ABJ anomaly as a U(1) symmetry, and Noether's theorem

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Based on recent papers with Valentin Benedetti and Javier Magán:

https://arxiv.org/abs/2309.03264
https://arxiv.org/abs/2205.03412

Two objectives of the talk:

Identify the reasons behind counterexamples of Noether's theorem in QFT :
The continuous symmetry mixes Haag duality violating sectors.
Some classification of simple cases.

Show the ABJ anomaly as an ordinary internal U(1) symmetry that, however, transforms the Haag duality violations of the theory.

This clarifies and unifies the origin of its main features in a perspective based on ordinary symmetry ideas:

1) It is a continuous symmetry but does not have a conserved Noether current

2) Goldstone theorem still applies (pions)

3) Anomaly quantization: general anomaly coefficient is proportional to an integer

4) Anomaly matching (the coefficient of the anomaly matches between the UV and IR)

Some examples of violations of Noether's theorem:

Free graviton

No stress tensor.

Two free Maxwell fields

No current for rotation symmetry.

Weinberg-Witten theorem

Duality symmetry Maxwell field

No duality current.

Maxwell field for dimension d≠4

Derivatives of free scalar d>2

No dilatation current.

Known counterexamples for DI implies CI

Free and massless. Why?

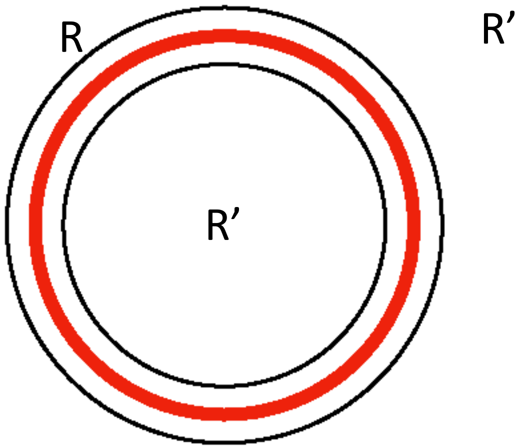
ABJ anomaly

No chiral current

Interacting case:
Is there a continuous chiral symmetry? In what sense?
Usually thought as an "anomalous symmetry",
not explictly broken, different from spontaneously broken.

Non local operators : two different notions of locality

It is "localized" in R: commutes with local operators in R'



Non local operators : two different notions of locality

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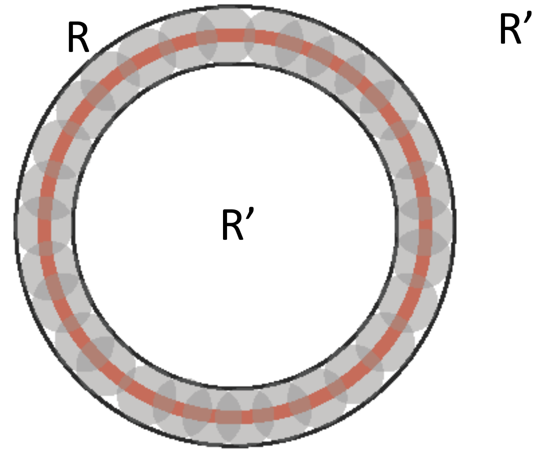
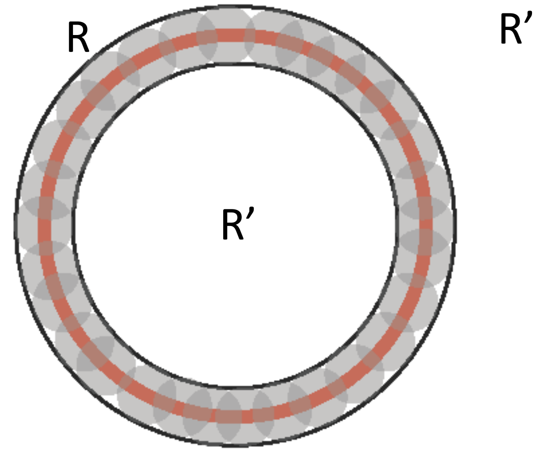
It cannot be generated by local operators in R



Non local operators : two different notions of locality



It is "localized" in R: commutes with local operators in R'

It cannot be generated by local operators in R

$$\mathcal{A}(R) \equiv \bigvee_{B \text{ ball}, \cup B = R} \mathcal{A}(B)$$ additive algebra

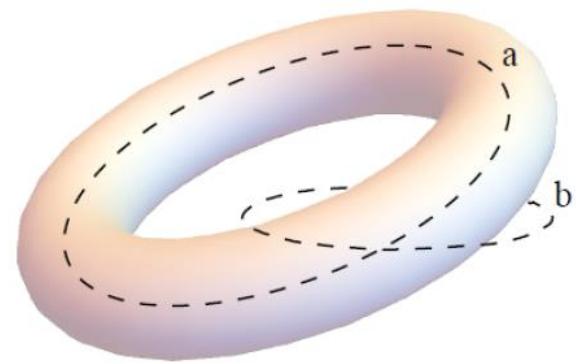
$$\mathcal{A}_{\max}(R) \equiv (\mathcal{A}(R'))'$$ maximal algebra

When there are non-local operators they always come in dual pairs:

$$\mathcal{A}_{\max}(R) = \mathcal{A}(R) \vee \{a\}$$

$$\mathcal{A}_{\max}(R') = \mathcal{A}(R') \vee \{b\}$$

a,b, dual ''non local'' operators.



Not all a's and b's commute to each other

When there are no non-local operators:

$$\mathcal{A}(R) = (\mathcal{A}(R'))'$$ Haag duality (for the additive algebra)

Remarks:

“Non local operator” is a notion relative to the region R. An operator (Wilson loop, t’ Hooft loop) can be non local in a region (ring) and is always additive on a different region (ball that contains the ring).

When there are no HDV for the additive algebra the theory is complete, in the usual sense of a complete set of charges. Haag duality also coincides with modular invariance for CFT, d=2.

H.C., J. Magán (2021)

Classification of HDV?
For disjoint balls or their complements in d>2 HDV are given by charge-anticharges and twists of a global symmetry group (DHR (Doplicher, Haag, Roberts) theorem).
These HDV can be eliminated by adding the charges to the additive algebra.
When disjoint region sectors have been eliminated, sectors of a ring form Abelian dual groups (d>3) and the commutation relations are fixed $a\,b = \chi_b(a)\,b\,a$

H.C., M. Huerta, J. Magan, D. Pontello (2020)
H.C, J. Magan (in preparation)

In the literature this subject is described by the idea of “Generalized symmetries” (Gaiotto, Kapustin, Seiberg, Willett (2015))
Usual description do not use Haag duality violations but continues by putting the QFT in topologically non trivial manifods, usually Euclidean description.

Global symmetry + HDV: obstructions to Noether's theorem

V. Benedetti, H.C., J. Magan (2022)

A global internal symmetry maps the local algebras in themselves (definition).
Can it change the non local operator classes between themselves? Many examples where it does.

$\longrightarrow$ Point-like transformation of HDV class labels

A Noether current for a continuous symmetry allows the construction of additive local charges for any region:

$$\tau(R) = e^{i\lambda\, Q(R)}\,, \quad Q(R) = \int \alpha_R(x)\, j^0(x)$$

An additive operator cannot change non local classes

If a continuos symmetry has a Noether current there cannot be charged non-local classes.
If a continuous symmetry changes the non local classes of a region it cannot have a Noether current

Some examples of violations of the Noether's theorem:

All known examples have charged non local sectors

Free graviton
No stress tensor.
Non local classes have Lorentz indices
Poincare symmetry mixes classes.

Two free Maxwell fields
No current for rotation symmetry.
Symmetry mixes Wilson loops.

Duality symmetry Maxwell field
No duality current
Symmetry mixes electric and magnetic fluxes.

Maxwell field for dimension $d \neq 4$
Derivatives of free scalar $d>2$
No dilatation current.
Symmetry mixes classes with dimensionfull labels.

Free and massless.

ABJ anomaly $\longrightarrow$ Interacting case?

What type of actions of a continuous 1-parameter symmetry on non local classes?

Classes and dual classes non invariant under a continuous symmetry: both must form a continuum.

Let us assume Abelian classes and a one dimensional group.

"One dimensional case": fusion $a_1 + a_2$ $b_1 + b_2$ Commutation relations $a\, b = e^{i\,a\,b}\, b\, a$

$\longrightarrow$ $a(\lambda) = e^{\lambda}\, a$ $b \to e^{-\lambda}\, b$

Dual classes are non-compact and continuous group R. (Example: dilatation symmetry for free Maxwell field d≠4)

"Two dimensional cases": $a = (a_1, a_2)$ $b = (b_1, b_2)$ $a\, b = b\, a\, e^{ia\cdot b}$

$\longrightarrow$ $a \to M(\lambda)\, a\,,$ $b \to (M(\lambda)^T)^{-1}\, b$

Apart from dilatations, there is rotation group U(1), for non compact sectors $A = \mathbb{R}\times\mathbb{R}$ $B = \mathbb{R}\times\mathbb{R}$

(Example: rotation between two free Maxwell fields, duality for Maxwell field)

"Two dimensional cases": there is also a unique possibility that allows for compact sectors, the "ABJ anomalous case"

$$A = \mathbb{Z}\times U(1) \qquad B = U(1)\times\mathbb{Z}.$$

$$(a_1, a_2) \to (a_1, a_2 + \lambda\, a_1)\,, \qquad (b_1, b_2) \to (b_1 - \lambda\, b_2, b_2)$$

$$a_1, b_2 \in \mathbb{Z}\,, \qquad a_2 \equiv a_2 + 2\pi\,, b_1 \equiv b_1 + 2\pi$$

Reason for free massless models: non compact sectors

V. Benedetti, H.C, J. Magán (2022)

Non-compact sectors (continuous dual sectors), independently of any global symmetry, in general lead to free massless theories:

$$\Phi_F = \int_{\Sigma_F} F\,, \qquad \Phi_G = \int_{\Sigma_G} G \qquad [\Phi_F, \Phi_G] = i$$

The cross correlator is necessarily a "linking number term", a massless term that cannot renormalize

$$\langle F(x)G(0)\rangle \;=\; \int \frac{d^d p}{(2\pi)^{d-1}}\,\theta(p^0)\,\delta(p^2)\,e^{ipx}\,(P^{(k)}\tilde{*})(p) \qquad \Box\,\langle F(x)G(0)\rangle = 0$$

"ABJ anomalous case:"

$$A = \mathbb{Z} \times U(1) \qquad B = U(1) \times \mathbb{Z}.$$

$$(a_1, a_2) \to (a_1, a_2 + \lambda\, a_1)\,, \qquad (b_1, b_2) \to (b_1 - \lambda\, b_2, b_2)$$

$$a_1, b_2 \in \mathbb{Z}\,, \qquad a_2 \equiv a_2 + 2\pi\,, b_1 \equiv b_1 + 2\pi$$

In this case we would have:

1) A continuous global U(1) symmetry without Noether current (it changes classes)

2) Goldstone theorem: it follows from existence of twists operators and does not require Noether's current (Buchholz, Doplicher, Longo, Roberts (1992))

3) "Quantization" of the group action: the compatibility of the cycle of the symmetry group $\lambda \equiv \lambda + \lambda_0$ with the one of the non-local sectors implies $\lambda_0 = 2\pi\, n$ (anomaly quantization)

4) If the non local operators exist in the IR limit then the symmetry has to exist in the IR: there must be massless local excitations charged under the symmetry and the rates of group action and group of non local operators must match between different scales (anomaly matching).

The ABJ anomaly

$$S = \int d^4x \left[-\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \overline{\psi}\, i\partial\!\!\!/\, \psi - \overline{\psi}\, A\!\!\!/\, \psi \right] \qquad J_5^\mu = \overline{\psi}\, \gamma^\mu \gamma^5\, \psi$$

$$\partial_\mu J_5^\mu = \frac{1}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \longrightarrow \tilde{J}_5^\mu = J_5^\mu - \frac{1}{4\pi^2} \tilde{F}^{\mu\nu} A_\nu \,, \qquad \partial_\mu \tilde{J}_5^\mu = 0$$

$$\tilde{Q} = \int d^3x \left[\psi^\dagger(x) \gamma^5 \psi(x) - \frac{B^i(x) A_i(x)}{4\pi^2} \right]$$

Taking into account Schwinger terms the new charge generates an internal U(1) symmetry (Adler 1969)

$$\left[\tilde{Q}, A_i(x)\right] = 0 \,, \qquad \left[\tilde{Q}, E^i(x)\right] = 0 \qquad \left[\tilde{Q}, \bar{\psi} \left(\frac{1 \pm \gamma^5}{2}\right) \psi(x)\right] = \pm 2 \bar{\psi}(x) \left(\frac{1 \pm \gamma^5}{2}\right) \psi(x)$$

It is an internal U(1) symmetry

Non trivial transformation of non local operators $\longrightarrow$ No Noether current

$\downarrow$

D.Harlow, H.Ooguri (2018)

How does the anomalous chiral symmetry changes non local classes?: Witten's effect

$$\partial_\mu\, j^\mu = \frac{\beta}{4}\, F_{\mu\nu}\, \tilde{F}^{\mu\nu}$$

Take electric charges with minimal charge q_0 the (non local) WL charge have range $q \in [0, q_0)$

The non local TL have integer charges given by the Dirac quantization condition $g = \frac{2\pi}{q_0} k$

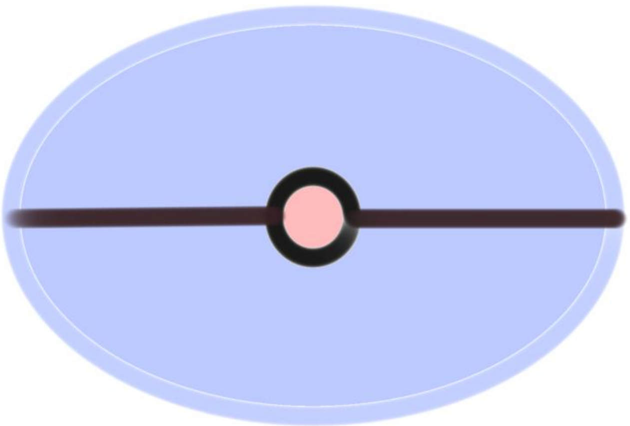
The current is normalized to have minimal charge equal to 1. Parameter $\lambda \in [0, 2\pi)$

Transformation with $\lambda(x)$ changes the action by $\delta S = \int \lambda(x)\, \partial_\mu\, j^\mu = \frac{\beta}{4} \int \lambda(x)\, F_{\mu\nu}\, \tilde{F}^{\mu\nu}$

$\longrightarrow$ $\nabla E = \beta\, (\nabla\lambda) \cdot B$

Monopole boundary conditions (TL) get mixed with
WL boundary conditions from classical equations of motion

$$(g, q) \to (g, q + \beta\, \delta\lambda\, g)$$



The possible values of the anomaly follow from compatibility
of the group cycle and the group of non local operators

$\longrightarrow$ $\beta \times (2\pi) \times \left(\frac{2\pi}{q_0}\right) = n\, q_0$

$\longrightarrow$ $\partial_\mu j^\mu = n\, \frac{q_0^2}{16\, \pi^2}\, F_{\mu\nu}\, \tilde{F}^{\mu\nu}$

Remarks

QED achieves the minimal possible value n=1.

Meaning of the integer n: The minimal chiral charge of non local operators is n

The TL transformation is fixed by that of the local ones. It transforms non trivially because its expression in terms of local operators in a ball must contain gauge invariant chiral fields.

$$G^{\mu\nu} \equiv \frac{1}{e^2} F^{\mu\nu} - \frac{\pi_0}{\mu} \tilde{F}^{\mu\nu}$$

When gauge and chiral charge are the same it does not make sense:
there is no gauge invariant local chirally charged operator, and hence no non-local operator can be charged.

Final remarks

Conjecture: (strong form of Noether's theorem): If there are no non local classes that are non invariant under a continuous global symmetry then there is a Noether current.

Requires UV complete theory.

Would improve WW theorem: gravitons in a QFT $\longrightarrow$ no stress tensor $\longrightarrow$ classes charged under Lorentz $\longrightarrow$ (non compact) free decoupled gravitons .

In the literature that this type of symmetry actions that mix with generalized symmetries are described as "non-invertible". However, the symmetry is a U(1) for the local physics. This invertibility is behind anomaly quantization.

The fact that the global internal symmetry is invertible (a group U(1)) is in accordance with the DHR theorem:
This can be rephrased as "All 0-form (internal) symmetries for d>2 come from a compact group".
Non invertible topological operators are combinations of invertible ones. (This is different in d=2)

Conjecture: An effective model with a continuous symmetry, no Noether current, and no non-invariant HDV class cannot be UV completed

Possible example: d=6 analogous to the pion electrodynamics

$$S = \frac{1}{2}\int d\pi_0 \wedge \star d\pi_0 + \frac{1}{2e^2}\int F \wedge \star F + \frac{1}{\mu}\int \pi_0 F \wedge F \wedge F$$

$\pi_0 \to \pi_0 + \mathrm{cons}$ is a symmetry with no current. However, TL and WL live in different topologies and the anomalous action does not hold, classes are invariant, and there is no current $\longrightarrow$ could not be UV completed