QUANTUM ENTANGLEMENT IN STRING THEORY

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CERN

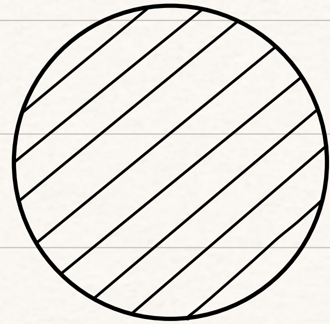
Entanglement Entropy is of fundamental importance in QM & QFT and even more so in Quantum Gravity

Finiteness of entanglement entropy is at the heart of the black Hole information paradox

Can we define a notion of entanglement entropy in string theory given its UV finiteness?

- Motivation
- Difficulties
- A method.

BLACK HOLE



$$\mathcal{H} \stackrel{?}{=} \mathcal{H}_I \otimes \mathcal{H}_O$$

Tracing over the interior naively gives a density matrix ρ_O

Its von Neumann entropy $S = -\mathrm{Tr}\,\rho_O \log \rho_O$ diverges

$\Rightarrow$ infinite q-bits for the black hole

We need a generalization of von Neumann entropy

Because usual notions from local QFT like the algebra of local observables are not available in string theory.

Quantum Entanglement in QM

Classic example is the Bell pair w/ "spooky" long distance EPR quantum correlations

Two photons in spin 0 state.

$$\xleftarrow{\quad\mathcal{H}_L\quad} \qquad\qquad \xrightarrow{\quad\mathcal{H}_R\quad}$$

$$|\Omega\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

Maximally entangled pure state in the Bipartite Hilbert space $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$

very different from unentangled pure state like $|U\rangle = |\uparrow\uparrow\rangle$

Entanglement Entropy

State $|\psi\rangle$ $\qquad\langle\psi|\psi\rangle = 1$

Density matrix $\rho = |\psi\rangle\langle\psi|$ $\qquad \mathrm{Tr}\,\rho = 1$

Reduced density matrix $\rho_R = \mathrm{Tr}_{\mathcal{H}_L}\,\rho$

$\mathrm{Tr}_{\mathcal{H}_R}\,\rho_R = 1$: sufficient to study

correlators like $\langle\psi|\mathcal{O}_R|\psi\rangle = \mathrm{Tr}_{\mathcal{H}_R}\,\mathcal{O}_R$

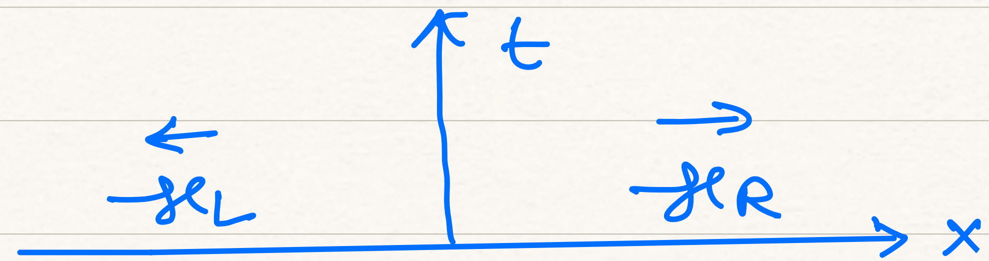
Fine-grained Von Neumann entropy

$S_{EE} = -\mathrm{Tr}_{\mathcal{H}_R}\,\rho_R \log \rho_R$ = Entanglement Entropy

For $|\Omega\rangle$, $\rho_R = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ $\qquad S_{EE} = \log 2$

Entanglement Entropy in QFT

$(d+1)$ spacetime, $\vec{y}$ = transverse



$t = 0$
Spatial slice

Divide space $x_L = x < 0$ $\quad \mathcal{H}_L$

$x_R = x > 0$ $\quad \mathcal{H}_R$

$\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$ (Not really true)

$|\Omega\rangle$ = Minkowski vacuum $\quad \rho = |\Omega\rangle\langle\Omega|$

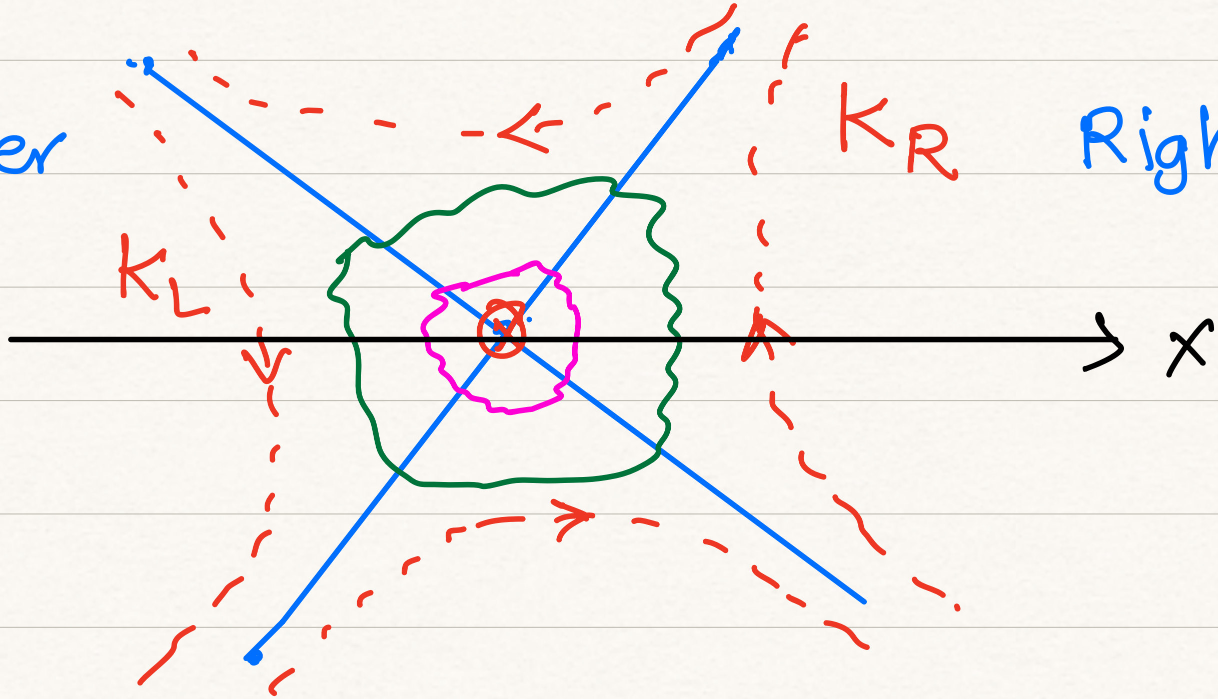
$$\rho_R = \mathrm{Tr}_{\mathcal{H}_L} \rho$$

$$S_{EE} = -\mathrm{Tr}\, \rho_R \log \rho_R = \frac{A(y)}{\varepsilon^{d-1}}$$

Area law with UV divergence $\quad \varepsilon$ = cutoff

Rindler spacetime

Left Rindler



Right Rindler

K_R = Right Lorentz boost $\rightarrow J$

$$ds^2 = -dt^2 + dx^2 = -\rho^2 d\tau^2 + d\rho^2$$

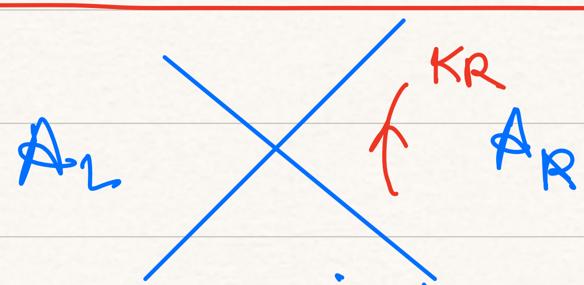
$$T(x) = \frac{1}{2\pi x} \qquad S = \int_\epsilon^\infty dx\, d\vec{y}\, T^d = \frac{A}{\epsilon^{d-1}}$$

$$\rho_R = \exp[-2\pi K_R]$$

Algebraic QFT

Its not quite correct to assume $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$

Because of strong correlations at the boundary, Hilbert space not really factorized, but the algebra of local observables is factorized.



$[A_R, A_L] = 0$

von Neumann algebra

Minkowski vacuum $|\Omega\rangle$ is a "cyclic separating" state

$K_R =$ Modular Hamiltonian in Tomita-Takesaki theory

UV divergence property of the algebra not of state

Type-III (QFT) does not admit irrep

Type-I (QM) admits irreps.

in QG?

Relative entropy $S(\rho|\sigma) = \mathrm{Tr}(\rho\log\rho - \rho\log\sigma)$ UV finite

(1) Generalized second law of thermodynamics

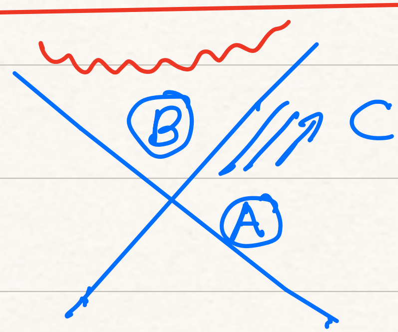
$$\Delta S_{gen} \equiv \Delta\left(\frac{A}{4G} + S_{out}\right) \geq 0 \qquad \text{Bekenstein}$$

Proof by Wall follows from monotonicty of relative entropy under inclusion.

Uses null Raichaudhari eqn to relate the change in "energy" to change in Area UV divergent.

$$\Delta S_{rel}(\rho|\sigma) = \Delta H_\sigma - \frac{\Delta S_\sigma}{2\pi} \leq 0$$

(2) Strong Subadditivity Paradox



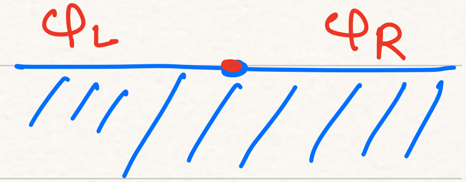
Mathur; (Almheiri, Marolf Polchinki, Sully)

Desirable to have a notion of finite entanglement entropy in many contexts in Quantum Gravity.

Path Integral

Wave functional of $|\Omega\rangle$ in field basis

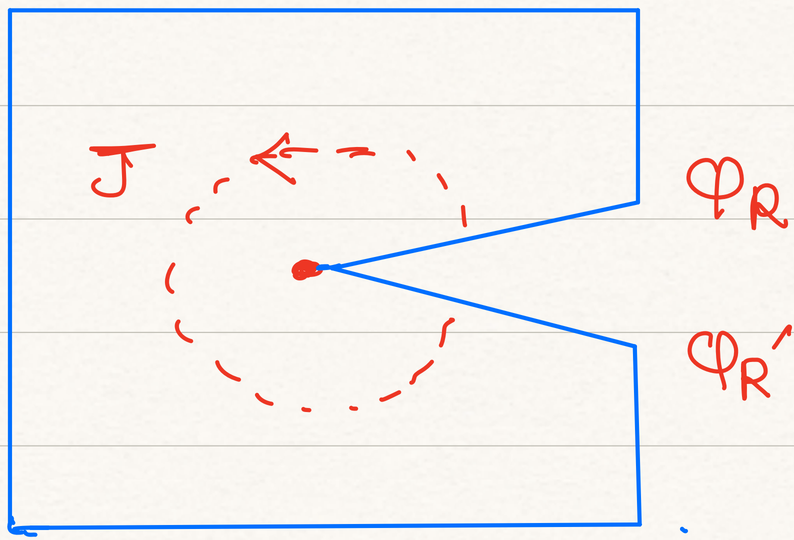
$$\langle \varphi_L, \varphi_R | \Omega \rangle = \Psi_\Omega(\varphi_L, \varphi_R) =$$



Reduced density matrix in field basis

$$\langle \varphi_R' | \rho_R | \varphi_R \rangle = \int D\phi_L \langle \varphi_L \; \varphi_R' | \rho | \varphi_L \; \varphi_R \rangle$$

Represented by a path integral on a cut plane



Euclidean Rindler plane

Rotation generator J

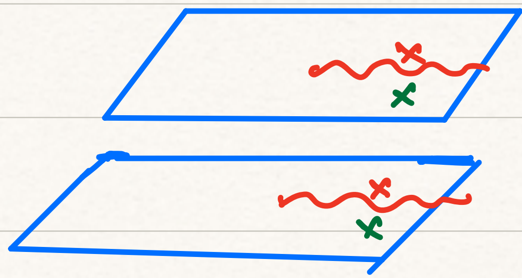
Minkowski vacuum is an ensemble of Bell pairs

Rényi Entropy

Given the density matrix ρ_R, compute

$$\hat{Z}(n) = \mathrm{Tr}\, \rho_R^n \qquad n \text{ integer}$$

path integral over n-sheeted cover (Replica)



$\theta = 2\pi n$

surplus opening angle

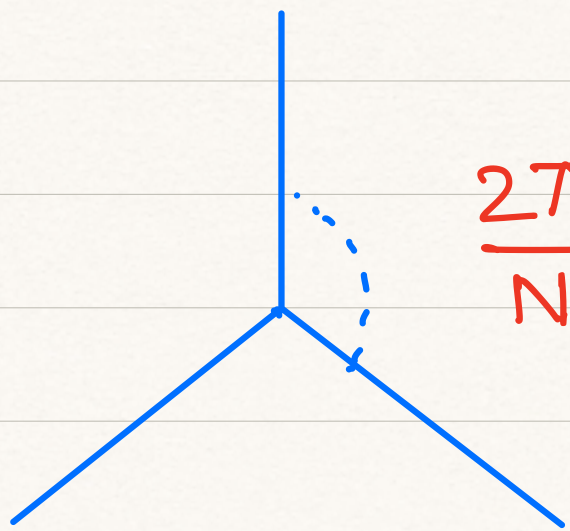
$$S_{EE} = -\frac{d}{dn}\hat{Z}(n)\Big|_{n=1} = -\frac{d}{dn}\mathrm{Tr}\left(e^{n\log\rho_R}\right)\Big|_{n=1}$$

Needs analytic continuation in n

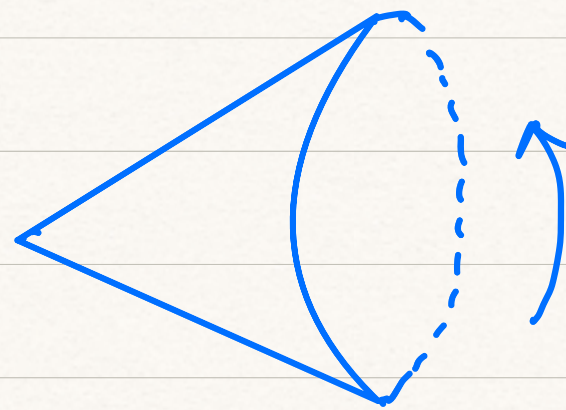
General formula: Much has been learnt in QFT

Orbifold Method

Instead of an n-fold cover of Euclidean Rindler plane with surplus opening angle consider an orbifold w/ deficit opening angle



$\frac{2\pi}{N}$



$\frac{2\pi}{N}$

Compute $\mathrm{Tr}\, \rho_R^N$ as the partion function of $\mathbb{C}/\mathbb{Z}_N$ orbifold of the Rindler plane

$N = \frac{1}{n}$

Strings on a Cone

$$\hat{Z}(N) = \mathrm{Tr}\left[\exp\left(-\frac{2\pi}{N} H_R\right)\right]$$

$$S(N) = -\beta \frac{\partial \log \hat{Z}}{\partial \beta} + \log \hat{Z} \qquad \beta = \frac{2\pi}{N}$$

Spacetime partition function

String Field Theory path integral on a cone

conical curvature singularity at the tip

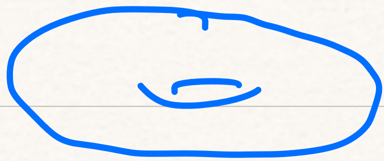
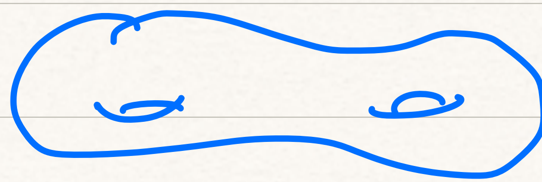
Ricci scalar $R(\vec{x}) = 4\pi\left(1-\frac{1}{N}\right)\delta^{(2)}(\vec{x})$

Deficit angle $\delta_N = 2\pi\left(1-\frac{1}{N}\right)$

$$S = S^{(0)} + S_q$$

$$\boxed{\log \hat{Z}_q(N) = Z_q(N)}$$

$$Z_q(N) = \sum_{g=1}^{N} Z^{(g)}(N)$$

 $+$  $+ \cdots$

$S^{(0)}$ = Classical Entanglement.

At classical level $\log \hat{Z}^{(0)} \neq Z^{(0)}(N)$

$$S = \frac{d}{dN}\left(N \log \hat{Z}(N)\right)\Big|_{N=1}$$

$\mathbb{C}/\mathbb{Z}_N$ Orbifold

Type-II in lightcone gauge on $\mathbb{R}^6 \times \mathbb{C}$

Green-Schwarz superstring $(X^i, S^a, \tilde{S}^{\dot{a}})$

$$Spin(8) \supset Spin(6) \times Spin(2)$$

$$X^i \qquad 8_v = 6(0) + \underbrace{1(1)} + \underbrace{1(-1)}$$

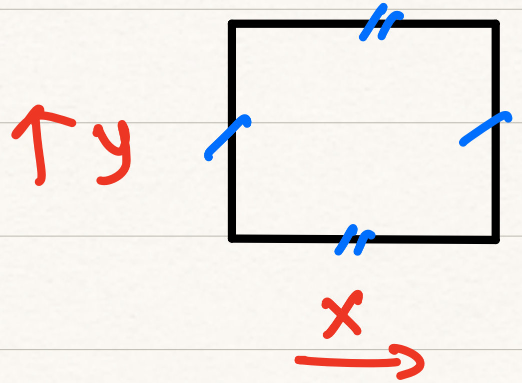
$$S^a \qquad 8_s = \underbrace{4\left(\tfrac{1}{2}\right)} + \underbrace{\bar{4}\left(-\tfrac{1}{2}\right)}$$

$$\tilde{S}^a \qquad 8_c = \underbrace{4\left(-\tfrac{1}{2}\right)} + \underbrace{\bar{4}\left(\tfrac{1}{2}\right)}$$

$$\mathbb{Z}_N = \{1, g, g^2 \ldots g^{N-1}\} \subset U(1) \simeq Spin(2)$$

$$\boxed{g = \exp\left[\frac{4\pi i}{N} J\right]} \qquad \underline{N \text{ odd}}$$

One-loop Partition function



$$\phi(x+1, y) = g^{k}\, \phi(x, y)$$

$$\phi(x, y+1) = g^{\ell}\, \phi(x, y)$$

$$\mathcal{Z}(\tau, N) = \sum_{k,\ell=0}^{N-1} \left| \frac{\vartheta^4\left(\frac{k\tau+\ell}{N} \middle| \tau\right)}{\eta^4(\tau)\, \vartheta\left(\frac{2k\tau+2\ell}{N} \middle| \tau\right)} \right|^2$$

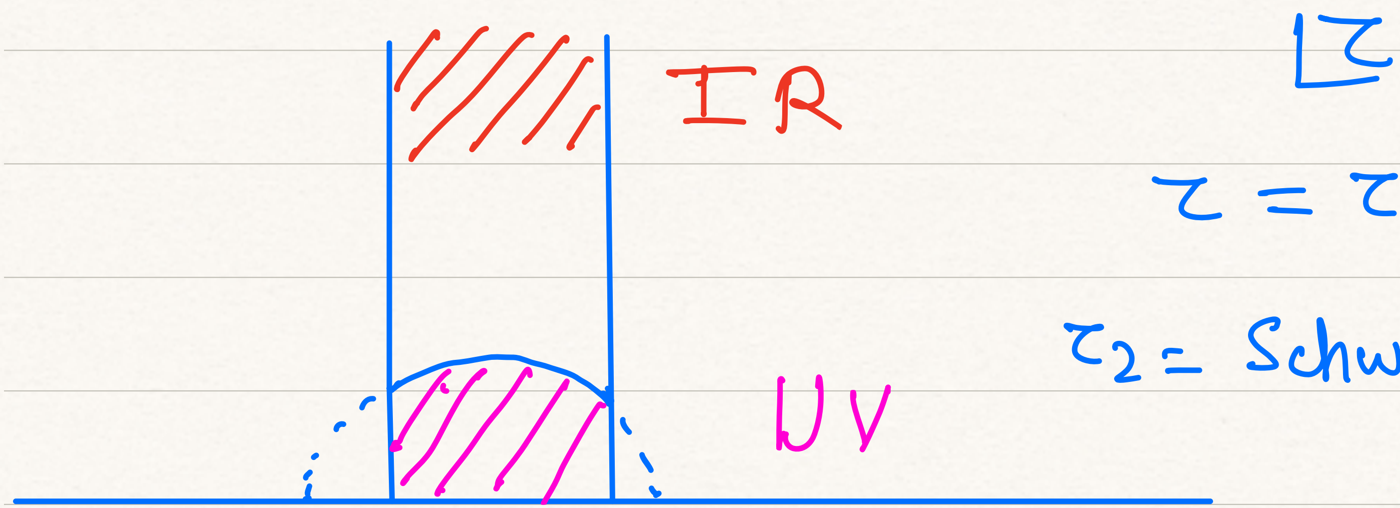
$$Z^{(1)}(N) = \frac{A_H}{N} \int \frac{d^2\tau}{\tau_2^5}\, \mathcal{Z}(\tau, N)$$

$$A_H = \text{Horizon Area} = \frac{V_8}{(2\pi \ell_s^2)^8}$$

$\vartheta(z|\tau)$: Jacobi theta $\underline{fn}$; $\tau = \tau_1 + i\tau_2$

UV and IR divergences

String theory provides a natural UV cutoff.
The $\mathbb{Z}_N$ orbifolds break supersymmetry.
The spectrum is replete with tachyons.
all localized at the tip of the cone



$\tau = \tau_1 + i\tau_2$

τ_2 = Schwinger parameter

$\exp\left[-2\pi m^2 \tau_2\right]$ diverges for $m^2 < 0$

Tachyons need not be a cause for despair

In QM, with $\mathrm{Tr}[\rho_R] = 1$

$\mathrm{Tr}[\rho_R^n]$ must converge.

$\mathrm{Tr}[\rho_R^{1/n}]$ may diverge.

We are computing $\mathrm{Tr}[\rho_R^N]$ with $N = \frac{1}{n}$

We need analytic continuation to the physical region $0 < N \leq 1$ given the data for $N > 1$ with N odd.

Under certain conditions, Carlson Theorem may guarantee uniqueness

$f(z) = \sum \frac{z^n f^{(n)}(0)}{n!}$	$f(N)$
countable data at $z=0$ gives analytic continuation	countable data at N odd gives analytic continuation
CAUCHY	CARLSON

For open strings on the Rindler horizon such an analytic continuation can be found

Witten 2018

Closed strings on Rindler horizon

Analytic continuation of the closed string partition fn is much harder.

Fortunately, the tachyonic terms have a very specific form and do admit an analytic continuation to a function that is finite for $0 < N \leq 1$

$$Z(N) = \int \frac{d^2\tau}{\tau_2^2} \mathcal{F}(\tau)$$

$$\mathcal{F}(\tau) = \sum_n \exp[2\pi i n \tau_1] \mathcal{F}_n(\tau_2)$$

$$Z(N) = \int \frac{d\tau_2}{\tau_2^2} \mathcal{F}_0(\tau_2)$$

Tachyonic Terms $T(\tau_2)$

Exponentially growing terms in $\mathcal{F}_0(\tau_2)$

$$T(\tau_2) = \frac{2}{\tau_2^3} \sum_{k=1}^{(N-1)/2} \exp\left[\frac{4\pi\tau_2 k}{N}\right] f_k(\tau_2)$$

$$f_k(\tau_2) = \sum_{n=0}^{\infty} \exp\left[\frac{-2\pi n \tau_2 (2N-4k)}{N}\right]$$

Complicated sum but can be resummed by first performing the k-sum.

Disappearing Tachyons

$$T(\tau_2, N) = \frac{-2}{\tau_2^3} \sum_{n=0}^{\infty} e^{-4\pi n \tau_2} \cdot \frac{1 - e^{\alpha_n (N-1)\tau_2}}{1 - e^{-2\alpha_n \tau_2}}$$

$$\alpha_n = \frac{\pi(4n+2)}{N} > 0 \quad ; \quad n \geq 0 \quad ; \quad N > 0$$

Remarkably, $T(\tau_2, N)$ has no "tachyonic" exponentially growing terms for $0 < N \leq 1$

We can write $\mathcal{F}_0(\tau, N) = T(\tau_2, N) + \mathcal{R}(\tau_2, N)$

$\mathcal{R}(\tau_2, N)$ has no divergences and can be easily integrated numerically & extrapolated

The analytic continuation of tachyons is not accidental. It depends on three 'just so' properties of string orbifolds

(1) There are exactly $(N-1)$ twisted sectors, each containing tachyons

(2) In the k-twisted sector there is precisely one leading tachyon whose mass-squared is linear in k

$$M^2 = -4\pi k/N$$

(3) There are many subleading tachyons but they all have unit multiplicity

Classical Entanglement Entropy

Classically $\quad \log \hat{Z}(N) \neq Z(N) \quad$ — 1

spacetime $\qquad\qquad$ worldsheet

nonzero $\qquad\qquad$ zero

Relation 1 true only for bulk action

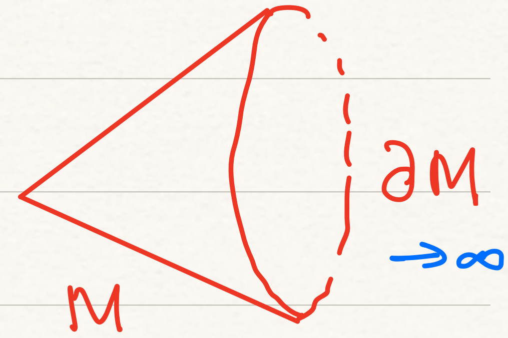
In general there are boundary terms that are not captured by sphere partition

We'll deduce them by spacetime reasoning

$$\log \hat{Z}^{(0)}(N) = I_M(N) + I_{\partial M}(N)$$

$$I_M = \frac{1}{16\pi G}\int_M e^{-2\phi}\left[R + 4(\nabla\phi)^2 - \delta^{2}(x)\, V(T) + \cdots\right]$$

+ Gibbons-Hawking boundary term depending on the extrinsic curvature.



Dilaton equations of motion are exactly satisfied to all orders for CFT background.

$\Rightarrow$ No dilaton tadpoles, $I_M = 0$

But there is a boundary contribution

$$\Rightarrow \quad \log\left(\hat{Z}^{(0)}(N)\right) = \frac{A}{4G\hbar}\left(\frac{1}{N} - 1\right)$$

Analytic continuation in N, $S^{(0)} = \frac{A}{4G\hbar}$

Calabi-Yau Compactifications

Our ability to obtain an analytic continuation that is finite in the physical region $0 < N \leq 1$ depended on a very specific structure of the tachyonic spectrum

May be this is a special feature of 10d superstring that is not more generally valid with less supersymmetry.

For example, in CY orbifolds there can be new tachyons in doubly-twisted sectors with quite different structure

$T^4/\mathbb{Z}_3$ K3 orbifold

Sectors twisted by $r = 1, 2$ & $k = 1, \cdots N-1$

In addition to 10 d tachyons, there are new tachyons with mass-squared:

$$M^2 = \begin{cases} 0 & 0 \leq \frac{k}{N} \leq \frac{1}{3} \\ 4\pi\left(\frac{1}{3} - \frac{k}{N}\right) & \frac{1}{3} < \frac{k}{N} < \frac{1}{2} \\ -4\pi\left(\frac{2}{3} - \frac{k}{N}\right) & \frac{1}{2} < \frac{k}{N} < \frac{2}{3} \\ 0 & \frac{2}{3} \leq \frac{k}{N} < 1 \end{cases}$$

Not a priori obvious how these tachyons contribute in the physical region.

Fortunately, we again find $T(\tau_2, N)$

$$= \sum_{n=0}^{\infty} \frac{1 - e^{2\pi\tau_2 (1+2n)(N-1)}}{1 - e^{-\frac{2\pi\tau_2(1+2n)}{N}}} \cdot e^{\frac{-4\pi\tau_2(3n+1)}{3}}$$

Again, finite for $0 < N \leq 1$ even though exponentially divergent for $N > 1$ as $\tau_2 \to \infty$

This feature continues for several other compactifications we investigated with less supersymmetry CY_2, CY_3.

Heterotic Strings

Worldsheet is now chiral with potential for gravitational anomalies. Additional new features: Gauge symmetry must be broken to obtain modular invariant partition functions for $\mathbb{Z}_N$ orbifolds.

For example, $\dfrac{Spin(32)}{\mathbb{Z}_2} \longrightarrow U(16)$

Structure of tachyons is different. New divergences that disappear after imposing level matching (τ_1- integral)

Entanglement on BH Horizons & Holography.

$$AdS_3 \times S^3 \times T^4$$

NS5 − F1 System

Exact worldsheet theory in bulk

WZW $SL(2,\mathbb{C})$ gauged model level k

BTZ Black Hole Mass M, spin J

Solvable but nontrivial SCFT

$Z(N, M, J, k)$ can be <u>exactly</u> computed for the $\mathbb{Z}_N$ orbifolds of the horizon

Once again tachyonic divergent terms get tamed in the physical region $0 < N \leq 1$

Exact Analytic Continuation of $Z(N)$

work in progress with Don Zagier

We need to perform finite sums of the form

$$\sum_{k,\ell} f(k,\ell) \quad \text{depending on } e^{\frac{2\pi i \ell}{N}}$$

root of unity $\omega^N = 1$

where N must be an integer to something where N can be treated as a complex number

Lemma

$$\frac{1}{N}\sum_{\omega^N=1} \underbrace{\frac{A+\omega}{A-\omega}\cdot\frac{B+\omega^{-1}}{B-\omega^{-1}}} + 1$$

$$= \frac{1+AB}{1-AB}\left(\frac{1+A^N}{1-A^N}+\frac{1+B^N}{1-B^N}\right)$$

Three one-line proofs. For example, using partial fraction decomposition of

$$= \frac{1+AB}{1-AB}\left(\frac{1+B\omega}{1-B\omega} - \frac{A+\omega}{A-\omega}\right)$$

then average over Nth roots of unity.

LHS $N \in \mathbb{Z}$ RHS $N \in \mathbb{C}$

Open string partition function

To compute entanglement entropy on the Dp brane worldvolume

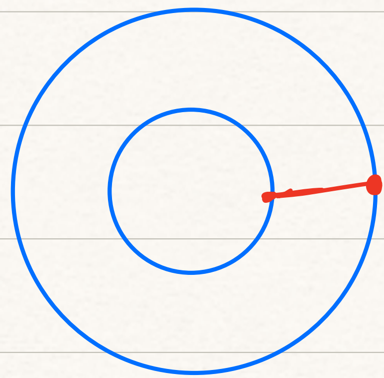
Using the lemma we obtain analytic continuation in agreement w/ Witten (18)

$$Z(N) = \int_0^\infty \frac{dT}{T^{\frac{p+1}{2}}} Z_W(N, iT)$$

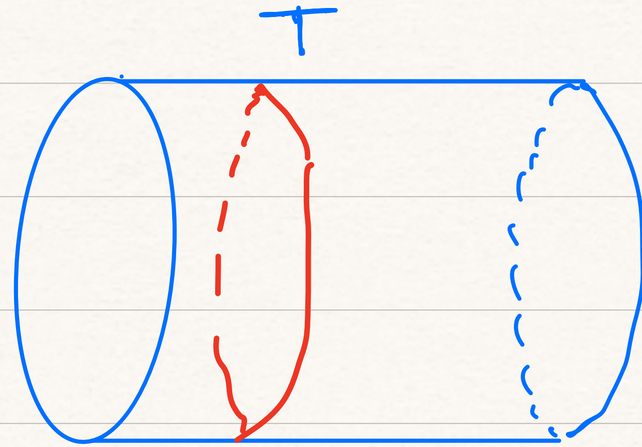
$$Z_W(N, \tau) = \frac{16\,\eta^8(2\tau)}{\eta^{16}(\tau)} \times$$

$$\times \sum_{r \in \mathbb{Z}} \left(\tanh \pi N r T - \frac{1}{N} \tanh \pi r T \right) \frac{1}{\sinh 2\pi r T} + \cdots$$

Disappearing Tachyons



open channel



closed channel

In the large T (IR) limit, one finds exponentially growing tachyonic terms in the closed string channel for $N > 1$

$$\boxed{\text{But no tachyons for } 0 < N \leq 1}$$

Summary

- The orbifold method offers a stringy generalization of the replica method.
- We have presented substantial evidence that this method could yield computable entropy that is finite both in UV and IR
- IR divergences have a very specific structure dictated by string theory.
- One obtains a natural order-by-order expansion. A generalization of von Neumann entropy. Tree level term included in the Euclidean calculation w/o statistical interpretation.

Perhaps a more fundamental notion than BH entropy

Open Problems

- Find the analytic continuation of $Z(N)$ for closed strings on cone.

- Can string field theory be used to compute the boundary terms & the classical entanglement entropy with $\mathbb{C}/\mathbb{Z}_N$ CFT as the starting data?

- Can we map the bulk computation in BTZ background to a quantity in the boundary CFT?

Thank you!