^\mu U^\dagger \partial^\nu U\right) + \cdots .$$

Dispersion relations

By the Regge behavior, $$\frac{1}{2\pi i}\oint_{\infty} ds' \frac{M(s',u)}{s'^{k+1}} = 0. \quad (k = 1,2, \dots)$$



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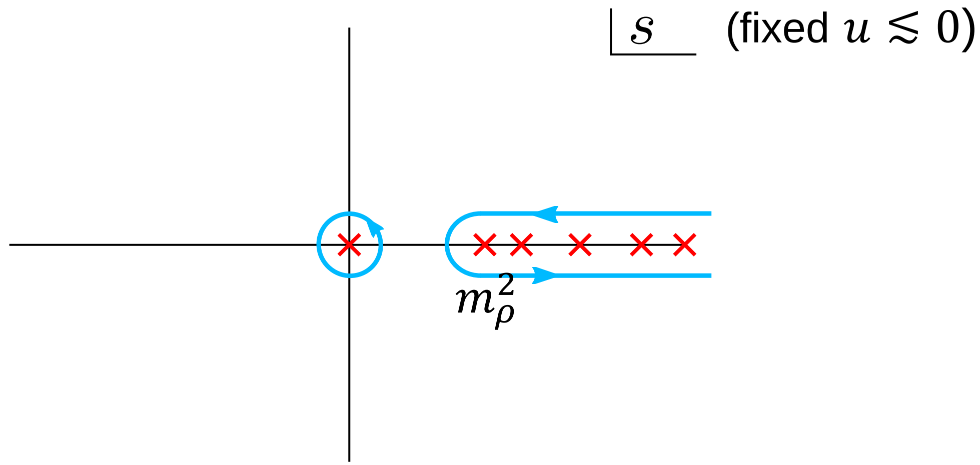

Deforming the contour yields a UV-IR link in the form of

Sum rules: $g_{n,\ell} = \sum_{i\,\in\,\mathrm{spect.}} \underbrace{\lambda^2_{\pi\pi i}}_{\geq 0 \text{ (unitarity)}} F_{n,\ell}(J_i, m_i^2) \equiv \langle\, F_{n,\ell}(J, m^2)\,\rangle\,.$

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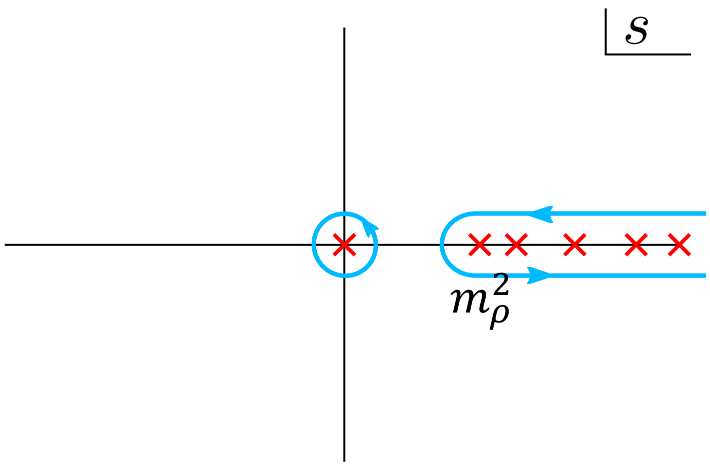

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Crossing symmetry is encoded in an infinite set of

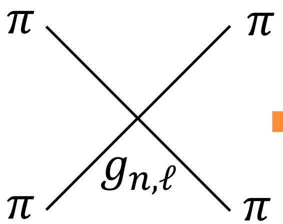
Null constraints: $0 = g_{n,\ell} - g_{n,n-\ell} = \left\langle \mathcal{N}_{n,\ell}(J,m^2) \right\rangle.$

[Caron-Huot, Van Duong 2021, Tolley, Wang, Zhou 2021]

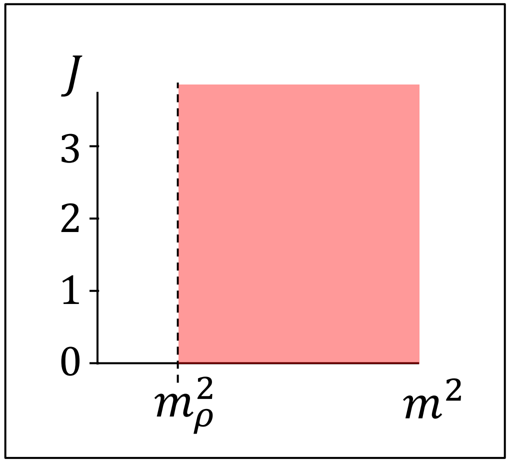
Bootstrap bounds

We choose a potential spectrum, and we recast sum rules and null constraints into a semidefinite program to place bounds on:

- **Wilson coefficients:** When being agnostic about the heavy mesons, we can bound the effective pion couplings from integrating them out.



$$\frac{g_{n,\ell}\, m_\rho^{2(n-1)}}{g_{1,0}}$$



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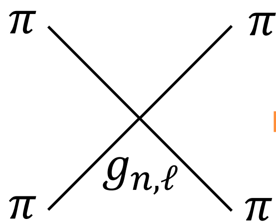
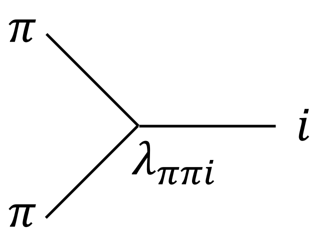
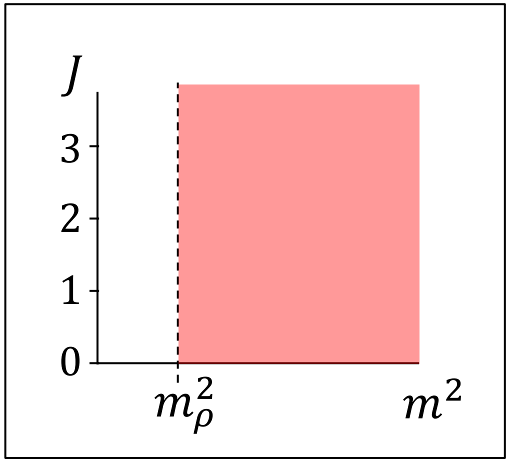


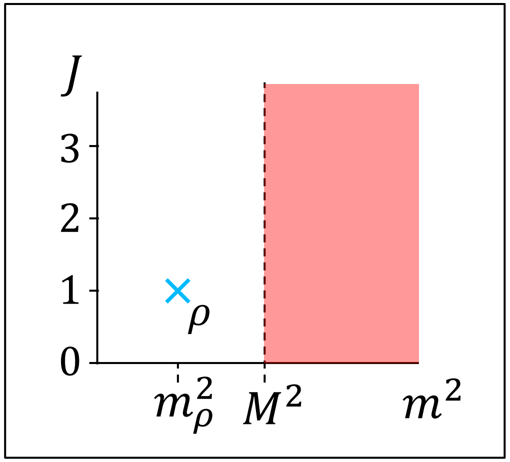
$$\frac{g_{n,\ell}\, m_\rho^{2(n-1)}}{g_{1,0}}$$

- **On-shell couplings:** If we refine our spectrum choice, we can directly probe their on-shell interactions.



$$\frac{\lambda_{\pi\pi i}^2}{g_{1,0}\, m_\rho^2}$$





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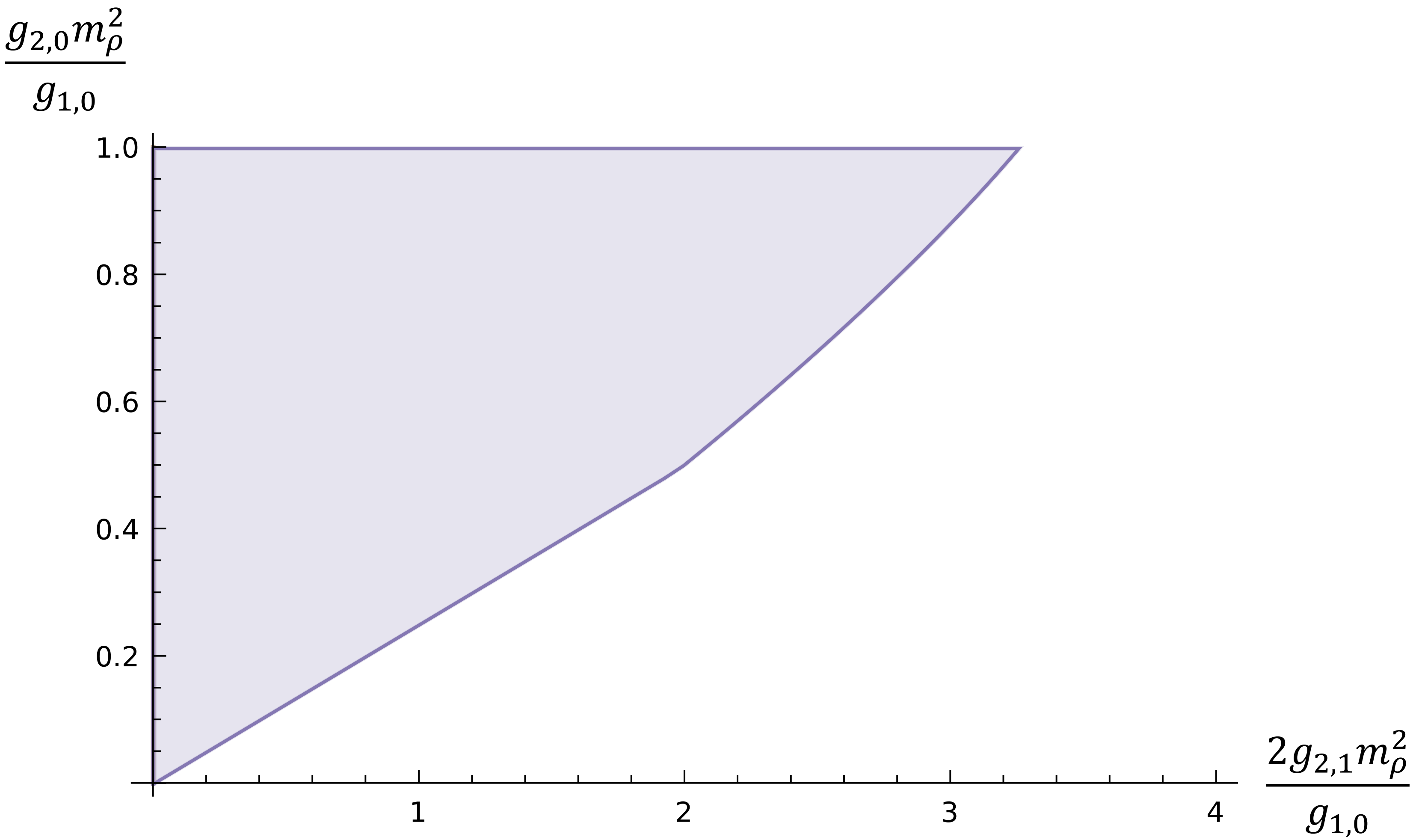
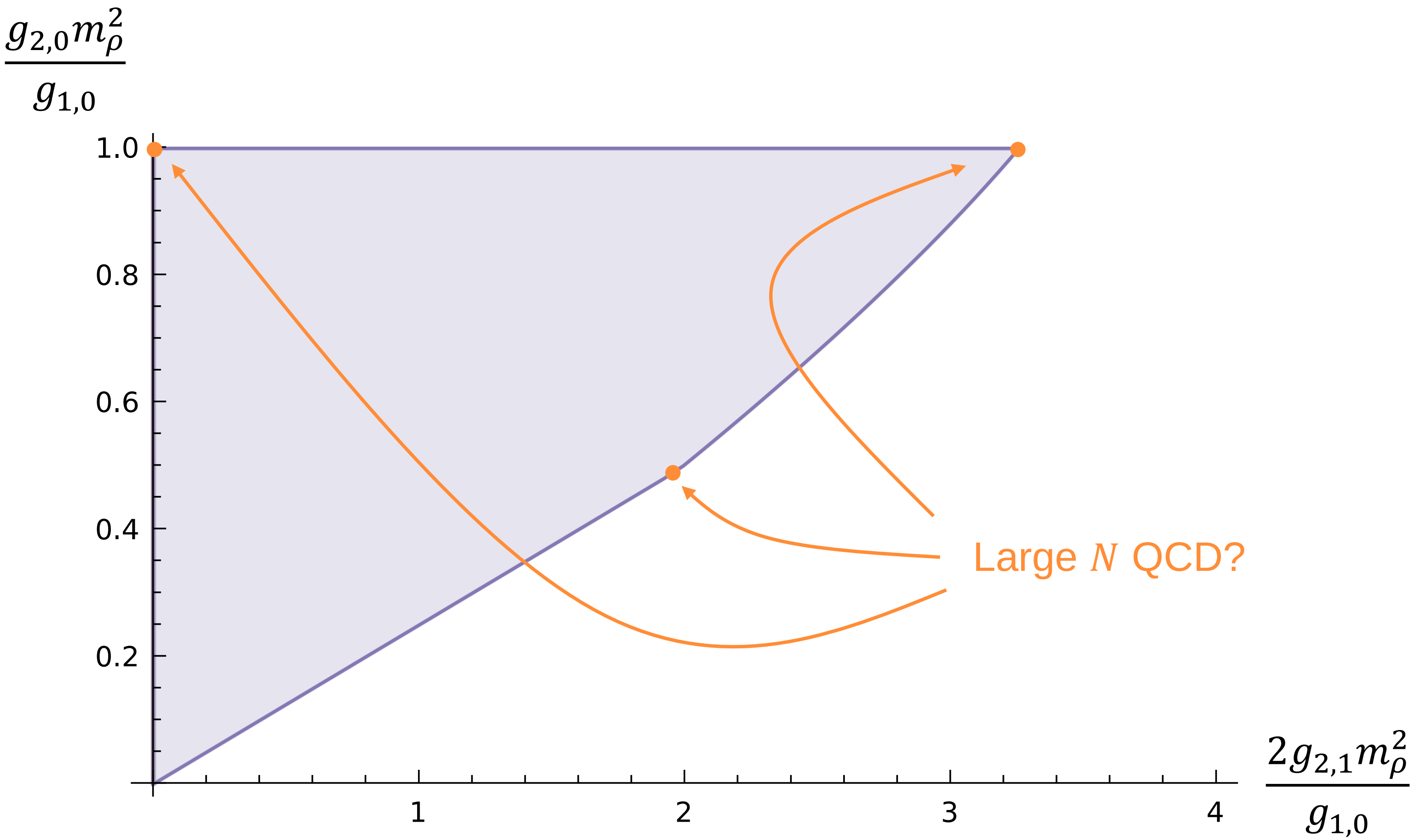


Figure 1: Allowed region in the space of (normalized) four-derivative couplings of the chiral Lagrangian.

Bounds on Wilson coefficients



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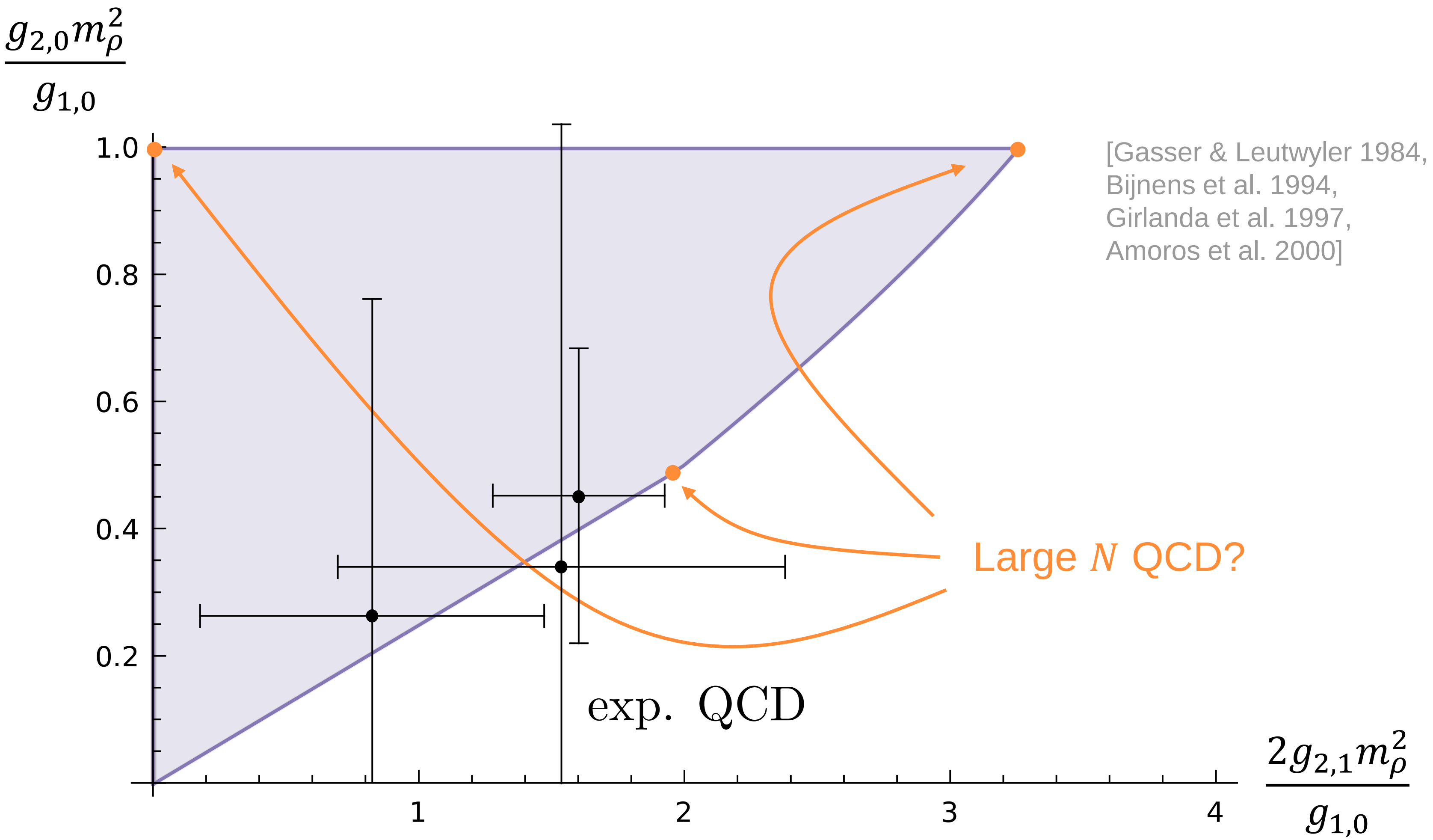

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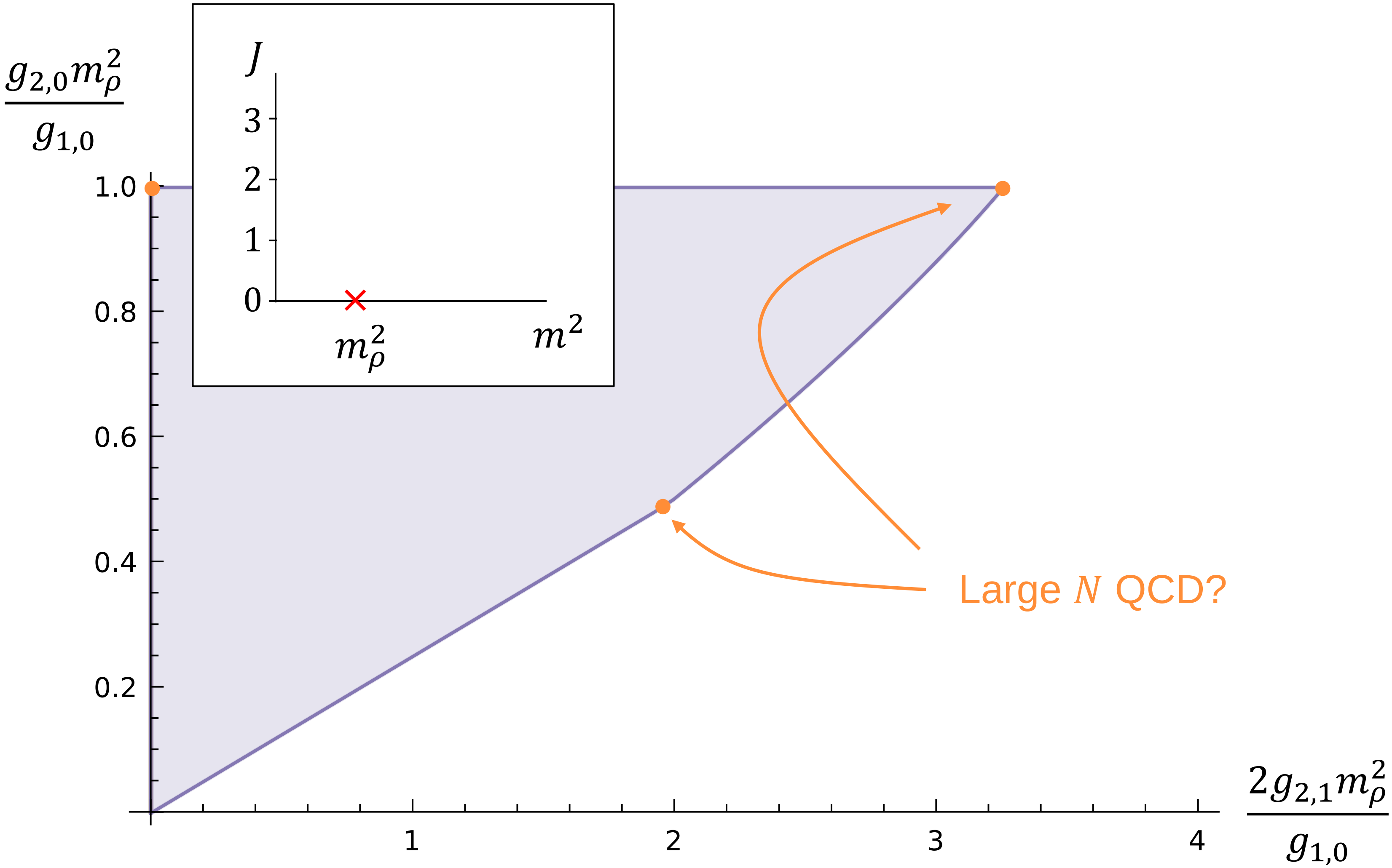
Extremal amplitudes



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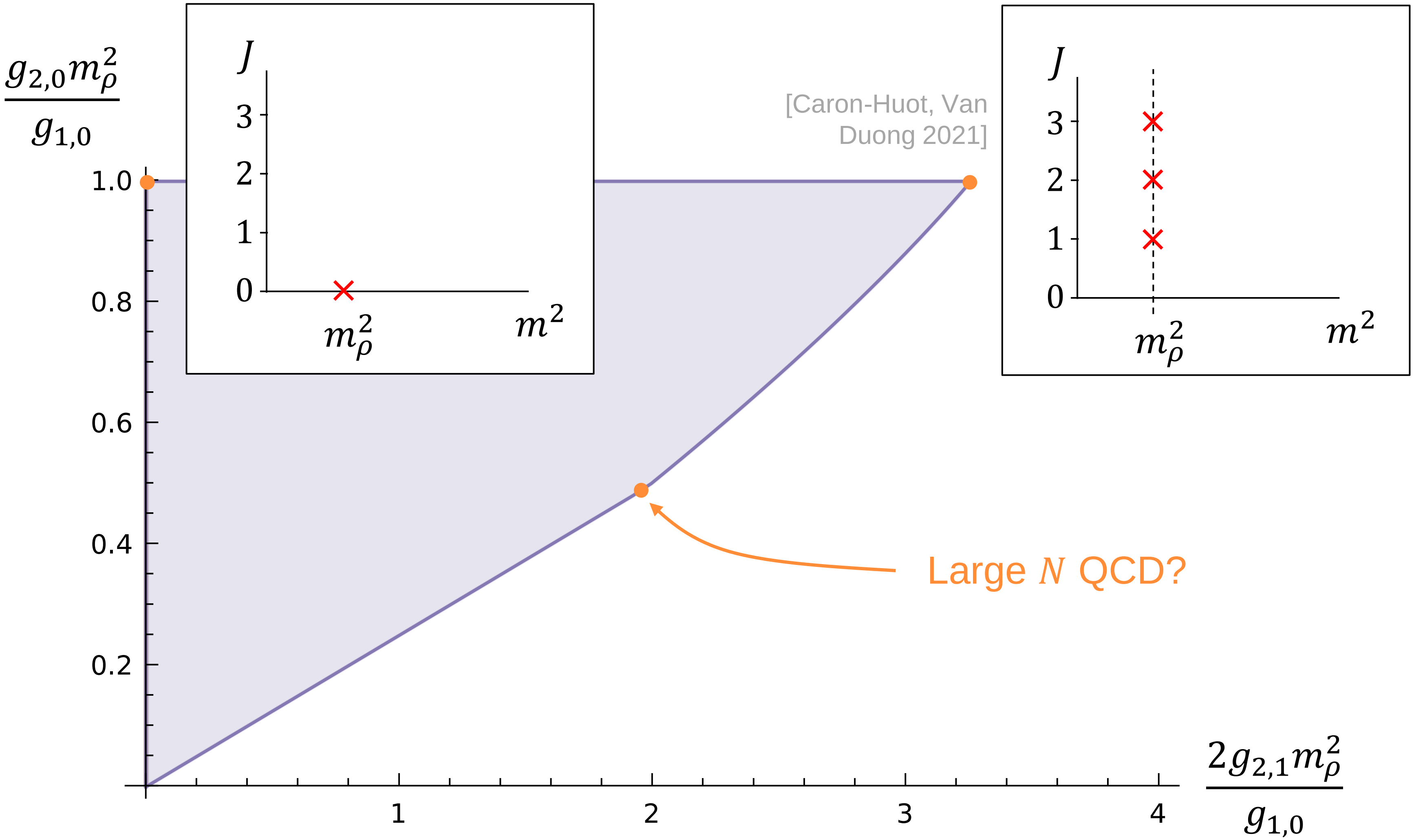

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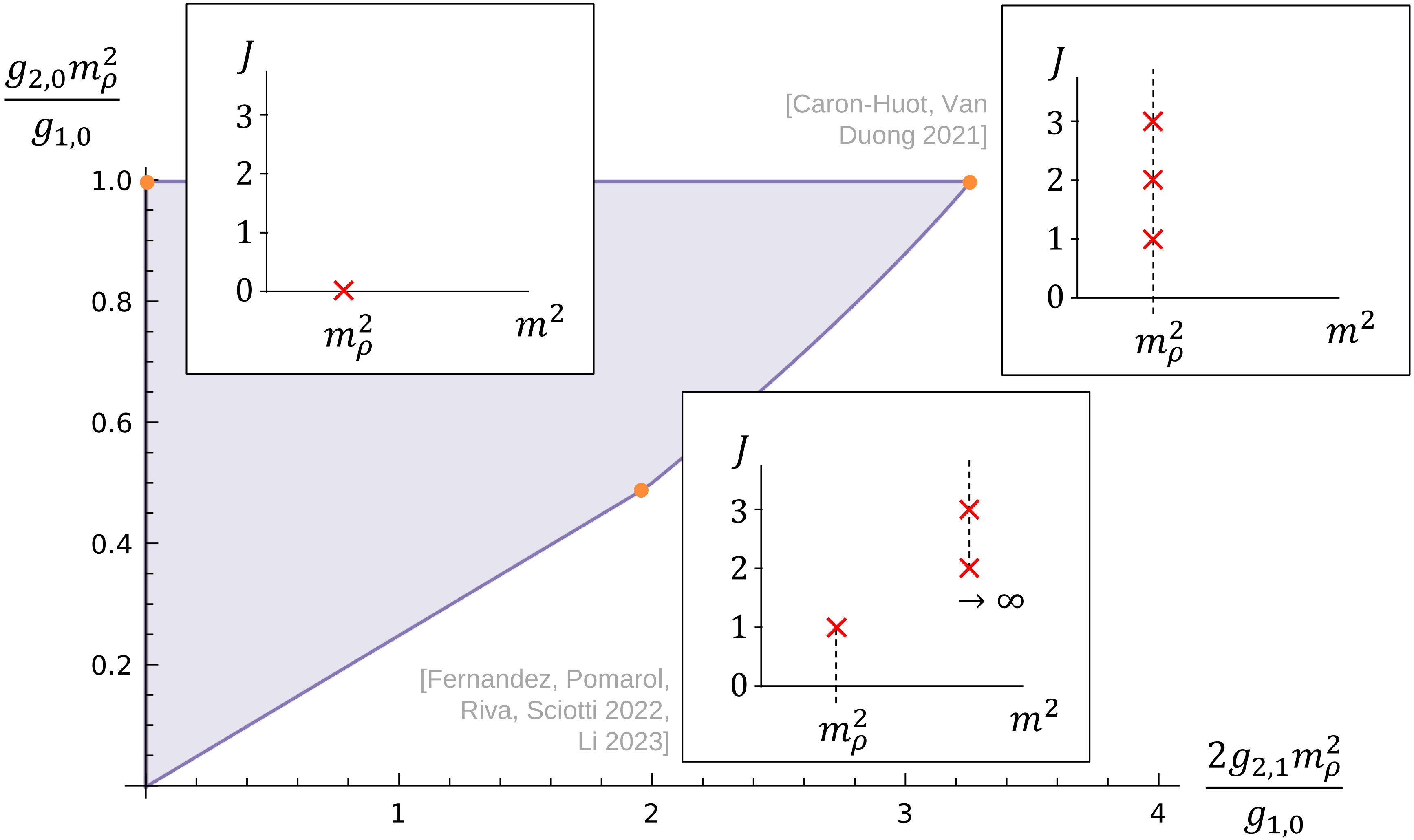

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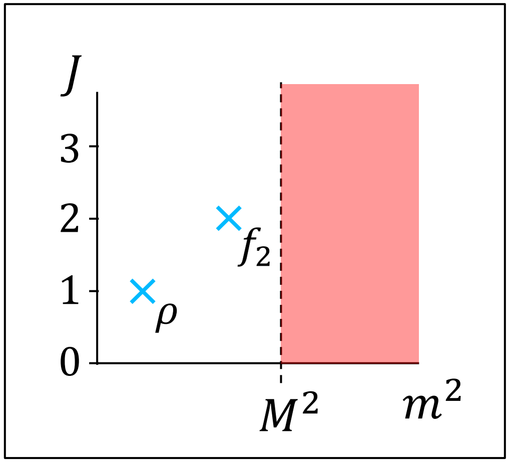

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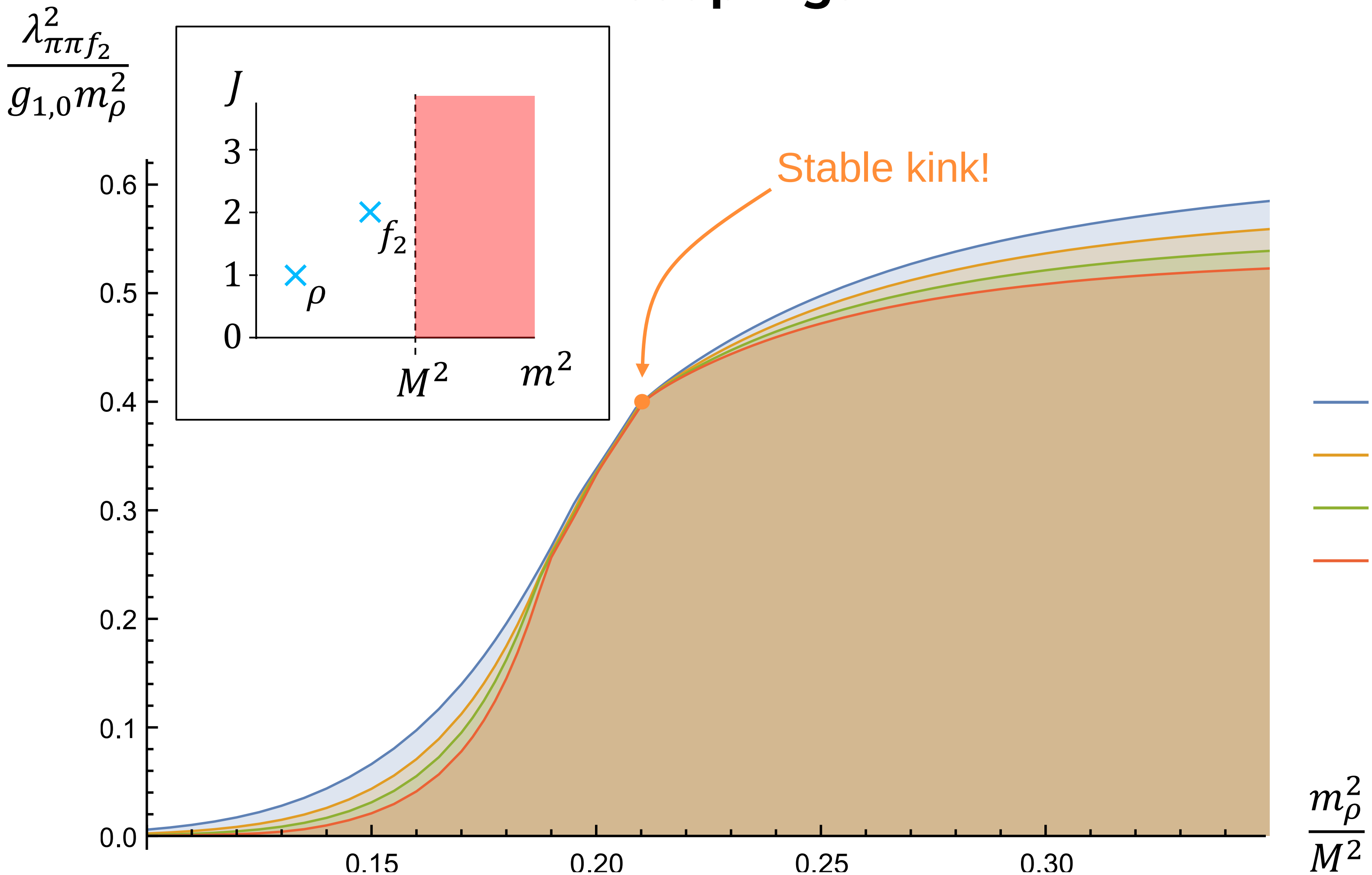
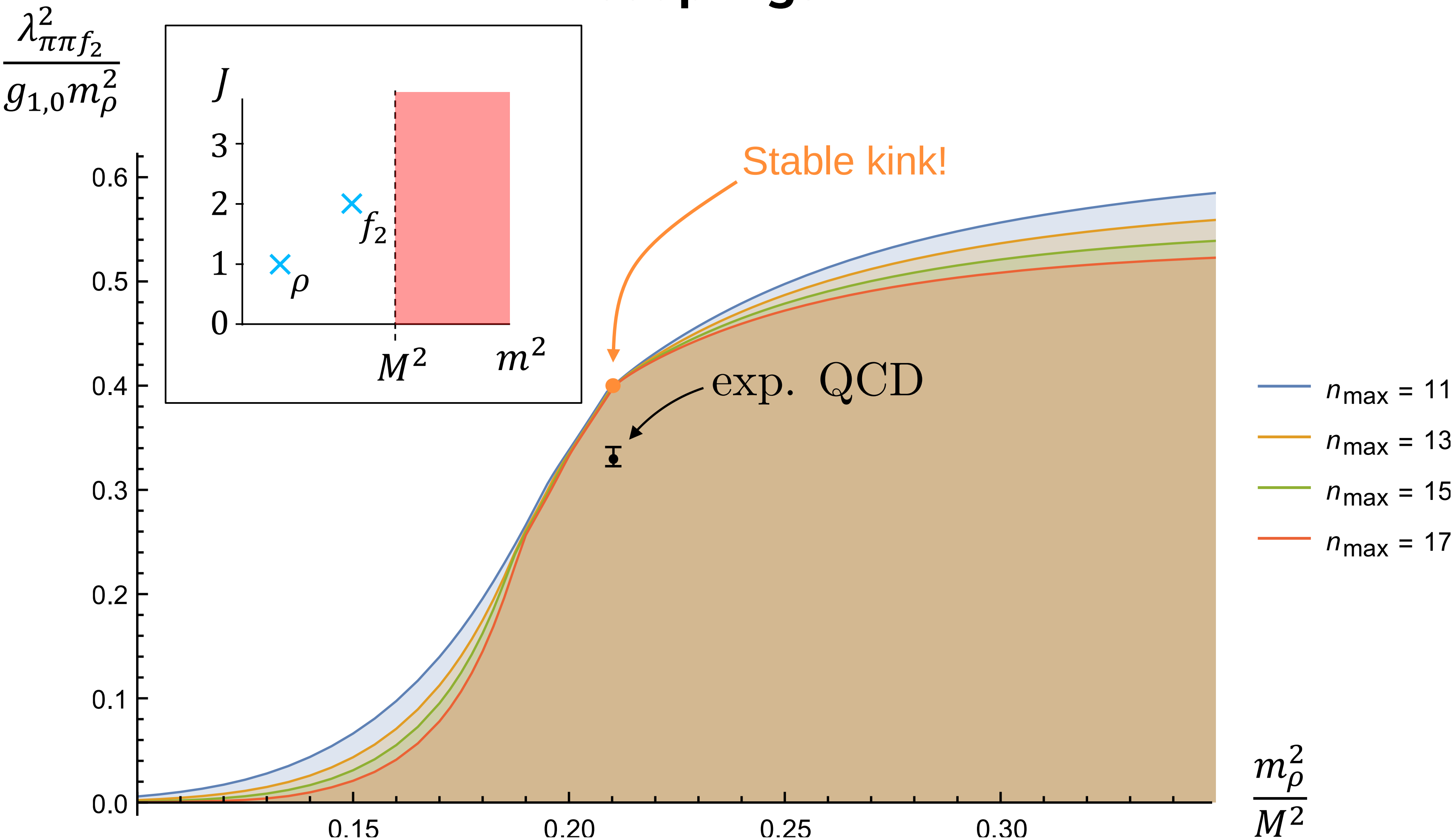


Figure 2: Upper bound on the (normalized) f_2 on-shell coupling as a function of the gap after the f_2.

Bounds on on-shell couplings



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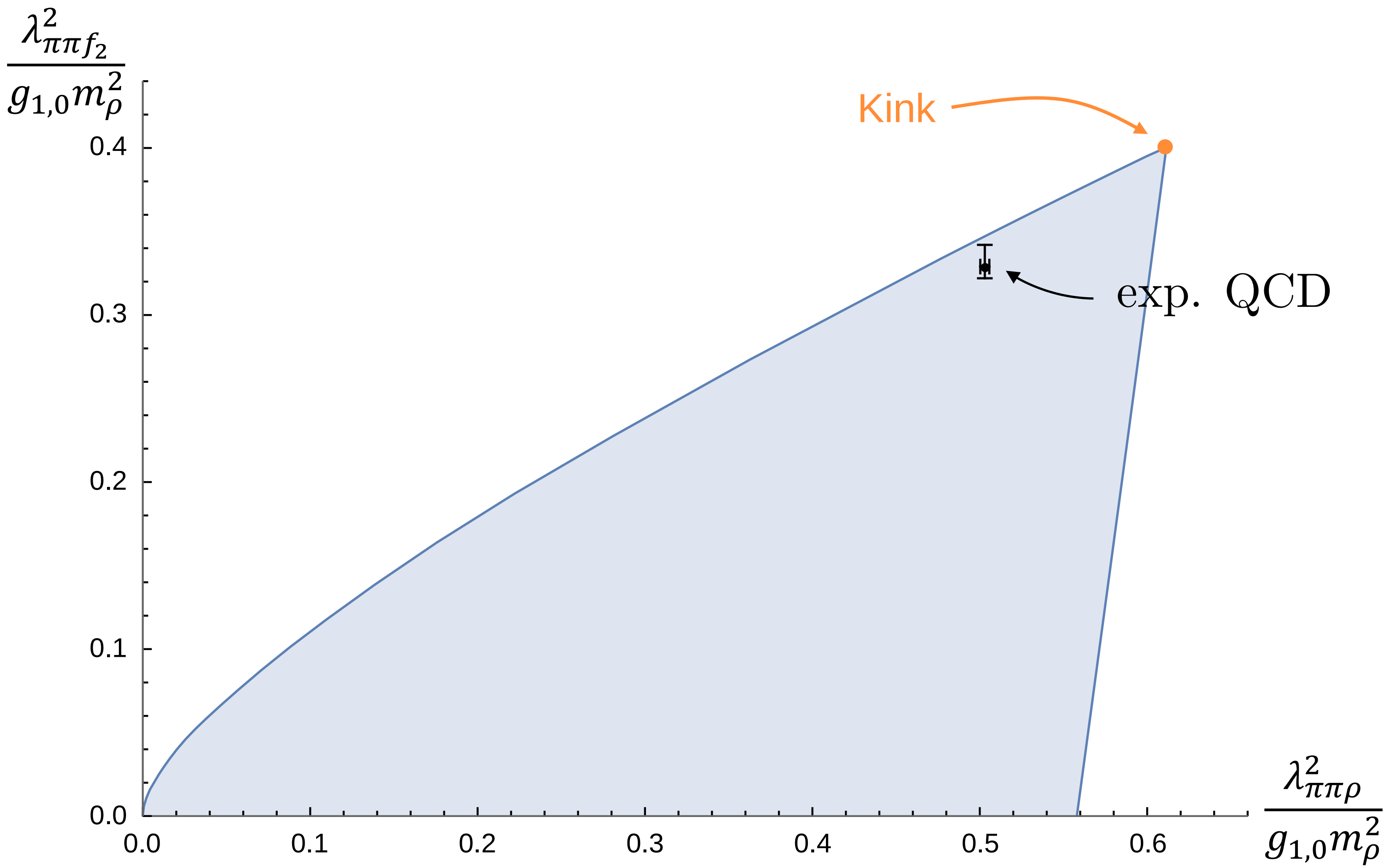

Figure 3: Exclusion plot in the space of (normalized) on-shell couplings for the rho and the f_2.

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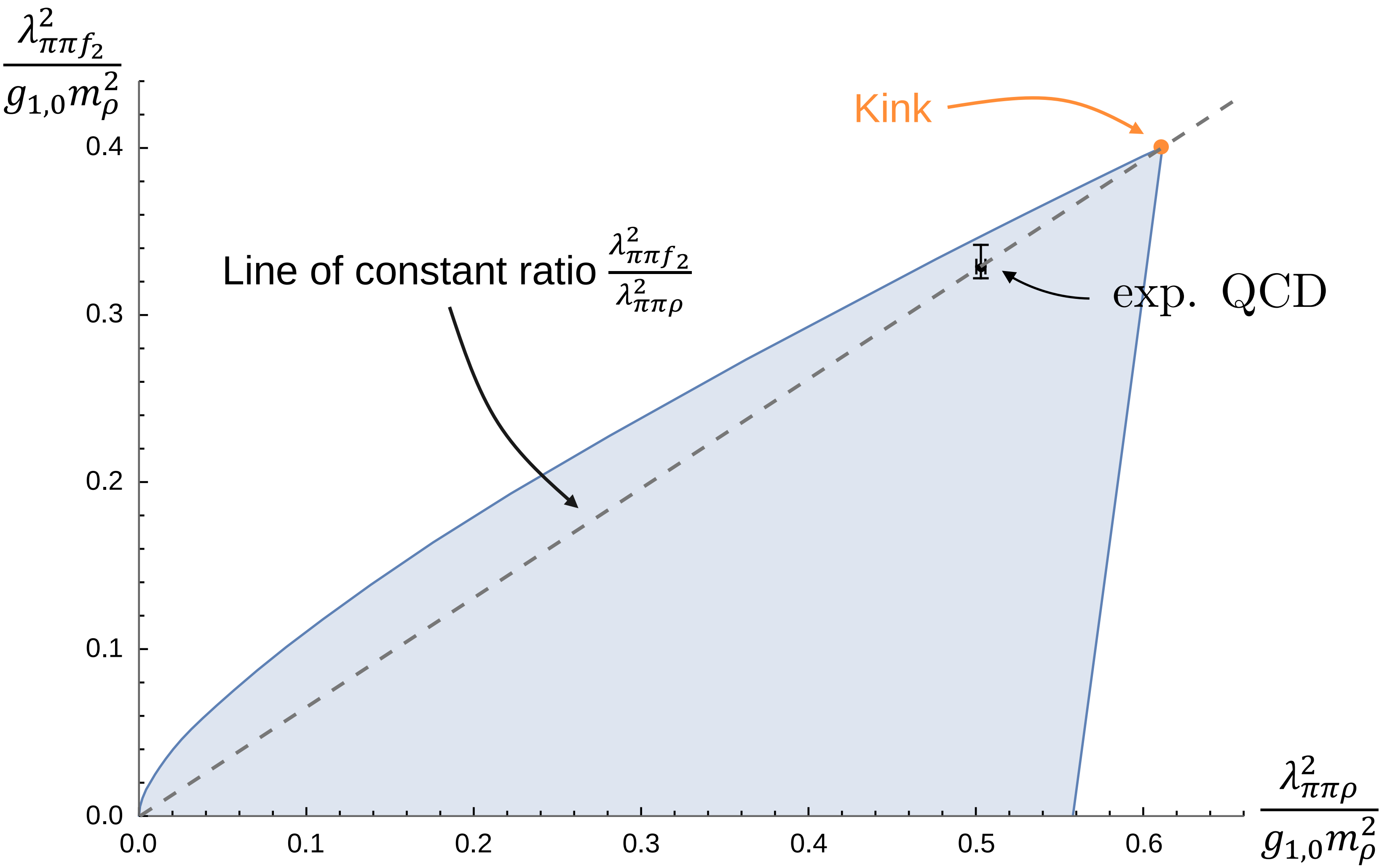

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Spectrum at the kink

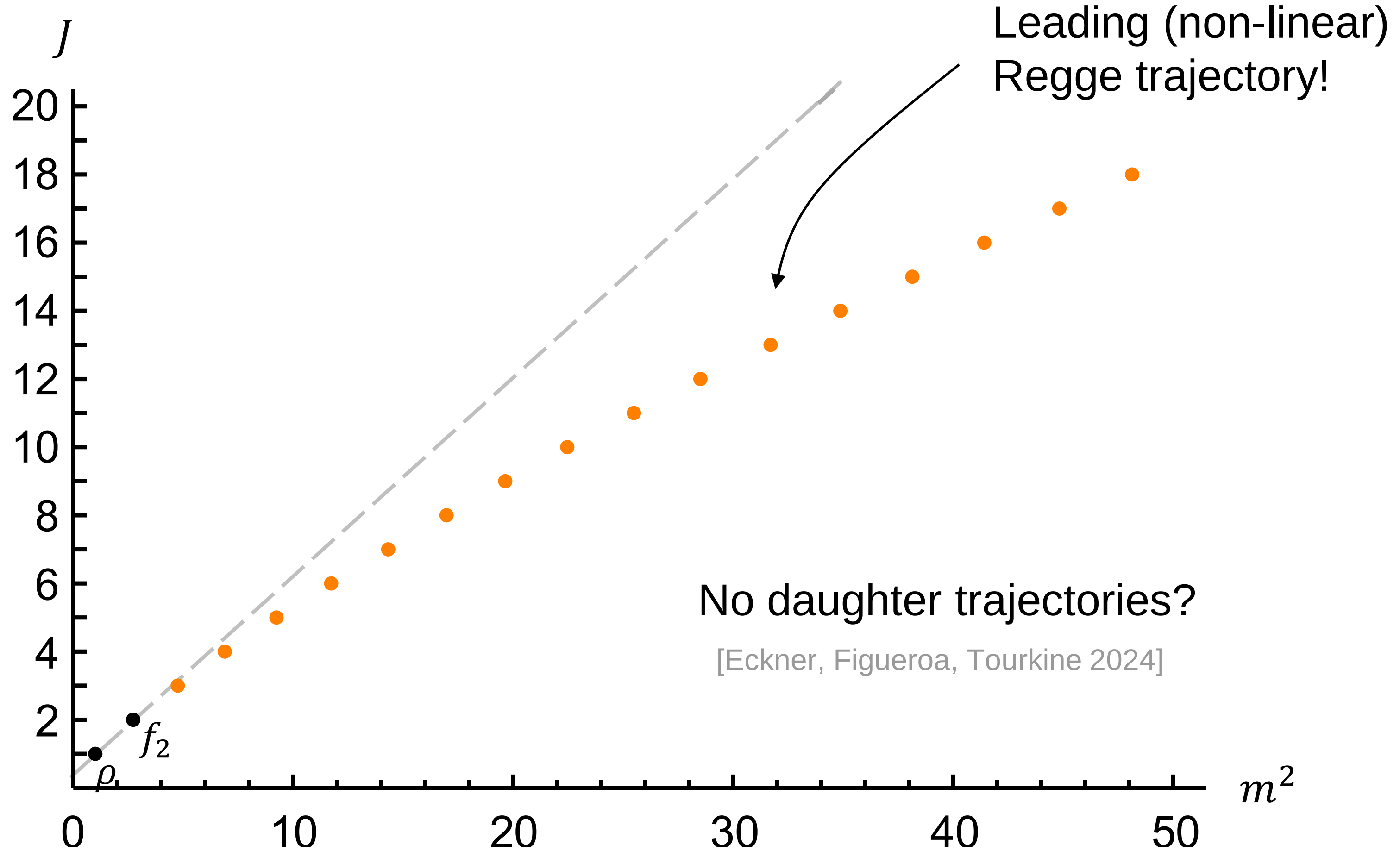


Figure 4: Spectrum of the numerical solution at the kink.

Spectrum of QCD

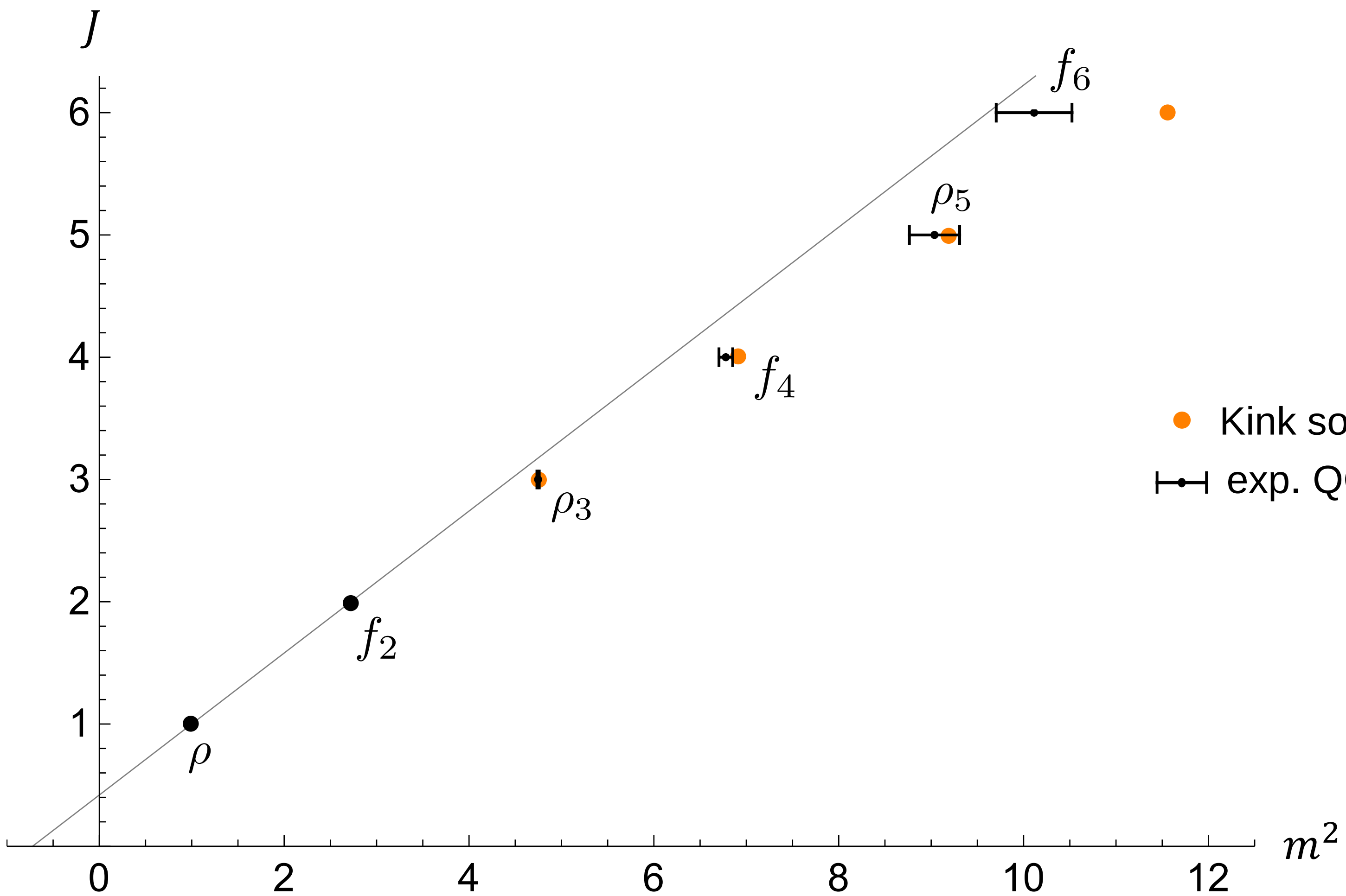
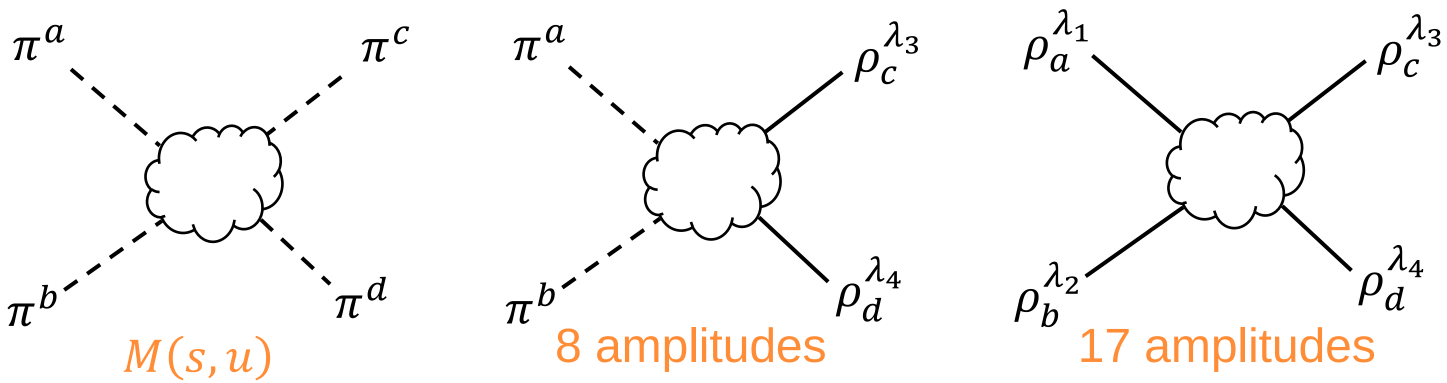


Figure 5: Comparison of the spectra of the extremal solution and real-world QCD.

Mixed *S*-matrices

Many more constraints come from considering a mixed scattering system of pions and rho mesons. But the difficulty grows quickly with spin.



[JA, Rastelli, 2023]

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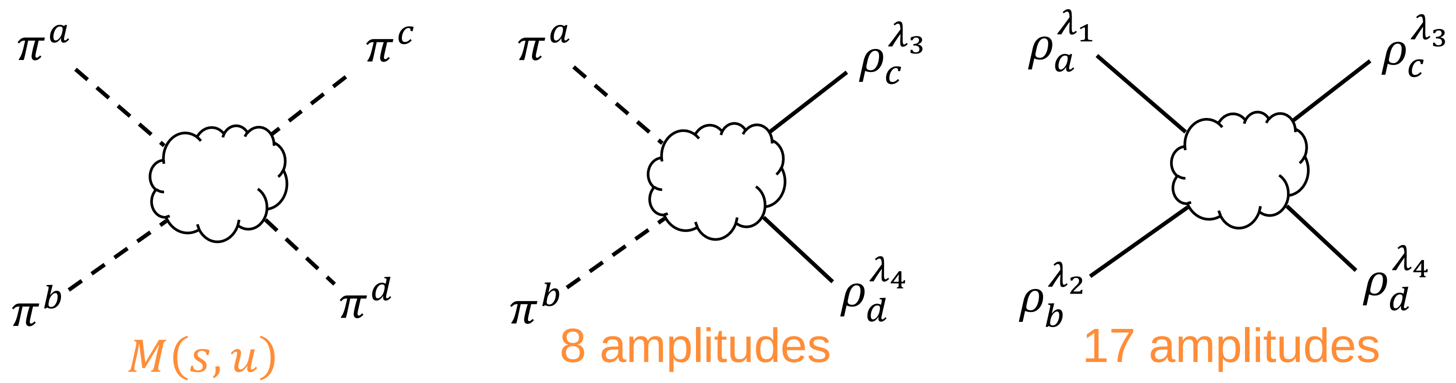
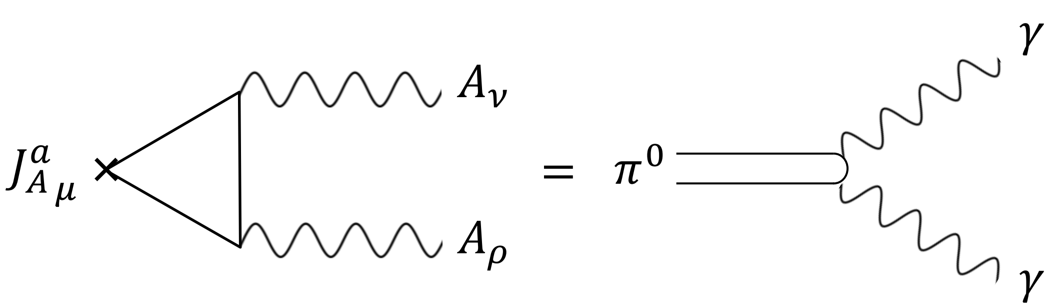


A simpler system is that of pions and photons, which captures the chiral anomaly. Matching the anomaly yields bounds with an explicit dependence on N/f_π^2.



$$\propto \frac{e^2 N}{16\pi^2 f_\pi} \pi^0 F \wedge F$$

Conclusions

Summary:

- We are carving out the space of large N confining gauge theories.
- Generic bounds on Wilson coefficients are saturated by simple solutions.
- The key seems to be to include explicit poles up to spin two.
- This reveals a kink with many features in common with real-world QCD.

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Not clear yet, but we're getting tantalizingly close."

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Future directions:

- Include external rho mesons. [in progress: JA, Henriksson, Rastelli, Vichi]
- Consider general background gauge fields and anomalies.
- Explore the glueball and baryon sectors.
- Target other weakly coupled systems.

  (e.g. tree-level string theory. [to appear: JA, Knop, Rastelli])
- Etc!

Thank you!