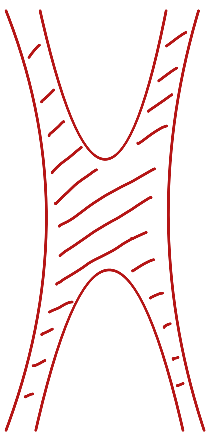
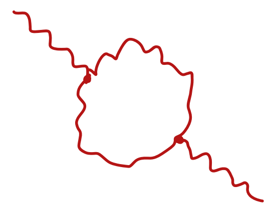
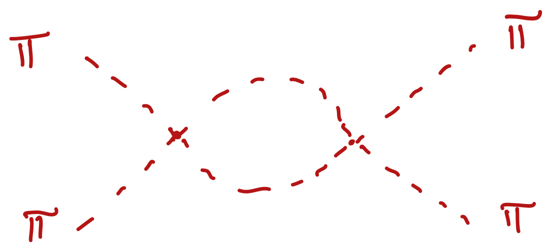
Real World Amplitudes from Curves on Surfaces



Carolina Figueiredo



Strings 2024
CERN
June 2024



w/ N. Arkani-Hamed
Q. Cao, J. Dong
S. He

Outline...

I. Surface Integral Formalism vs. Standard Worldsheet

[Built around manifesting singularities].

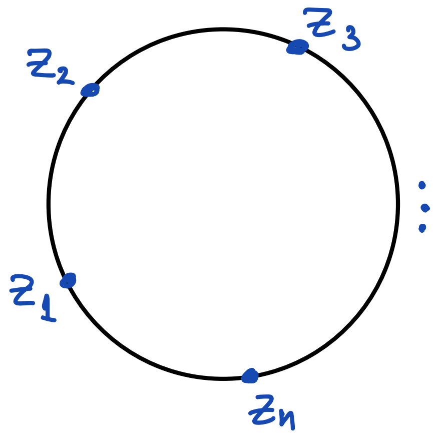
II. Revealing Qualitatively New Features

* Hidden Factorizations away from Poles + Zeros.

* Kinematic connection between different ths.: $\mathrm{Tr}\phi^3 \leftrightarrow$ NLSM $\leftrightarrow$ YM

* "Perfect" Integrands for real world amplitudes.

Bosonic String Theory $\xrightarrow{\alpha' p^2 \ll 1}$ Field Theory: $\mathrm{Tr}\,\varphi^3$



$z_1 < z_2 < z_3 < \ldots < z_n$

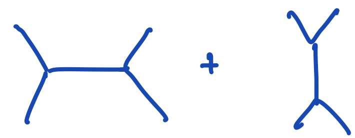
$$A_n = \int \frac{d^n z}{SL(2,\mathbb{R})} \left\langle e^{iP_1 X(z_1)} \cdots e^{iP_n X(z_n)} \right\rangle$$

$$= \int \frac{d^n z}{SL(2,\mathbb{R})} \frac{1}{z_{12} z_{23} \cdots z_{n,1}} \prod_{i<j} z_{ij}^{2\alpha' p_i \cdot p_j}$$

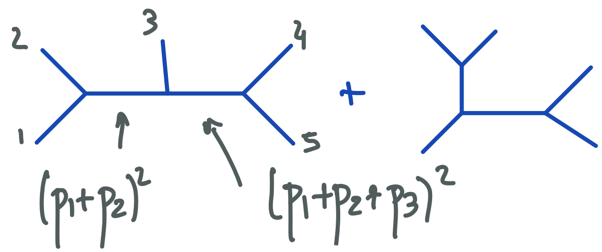
$$\mathcal{L}_{\mathrm{Tr}\varphi^3} = \frac{1}{2}\mathrm{Tr}(\partial\varphi)^2 + \frac{g}{3}\mathrm{Tr}(\varphi^3)$$

$$A^{\mathrm{Tr}\varphi^3} = \sum_{\mathcal{D},\ \text{diagrams}} \left(\prod_{P \in \mathcal{D}} \frac{1}{P^2} \right)$$

$A_4 =$



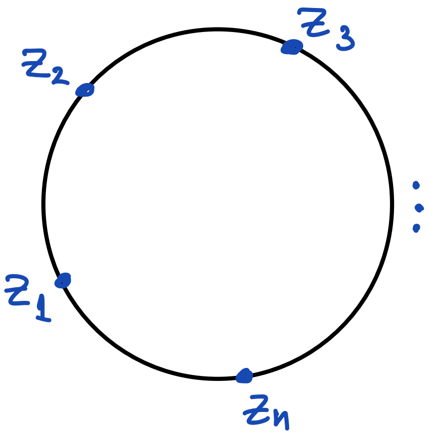
$A_5 =$



Singularities $\equiv$

$$(p_i + p_{i+1} + \cdots + p_{j-1})^2 = 0$$

Bosonic String Theory $\xrightarrow[\alpha' p^2 \ll 1]{}$ Field Theory: $\mathrm{Tr}\,\varphi^3$



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z_{ij}

Need Blow Ups $\longleftrightarrow$

Kin. dependence

poles NOT manifest

$$\mathcal{L}_{\mathrm{Tr}\varphi^3} = \frac{1}{2}\mathrm{Tr}(\partial\varphi)^2 + \frac{g}{3}\mathrm{Tr}(\varphi^3)$$

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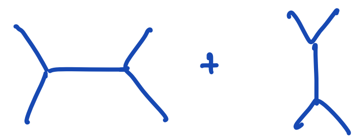
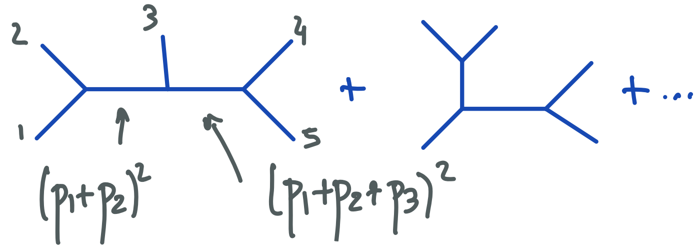
$A_4 =$ 

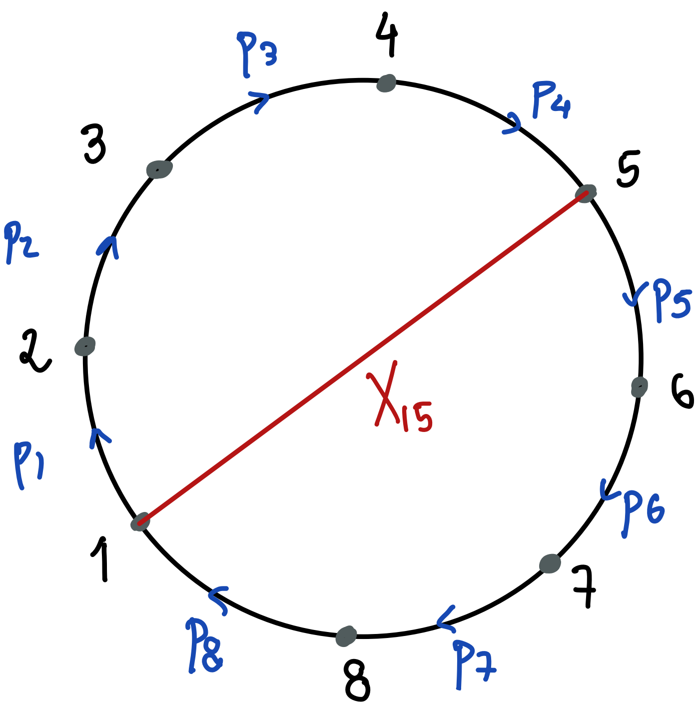
$A_5 =$ 

Singularities $\equiv$

$$(p_i + p_{i+1} + \cdots + p_{j-1})^2 = 0$$

[Kinematics and curves on Surfaces]

Color-ordered amplitude:



Singularities $\equiv (p_i + p_{i+1} + \cdots + p_{j-1})^2 = 0$

$$|||$$

Curves, $\mathcal{C}$, on the Surface. $\mathcal{S}$

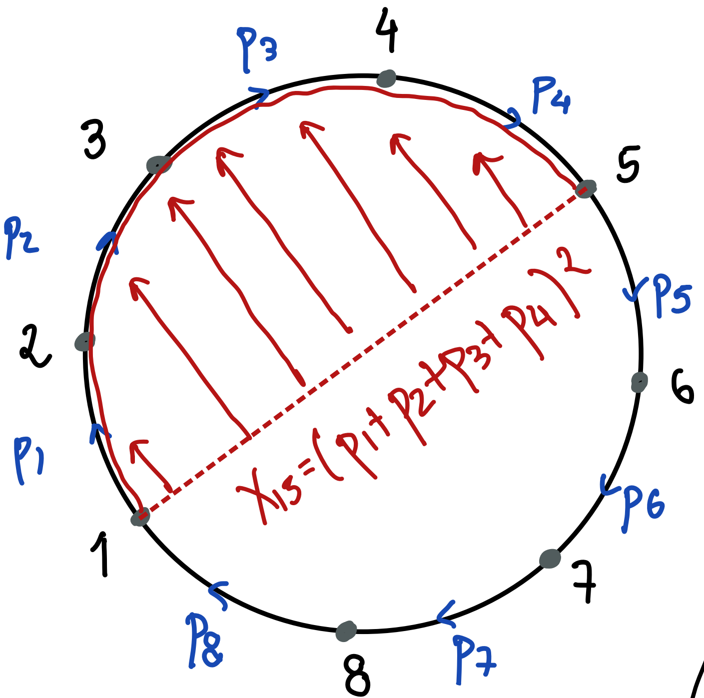
$X_{\mathcal{C}}$ = Kinematics associated to the curve read off by Homology !

$$|||$$

$$X_{i,j} = (p_i + p_{i+1} + \cdots + p_{j-1})^2$$

[Momentum ↔ Homology]

Color-ordered amplitude: Singularities $\equiv (p_i + p_{i+1} + \cdots + p_{j-1})^2 = 0$



$$|||$$

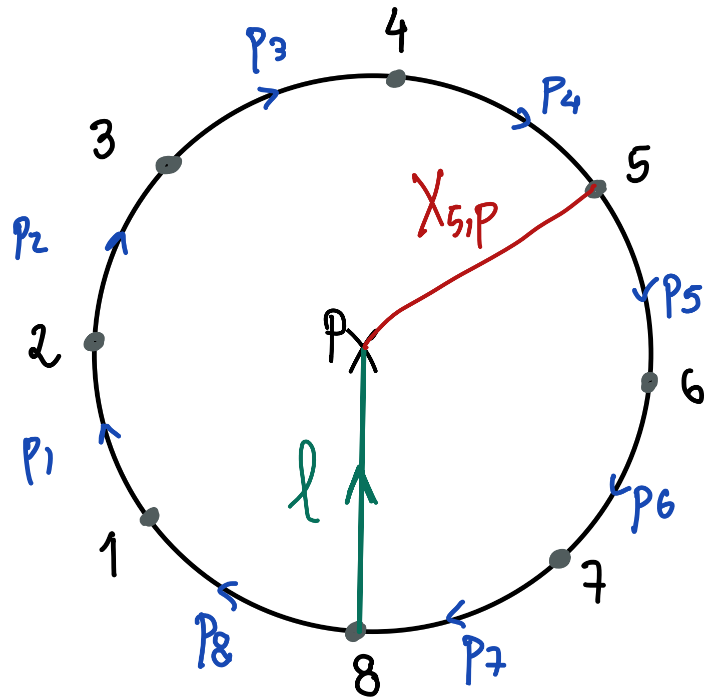
Curves, $\mathcal{C}$, on the Surface. $\mathcal{S}$

$X_{\mathcal{C}}$ = Kinematics associated to the curve read off by Homology !

$A^{\mathrm{Tr}\,\phi^3}\left[X_{\mathcal{C}} \equiv X_{i,j}\right]$ manifest singularities. ✓

[Momentum ↔ Homology]

Loop-level : $\mathcal{S}$ punctures + more interesting topology.

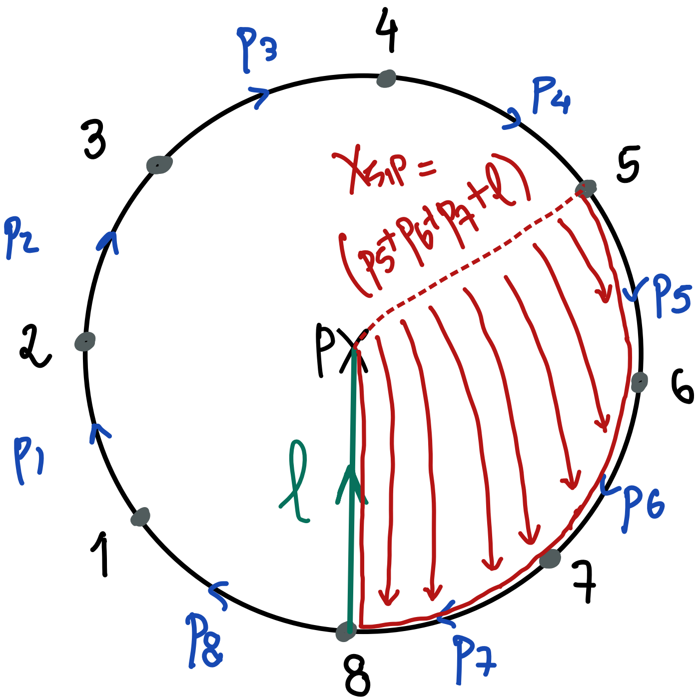


To each curve $\mathcal{C}$ on $\mathcal{S}$ associate:

$$X_{\mathcal{C}} = \text{Homology}!$$

[Momentum ↔ Homology]

Loop-level : $\mathcal{S}$ punctures + more interesting topology.



To each curve $\mathcal{C}$ on $\mathcal{S}$ associate:

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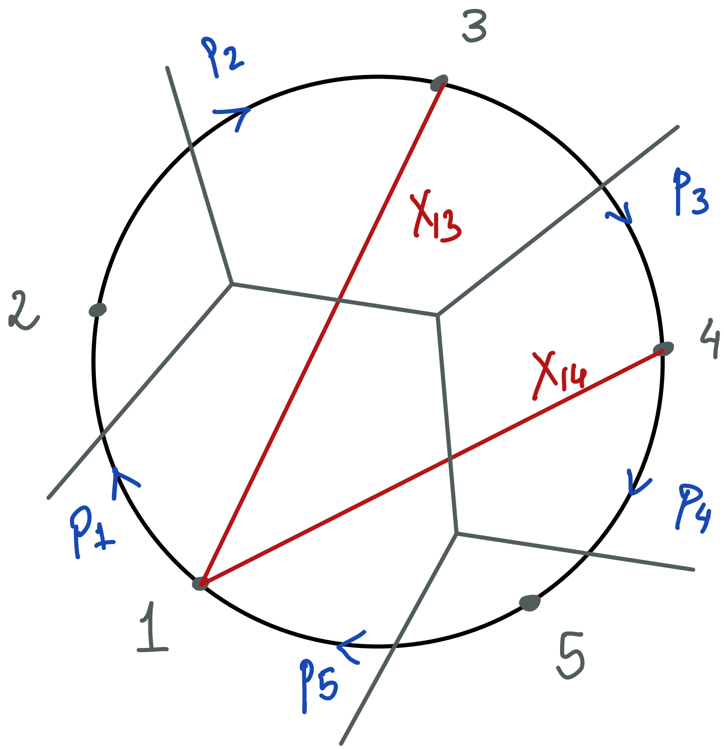
$$|||$$

Propagators in Feynman Diagrams.

$$\mathcal{A}^{\mathrm{Tr}\varphi^3}\left[X_{\mathcal{C}} \equiv X_{i,j}\right]$$ manifest singularities. ✓

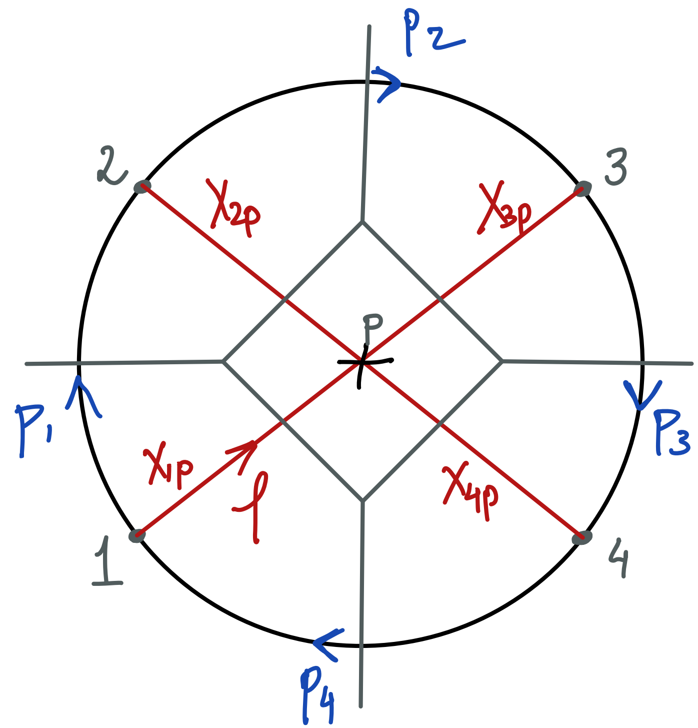
[Feynman Diagrams as Triangulations of S]

5-point Tree



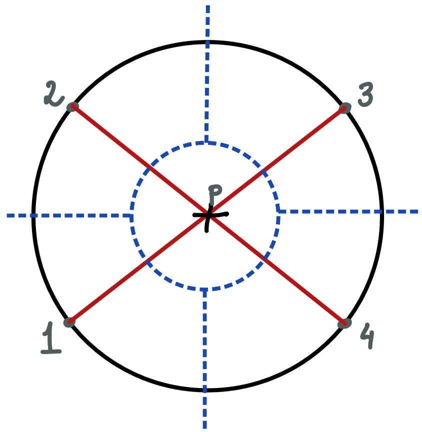
$$\frac{1}{X_{1,3}\,X_{1,4}}$$

4-point 1-loop



$$\frac{1}{X_{1,p}\,X_{2,p}\,X_{3,p}\,X_{4,p}}$$

[Amplitude $\leftrightarrow$ Σ Triangulations of S]



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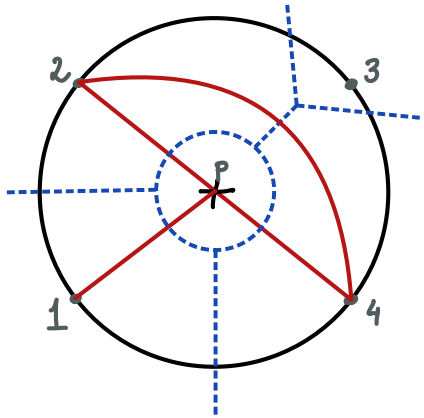
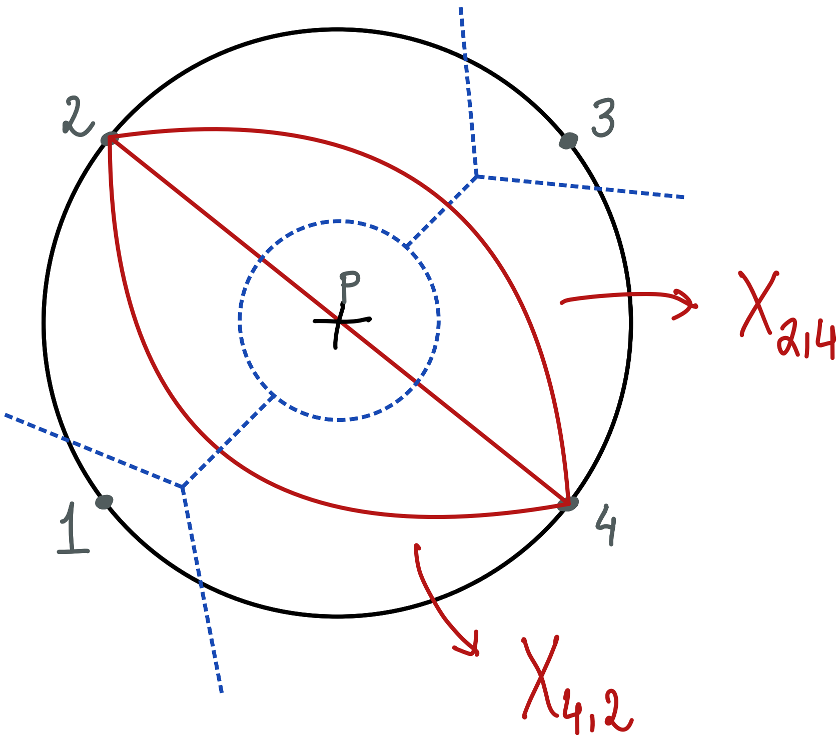


+ ... +



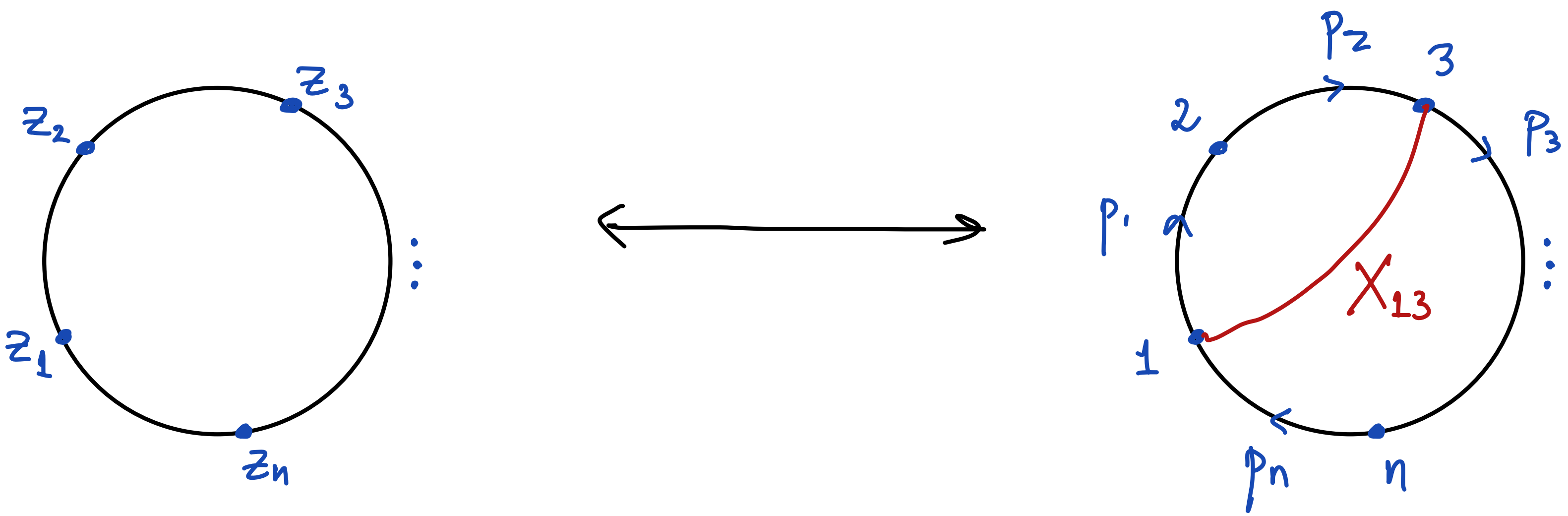
Keep Curves up to Homotopy!

$X_{2,4} \neq X_{4,2}$ as CURVES

Generalizing away from Momenta $\leftrightarrow$ Homology.

$(X_{24} = X_{42})$.

Q: Can we formulate string amplitudes in a way that:



Points in Worldsheet

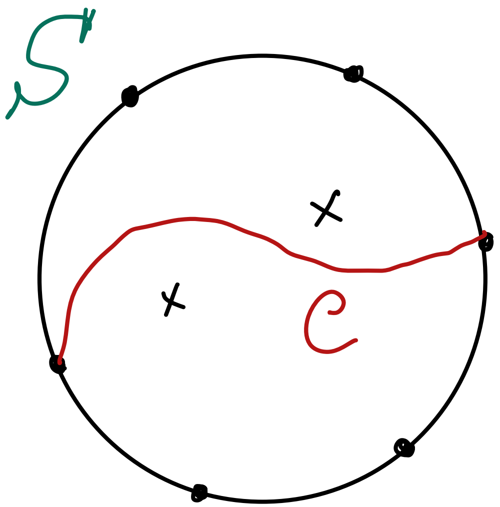
$z_i \Leftrightarrow p_i^{\mu}$

Curves on $\mathcal{S}$

$\Updownarrow$

Manifest location of poles.

Surface Integral Formalism



u-variables:

$$u_C[y] = \frac{F^{1,2}(y)\, F^{2,1}(y)}{F^{1,1}(y)\, F^{2,2}(y)}$$

Positive coordinates y

Counting Problem associated to C on S.

$$A = \int_0^{+\infty} \underbrace{\left(\prod \frac{dy}{y} \right)}_{\text{dlog form}} \times \prod_{C \in S} u_C[y]^{\alpha' X_C}$$

Product over all curves on S

[N. Arkani-Hamed, H. Frost, G. Salvatori, P-G. Plamondon, H. Thomas 23']

Surface Integral Formalism: Tree-level.

$$A_n^{\text{tree}} = \underbrace{\int \frac{d^n z}{SL(2,\mathbb{R})} \times \frac{1}{z_{12}\, z_{23} \cdots z_{n1}}}_{\displaystyle \parallel \atop \displaystyle \prod_{i=1}^{n-3} \frac{dy_i}{y_i}} \times \underbrace{\prod_{i<j} z_{ij}^{-2\alpha' p_i \cdot p_j}}_{\displaystyle \prod_{i<j} u_{ij}^{\alpha' X_{ij}}} \qquad [\text{Worldsheet}]$$

$$\prod_{i=1}^{n-3} \frac{dy_i}{y_i} \quad \times \quad \prod_{i<j} u_{ij}^{\alpha' X_{ij}}$$

Surface Integral Formalism: Tree-level.

$$A_n^{tree} = \underbrace{\int \frac{d^n z}{SL(2,\mathbb{R})} \times \frac{1}{z_{12}\, z_{23} \cdots z_{n1}}}_{= \prod_{i=1}^{n-3} \frac{dy_i}{y_i}} \times \underbrace{\prod_{i<j} z_{ij}^{-2\alpha' p_i \cdot p_j}}_{\prod_{i<j} u_{ij}^{\alpha' X_{ij}}} \qquad [\text{Worldsheet}]$$

$$u_{ij} = \frac{z_{i-1,j}\, z_{i,j-1}}{z_{i,j}\, z_{i-1,j-1}}$$

Manifest all poles
$[u_{ij} \to 0]$

Surface Integral Formalism: Tree-level.

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$$u_{ij} = \frac{z_{i-1,j}\, z_{i,j-1}}{z_{i,j}\, z_{i-1,j-1}}$$

Blows up ALL Singularities

Manifest all poles $[u_{ij} \to 0]$

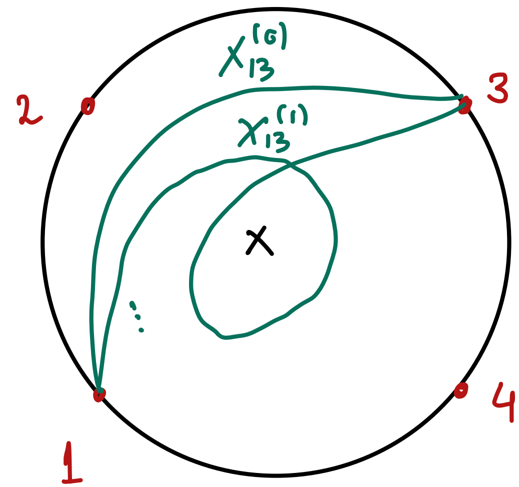
* No Gauge Redundancy / Fixing

* Trivial to extract Field Theory Limit.

Surface Integral Formalism: Loop-level.

$$A_n = \int_0^{+\infty} \underbrace{\prod \frac{dy_p}{y_p}}_{\text{dlog form}} \times \underbrace{\prod_{C \in S'} u_C[y_p]^{\alpha' X_C}}$$

Now ∞ many Curves!



∞ self intersection

$$\prod_{C^{(0)} \in S} u_C^{\alpha' X_{C^{(0)}}} \times \underbrace{\prod_{C^{(q)} \in S} u_C^{\alpha' X_{C^{(q)}}}}_{\infty \text{ product}}$$

$$|||$$

1-loop Bosonic String.

Surface Integral Formalism: Loop-level.

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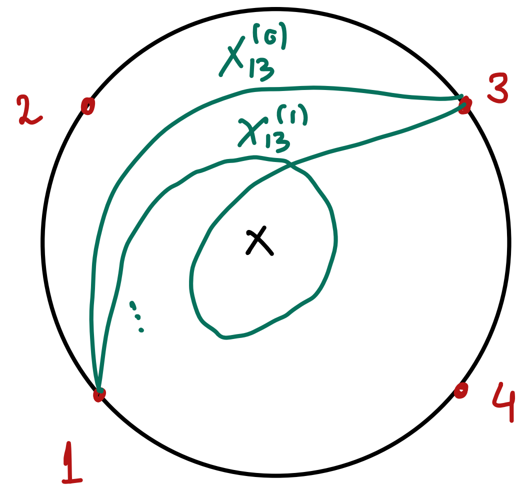

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@ low-energies:

$(\mathrm{Tr}\,\varphi^3)$

Surface Integral Formalism: Loop-level.

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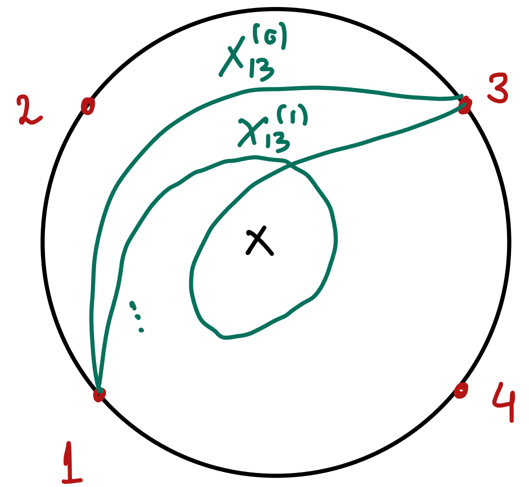
→ Now ∞ many Curves!



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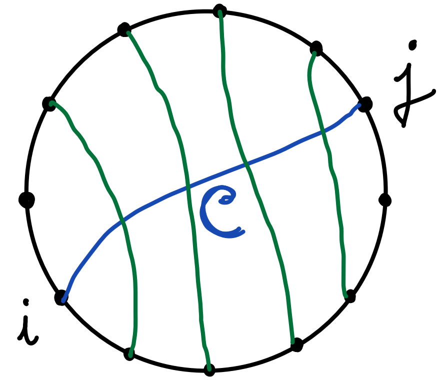
@ low-energies: $(\mathrm{Tr}\,\varphi^3)$

$$\int_0^{+\infty} \prod \frac{dy_p}{y_p} \prod_{C^{(0)}\in S} u_C^{\alpha' X_{C^{(0)}}}$$

* Much simpler!
* "Stringy" UV regularization

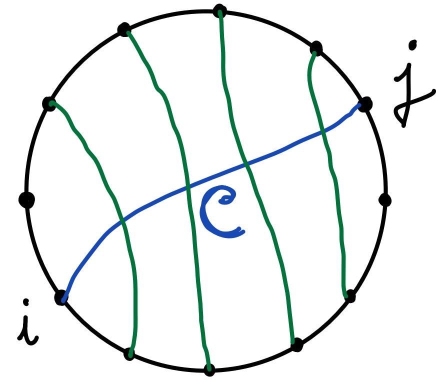
Factorization $\leftrightarrow$ U-variables

$$\boxed{u_e + \prod_{e' \in S} u_{e'}^{\mathrm{Int}(e';e)} = 1}$$



U- equations $\qquad u_e \geqslant 0 \Rightarrow u_e \in [0,1]$

Factorization $\leftrightarrow$ U-variables



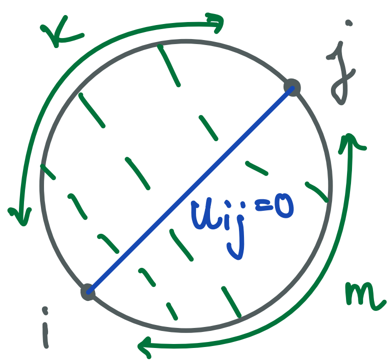
$$\boxed{u_e + \prod_{e' \in S} u_{e'}^{\mathrm{Int}(e';e)} = 1}$$

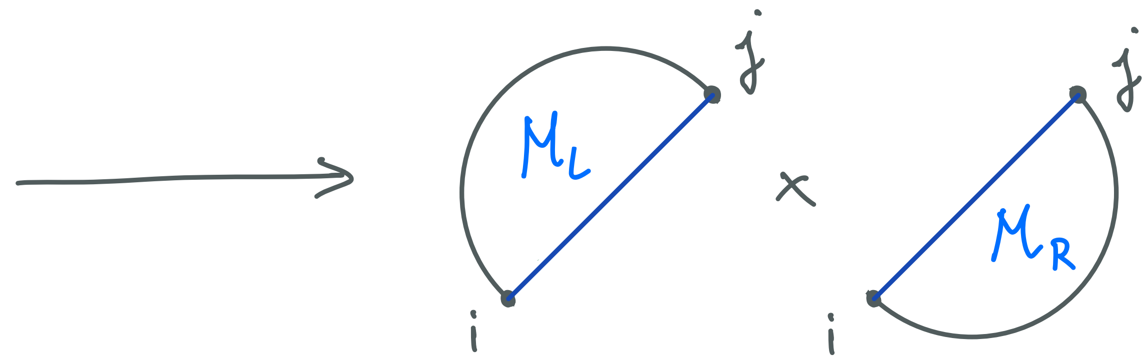
U- equations $\quad u_e \geq 0 \Rightarrow u_e \in [0,1]$

* Factorization: [$X_{ij} = -n \Rightarrow$ singularity $u_{ij} = 0$]

$$u_{ij} \to 0 \Rightarrow \forall_{(km)\ \text{incomp.}\ (ij)} \ u_{km} \to 1 \qquad [\text{Binary}]$$

$$\prod_c u_c^{\alpha' X_c}$$





Revealing New Features.

* Hidden Factorizations away from poles

+ Zeros of string/particle amplitudes

[2312.16282, 2405.09608 w/ N. Arkani-Hamed, Q. Cao, J. Dong, S. He]

New Kind of Factorizations ($\Rightarrow$ Zeros)

$$\mathcal{A}_S\left[\{p_i \cdot p_j \to 0\}\right] \longrightarrow \mathcal{A}_{S_1} \times \mathcal{A}_{S_2}$$

w/ i, j not-adjacent

(no poles)

New Kind of Factorizations ($\Rightarrow$ Zeros)

$$\mathcal{A}_{\mathcal{S}}\left[\{p_i \cdot p_j \to 0\}\right] \longrightarrow \mathcal{A}_{\mathcal{S}_1} \times \boxed{\mathcal{A}_{\mathcal{S}_2}}$$

w/ i,j not-adjacent

(no poles)

$\to 0$

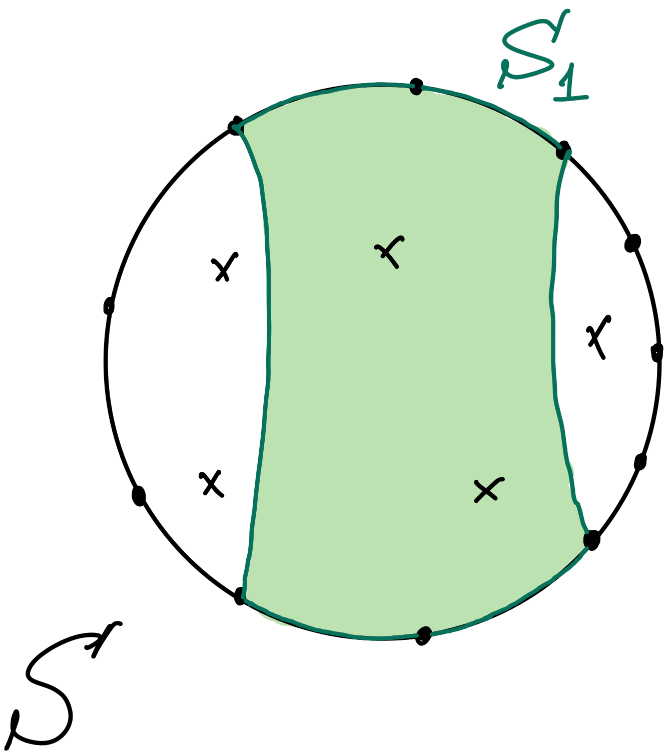
$p'_i \cdot p'_j \to 0$

SPLITs: AWAY FROM Poles (Near Zeros)

u-variables for Subsurfaces

Simple fact about u-variables!

u-variables for subsurface $S_1 \subset S$ can be written in terms of u-variables of S



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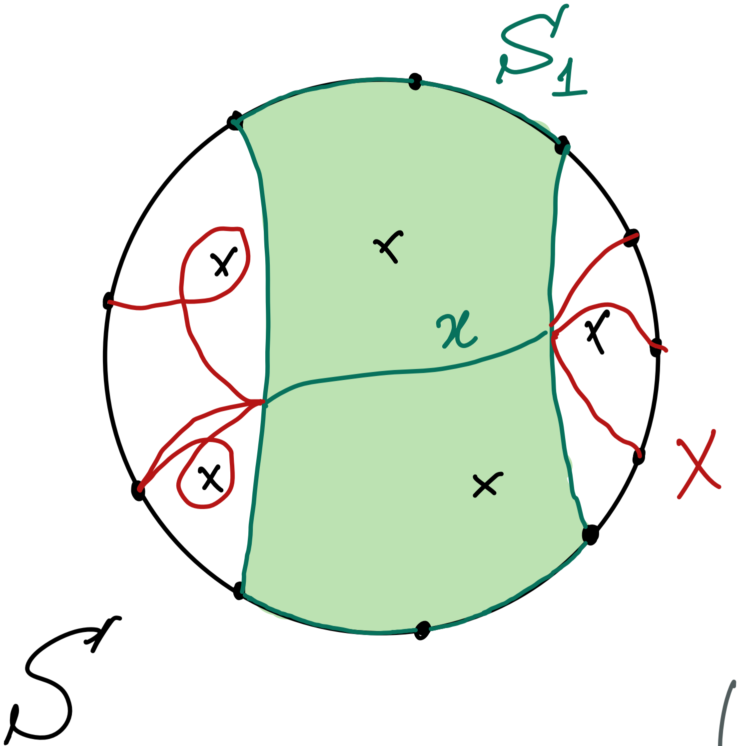
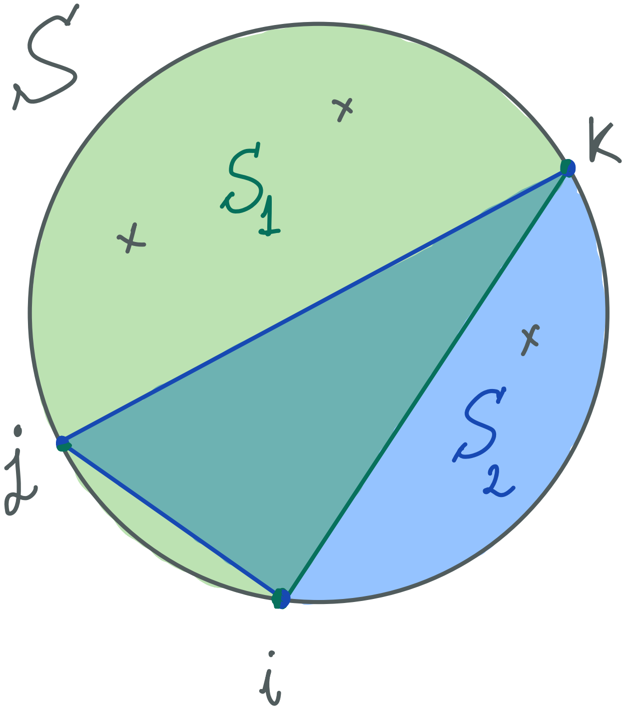


$$u_x = \prod U_X^{\#(x \subset X)}$$

Extension Formula

(all ways of extending x into a curve in S)

Split Factorization: Choose $\mathcal{S}_1$ and $\mathcal{S}_2$ such that all curves X in $\mathcal{S}$ belong to at least one of subsurfaces.

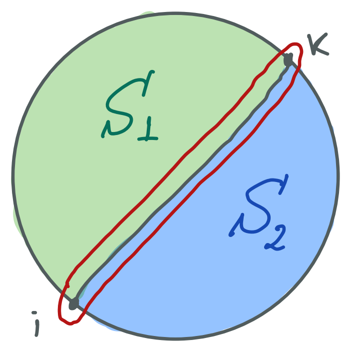


$$A_{\mathcal{S}_1} = \int \prod \frac{dy}{y} \times \prod_{C_1 \in \mathcal{S}_1} u_{C_1}^{\alpha' x_{C_1}} \rightarrow \text{Kinematics of } \mathcal{S}_1$$

$$A_{\mathcal{S}_2} = \int \prod \frac{dy}{y} \times \prod_{C_2 \in \mathcal{S}_2} u_{C_2}^{\alpha' y_{C_2}} \rightarrow \text{Kinematics of } \mathcal{S}_2$$

OVERLAP ON TRIANGLE!

[



Factorization.

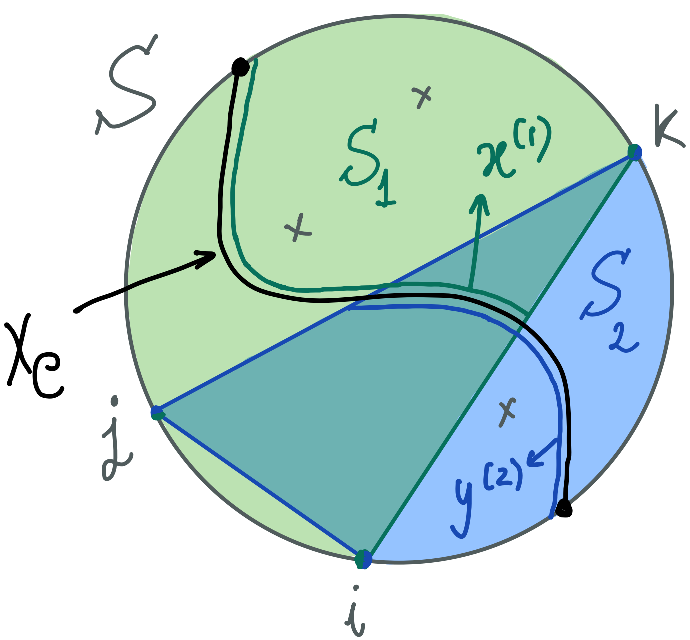
$X_{i,k}$ NOT in $\mathcal{S}_1 / \mathcal{S}_2$.

]

Extension Formula $\Rightarrow$ Split Kinematics

$$A_{S_1} \times A_{S_2} = \left(\int \prod \frac{dy}{y} \prod_{c_1 \in S_1} u_{c_1}^{\alpha' x_{c_1}} \right) \times \left(\int \prod \frac{dy}{y} \prod_{c_2 \in S_2} u_{c_2}^{\alpha' y_{c_2}} \right) = \int \prod \frac{dy}{y} \prod_{c \in S} u_c^{\alpha' \tilde{X}_c}$$

$$\text{w/} \quad \tilde{X}_c = \sum_{c_1\, c_2} [\# c_1 \subset c]\, x_{c_1} + [\# c_2 \subset c]\, y_{c_2}$$



$$\tilde{X}_c \to x^{(1)} + y^{(2)}$$

$$\left(\Rightarrow \tilde{X}_c \neq 0 \quad \text{NO POLES} \right)$$

$$A_S(\tilde{X}_c) \to A_{S_1}(x_{c_1}) \times A_{S_2}(y_{c_2})$$

From Splits to Zeros:

Consider a Split in which one subsurface is a 4 point:

$$A_4(s,t) \times A_{S_2} \quad \text{or} \quad A_4(s,t) \times A_{S_1} \times A_{S_2} \times \cdots$$

$$\left(\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}\right)$$

which vanishes for $s+t=0$ (or $s+t=-n$)

$\Rightarrow$ Broad class of Zeros for particle / string amplitudes

* Standard Physical Interpretation

Zeros? Factorization near Zeros?

Split Factorizations & Zeros

* Surface Integrals [u-variables] $\xrightarrow{\alpha' X \ll 1}$ Field Theory
(string amplitudes) $\quad$ $\mathrm{Tr}\,\varphi^3$ amplitudes.

Q: Are they also present for other [colored] theories ?

* Non-linear Sigma model (NLSM) ✓✓ (Zeros $\Downarrow$ Adler Zero)

* Non-SUSY Yang-Mills Theory ✓✓

Revealing New Features.

* Kinematic Connection between different theories: $\mathrm{Tr}\,\varphi^3 \leftrightarrow \mathrm{NLSM} \leftrightarrow \mathrm{YM}$

[2401.00041, 2401.05483, 2403.04826]
w/ N. Arkani-Hamed, Q. Cao, J. Dong, S. He]

Hint: NLSM, YM share all the Zeros (and Splits) found for $\mathrm{Tr}\varphi^3$.

Build simplest deformation of $\mathrm{Tr}\varphi^3$

$$A^{\alpha'}_{\mathrm{Tr}\varphi^3}\left[X_{ij} \to X_{i,j} + \delta_{i,j}\right]$$

preserving ALL ZEROS.

Hint: NLSM, YM share all the Zeros (and Splits) found for $\mathrm{Tr}\varphi^3$.

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preserving ALL ZEROS.

UNIQUE SOLUTION!

$$\delta_{i,j} = \begin{cases} +\delta & (i,j) \text{ even} \\ -\delta & (i,j) \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

Hint: NLSM, YM share all the Zeros (and Splits) found for $\mathrm{Tr}\varphi^3$.

Build simplest deformation of $\mathrm{Tr}\varphi^3$

$$A^{\alpha'}_{\mathrm{Tr}\varphi^3}\left[X_{ij} \to X_{i,j} + \delta_{i,j}\right]$$

preserving ALL ZEROS.

UNIQUE SOLUTION!

$$\delta_{i,j} = \begin{cases} +\delta & (i,j) \text{ even} \\ -\delta & (i,j) \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

$$A^{\alpha'}_{\mathrm{Tr}\varphi^3}\left[X_{i,j} + \delta_{i,j}\right] = \int_0^{+\infty} \prod \frac{dy}{y} \prod u_{i,j}^{\alpha' X_{i,j}} \times \left(\frac{\prod u_{e,e}}{\prod u_{o,o}}\right)^{\alpha'\delta}$$

$$= \int_0^{+\infty} \prod \frac{dy}{y} \prod u_{i,j}^{\alpha' X_{i,j}} \times \left(\frac{1}{\prod y}\right)^{\alpha'\delta}$$

Only Changing Measure!

δ-Shifted $\mathrm{Tr}\,\varphi^3$

$$\mathcal{A}_{2n}[\alpha' X_{ij}] \propto \underbrace{\int \prod \frac{dy}{y} \times \prod_{i<j} u_{ij}^{\alpha' X_{ij}}}_{\mathrm{Tr}\,\varphi^3} \times \underbrace{\left(\frac{\prod u_{\text{even,even}}}{\prod u_{\text{odd,odd}}}\right)^{\alpha'\delta}}_{\delta\text{-shift}}$$

$\alpha'\delta$

1

@ low energies

$0 \to \alpha'\delta = 0$ $\quad$ $\mathrm{Tr}\,\varphi^3$ theory

δ-Shifted $\mathrm{Tr}\,\varphi^3$

$$\mathcal{A}_{2n}[\alpha' X_{ij}] \propto \underbrace{\int \prod \frac{dy}{y} \times \prod_{i<j} u_{ij}^{\alpha' X_{ij}}}_{\mathrm{Tr}\,\varphi^3} \times \underbrace{\left(\frac{\prod u_{\text{even,even}}}{\prod u_{\text{odd,odd}}} \right)^{\alpha'\delta}}_{\delta\text{-shift}}$$

$\alpha'\delta$

1

$\alpha'\delta \in \mathbb{R}^+ \setminus \mathbb{Z}$ NLSM @ low energies

$0 \rightarrow \alpha'\delta = 0$ $\mathrm{Tr}\,\varphi^3$ theory

δ- Shifted $\mathrm{Tr}\,\varphi^3$

$$\mathcal{A}_{2n}[\alpha' X_{ij}] \propto \underbrace{\int \prod \frac{dy}{y} \times \prod_{i<j} u_{ij}^{\alpha' X_{ij}}}_{\mathrm{Tr}\,\varphi^3} \times \underbrace{\left(\frac{\prod u_{\text{even,even}}}{\prod u_{\text{odd,odd}}} \right)^{\alpha'\delta}}_{\delta\text{-shift}}$$

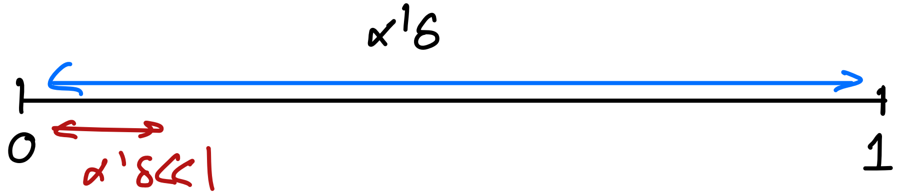
$\alpha'\delta$

$1 \rightarrow \alpha'\delta = 1$ YM theory

$\alpha'\delta \in \mathbb{R}^+ \setminus \mathbb{Z}$ NLSM @ low energies

$0 \rightarrow \alpha'\delta = 0$ $\mathrm{Tr}\,\varphi^3$ theory

Claim:

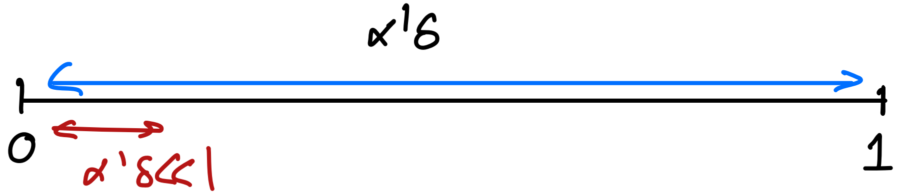


$\Longrightarrow$ NLSM

$\alpha'\delta \ll 1$, @ low energies: $A_\delta^{\mathrm{tr}\varphi^3}[X_{ij} \to X_{ij} + \delta_{ij}]$

Ex. 4 pts

$$A_4^\delta = \frac{1}{X_{13} - \delta} + \frac{1}{X_{24} + \delta} \xrightarrow[X \ll \delta]{} 1/\delta^2 \underbrace{\left(X_{13} + X_{24}\right)}_{A_4^{NLSM}} + \mathcal{O}\left(\delta^{-3}\right)$$

Claim: 

$\Longrightarrow$ NLSM

$\alpha'\delta \ll 1$, @ low energies: $A_\delta^{\mathrm{tr}\varphi^3}[X_{ij} \to X_{ij} + \delta_{ij}]$

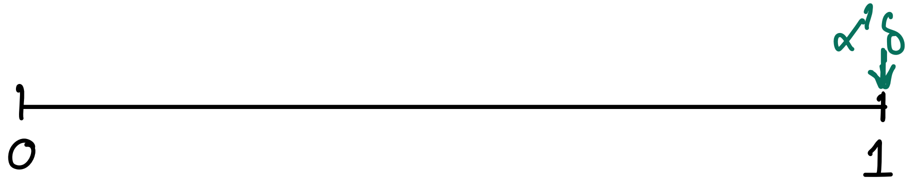
Ex. 4 pts

$$A_4^\delta = \frac{1}{X_{13}-\delta} + \frac{1}{X_{24}+\delta} \xrightarrow[X \ll \delta]{} \frac{1}{\delta^2} \underbrace{(X_{13}+X_{24})}_{A_4^{NLSM}} + \mathcal{O}(\delta^{-3})$$

Why NLSM?

* Factorization ✓
* Split Zeros $\Longrightarrow$ Adler Zero
* Lagrangian derivation of $\mathrm{Tr}\varphi^3 \Leftrightarrow$ NLSM.
* Useful (Tree + Loop-level)!

Claim:



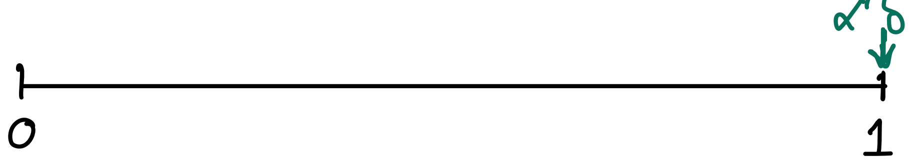
$\Longrightarrow$ YM

$$\alpha' \delta = 1$$

$$A_{2n}[\alpha' X_{ij}] \propto \int \prod \frac{dy}{y^2} \times \prod_{i<j} u_{ij}^{\alpha' X_{ij}}$$

different measure

Claim:



$\Rightarrow$ YM

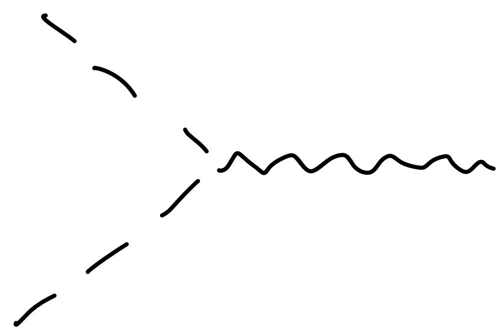
$$\alpha'\delta = 1$$

$$A_{2n}[\alpha' X_{ij}] \propto \int \prod \frac{dy}{y^2} \times \prod_{i<j} u_{ij}^{\alpha' X_{ij}}$$

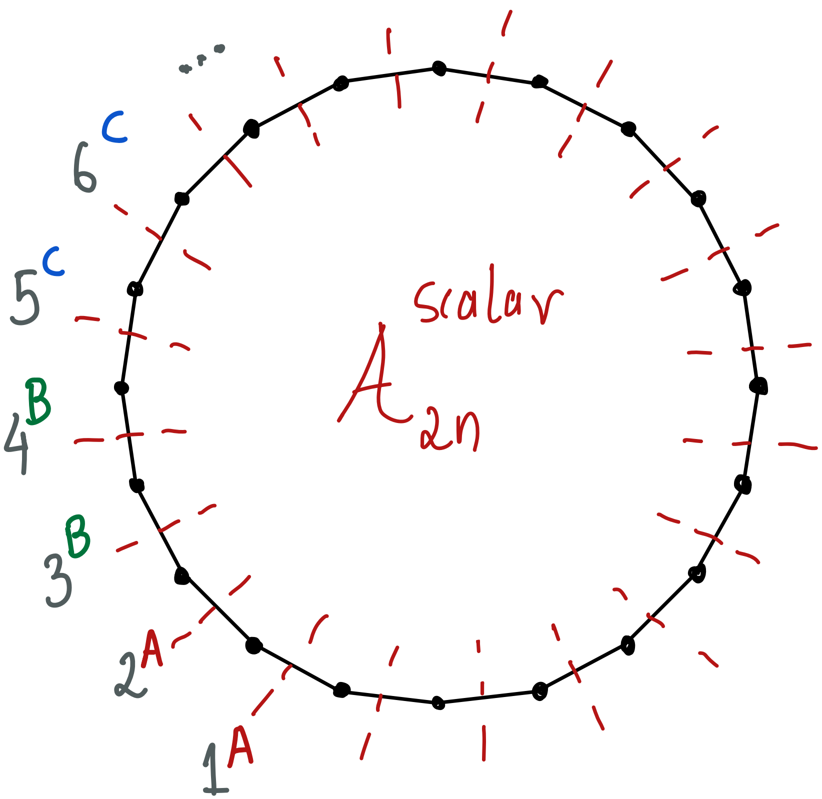
different measure

Q: Where are the polarizations?

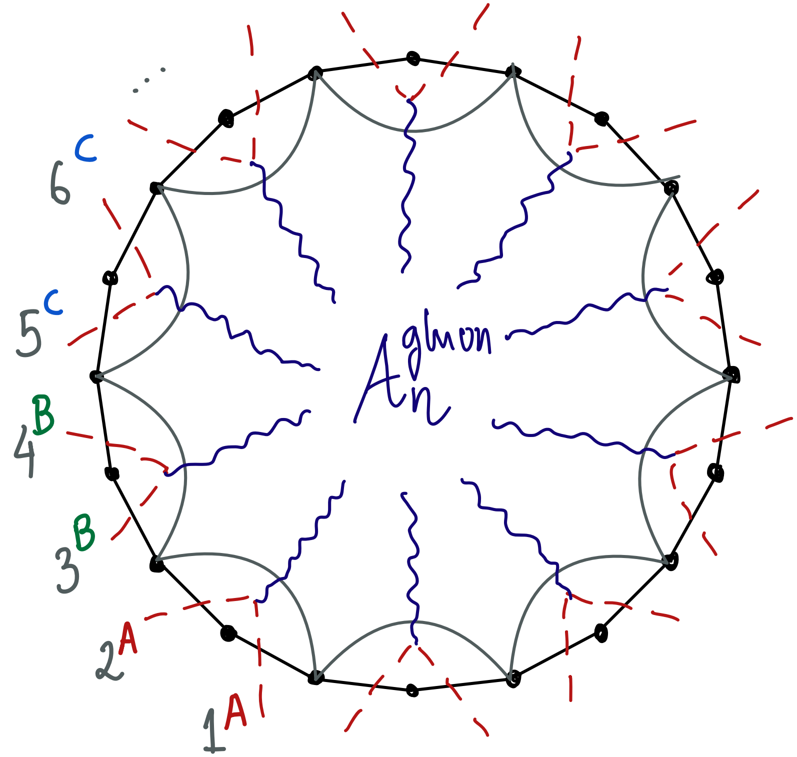
$A_{2n}[X] \equiv$ scattering of scalars that couple to gluons



General 2n-scattering scalars $\longrightarrow$ n-gluons



$\longrightarrow$



$$A_{2n} = \int \prod \frac{dy}{y_2} \prod u_{ij}^{\alpha' X_{ij}} \longrightarrow A_n^{\text{gluon}} = \underset{\text{Scaffolding } X=0}{\text{Res}} \left(A_{2n} \right) [X_{ij}].$$

Getting $A_{2n} = \int \prod \frac{dy}{y_2} \prod u_{ij}^{\alpha' X_{ij}}$ from Bosonic String (Tree-level)

$$A_{2n}^{tree}(1,2,\ldots,2n) = \int \frac{dz_1 \cdots dz_{2n}}{SL(2,\mathbb{R})} \times \prod_{i<j} z_{ij}^{2\alpha' p_i \cdot p_j} \exp\left\{-\sum_{i\neq j}\left(\frac{\sqrt{\alpha'}\,\varepsilon_i \cdot p_j}{z_{ij}} - 2\,\frac{\varepsilon_i \cdot \varepsilon_j}{z_{ij}^2}\right)\right\}\Bigg|_{\text{multilinear in } \varepsilon_i}$$

$$\Big\downarrow \quad \varepsilon_1 \cdot \varepsilon_2 = \varepsilon_3 \cdot \varepsilon_4 = \ldots = \varepsilon_{2n-1} \cdot \varepsilon_{2n} = 1 \,; \quad p_i \cdot \varepsilon_j = 0, \quad \varepsilon_i \cdot \varepsilon_j = 0$$

(Gluons $\to$ Scalars)

$$A_{2n\ \text{scalars}}^{tree}(1,2,\ldots,2n) \propto \int \frac{dz_1 \cdots dz_{2n}}{SL(2,\mathbb{R})} \times \prod_{i<j} z_{ij}^{2\alpha' p_i \cdot p_j} \times \frac{1}{z_{12}^2\, z_{34}^2 \cdots z_{2n-1,2n}^2}$$

$$\Big\downarrow \quad z_{ij} \to u[y]$$

$$A_{2n}(1,2,\ldots,2n) \propto \int \prod \frac{dy}{y} \prod u_{ij}^{\alpha' X_{ij}} \times \left(\frac{\prod u_{ee}}{\prod u_{oo}}\right)$$

Gluons @ Loop-level.

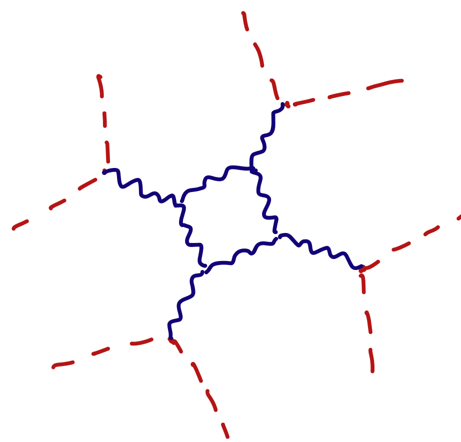
$$A^{\mathrm{Tr}\varphi^3}_{\mathrm{Loop}} = \int \prod_{y} \frac{dy}{y} \prod_{C\in S} u_C^{\alpha' X_C} \longrightarrow \int \prod \frac{dy}{y^{2}} \prod_{C\in S'} u_C^{\alpha' X_C} \equiv A^{\mathrm{Gluons}}_{\mathrm{Loop}}.$$

Surface Gluon Integrands.

* General Dimensions

* Leading Singularities ✓ (2 loops)

$- X14\ X15\ X27\ X36 + X15^2\ X27\ X36 + X14\ X15\ X28\ X36 - X15^2\ X28\ X36 + X14\ X15\ X27\ X37 - X15^2\ X27\ X37 - X14\ X16\ X27\ X37 + X15\ X16\ X27\ X37 - X14\ X15\ X28\ X37 + X15^2\ X28\ X37 +$
$X14\ X16\ X28\ X37 - X15\ X16\ X28\ X37 + X14\ X15\ X36\ X37 - X15^2\ X36\ X37 - X15\ X24\ X36\ X37 + X15\ X25\ X36\ X37 - X14\ X15\ X37^2 + X15^2\ X37^2 + X14\ X16\ X37^2 - X15\ X16\ X37^2 - X15\ X24\ X37^2 -$
$X16\ X24\ X37^2 - X15\ X25\ X37^2 + X16\ X25\ X37^2 - X14\ X15\ X36\ X38 + X15^2\ X36\ X38 + X15\ X24\ X36\ X38 - X15\ X25\ X36\ X38 + X14\ X15\ X37\ X38 - X15^2\ X37\ X38 - X14\ X16\ X37\ X38 +$
$X15\ X16\ X37\ X38 - X15\ X24\ X37\ X38 + X16\ X24\ X37\ X38 + X15\ X25\ X37\ X38 - X16\ X25\ X37\ X38 - X15^2\ X27\ X46 + X15^2\ X28\ X46 + X15^2\ X37\ X46 - X15\ X25\ X37\ X46 - X15^2\ X38\ X46 +$
$X15\ X25\ X38\ X46 + X15^2\ X27\ X47 - X15\ X16\ X27\ X47 - X15^2\ X28\ X47 + X15\ X16\ X28\ X47 - X15^2\ X37\ X47 + X15\ X16\ X37\ X47 + X15\ X25\ X37\ X47 - X16\ X25\ X37\ X47 + X15^2\ X38\ X47 -$
$X15\ X16\ X38\ X47 - X15\ X25\ X38\ X47 + X16\ X25\ X38\ X47 + X14\ X27\ X36\ X58 - X15\ X27\ X36\ X58 - X14\ X27\ X37\ X58 + X15\ X27\ X37\ X58 - X14\ X36\ X37\ X58 + X15\ X36\ X37\ X58 + X24\ X36\ X37\ X58 -$
$X25\ X36\ X37\ X58 + X14\ X37^2\ X58 - X15\ X37^2\ X58 - X24\ X37^2\ X58 + X25\ X37^2\ X58 + X15\ X27\ X46\ X58 - X15\ X37\ X46\ X58 + X25\ X37\ X46\ X58 - X15\ X27\ X47\ X58 + X15\ X37\ X47\ X58 -$
$X25\ X37\ X47\ X58 + X14\ X27\ X37\ X68 - X15\ X27\ X37\ X68 - X14\ X37^2\ X68 + X15\ X37^2\ X68 + X24\ X37^2\ X68 - X25\ X37^2\ X68 + X15\ X27\ X47\ X68 - X15\ X37\ X47\ X68 + X25\ X37\ X47\ X68 +$
$X15\ X37\ X46\ Y2 - X15\ X38\ X46\ Y2 - X15\ X36\ X47\ Y2 + X15\ X38\ X47\ Y2 - X16\ X38\ X47\ Y2 + X15\ X36\ X48\ Y2 - X15\ X37\ X48\ Y2 + X16\ X37\ X48\ Y2 - X14\ X36\ X58\ Y2 + X14\ X37\ X58\ Y2 -$
$X16\ X37\ X58\ Y2 - X37\ X46\ X58\ Y2 + X16\ X47\ X58\ Y2 + X36\ X47\ X58\ Y2 - X14\ X37\ X68\ Y2 + X15\ X37\ X68\ Y2 - X15\ X47\ X68\ Y2 + X15\ X27\ X36\ Y4 - X15\ X28\ X36\ Y4 + X16\ X25\ X37\ Y4 -$
$X15\ X26\ X37\ Y4 + X15\ X28\ X37\ Y4 - X16\ X28\ X37\ Y4 - X16\ X25\ X38\ Y4 + X15\ X26\ X38\ Y4 - X15\ X27\ X38\ Y4 + X16\ X27\ X38\ Y4 + X16\ X27\ X58\ Y4 - X27\ X36\ X58\ Y4 - X16\ X37\ X58\ Y4 +$
$X26\ X37\ X58\ Y4 - X15\ X27\ X68\ Y4 + X15\ X37\ X68\ Y4 - X25\ X37\ X68\ Y4 - 2\ X16\ X58\ Y2\ Y4 + 2\ X15\ X68\ Y2\ Y4 + X15\ X24\ X37\ Y6 - X14\ X25\ X37\ Y6 - X14\ X28\ X37\ Y6 + X15\ X28\ X37\ Y6 -$
$X15\ X24\ X38\ Y6 + X14\ X25\ X38\ Y6 + X14\ X27\ X38\ Y6 - X15\ X27\ X38\ Y6 - X15\ X28\ X47\ Y6 + X15\ X38\ X47\ Y6 - X25\ X38\ X47\ Y6 + X15\ X27\ X48\ Y6 - X15\ X37\ X48\ Y6 + X25\ X37\ X48\ Y6 -$
$X14\ X27\ X58\ Y6 + X14\ X37\ X58\ Y6 - X24\ X37\ X58\ Y6 + 2\ X14\ X37\ Y2\ Y6 - 2\ X15\ X37\ Y2\ Y6 - 2\ X14\ X38\ Y2\ Y6 + 2\ X15\ X38\ Y2\ Y6 + 2\ X15\ X47\ Y2\ Y6 + 2\ X38\ X47\ Y2\ Y6 - 2\ X15\ X48\ Y2\ Y6 -$
$2\ X37\ X48\ Y2\ Y6 + 2\ X14\ X58\ Y2\ Y6 - 2\ X37\ X58\ Y2\ Y6 - 2\ X47\ X58\ Y2\ Y6 + 2\ X28\ X37\ Y4\ Y6 - 2\ X27\ X38\ Y4\ Y6 - X15\ X24\ X36\ Y8 + X14\ X25\ X36\ Y8 - X14\ X27\ X36\ Y8 + X15\ X27\ X36\ Y8 +$
$X15\ X24\ X37\ Y8 - X16\ X24\ X37\ Y8 - X14\ X25\ X37\ Y8 + X16\ X25\ X37\ Y8 + X14\ X26\ X37\ Y8 - X15\ X26\ X37\ Y8 - X15\ X27\ X46\ Y8 + X15\ X37\ X46\ Y8 - X25\ X37\ X46\ Y8 - X16\ X25\ X47\ Y8 +$
$X15\ X26\ X47\ Y8 - X15\ X36\ X47\ Y8 + X25\ X36\ X47\ Y8 + 2\ X37\ X46\ Y2\ Y8 - 2\ X36\ X47\ Y2\ Y8 + 2\ X16\ X25\ Y4\ Y8 - 2\ X15\ X26\ Y4\ Y8 + 2\ X15\ X27\ Y4\ Y8 - 2\ X16\ X27\ Y4\ Y8 + 2\ X15\ X36\ Y4\ Y8 -$
$2\ X25\ X36\ Y4\ Y8 + 2\ X27\ X36\ Y4\ Y8 - 2\ X15\ X37\ Y4\ Y8 + 2\ X16\ X37\ Y4\ Y8 + 2\ X25\ X37\ Y4\ Y8 - 2\ X26\ X37\ Y4\ Y8 + 2\ X15\ X24\ Y6\ Y8 - 2\ X14\ X25\ Y6\ Y8 - Y2\ Y4\ Y6\ Y8 - Y2\ Y4\ Y6\ Y8$



A_4^{Gluon}

Revealing New Features.

* Surface Kinematics → Determining "Perfect" Integrands for real world amplitudes.

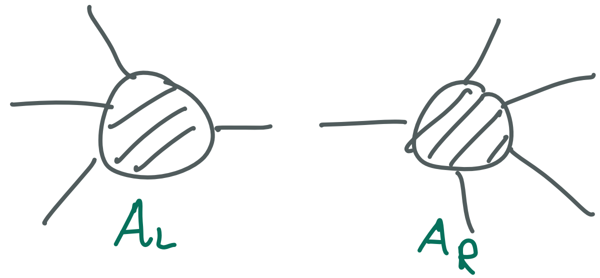
[2403.048261, 2401.00041, 2401.05483, to appear
w/ N. Arkani-Hamed, Q. Cao, J. Dong, S. He]

CHALLENGE: Determine real world amplitudes from

SINGLE CUTS $\equiv$ single poles ?

Tree-level ✓

Rational Function + On poles:



$\Rightarrow$ Can be used to recursevely construct A. [BCFW].

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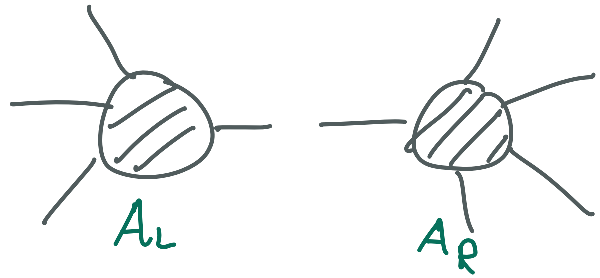
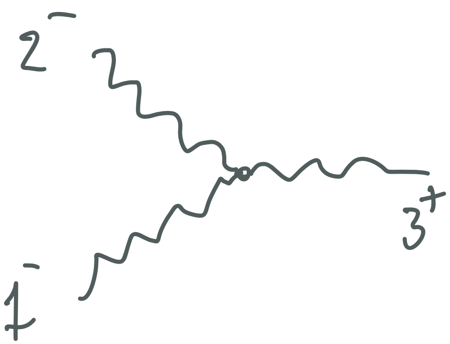


$\Rightarrow$ Can be used to recursively construct A. [BCFW].

BUT, Needs extension of momenta $\mathbb{R} \to \mathbb{C}$.



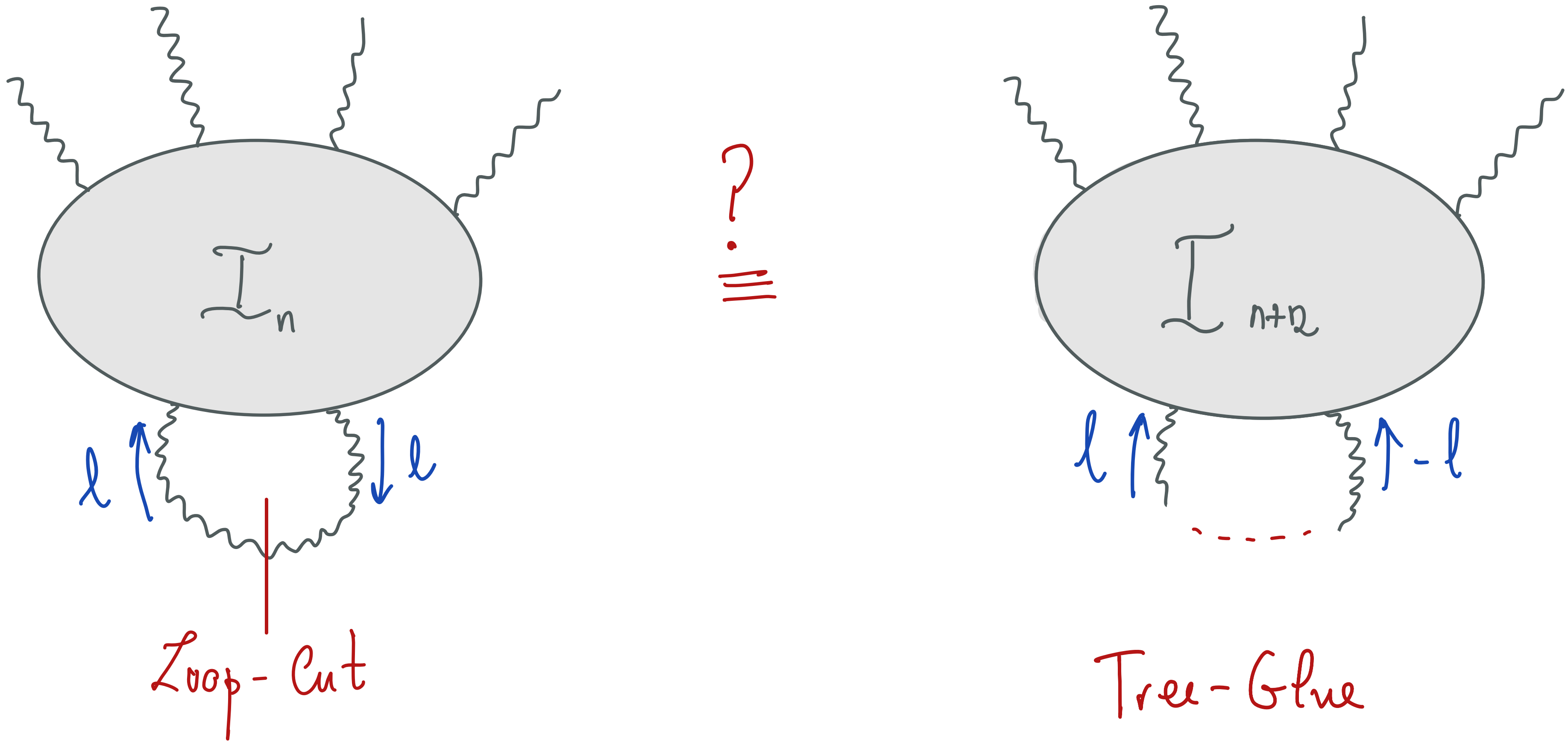
3 point amplitude $= 0$ Lorentzian

$\neq 0$ Complex Kinematics!

CHALLENGE: Determine real world amplitudes from

SINGLE CUTS $\equiv$ single poles ?

Loop-level? No!



CHALLENGE: Determine real world amplitudes from

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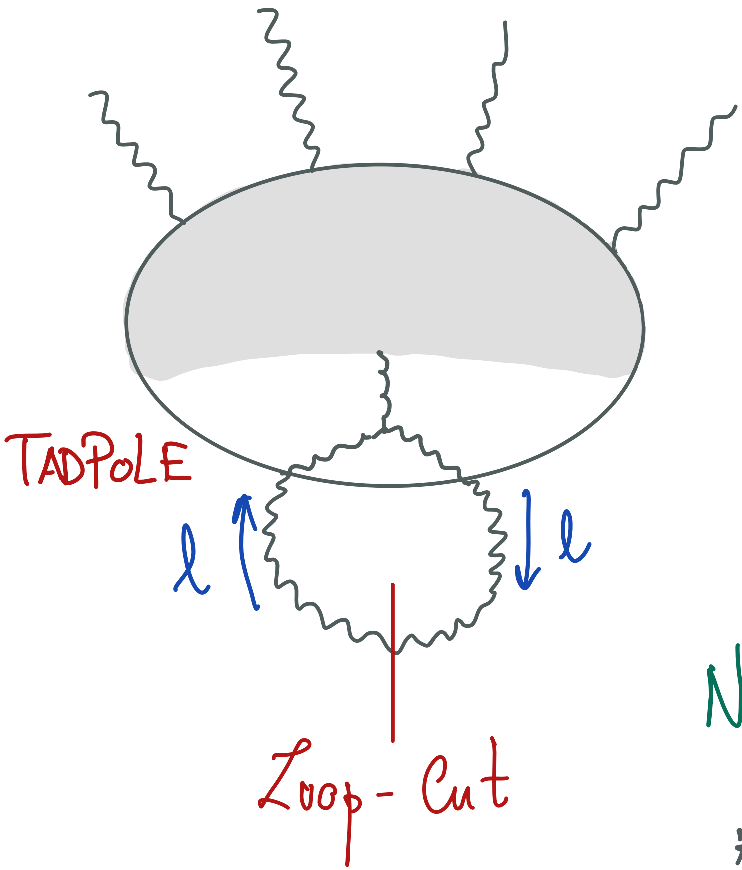


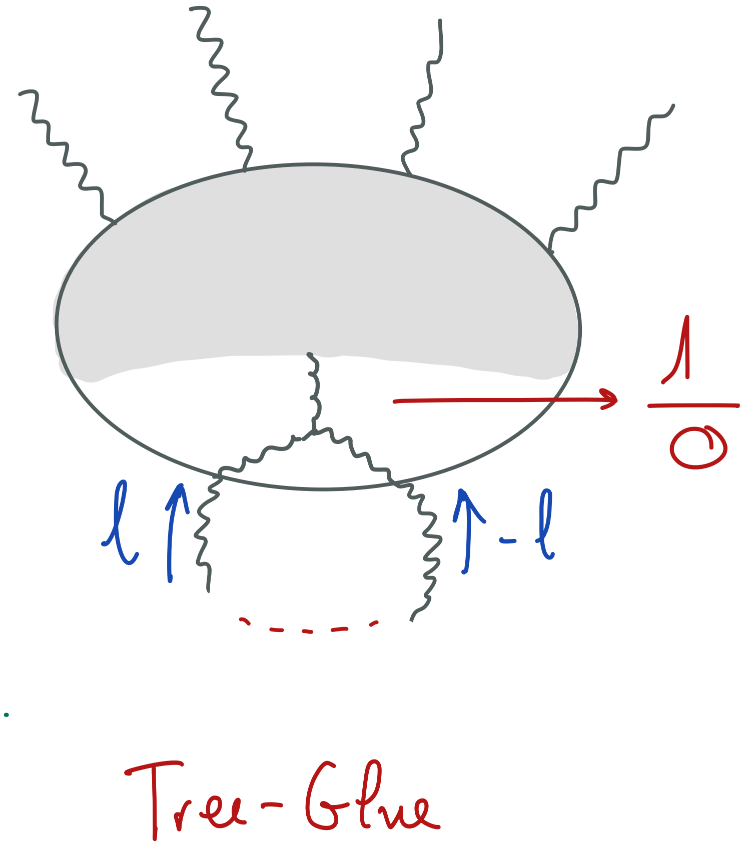
?
≡

DIVERGENT!

⇓

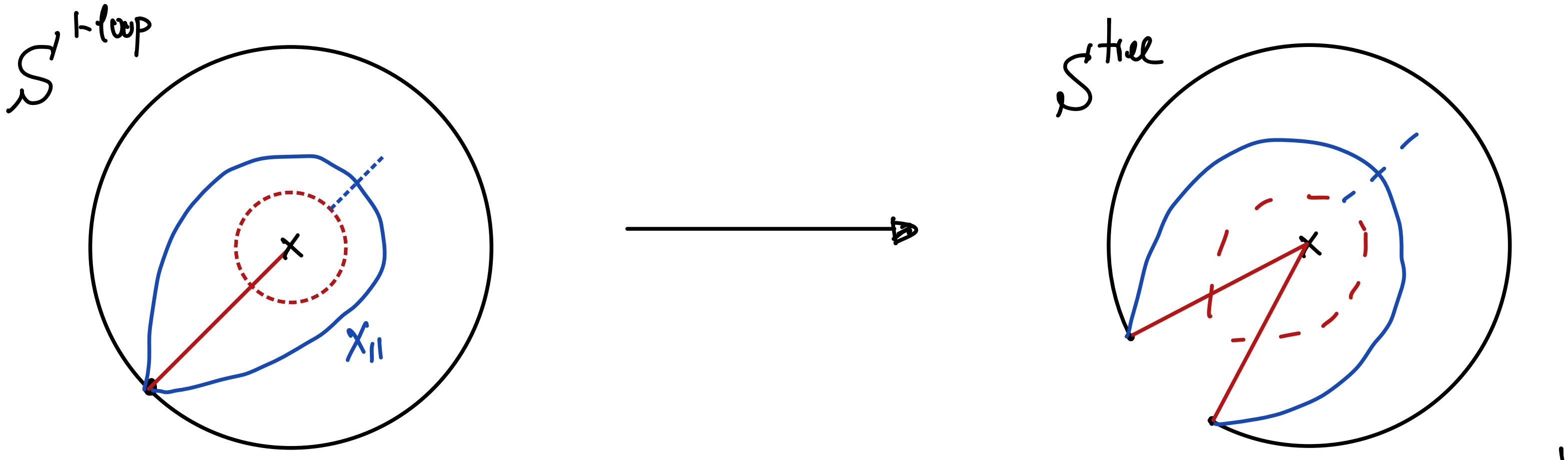
No "The" Integrand.

* Adler Zero ✗
* Gauge Invariant ✗



[cancels in maximal SUSY]

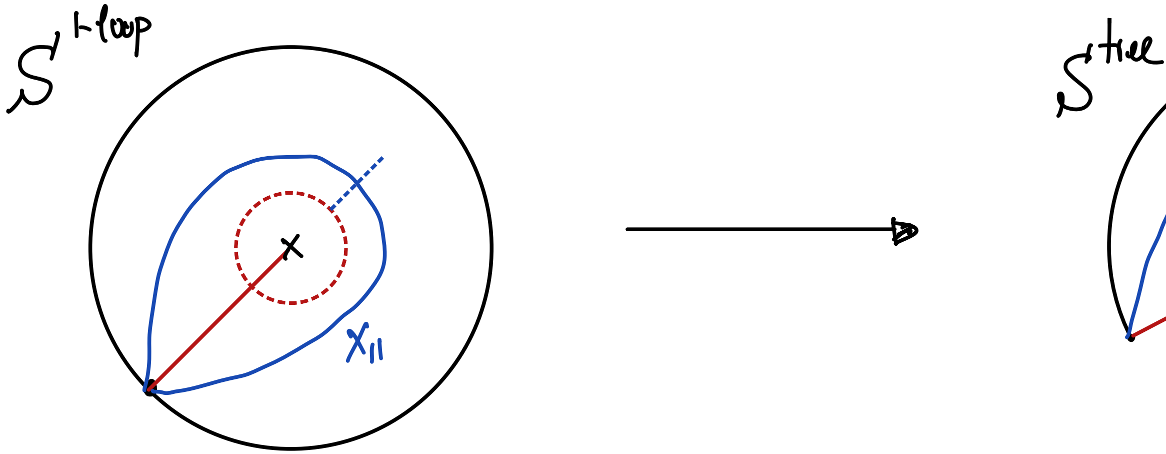
In Surface language: Loop-cut



$X_{11} \equiv 0$ in momentum space

$A^{\text{tree}}\big|_{X_{11}=0}$ blows up!

In Surface language: Loop-cut



$X_{11} \equiv 0$ in momentum space

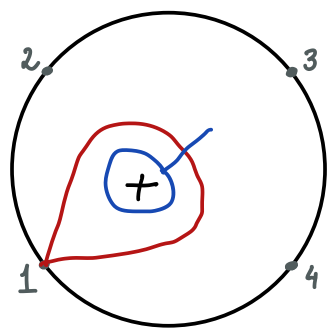
$A^{tree}\big|_{X_{11}=0}$ blows up!

BUT,

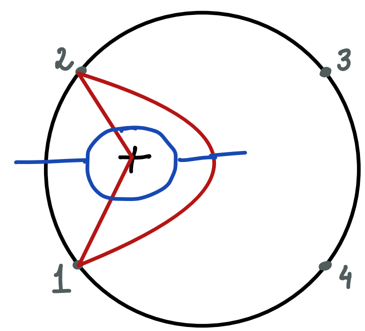
in Surface integral CAN keep curves w/o standard momentum

$$A^{\delta}_{2n} = \int \prod \frac{dy}{y^{1+\delta}} \prod_{C\in S} u_{ij}^{X_C}$$

HOMOLOGY → HOMOTOPY!



$X_{11} \sim$ tadpoles



$X_{2,1} \sim$ Bubbles

Lorentzian $\xrightarrow{\text{Tree}}$ Complex $\xrightarrow{\text{Loop}}$ "Surface" Kinematics

(Curves on surfaces)

"The" (Surface) Integrand for Pions & Gluons

* Exists! ⇒ Can be obtained Recursively.

[Tree, One-loop, general D Gluons
(higher-loops in progress)
All-loop Pions]

$$A_{2n}^{\delta} = \int \prod \frac{dy}{y^{1+\delta}} \prod_{C \in S} u_{ij}^{X_C}$$

Lorentzian $\xrightarrow{\text{Tree}}$ Complex $\xrightarrow{\text{Loop}}$ "Surface" Kinematics

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"The" (Surface) Integrand for Pions & Gluons

* Exists! ⇒ Can be obtained Recursively.

[Tree, One-loop, general D Gluons
(higher-loops in progress)
All-loop Pions]

* Adler Zero (Pions)

* Gauge Invariance (Gluons).

$$A^{\delta}_{2n} = \int \prod \frac{dy}{y^{1+\delta}} \prod_{C \in S} u_{ij}^{X_C}$$

* Splits ~ Multi-soft limits. ⇒ All-order statements for pions and Gluons.

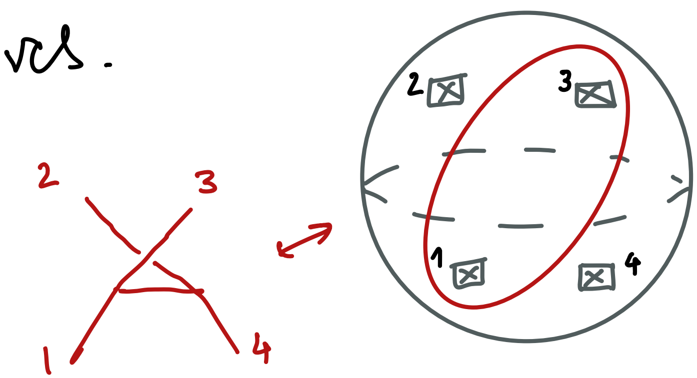
Outlook...

* Fermions.

* Realistic particle content. (Beyond) SM

$$\phi^i_j \overset{u(N)}{\rightleftarrows} \longrightarrow \overset{SU(3)\times SU(2)\times U(1)}{\begin{matrix} Q\,(3, 2, +1/6) \\ H\,(1, 2, -1/2) \end{matrix}} \longrightarrow \text{Extension of SM?}$$

* Gravity from Closed Curves.



Thank You !