A matrix model for 2d de Sitter

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MIT CTP

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based on **WIP** with **Lorenz Eberhardt**, **Beatrix Mühlmann** and **Victor Rodriguez**

Motivation

- There is a **dearth of precise, controllable models of de Sitter quantum gravity**
- Many recent approaches to dS_2 gravity [Anninos Hofman 17; Maldacena Turiaci Yang 19; Cotler Jensen Maloney 19; Anninos Mühlmann 21, 23; Cotler Jensen 23, 24; …] but no unifying picture has emerged
- Recent stringy embeddings of AdS_2 JT gravity [Stanford & Turiaci discussion today] via semiclassical limits of the $(2,p)$ **minimal string** [Seiberg Stanford; …] and of the **Virasoro minimal string** [SC Eberhardt Mühlmann Rodriguez 23] contextualize the holographic duality with a double-scaled matrix integral [Saad Shenker Stanford 19]
- Stringy realization of dS_2?

A two-dimensional string landscape

Persistent paradigm:

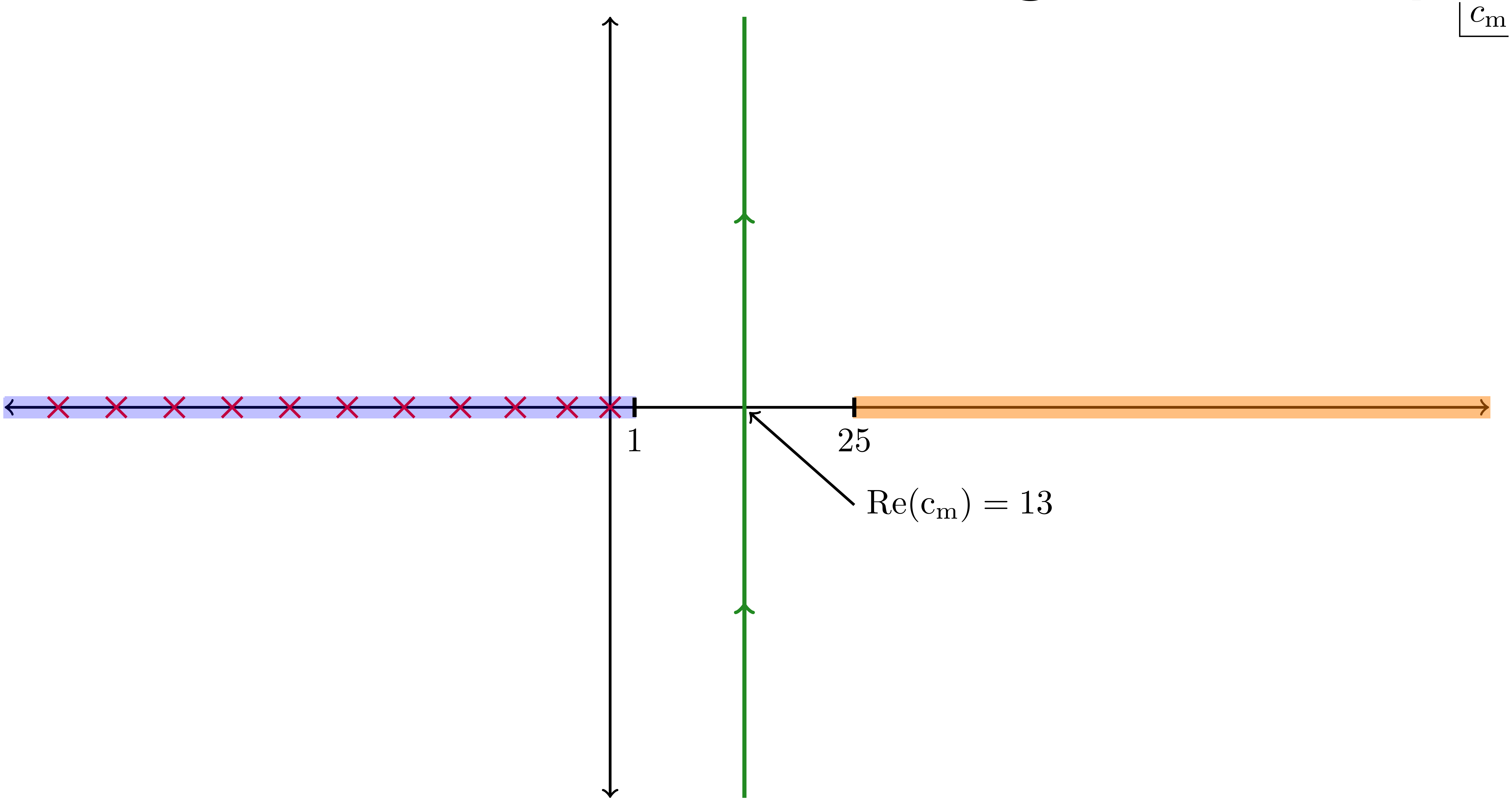
Worldsheet:	matter CFT $\oplus$	Liouville CFT $\oplus$	b c ghosts
central charge:	$c_{\rm m}$	$c_{\rm L} = 26 - c_{\rm m}$	$c_{\rm gh} = -26$

$\Updownarrow$

Double-scaled matrix integral

$$\int_{\mathbb{R}^{N^2}} [dM] e^{-N \,\mathrm{Tr}\, V(M)}$$

A two-dimensional string landscape



A landscape of string/matrix model holographic dualities

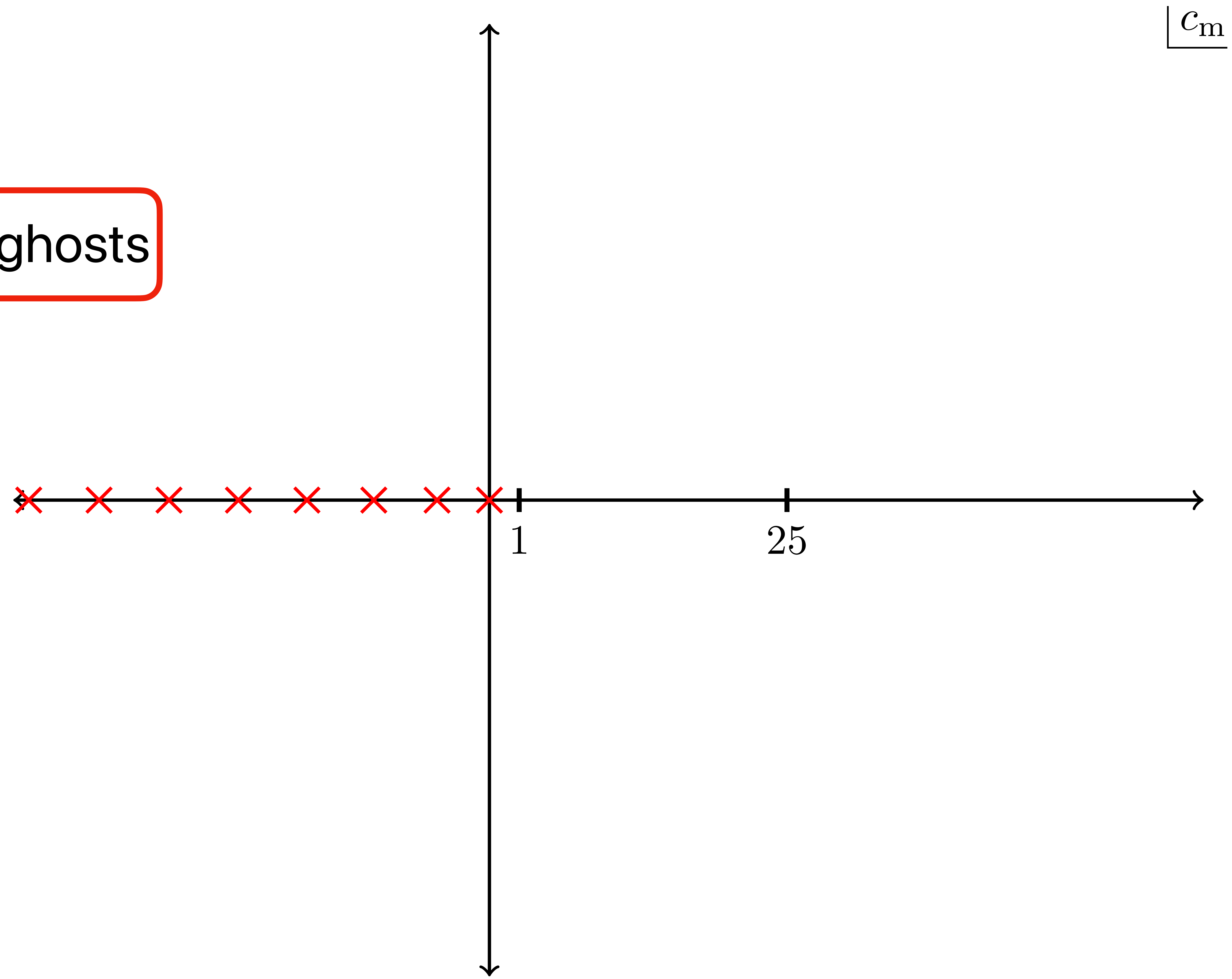
$(2,p)$ minimal string

$(2,p)$ minimal model ⊕ Liouville CFT ⊕ ghosts

↕

Double-scaled one-matrix integral

$$\rho_0(E) = \sinh\left(\frac{p}{2}\text{arccosh}(1+E)\right)$$



[Brezin Kazakov 90; Gross Migdal 90; Douglas Shenker 90; ...]

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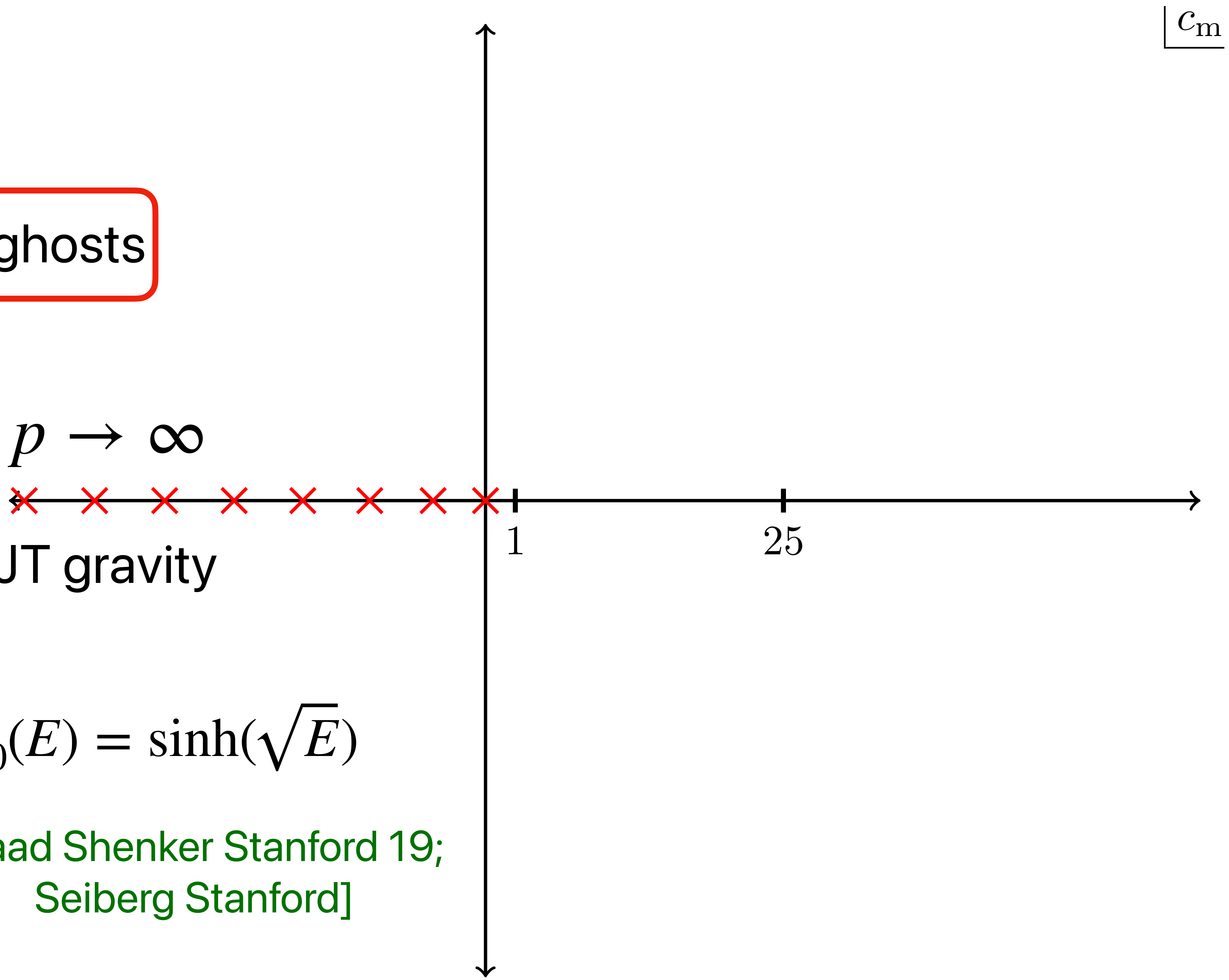


$p \to \infty$

JT gravity

$$\rho_0(E) = \sinh(\sqrt{E})$$

[Saad Shenker Stanford 19; Seiberg Stanford]

[Brezin Kazakov 90; Gross Migdal 90; Douglas Shenker 90; ...]

A landscape of string/matrix model holographic dualities

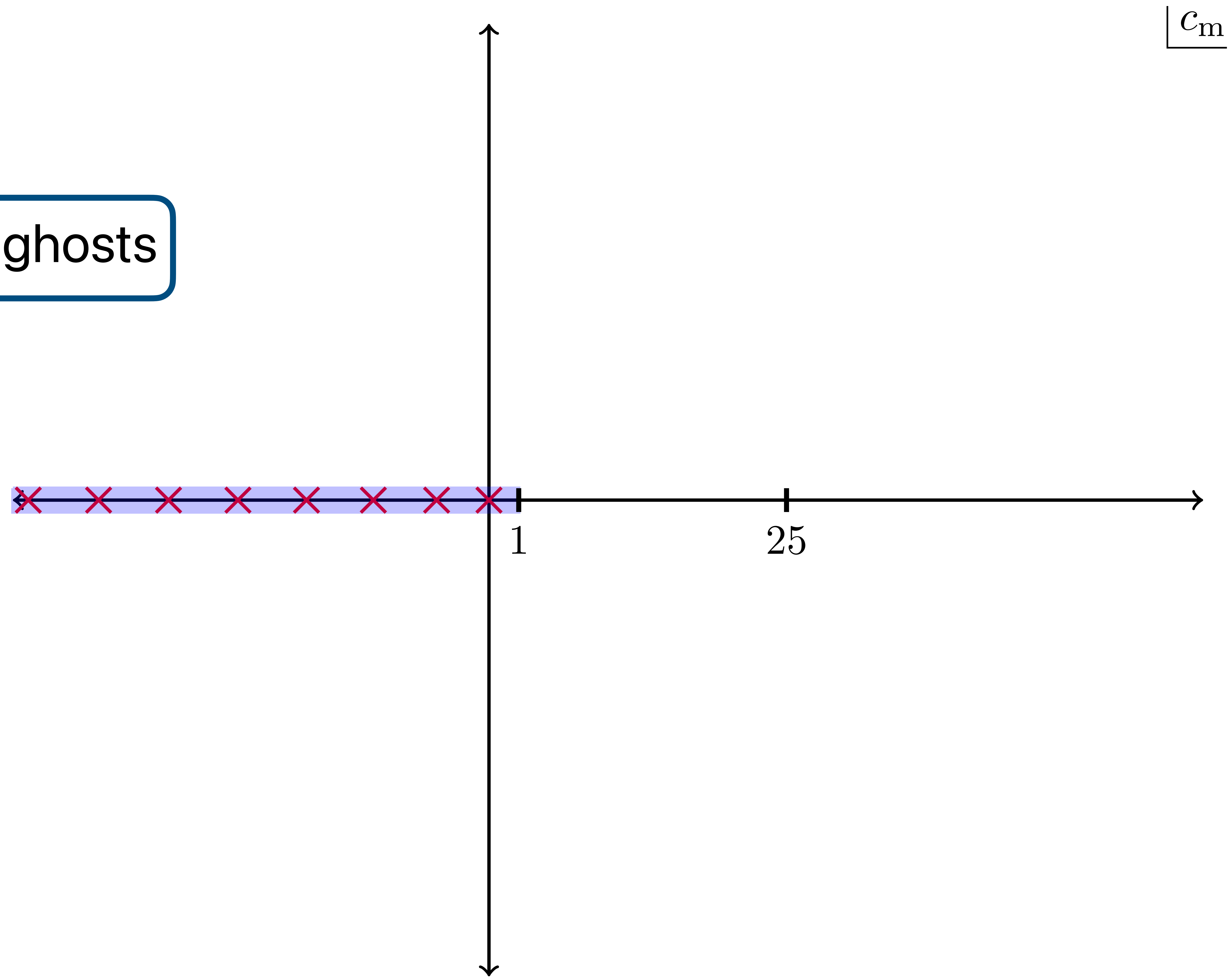
(p,q) minimal string

(p,q) minimal model $\oplus$ Liouville CFT $\oplus$ ghosts

$\Updownarrow$

Double-scaled **two-matrix integral**

$$x(z) = T_p(z), \quad y(z) = T_q(z)$$



[Kazakov 86; Boulatov Kazakov 87; …; Eynard 02; Seiberg Shih 04; …]

A landscape of string/matrix model holographic dualities

“Virasoro minimal string (VMS)”

Liouville CFT $\oplus$ timelike Liouville CFT $\oplus$ ghosts
$c \geq 25$ $\quad$ $26 - c \leq 1$

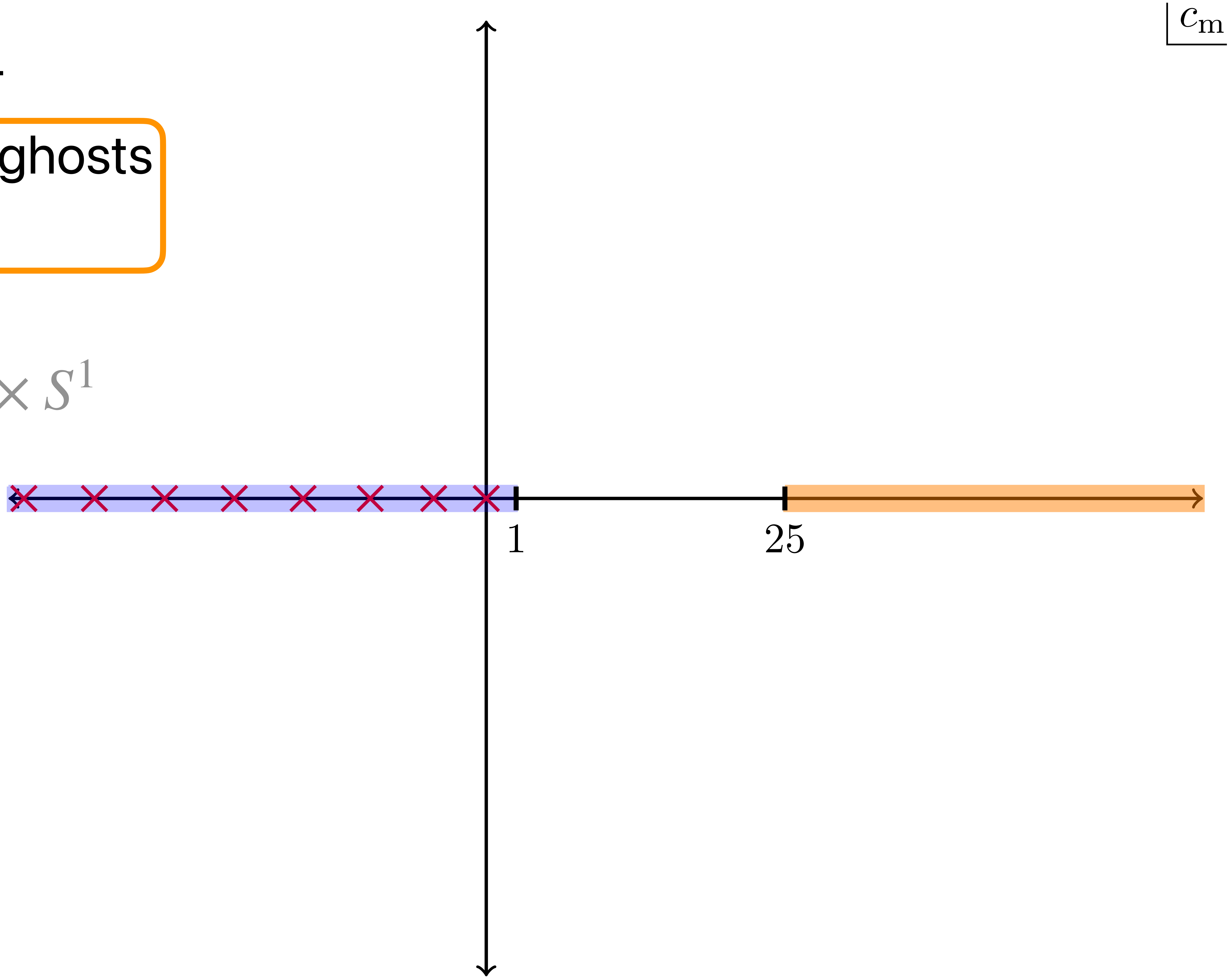
↕ 3d gravity/Virasoro TQFT on $\Sigma \times S^1$

Double-scaled **one-matrix integral**

$$\rho_0^{(b)}(E) = \frac{\sinh(2\pi b\sqrt{E})\sinh(2\pi b^{-1}\sqrt{E})}{\sqrt{E}}$$

String amplitudes $\mathsf{V}^{(b)}_{g,n}$: polynomials deformations of Weil-Petersson volumes

[SC Eberhardt Mühlmann Rodriguez 23]



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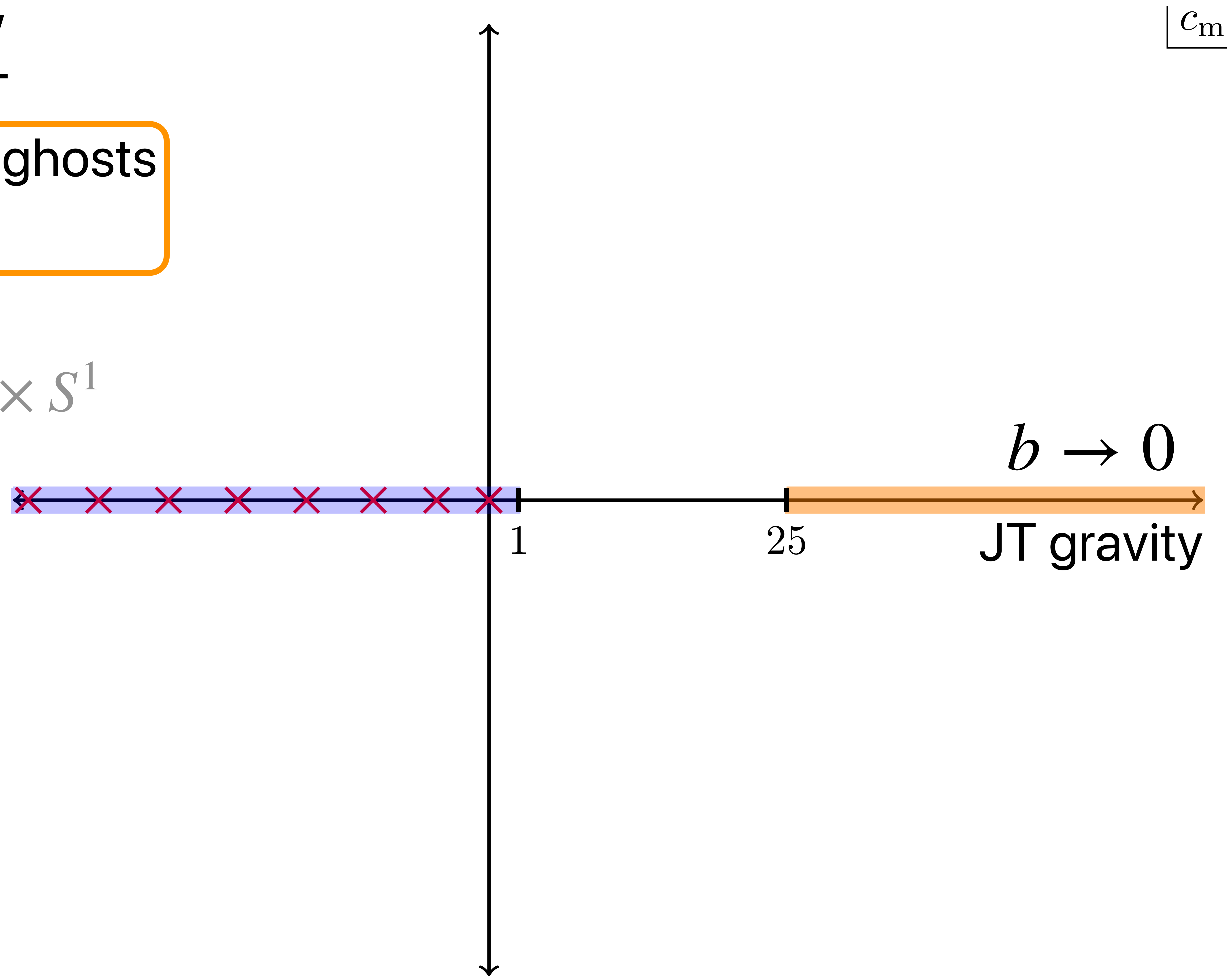


A landscape of string/matrix model holographic dualities

"$|$Liouville$|^2$ string theory"

Liouville CFT ⊕ Liouville CFT ⊕ ghosts
$c_+ = 13 + i\mathbb{R}$ $\quad$ $c_- = 13 - i\mathbb{R}$

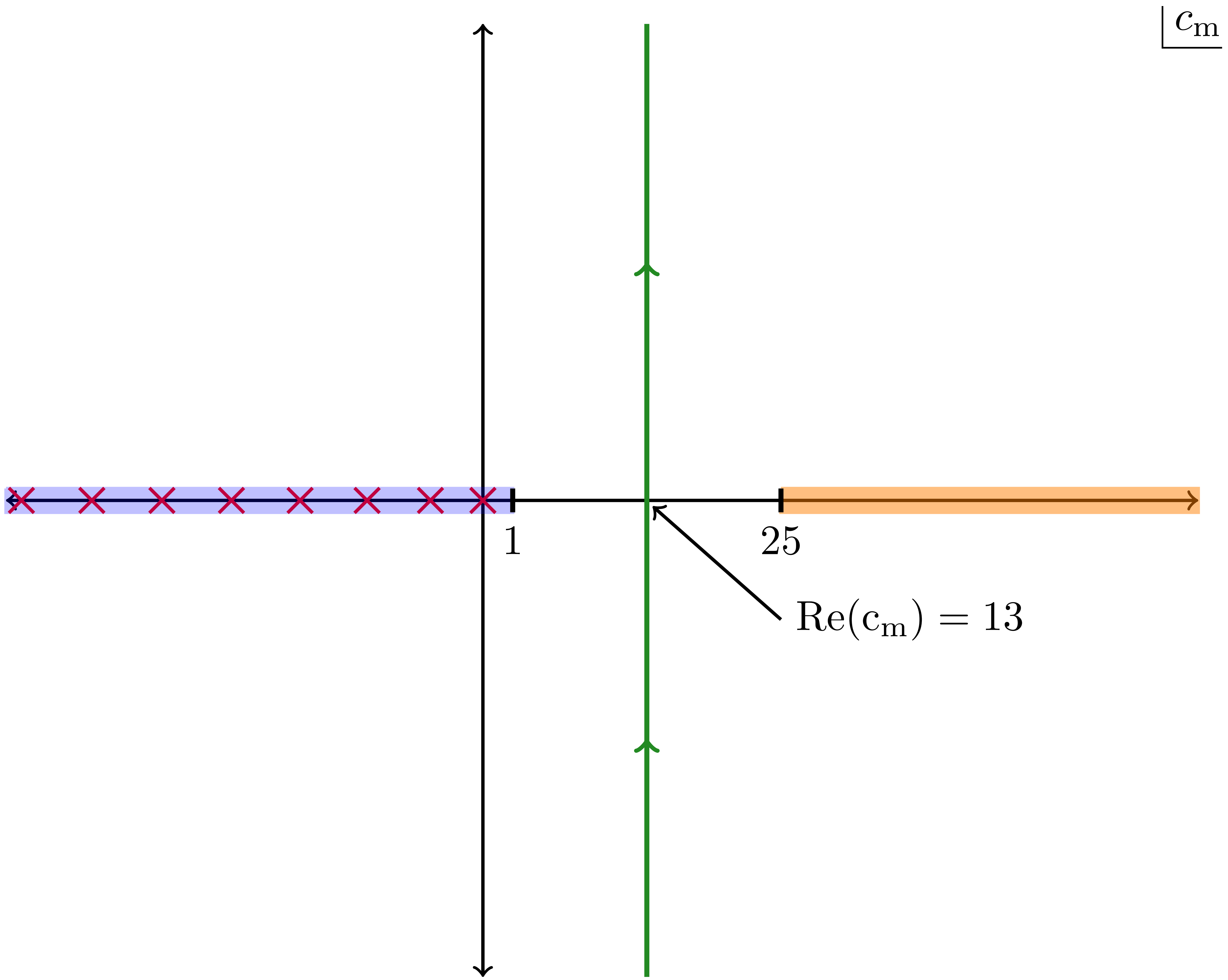
↕

Double-scaled **two-matrix integral**

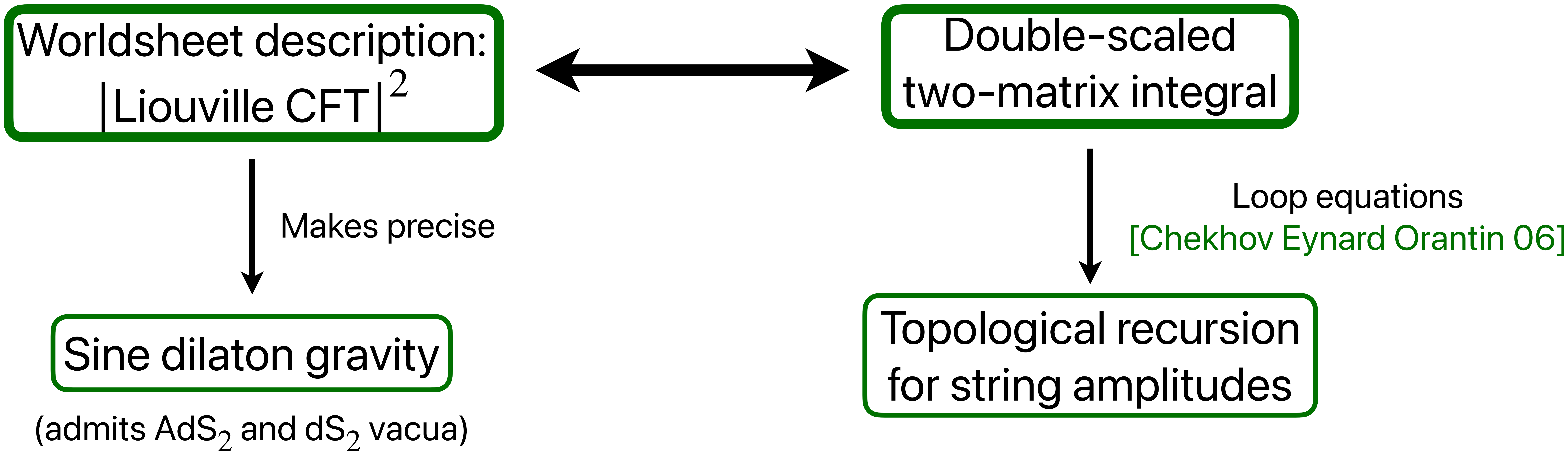
$$x(z) = -2\cos(\pi b^{-1}\sqrt{z}), \quad y(z) = 2\cos(\pi b\sqrt{z})$$

String amplitudes $\mathsf{A}_{g,n}^{(b)}$ computed by topological recursion

[SC Eberhardt Mühlmann Rodriguez WIP]



Today: $|\text{Liouville}|^2$ string theory



- A precise and controllable holographic duality that includes dS_2

Plan

- The worldsheet theory and sine dilaton gravity
- String amplitudes
- The dual matrix integral and topological recursion

The worldsheet theory

Sine dilaton gravity

- A field redefinition maps the worldsheet theory to **2d dilaton gravity** with a **sine potential** (see also [Blommaert Mertens Papalini 24; Blommaert talk today])

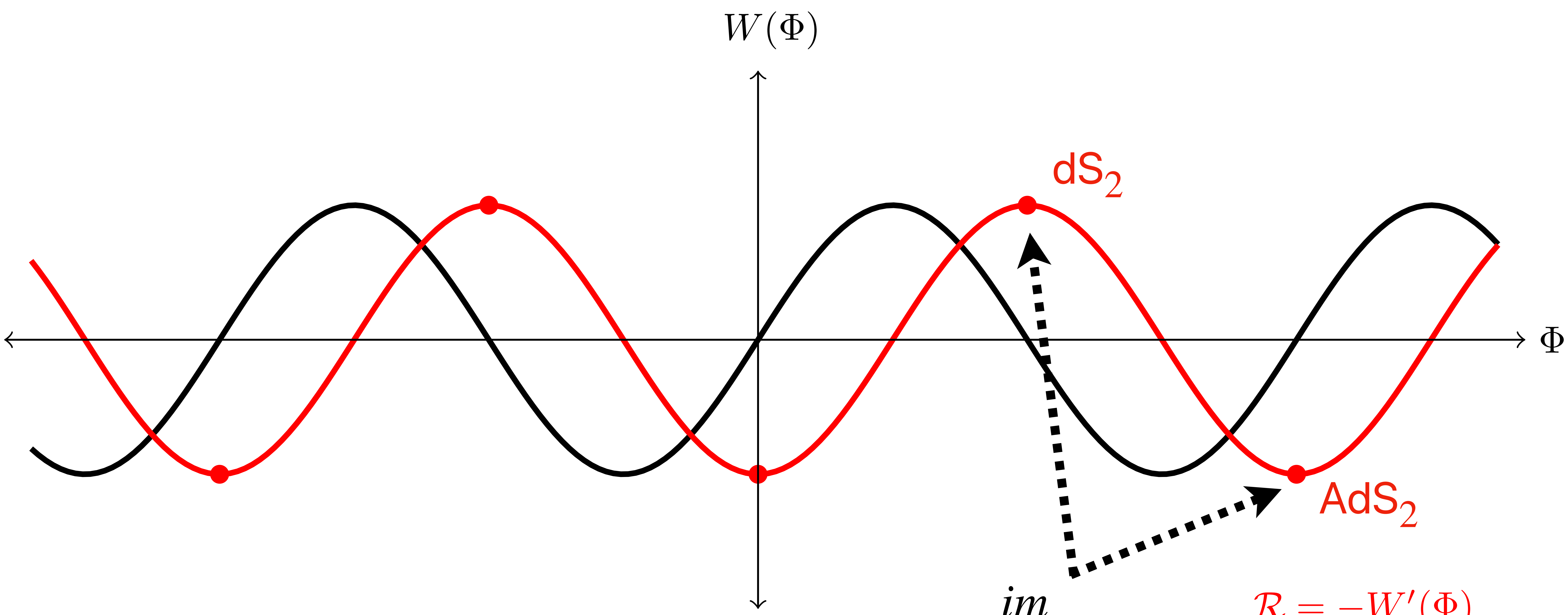
$$S_\Sigma[\Phi, g] = \frac{1}{2}\int_\Sigma d^2x \sqrt{g}(\Phi\mathscr{R} + W(\Phi)) + (\text{bdy term} + \text{Euler term})\,,$$

$$W(\Phi) \propto \sin(2\pi i b^2 \Phi) \qquad (ib^2 \in \mathbb{R})$$

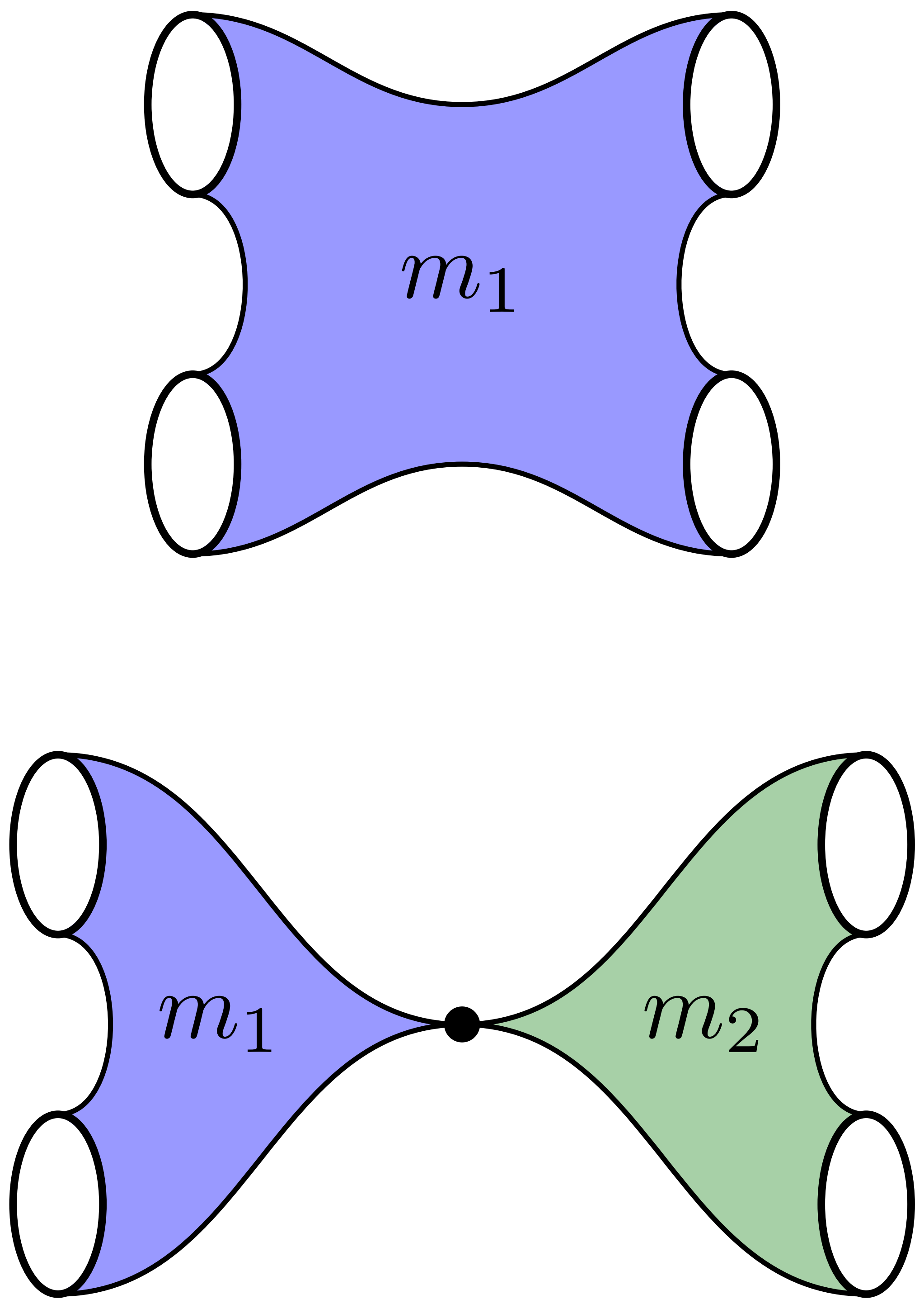
- Classical solutions: constant $\Phi_* = \dfrac{im}{2b^2}$, $(m \in \mathbb{Z})$

$$\mathscr{R} = -W'(\Phi_*)$$

dS_2 and AdS_2 vacua

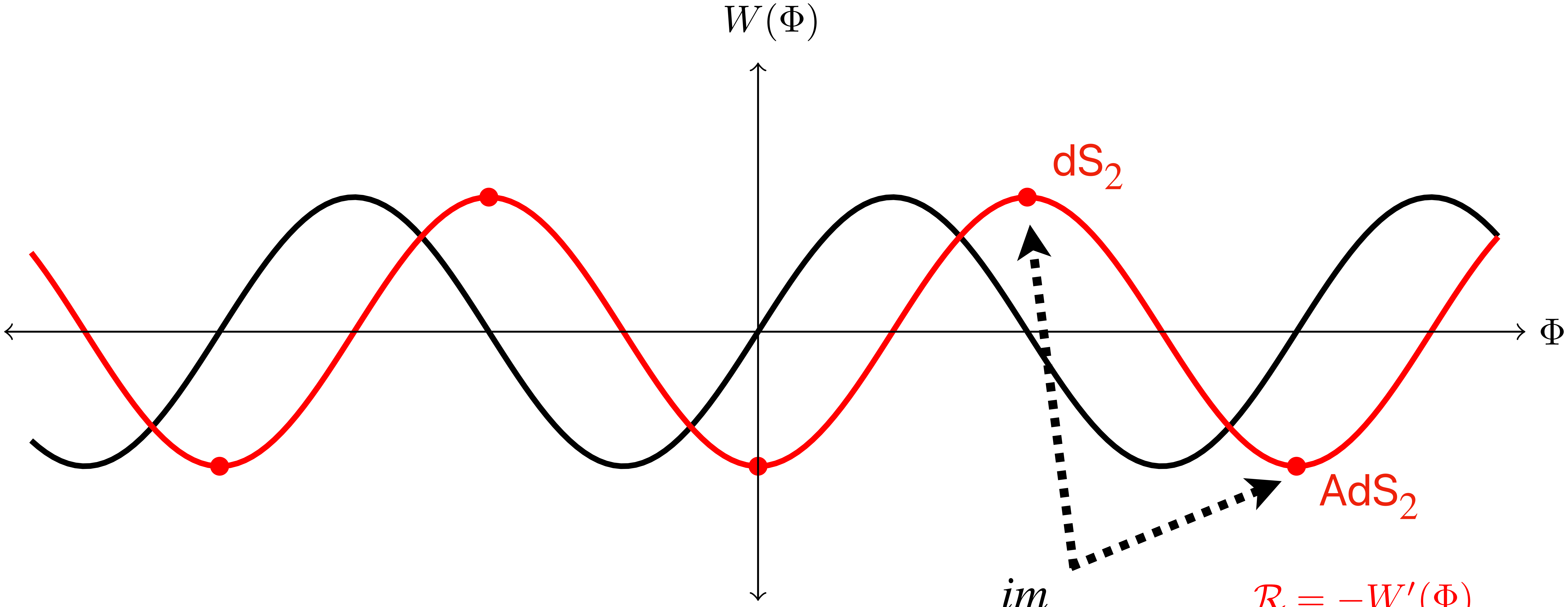


$$\Phi_* = \frac{im}{2b^2}, \; m \in \mathbb{Z}$$

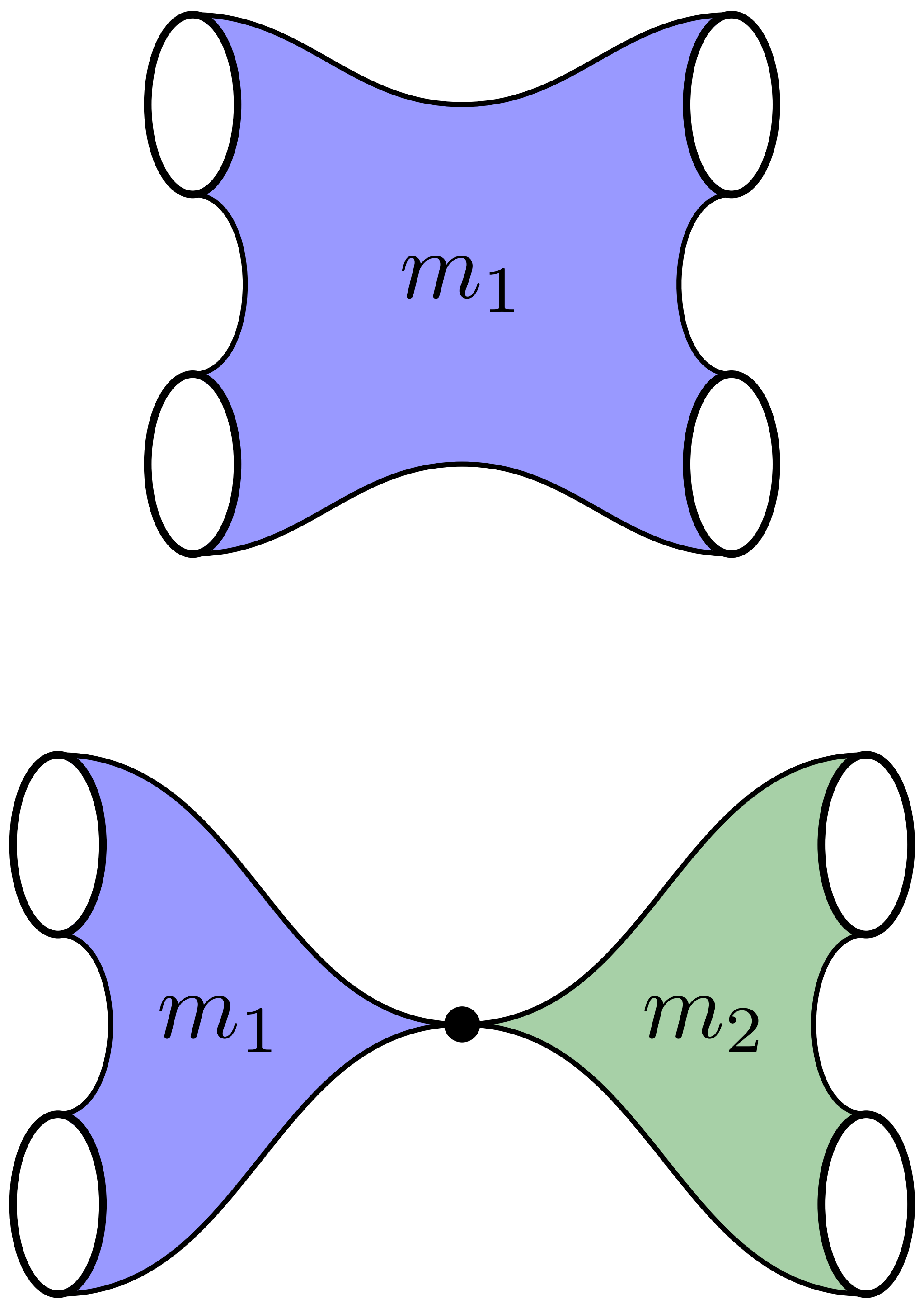


"Transitions between universes"

dS_2 and AdS_2 vacua



$$\Phi_* = \frac{im}{2b^2},\ m \in \mathbb{Z}$$



"Transitions between universes"

(dS_2 solutions in (−,−) signature as in [Cotler Jensen 24])

The worldsheet CFT

Liouville CFT $\oplus$ (Liouville CFT)* $\oplus$ b c ghosts

$c_+ = 13 + i\mathbb{R}$ $\quad$ $c_- = 13 - i\mathbb{R}$ $\quad$ $c_{\text{gh}} = -26$

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- The worldsheet theory is defined by the non-perturbative **CFT data**:

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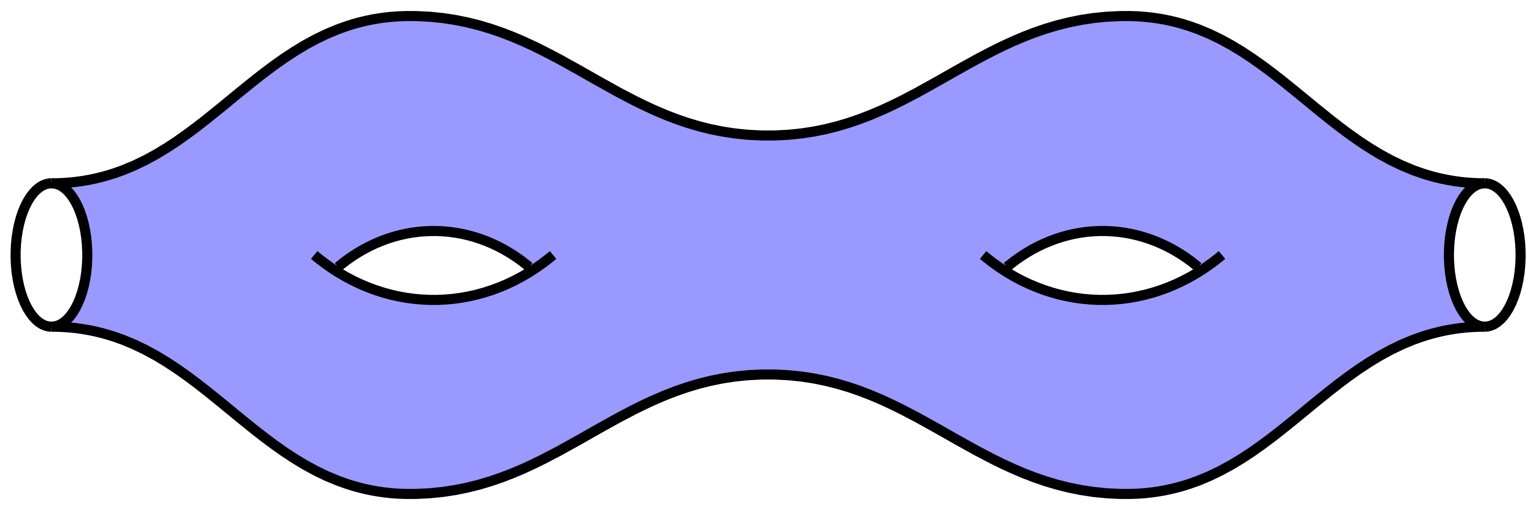
- The worldsheet theory is defined by the non-perturbative **CFT data**:

  - Central charge:
    $$c = 1 + 6(b + b^{-1})^2$$
    $$b \in e^{\frac{\pi i}{4}}\mathbb{R}$$
  - Spectrum:
    continuum $\{V_p\}$
    $$h_p = \bar{h}_p = \frac{c-1}{24} - p^2$$
  - OPE data:
    $$\langle V_{p_1} V_{p_2} V_{p_3} \rangle^{(b)}_{g=0} = C_b(p_1, p_2, p_3)$$
    "DOZZ formula"



[Dorn Otto 95; Zamolodchikov^2 95; Teschner 01]

String amplitudes

String amplitudes

- We will focus on the particular choice:

$$b \in e^{\frac{\pi i}{4}}\mathbb{R} \;\leftrightarrow\; \boxed{c \in 13 + i\mathbb{R}}$$

$$p_j \in e^{-\frac{\pi i}{4}}\mathbb{R} \;\leftrightarrow\; \boxed{h_j \in \frac{1}{2} + i\mathbb{R}}$$

- On-shell vertex operators:

$$\mathscr{V}_p = \mathrm{c}\,\bar{\mathrm{c}}\, V_p^{(b)}\, V_{ip}^{(-ib)}$$

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$$\boxed{\mathsf{A}_{g,n}^{(b)}(p_1,\ldots,p_n) = \left(\prod_{j=1}^{n} \mathscr{N}_b(p_j)\right) \int_{\mathscr{M}(\Sigma_{g,n})} \left| \left\langle \prod_{j=1}^{n} V_{p_j} \right\rangle_g^{(b)} \right|^2 \times \left(\mathsf{b}\,\mathsf{c}\text{ ghosts}\right)}$$

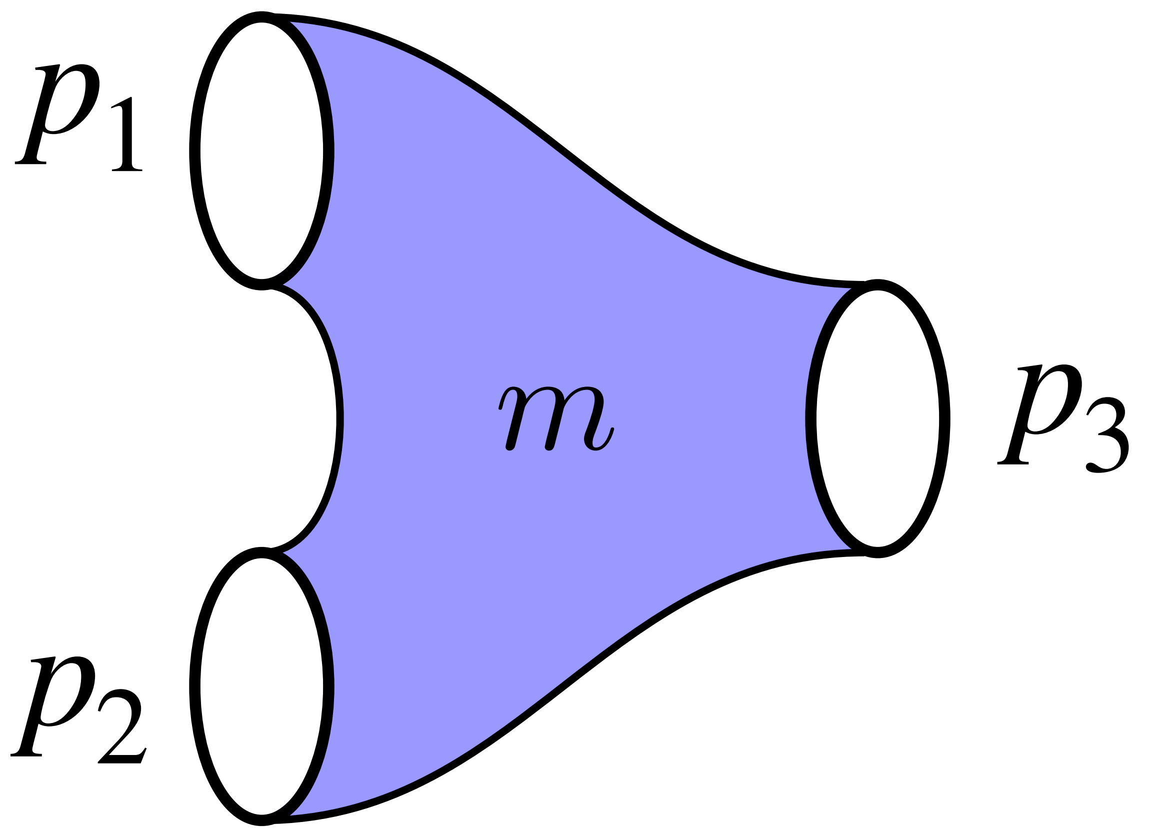
- **Absolutely convergent** integral over moduli space
- Invariant under **swap symmetry** $b \to -ib, p_j \to ip_j$

Sphere three-point amplitude

- The simplest observable in the theory is the sphere three-point amplitude

$$\mathsf{A}_{0,3}^{(b)}(p_1,p_2,p_3) = \left(\prod_{j=1}^{3} \mathcal{N}_b(p_j)\right) C_b(p_1,p_2,p_3) C_{-ib}(ip_1,ip_2,ip_3)$$

$$= \sum_{m=1}^{\infty} \frac{2b(-1)^m \sin(2\pi m b p_1)\sin(2\pi m b p_2)\sin(2\pi m b p_3)}{\sin(\pi m b^2)}$$



[Zamolodchikov 05]

General string amplitudes

$$\mathsf{A}^{(b)}_{g,n}(p_1,\ldots,p_n) = \left(\prod_{j=1}^{n}\mathcal{N}_b(p_j)\right)\int_{\mathcal{M}(\Sigma_{g,n})}\left|\left\langle\prod_{j=1}^{n}V_{p_j}\right\rangle_g^{(b)}\right|^2 \times \left(\mathsf{b\,c}\text{ ghosts}\right)$$

Complicated!

- Harness **analytic structure** & **swap symmetry** to bootstrap amplitude
  - **Poles** associated with resonances of Liouville CFT correlators
  - **Discontinuities** when moduli integral ceases to converge

String amplitudes from Feynman diagrams

- Bootstrap systematically implemented with **simple diagrammatic rules**
  - Diagrams correspond to different **degenerations of the worldsheet**
  - **VMS quantum volume** $\mathsf{V}^{(b)}_{g,n}$ [SC Eberhardt Mühlmann Rodriguez 23] arises as a **string vertex**

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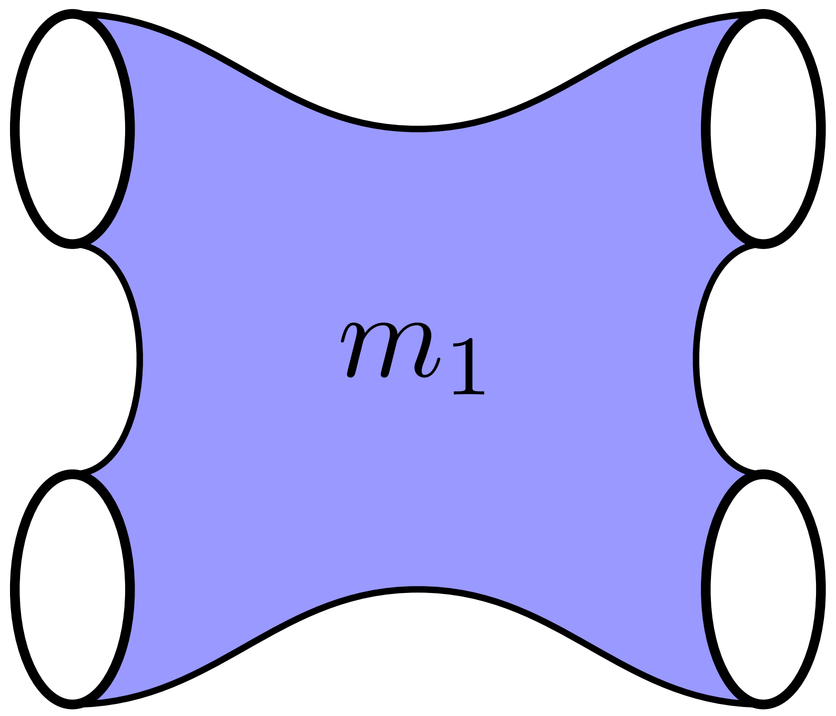
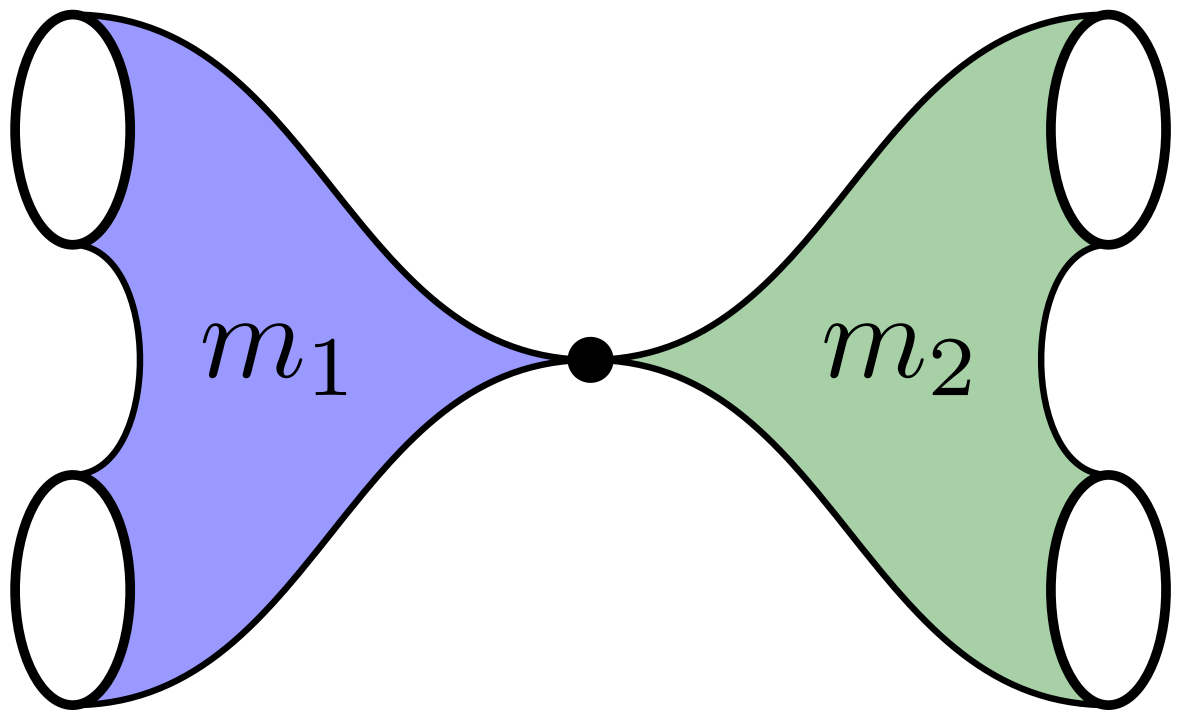


$$+ \sum_{m_1, m_2=1}^{\infty}$$



$+ 2$ permutations

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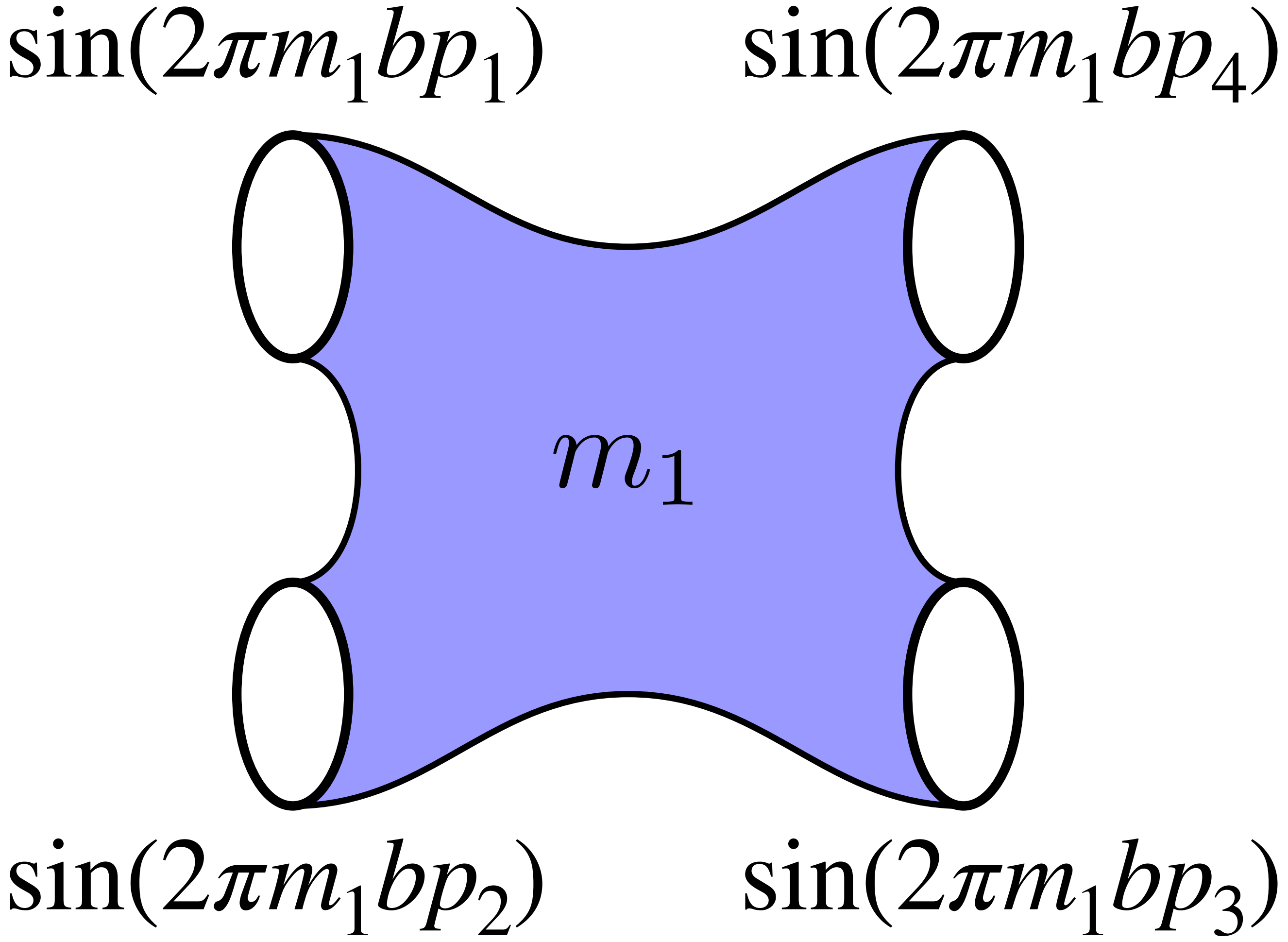

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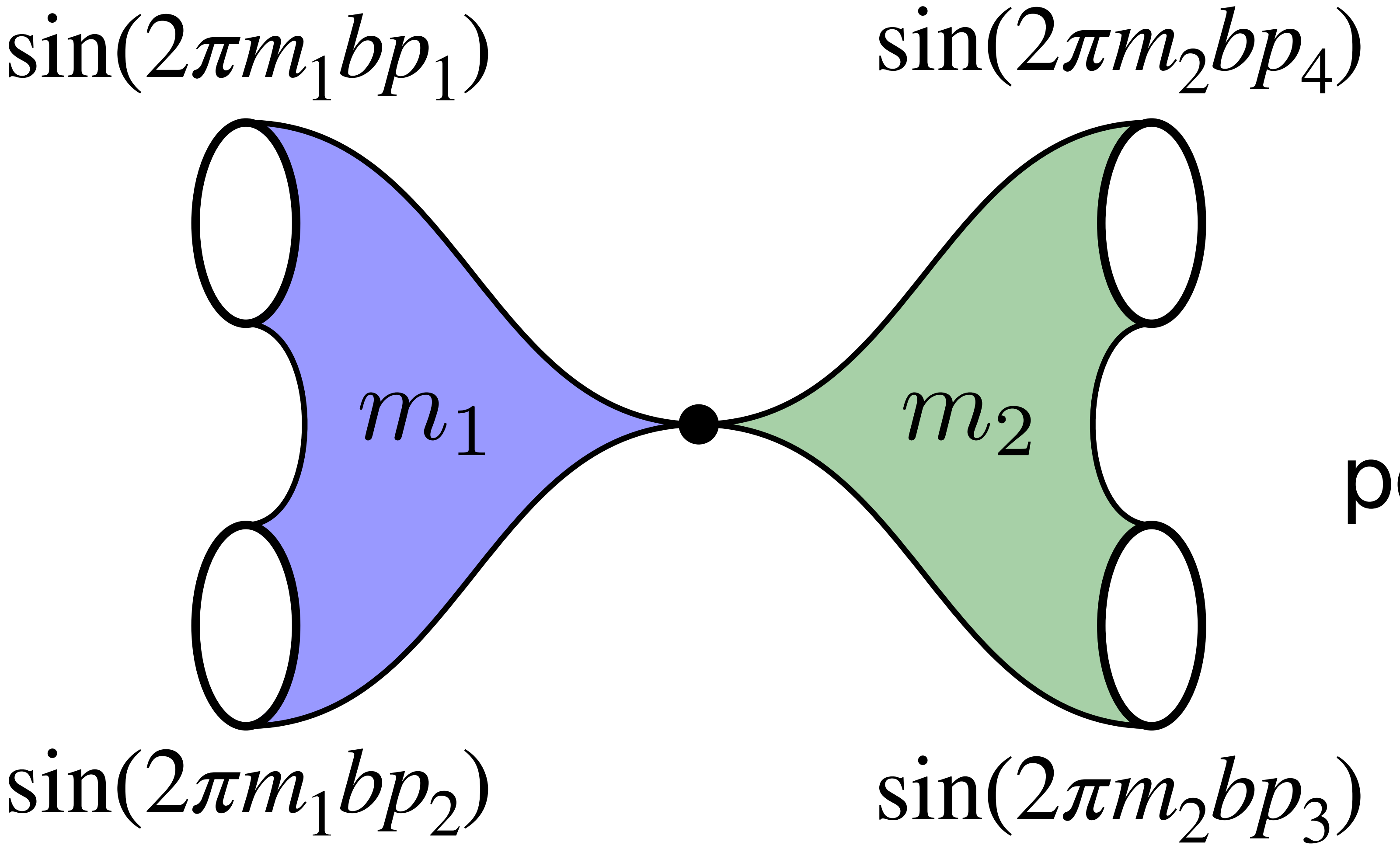

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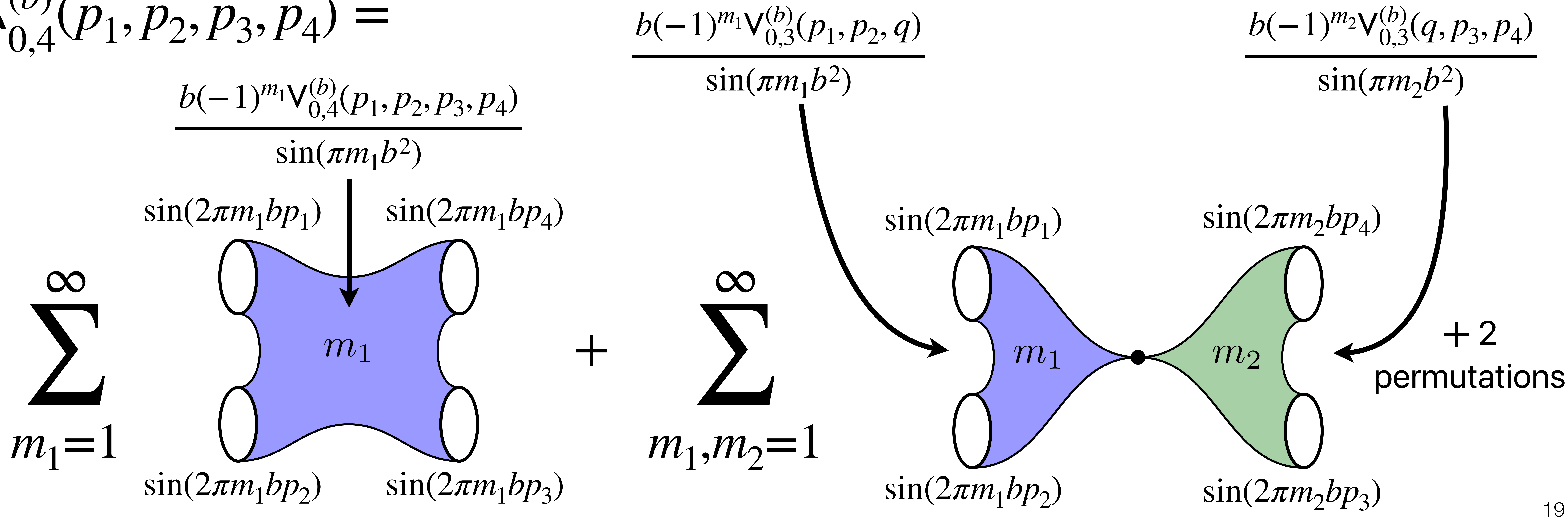

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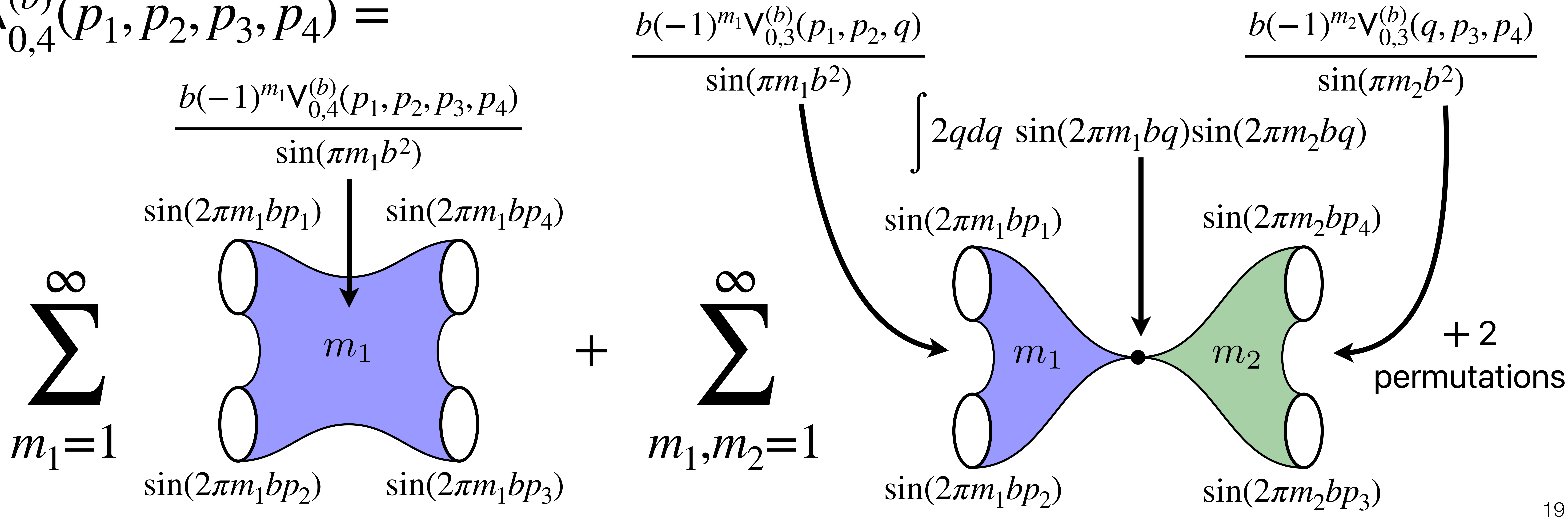

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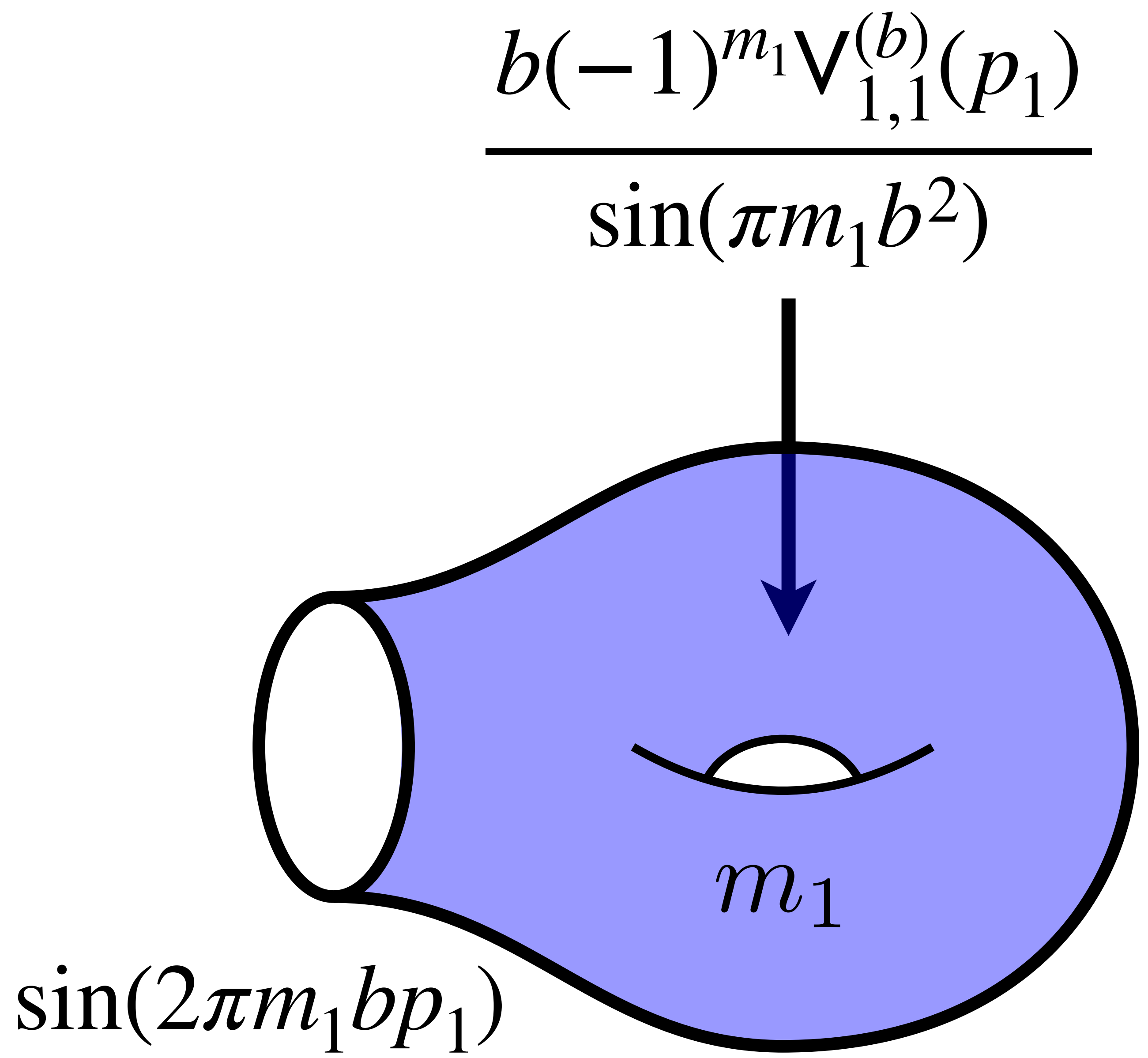
$$\mathsf{A}^{(b)}_{1,1}(p_1) = \sum_{m_1=1}^{\infty}$$



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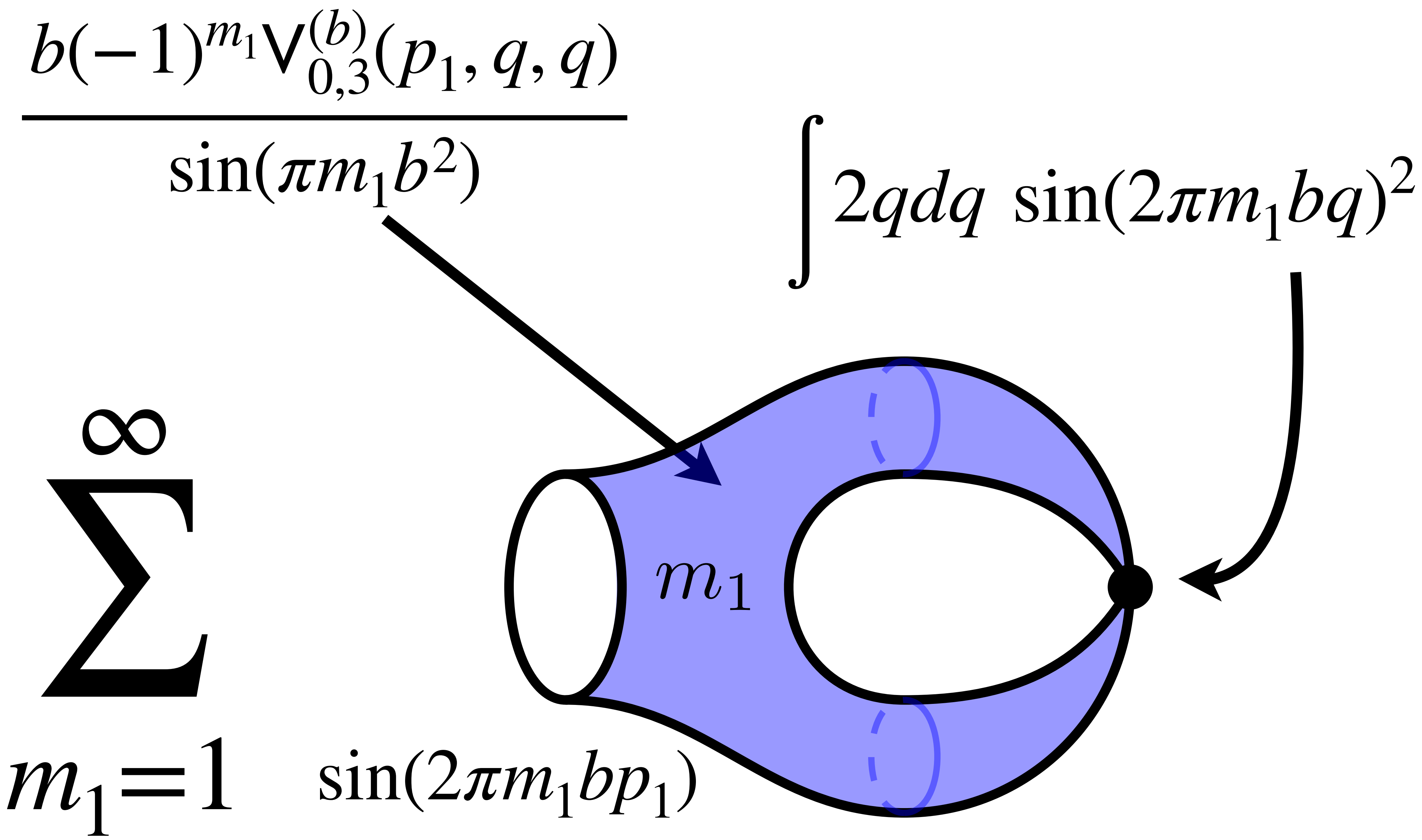
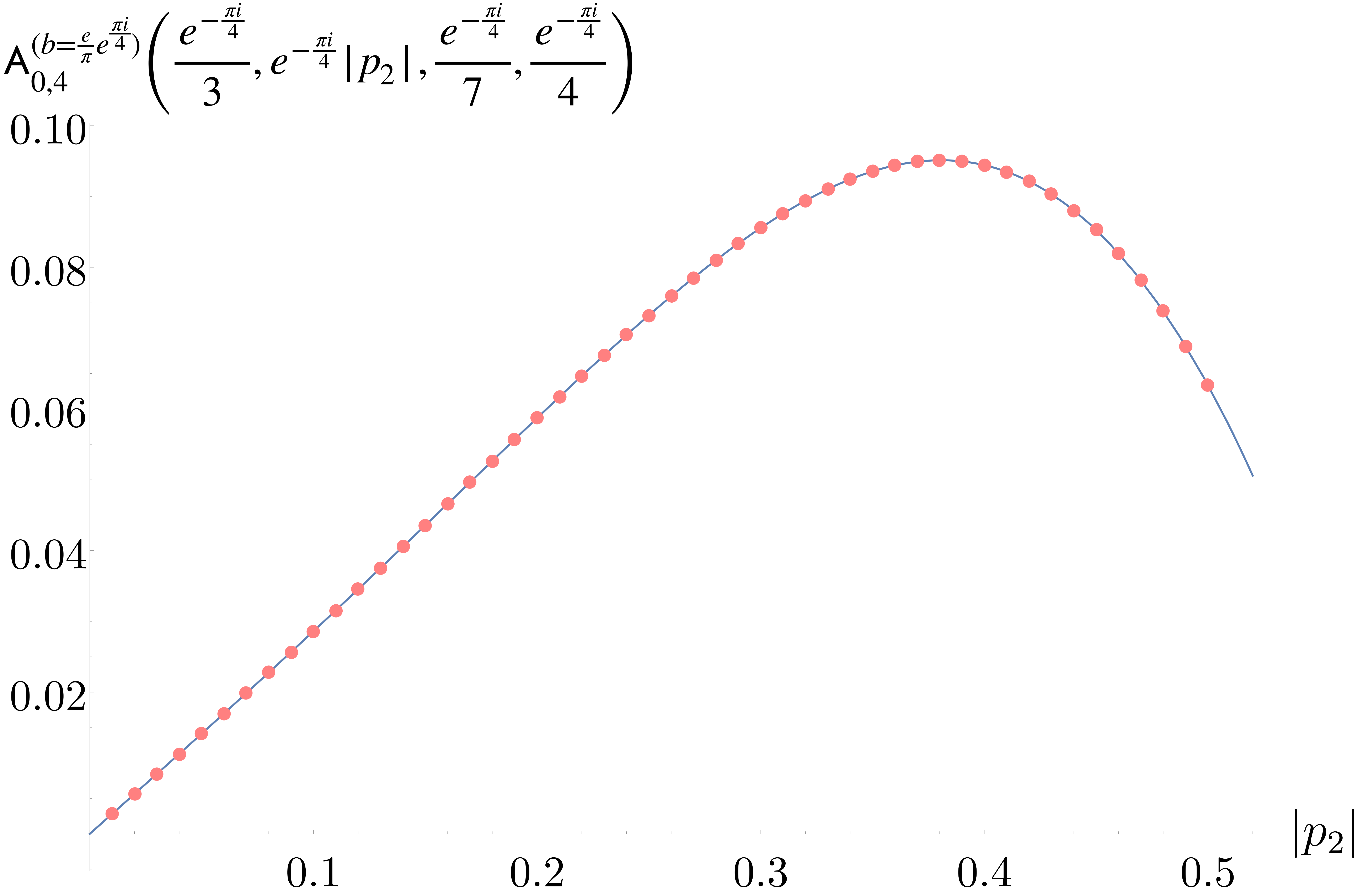


Sphere four-point amplitude: numerical verification

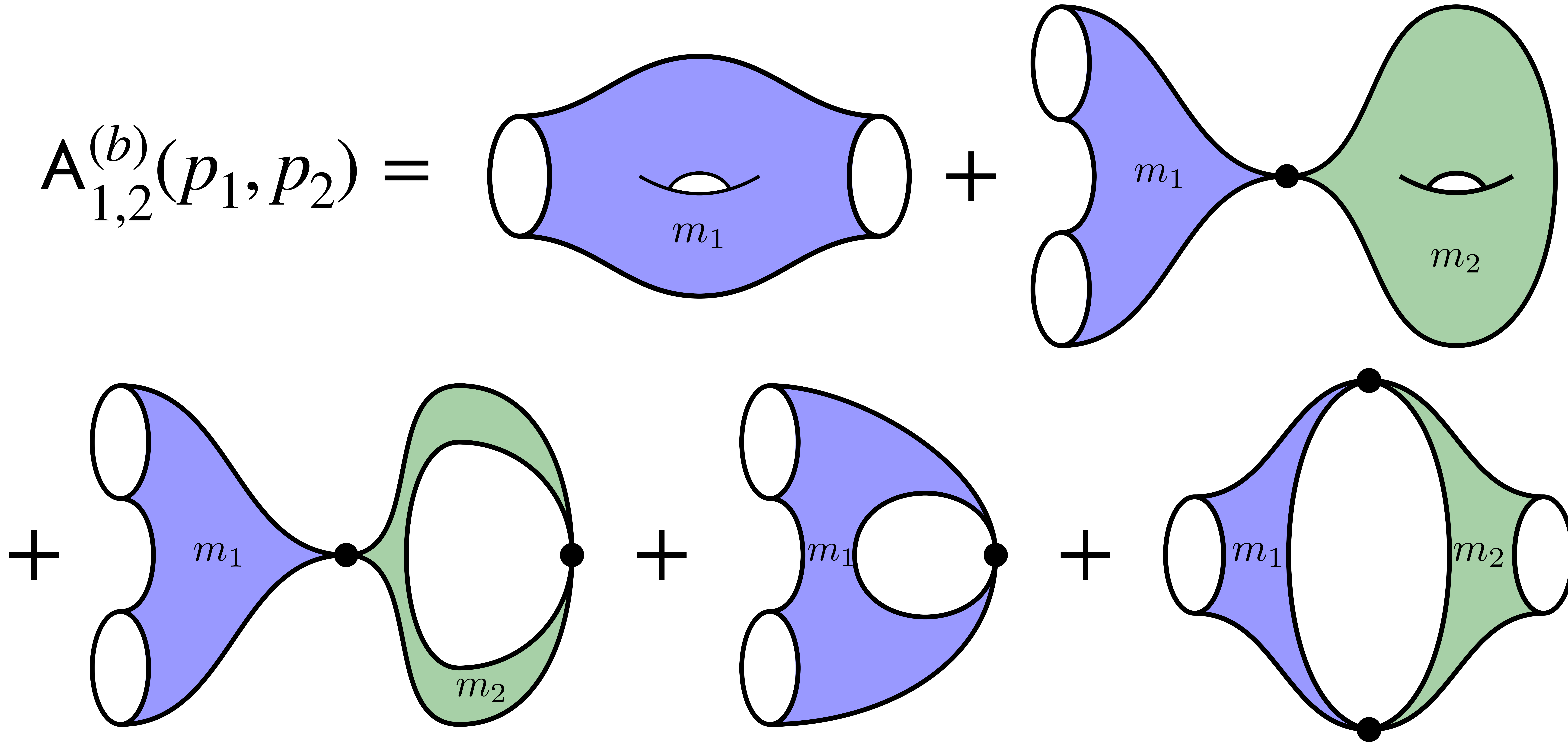
- Direct numerical integration over moduli space



Error $< 10^{-5}$ %

Example: torus two-point amplitude

$$\mathsf{A}^{(b)}_{1,2}(p_1, p_2) =$$



The dual matrix integral and topological recursion

The dual matrix integral

- Claim: $|\text{Liouville}|^2$ string theory is precisely dual to a **double-scaled two-matrix integral**

$$\int_{\mathbb{R}^{2N^2}} [dM_1][dM_2] e^{-N\,\mathrm{Tr}\left(V_1(M_1)+V_2(M_2)-M_1M_2\right)}$$

  - Characterized by the spectral curve:

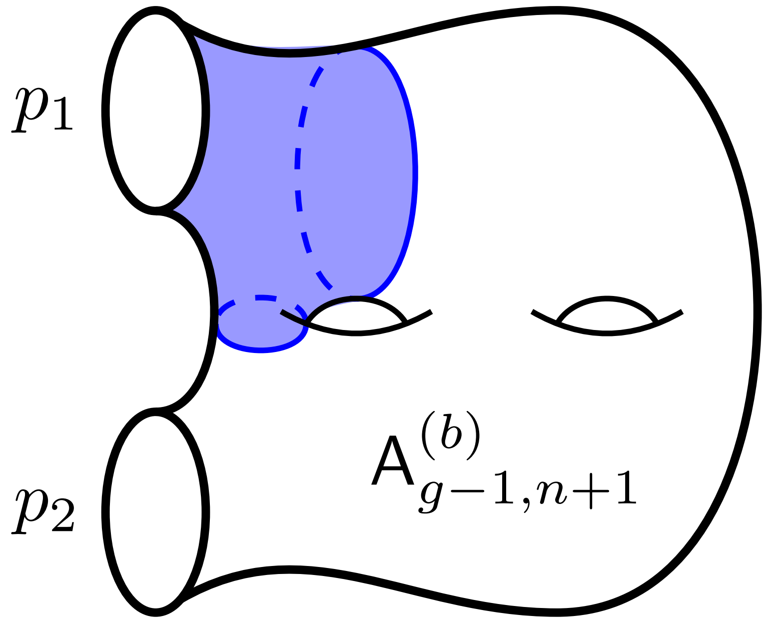
$$x(z) = -\,2\cos(\pi b^{-1}\sqrt{z}), \quad y(z) = 2\cos(\pi b\sqrt{z})$$

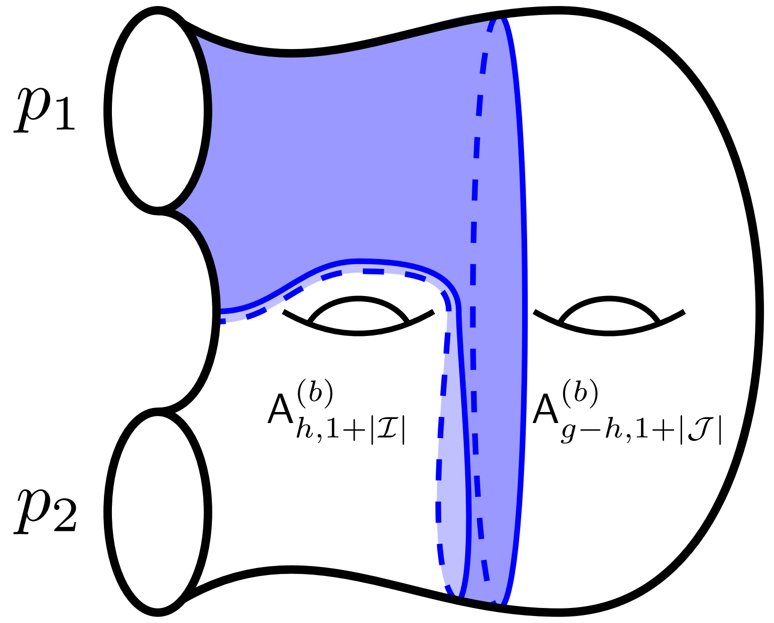
Remarkably similar to the ordinary (p,q) minimal string!
cf. e.g. [Seiberg Shih 04]

  - Leading density of eigenvalues: $\rho_0(x) = \frac{2}{\pi}\sinh(-\pi i b^2)\,\sin\left(-\,ib^2\,\mathrm{arccosh}\left(\frac{x}{2}\right)\right), \quad x \geq 2$

Topological recursion for string amplitudes

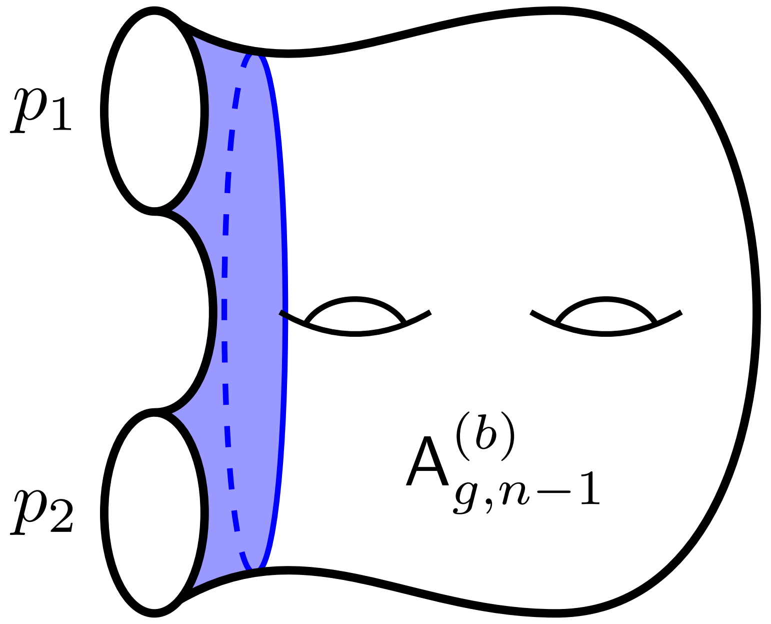
- The loop equations for the matrix integral [Chekhov Eynard Orantin 06] translate into a **recursion relation** for the string amplitudes (cf. [Mirzakhani 06; Eynard Orantin 07])





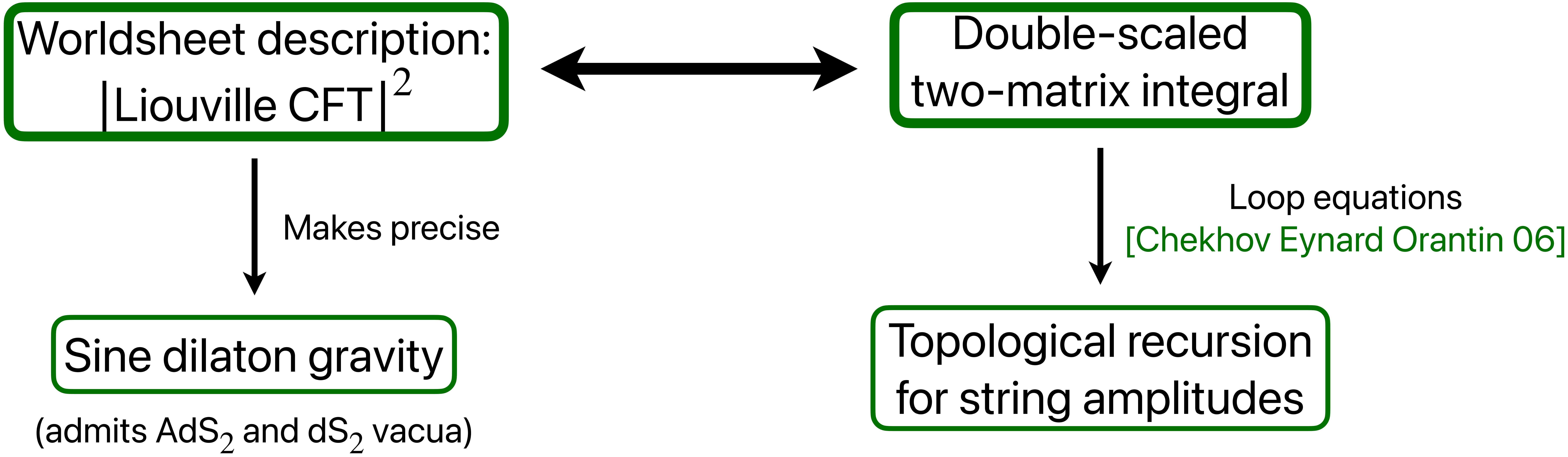
$$p_1 \mathsf{A}^{(b)}_{g,n}(p_1,\mathbf{p}) = \int 2q dq\, 2q' dq'\, H_b(q+q',p_1)\mathsf{A}^{(b)}_{0,3}(p_1,q,q')\Bigg(\mathsf{A}^{(b)}_{g-1,n+1}(q,q',\mathbf{p}) + \sum_{h=0}^{g} \sum_{I\sqcup J=\{2,\dots,n\}} \mathsf{A}^{(b)}_{h,1+|I|}(q,\mathbf{p}_I)\mathsf{A}^{(b)}_{g-h,1+|J|}(q',\mathbf{p}_J)\Bigg)$$

$$-\sum_{j=2}^{n} \int 2q dq\, \Big(H_b(q,p_1+p_j) + H_b(q,p_1-p_j)\Big)\, \mathsf{A}^{(b)}_{0,3}(p_1,p_j,q)\mathsf{A}^{(b)}_{g,n-1}(q,\mathbf{p}\backslash p_j)$$



$$H_b(x,y) = \frac{y}{2} - \frac{1}{2}\int_\Gamma du \frac{\sin(4\pi u x)\sin(4\pi u y)}{\sin(2\pi b u)\sin(2\pi b^{-1} u)}$$

⟵ Essentially identical to recursion kernel for quantum volumes $\mathsf{V}^{(b)}_{g,n}$ in VMS!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Double-scaled
two-matrix integral
Makes precise
Loop equations
[Chekhov Eynard Orantin 06]
Sine dilaton gravity
(admits AdS_2 and dS_2 vacua)
Topological recursion
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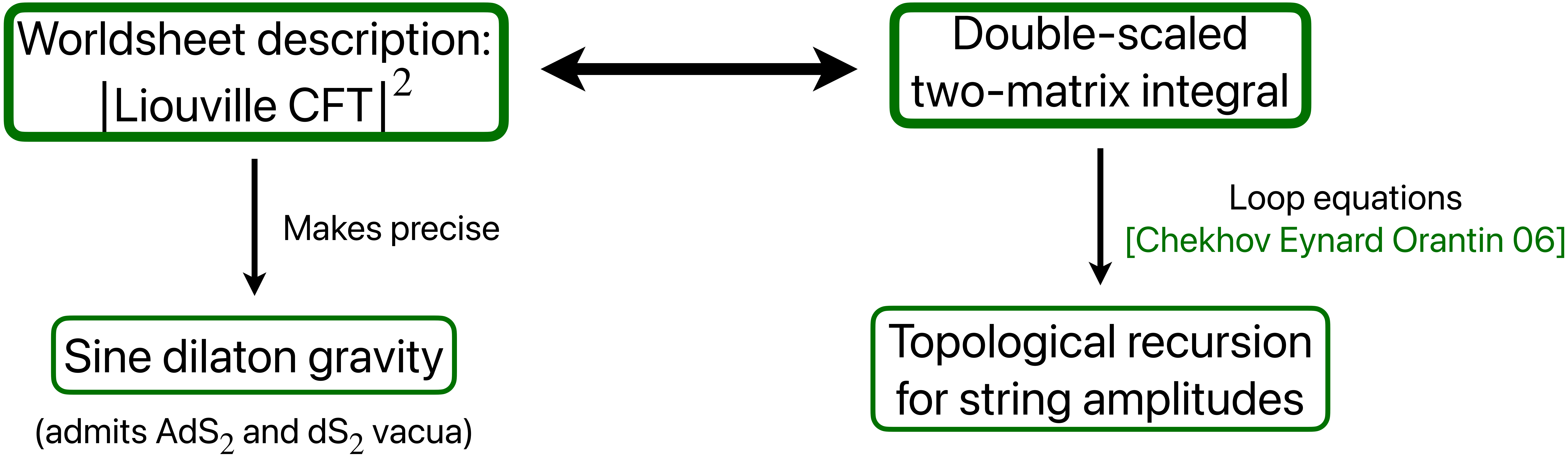
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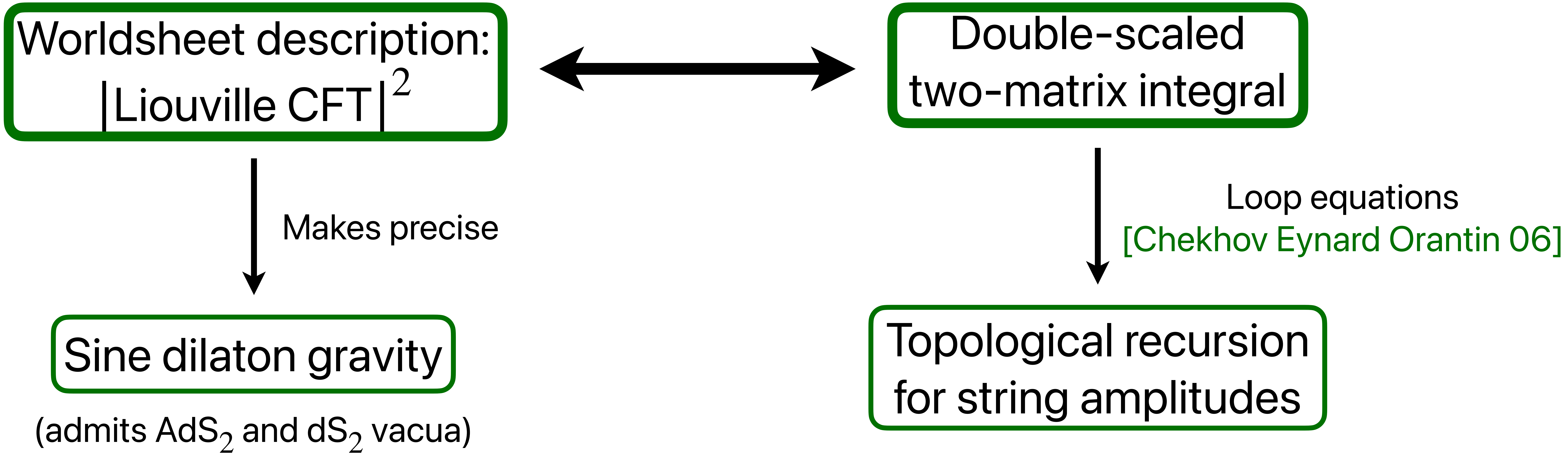
Thank you!



- Some questions:
  - How to connect these discussions to the $c = 1$ matrix model? (**Matrix quantum mechanics** vs. matrix integrals) [...; Moore Plesser Ramgoolam; ...; Balthazar Rodriguez Yin; Sen; ...]
  - Is there a 3d description? ($SL(2,\mathbb{C})$ structure on worldsheet) Relation to [Narovlansky Verlinde Zhang 23, 24]?

Thank you!

Bonus slides

Liouville theory and sine dilaton gravity

- Lagrangian formulation of the worldsheet theory:

$$S_L^+[\phi] = \frac{1}{4\pi}\int d^2x\sqrt{\tilde{g}}\left(\tilde{g}^{ij}\partial_i\phi\partial_j\phi + (b_+ + b_+^{-1})\tilde{R}\phi + \mu e^{2b_+\phi}\right)$$

$b_+ \in e^{\frac{\pi i}{4}}\mathbb{R}$

$\mu \in i\mathbb{R}$

$$S_L^-[\bar{\phi}] = \frac{1}{4\pi}\int d^2x\sqrt{\tilde{g}}\left(\tilde{g}^{ij}\partial_i\bar{\phi}\partial_j\bar{\phi} + (b_- + b_-^{-1})\tilde{R}\bar{\phi} + \bar{\mu}e^{2b_-\bar{\phi}}\right)$$

$b_- = -ib_+$

$\in e^{-\frac{\pi i}{4}}\mathbb{R}$

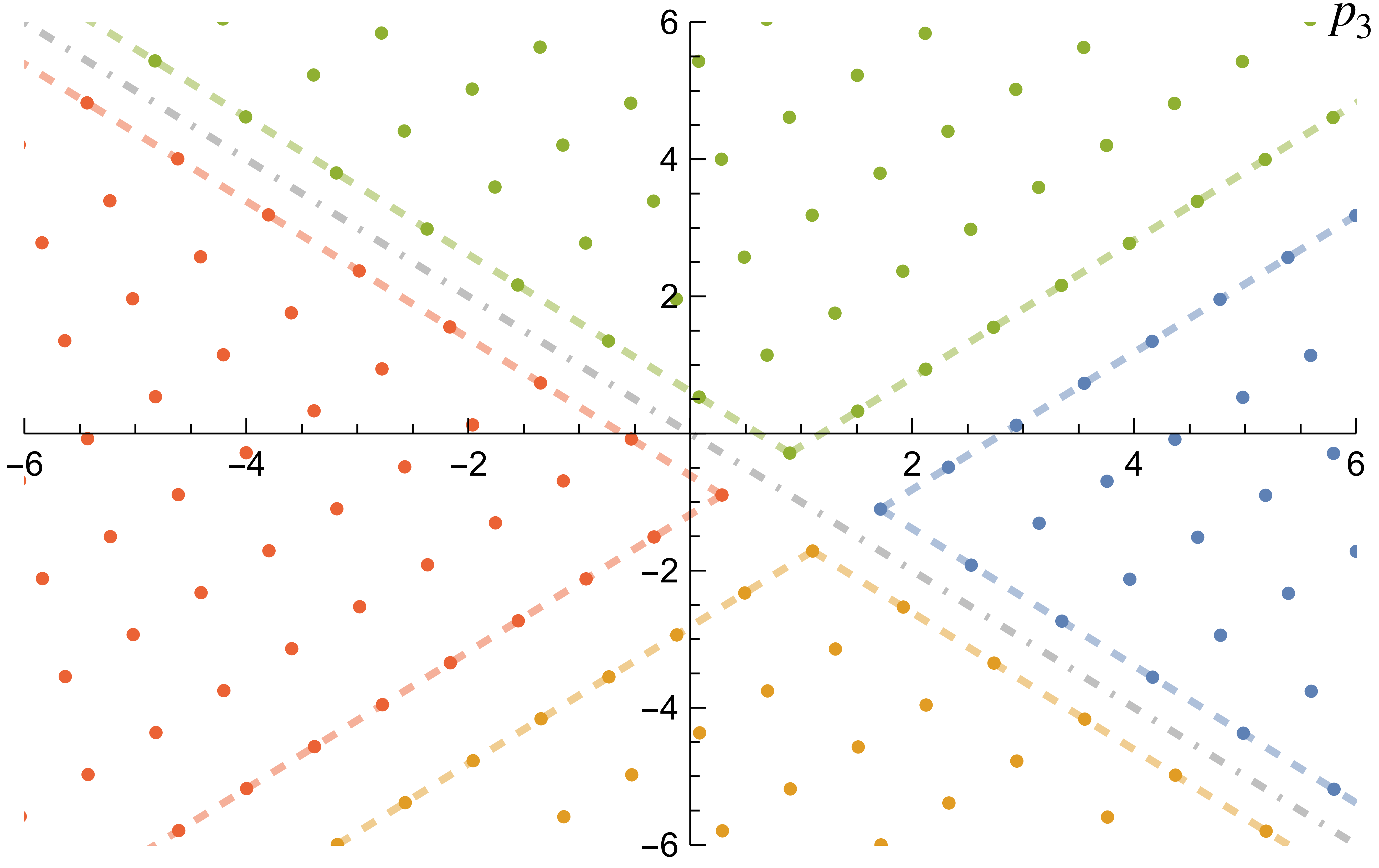
- Field redefinition:

$$\phi = b_+^{-1}\rho + \pi b_+\Phi\,, \qquad \bar{\phi} = b_-^{-1}\rho + \pi b_-\Phi\,, \qquad g = e^{2\rho}\tilde{g}$$

Sphere three-point amplitude

- $\mathsf{A}^{(b)}_{0,3}(p_1, p_2, p_3)$ has simple poles for

$$p_1 \pm p_2 \pm p_3 = \left(r + \frac{1}{2}\right)b + \left(s + \frac{1}{2}\right)b^{-1},\ r, s \in \mathbb{Z}$$



$b \in e^{\frac{\pi i}{4}}\mathbb{R}$

$p_1, p_2 \in e^{-\frac{\pi i}{4}}\mathbb{R}$

Sphere three-point amplitude

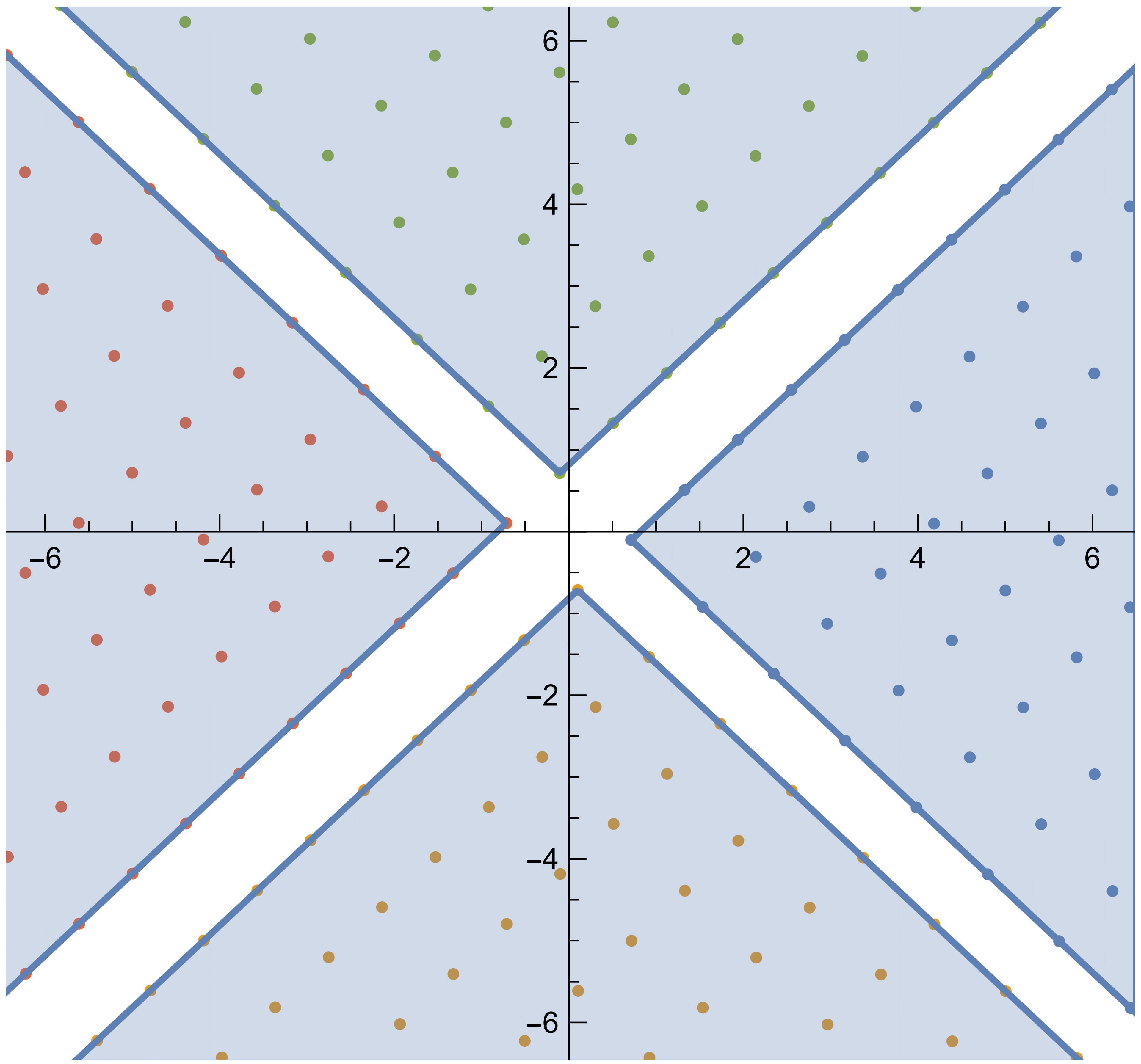
$$\mathsf{A}_{0,3}^{(b)}(p_1, p_2, p_3) = \frac{ib\,\eta(b^2)^3\,\prod_{j=1}^{3}\vartheta_1(2bp_j\,|\,b^2)}{2\vartheta_3(bp_1 \pm bp_2 \pm bp_3\,|\,b^2)}$$

[Zamolodchikov 05]

- With a suitable identification of the parameters, this combination has recently appeared as the boundary two-point function in double-scaled SYK (cf. [Narovlansky, Verlinde, Zhang 23, 24])

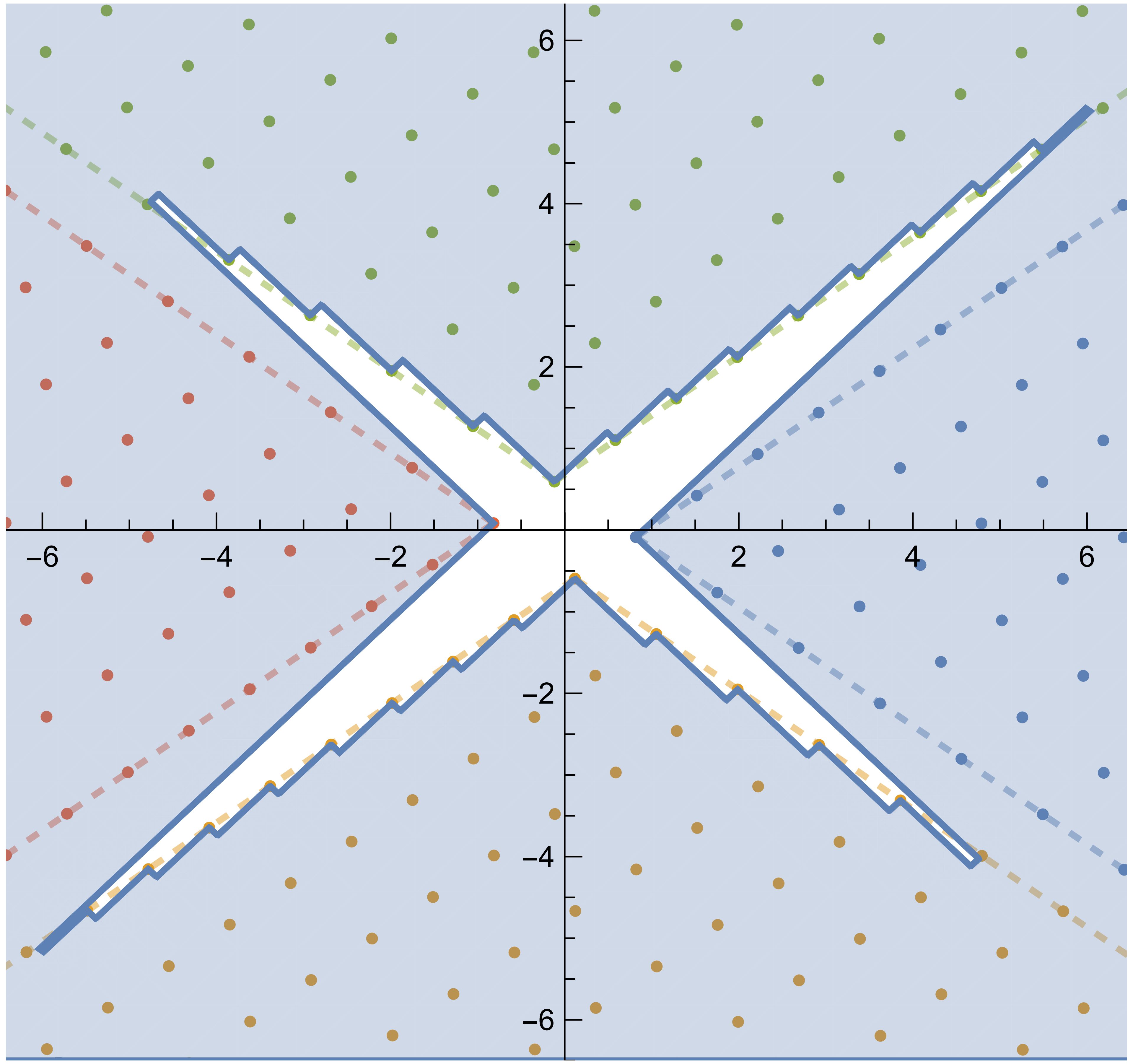
Sphere four-point amplitude: domain of analyticity

- The wedges of divergence associated with the branch cuts carve out a domain of analyticity in the $p_i \pm p_j$ plane that is non-compact if $b \in e^{\frac{\pi i}{4}}\mathbb{R}$, and compact otherwise



$b \in e^{\frac{\pi i}{4}}\mathbb{R}$

$p_i \pm p_j$



$b \in e^{\frac{\pi i}{5}}\mathbb{R}$

Sphere four-point amplitude: solution

- A solution is given by:

$$\mathsf{A}^{(b)}_{0,4}(p_1,p_2,p_3,p_4) = -\sum_{m_1,m_2=1}^{\infty} \frac{(-1)^{m_1+m_2}\sin(2\pi m_1 b p_1)\sin(2\pi m_1 b p_2)\sin(2\pi m_2 b p_3)\sin(2\pi m_2 b p_4)}{\pi^2 \sin(\pi m_1 b^2)\sin(\pi m_2 b^2)}$$

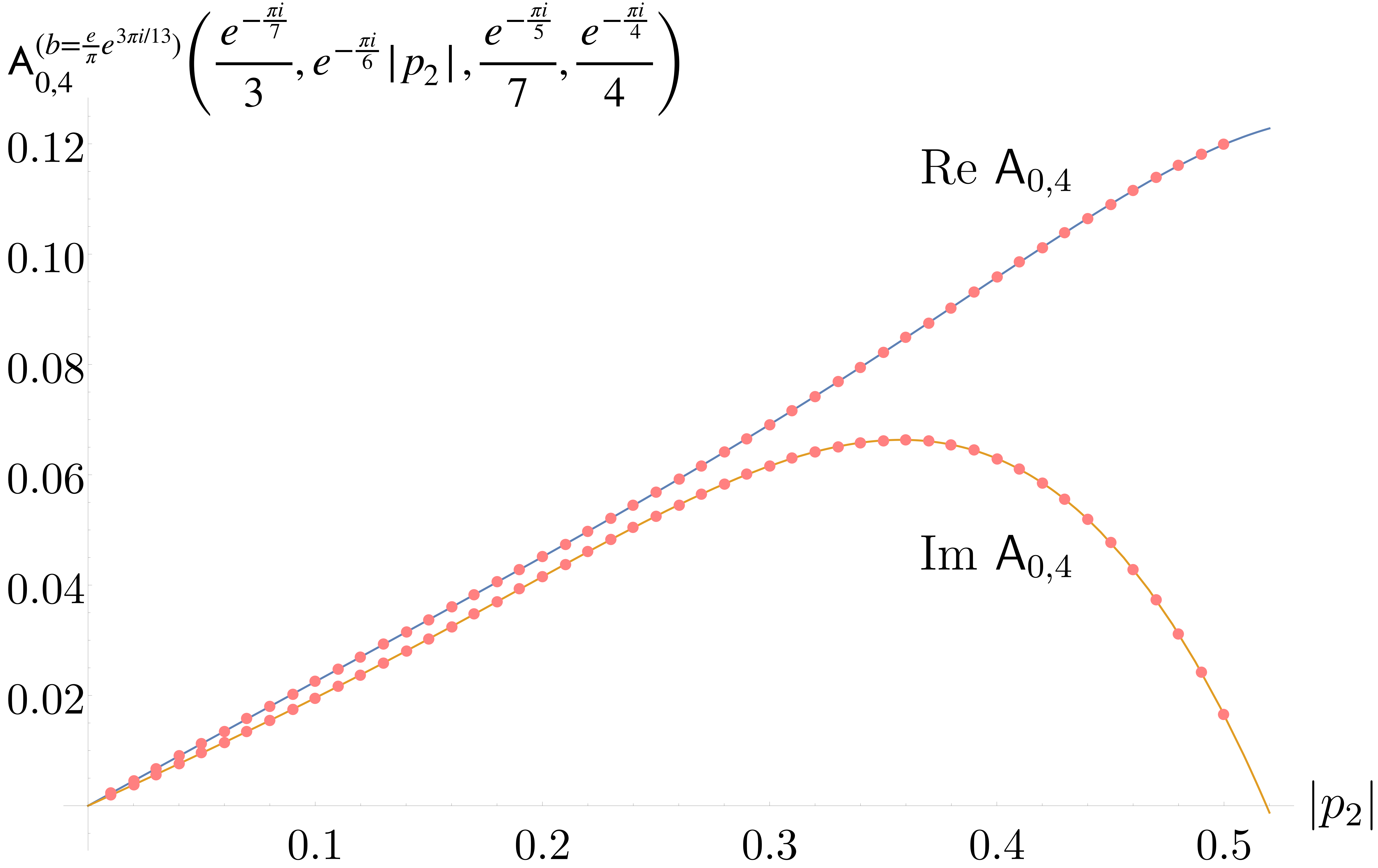
$$\times\left(\frac{1}{(m_1+m_2)^2} - \frac{1-\delta_{m_1,m_2}}{(m_1-m_2)^2}\right) + \text{2 perms}$$

$$+\sum_{m_1=1}^{\infty} \frac{2b^2\left(\prod_{j=1}^{4}\sin(2\pi m_1 p_j)\right)\mathsf{V}^{(b)}_{0,4}(p_1,p_2,p_3,p_4)}{\sin(\pi m_1 b^2)^2}$$

Discontinuities $\sim \int' 2p\,dp\, \mathsf{A}^{(b)}_{0,3}(p_1,p_2,p)\mathsf{A}^{(b)}_{0,3}(p,p_3,p_4)$

Sphere four-point amplitude: numerical verification

- Direct numerical integration over moduli space for other values of the parameters

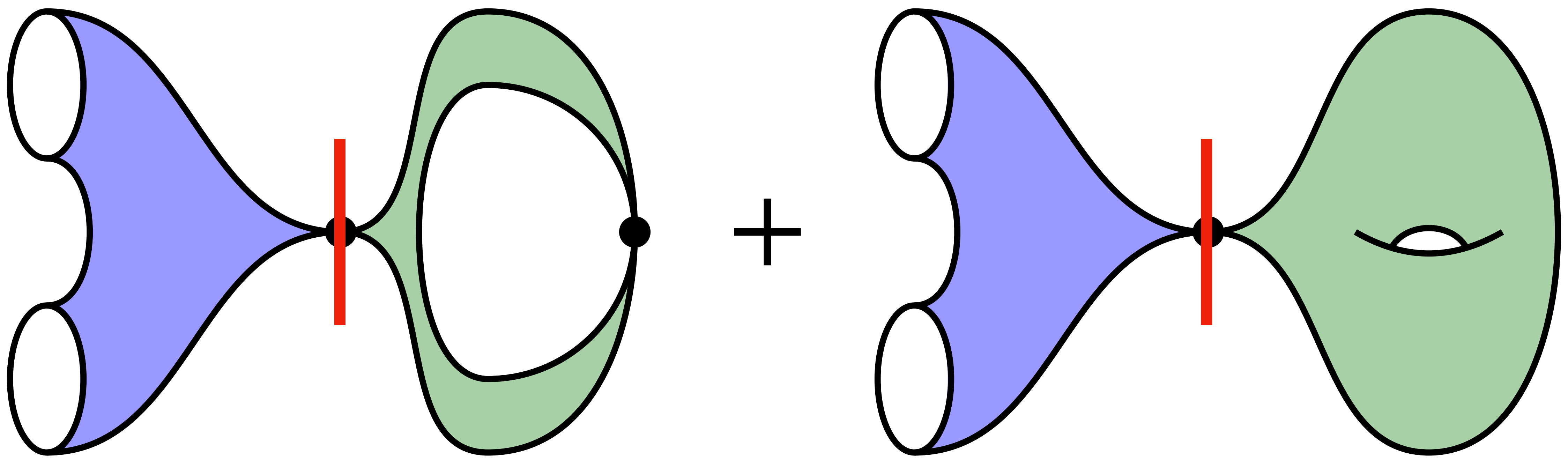


Error $< 10^{-5}$ %

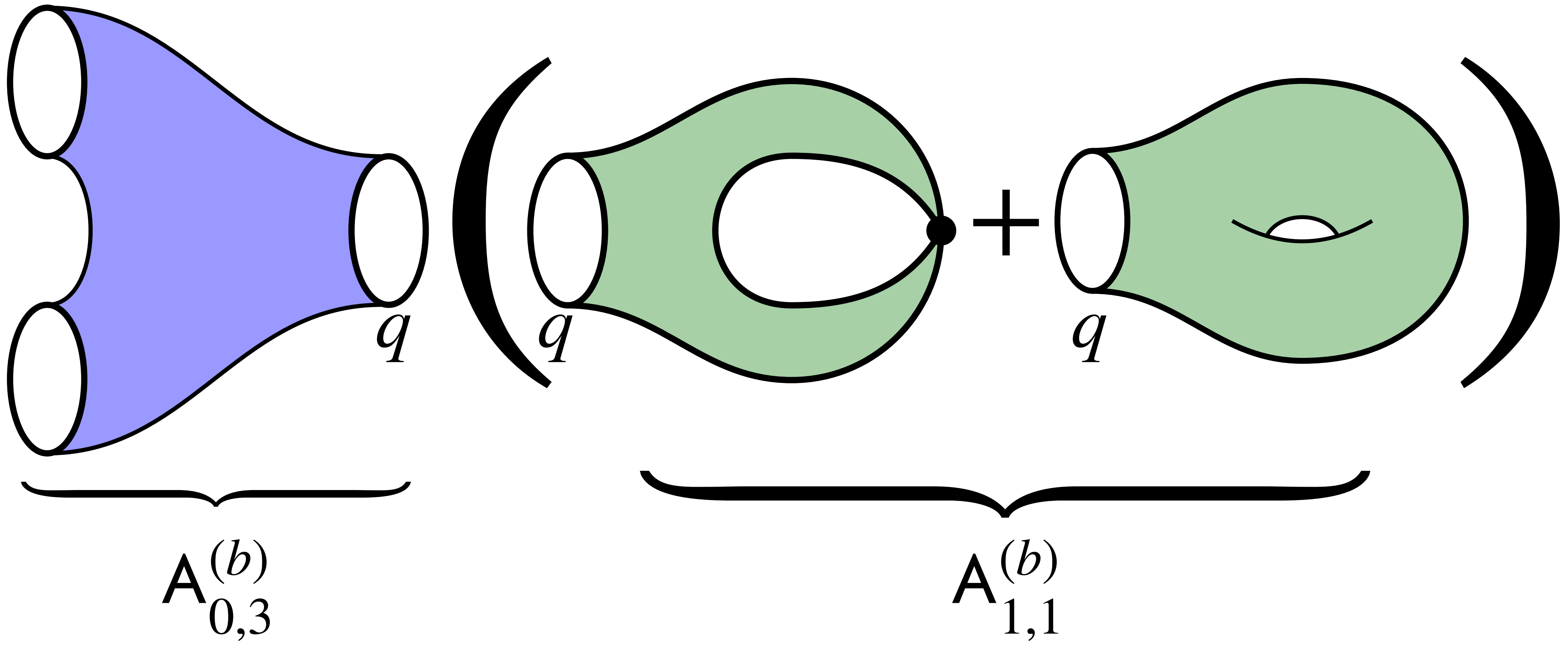
Discontinuities of string amplitudes from cutting Feynman diagrams

$$\underset{q_*=0}{\mathrm{Disc}}\, \mathsf{A}^{(b)}_{g,n}(\mathbf{p}) = 2\pi i q_* \left(\underset{q=\frac{1}{2}q_*}{\mathrm{Res}}\ \mathsf{A}_{g-1,n+2}(q,q,\mathbf{p}) + 2\sum_{h=0}^{g}\ \sum_{I\sqcup J=\{p_1,\ldots,p_n\}} \underset{q=q_*}{\mathrm{Res}}\, \mathsf{A}^{(b)}_{h,1+|I|}(q,\mathbf{p}_I)\mathsf{A}^{(b)}_{g-h,1+|J|}(q,\mathbf{p}_J) \right)$$

e.g.



$$= \quad 4\pi i q_* \underset{q=q_*}{\mathrm{Res}}$$



Topological recursion and string amplitudes

- Initial data for topological recursion:

$$\omega_{0,1}^{(b)}(z) = -\frac{2\pi \sin(\pi b^{-1}\sqrt{z})\cos(\pi b\sqrt{z})}{b\sqrt{z}}dz \qquad \omega_{0,2}^{(b)}(z_1, z_2) = \frac{dz_1\, dz_2}{(z_1 - z_2)^2}$$

- String amplitudes are simply related to the $\omega_{g,n}^{(b)}(z_1, \ldots, z_n)$ differentials by inverse Laplace transform

$$\mathsf{A}_{g,n}^{(b)}(p_1, \ldots, p_n) = \int_\gamma \left(\prod_{j=1}^{n} \frac{1}{4\pi i}\frac{e^{2\pi i p_j w_j}}{p_j}\right)\omega_{g,n}^{(b)}(w_1, \ldots, w_n) \qquad \left(w_j = \sqrt{z_j}\right)$$

$$= \sum_{m_1,\ldots,m_n=1}^{\infty} \operatorname*{Res}_{z_1=m_1^2 b^2} \cdots \operatorname*{Res}_{z_n=m_n^2 b^2} \prod_{j=1}^{n} \frac{\cos(2\pi p_j\sqrt{z_j})}{p_j}\omega_{g,n}^{(b)}(z_1, \ldots, z_n)$$