3d gravity as a random ensemble

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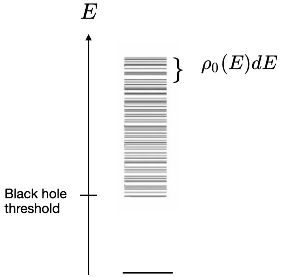
To appear soon with Dan Jafferis and Liza Rozenberg

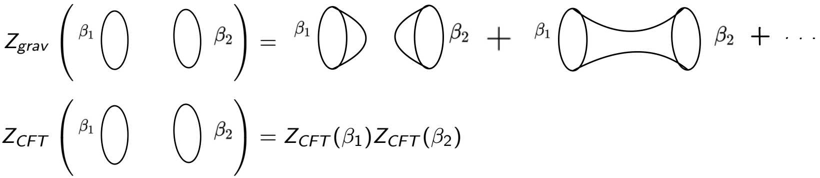
- Semi-classical gravity averages over UV microstates. This is due to black hole solutions, whose horizon area gives the high energy density of states:

$$\rho_0(E) = \exp \frac{\text{Area}(E)}{4G}$$

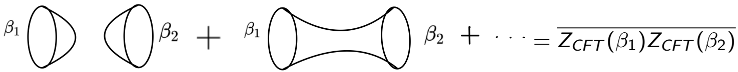
- This density of states is continuous because it is averaged over a high energy window. In AdS3, it corresponds to the Cardy density of the dual CFT.



- Gravity also computes ensemble averages. This is because of wormholes.



- The inclusion of wormhole topologies provides a coarse grained description in terms of an ensemble of quantum systems (see Monday's review talk).



- These two types of coarse graining are related. For a chaotic system like a black hole, averaging over a high energy window is indistinguishable from averaging over an ensemble.

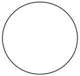
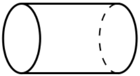
Gravity as a maximum ignorance ensemble

- Just like in statistical physics, the ensemble interpretation of quantum gravity provides the "best" description of a system given a set of constraints (de Boer)
- For instance, the micro-canonical ensemble maximizes entropy subject to the constraint of fixed energy.
- A random matrix model, describing a random Hamiltonian H, is the maximum entropy ensemble for a given average density of states.

$$\mathcal{Z} = \int dH \quad e^{-V(H)} \qquad\qquad V(H) \longleftrightarrow \overline{\rho(E)}_{\text{ensemble}}$$

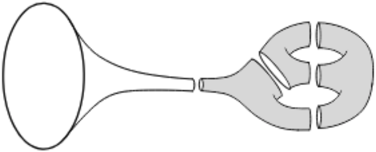
JT gravity as a maximum ignorance ensemble

- Saad, Shenker, Stanford (SSS) showed that two dimensional JT gravity is equivalent to a double scaled matrix model.
- Inputs:

$\overline{\rho(E)}_{\text{disk}} = E$  $\overline{\rho(E)\rho(E')}_{cylinder} = E$  E' (1)

  The cylinder determines the eigenvalue statistics associated to the symmetry class of the ensemble.

- This determines the gravity path integral on all 2D geometries via the machinery of topological recursion



[figure from SSS].

AdS3 gravity as a maximum ignorance ensemble

- We consider a generalization of the SSS model and study an ensemble of CFT2's dual to AdS3 gravity
- CFT2 data is given by the Dilatation operator Δ_s graded by spin, and the OPE coefficients C_{ijk}. Assuming integers spins, we get a set of random matrices and a random tensor:

$$(\Delta_s, C), \qquad s \in \mathbb{Z}$$

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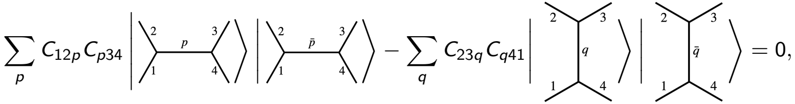

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  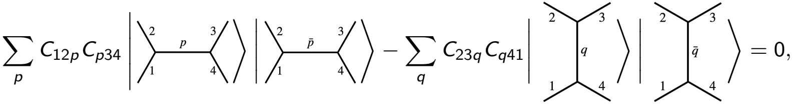

  and torus modular invariance.
- We will argue that the sum over topologies in AdS3 gravity produces the maximum ignorance ensemble consistent with the modular bootstrap.

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Ensemble of approximate CFT's

- An ensemble of exact CFT2's would have a potential V_0 defined by the Cardy density, and a delta function imposing the bootstrap constraints:

$$\mathcal{Z}_0 \equiv \sum_{s\in\mathbb{Z}} \int D[\Delta_s]D[C]e^{-V_0(\Delta_s)}\delta(\text{constraints})$$

- For irrational CFT's with only Virasoro symmetry, we don't know how to impose these constraints. Following (BdBJNS), we relax the constraints by smearing the delta function with a parameter $\hbar$

$$\mathcal{Z}_\hbar \equiv \sum_{s\in\mathbb{Z}} \int D[\Delta_s]D[C]\exp\left(-V_0(\Delta_s) - \frac{1}{\hbar}V(\Delta_s, C)\right)$$

- $V(\Delta_s, C)$ is a "constraint squared" potential that is minimized on solutions of the bootstrap. This defines an ensemble of *approximate* CFT's, with $\hbar$ parametrizing the violation of the bootstrap.

- Since $\hbar = 0$ localizes to exact CFT's, computing $\mathcal{Z}_\hbar$ and then taking $\hbar \to 0$ should teach us something about the space of CFT's

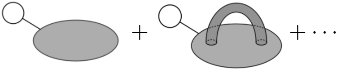
3d gravity from the ensemble of approximate CFT's

- We first define the ensemble as an integral over a finite tensor and matrices by truncating to N primaries. Then take a double scaling limit of Δ_s that sends $N \to \infty$

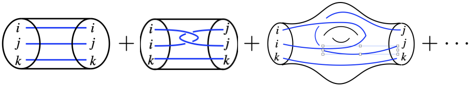
- Apriori, we want $\hbar \to 0$ at fixed central charge c. However, we will first re-organize the Feynman diagrams of the ensemble into an e^{-c} expansion, then take $\hbar \to 0$

- We will argue that the e^{-c} expansion of the ensemble reproduces the topological expansion of 3d gravity.

$$\overline{\rho(\Delta, s)} =$$



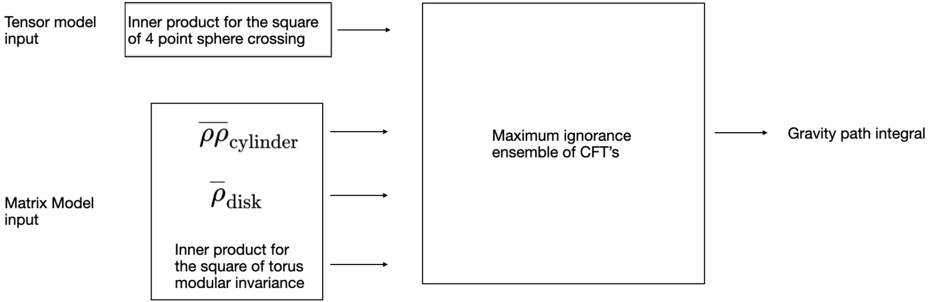
$$\overline{C_{ijk} C_{kji}} =$$



$$\sum_i \overline{C_{ijk} C_{kji}} \longrightarrow \text{surgery on the manifolds above}$$

- Unifies many recent works on 3d gravity and random ensembles: Mertens/Turiaci,Collier/Maloney Maxfield/Tsaires, Cotler/Jensen, Belin/de Boer, Anous/Belin/de Boer/ Liska, Chandra/Collier/Hartman/ Maloney,Jafferis/Kolchmeyer/Mukhametzhanov/Sonner,Yan...

A flow chart



Tensor model input
Inner product for the square of 4 point sphere crossing
Matrix Model input
$\overline{\rho}\rho_{\text{cylinder}}$
$\overline{\rho}_{\text{disk}}$
Inner product for the square of torus modular invariance
Maximum ignorance ensemble of CFT's
Gravity path integral

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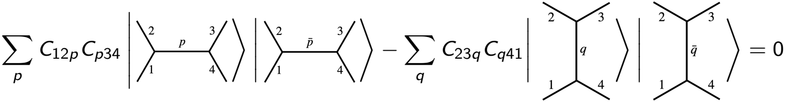
Verlinde inner product and 4 point crossing

- 4 point crossing = the vanishing of a vector.



We want an inner product to take its square.

Verlinde inner product and 4 point crossing

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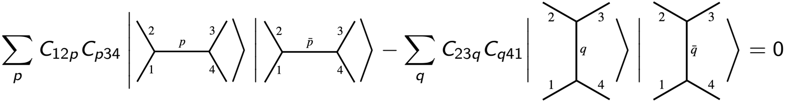

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- The space of Virasoro conformal blocks forms a Hilbert space equipped with the Verlinde inner product and a representation of crossing transformations.

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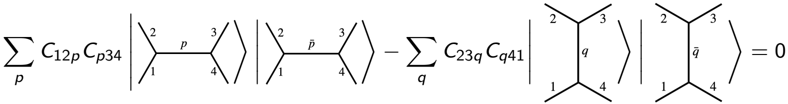

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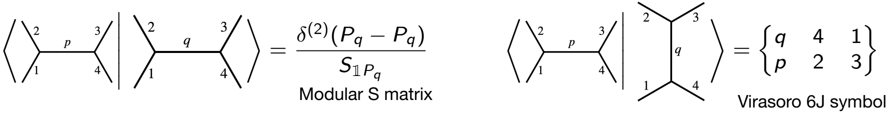
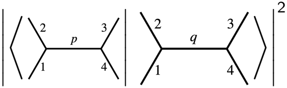


- This is the Hilbert space structure of Virasoro TQFT (Verlinde/Collier/Eberhardt/Mengyang).

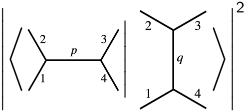
Constraint squared potential

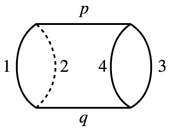
- The square of 4 point crossing defines a quartic potential

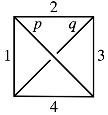
$$V_4 = \sum_{i_1 \cdots i_4} \sum_{p,q} \left(C_{i_1 i_2 p} C_{p i_3 i_4} C_{q i_2 i_1} C_{i_4 i_3 q} \right)$$



$$- \left(C_{i_1 i_2 p} C_{p i_3 i_4} C_{i_1 i_4 q} C_{i_3 i_2 q} \right)$$



Pillow 



6J

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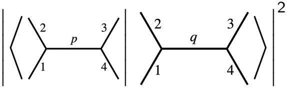

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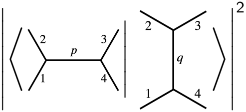

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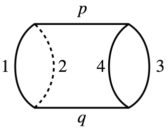
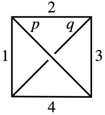
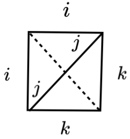




6J

- The kinetic term comes from a special case of the 6J:

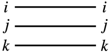
$$C_{ii\mathbb{1}} = 1 \longrightarrow V_2 = -\sum_{ijk} C_{ijk} C_{jik}$$



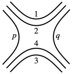
theta

Triple line Feynman diagrams and theirgravity interpretation

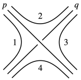
- The triple line Feynman rules are defined by removing the vertices from the graphs



$= \hbar\delta_{il}\delta_{jm}\delta_{kp}$,

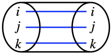


$= \frac{1}{\hbar}\frac{\delta(P_p - P_q)}{S_{\mathbb{1}P_q}}$,

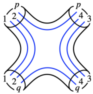


$= \frac{1}{\hbar}\begin{Bmatrix} q & 4 & 3 \\ p & 2 & 2 \end{Bmatrix}$

- We interpret these diagrams as multiboundary wormholes with Wilson lines inserted:



Theta



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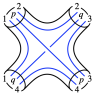

6J

- They come from Wilson line networks in the three sphere, with solid balls removed around the junctions.
- These Feynman rules generate a sum over 3- manifolds. On a fixed manifold they agree with 3d gravity as defined by 2 copies of Virasoro TQFT .

$$Z_{\text{tensor}}(M) = |Z_{\text{Vir}}(M)|^2$$

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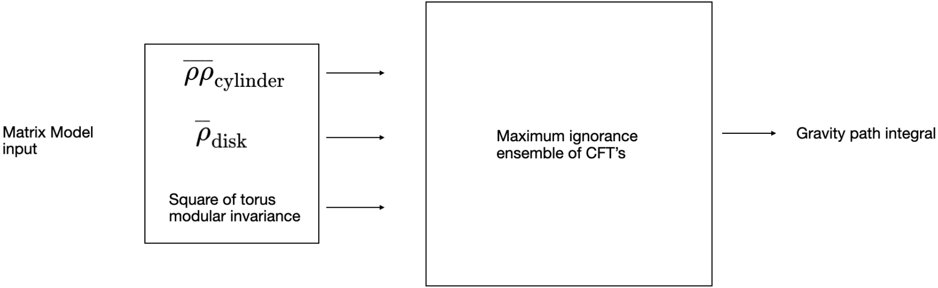
A flow chart


Matrix Model input
$\overline{\rho\rho}_{\text{cylinder}}$
$\overline{\rho}_{\text{disk}}$
Square of torus modular invariance
Maximum ignorance ensemble of CFT's
Gravity path integral

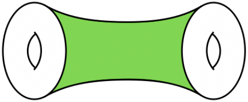
Random matrix statistics and the torus wormhole

- The random Hamiltonians Δ_s belong to the GOE ensemble (Yan). The GOE measure exponentiates to the Vandermonde potential

$$K(\Delta_1, s_1; \Delta_2, s_2) = \delta(s_1 - s_2)\mathrm{Log}|\Delta_1 - \Delta_2|$$ [1]

- The "cylinder" contribution to $\overline{\rho\rho}$ is the inverse of the Vandermonde kernel. It is the propagator for ρ.

$$\overline{\rho(\Delta_1, s_1)\rho(\Delta_2, s_2)}_{cylinder} = K^{-1}(\Delta_1, s_1; \Delta_2, s_2) =$$

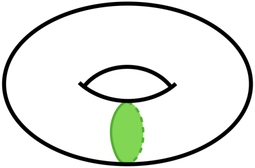


It agrees with the 3d gravity path integral on the torus wormhole computed in (Cotler/Jensen 20), up to an extra factor of 2 due to extra wormholes needed to get agreement with the GOE ensemble (Yan, Jensen) .

Cardy density and the BTZ black hole

- The disk density for Δ_s is given by the Cardy formula. This is a product of Virasoro S matrix elements in the left and right moving sector.

$$\overline{\rho(h,\bar{h})}_{\text{disk}} = \mathbb{S}_{\mathbb{1}h}\mathbb{S}_{\mathbb{1}\bar{h}} =$$



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$$\Delta \geq s + \frac{c-1}{12}$$

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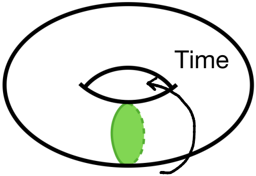

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$$Z_{\text{BTZ}}(\tau,\bar{\tau}) = \chi_{\mathbb{1}}\left(-\frac{1}{\tau}\right)\chi_{\mathbb{1}}\left(-\frac{1}{\bar{\tau}}\right) = \int_{\frac{c-1}{24}}^{\infty}\int_{\frac{c-1}{24}}^{\infty} dh d\bar{h}\; \mathbb{S}_{\mathbb{1}h}\mathbb{S}_{\mathbb{1}\bar{h}}\; \chi_h(\tau)\chi_{\bar{h}}(\bar{\tau})$$

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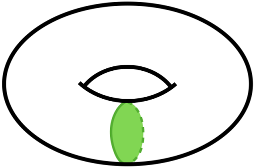

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- To get all spins, write $\sum_{s\in\mathbb{Z}} = \int ds \sum_{n=-\infty}^{\infty} e^{2\pi i n s}$

$$\mathcal{Z}_0 \equiv \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} ds\ e^{2\pi i n s} \int D\Delta_s D[C]\exp(-V_0(\Delta_s))$$

↑ T transform

Constraint squared potential for S-modular invariance I:

- The (approximate) CFT partition function is a vector on the Hilbert space $\mathcal{H}_{T^2} \otimes \mathcal{H}_{T^2}$ of Virasoro characters

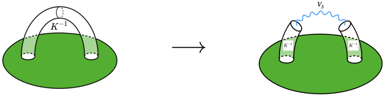
$$|Z(\tau,\bar{\tau})\rangle = \sum_i |h_i\rangle\, |\bar{h}_i\rangle$$

- Given an inner product, we can define a double trace potential for modular invariance

$$V_S \equiv |(\mathbb{1} - \mathbb{S})\,|Z(\tau,\bar{\tau})\rangle|^2 = \sum_{i,j} \langle h_i, \bar{h}_i | \hat{V}_S | h_j, \bar{h}_j \rangle$$

- We then expand in powers of the potential $\frac{1}{\hbar} V_S$:

$$\mathcal{Z} \equiv \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} ds \;\; e^{2\pi i n s} \int D\Delta_s D[C] \exp(-V_0(\Delta_s) - \frac{1}{\hbar} V_S)$$



- We need to find the gravitational inner product: it is not the one defined by VTQFT.

Gravity inner product:Torus wormhole

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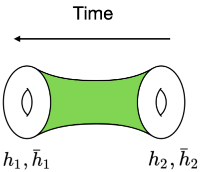

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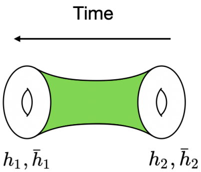
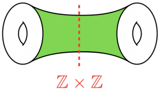


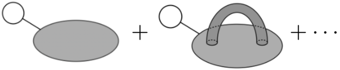
- Because 3d gravity is topological, this propagator is the identity on $\mathcal{H}_{T^2} \otimes \mathcal{H}_{T^2}$. K^{-1} gives the resolution of identity in an non-orthogonal basis of Virasoro characters
- The non orthogonality comes from the the gauging of the bulk mapping class group.



- The VTQFT inner product $\longrightarrow$ orthogonal Virasoro characters. VTQFT has different random matrix statistics than gravity.

$SL(2,\mathbb{Z})$ modular invariance from sum over topologies

- Before adding V_S, we have a simple set of 3 manifolds obtained from the genus expansion of the double scaled matrices Δ_s. The expansion parameter the level spacing e^{-c}.



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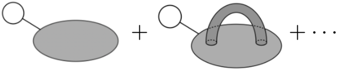
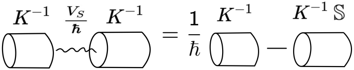
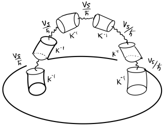


- Introducing V_S using the gravity inner product inserts S transforms into the propagation of a bulk toriodal slice





In the $\hbar \to 0$ limit this produces a projector onto the $\mathbb{S} = 1$ states $\Rightarrow$ S modular invariance.

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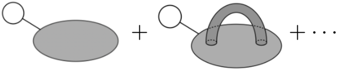

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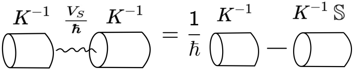
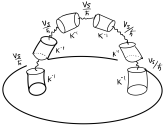




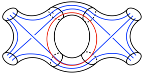
In the $\hbar \to 0$ limit this produces a projector onto the $\mathbb{S} = 1$ states $\Rightarrow$ S modular invariance.

- Full $SL(2,\mathbb{Z})$ modular invariance is achieved when combined with the sum over T transforms.
- These new wormholes produces Seifert manifolds that are needed to cure the negative density of states that would arise from the $SL(2,\mathbb{Z})$ sum over BTZ black holes (Maxfield-Turiaci).

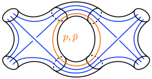
Table of Contents

Surgery = averaging over tensor model Loops

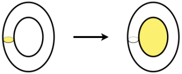
- Loops in the tensor model diagrams ↔ insertion of the Cardy density



$$= \int d^2p S_{\mathbb{1}p} S_{\mathbb{1}p}$$



- This implements toroidal surgery, which is equivalent to Moore-Seiberg identities



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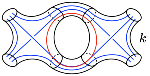
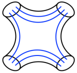
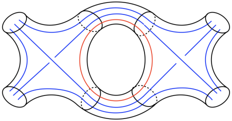


k $= l$



k

- Gluing 6J's with a relative rotation creates arbitrary braids



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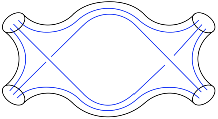

This allows us to build all 3 manifolds (Lickorish)

Schwinger Dyson and the sum over topologies

- Due to surgery relations, we obtain many tensor model diagrams with the same topology, weighted by different powers of $\hbar$.

$$\langle C \cdots C \rangle = \sum_M Z_{\text{grav}}(M, c) f(M, \hbar). \qquad\qquad Z_{\text{grav}}(M, c) \sim e^{-c\text{Vol}M}$$

- To match with 3d gravity, we want $f(M, \hbar) \to 1$ as $\hbar \to 0$

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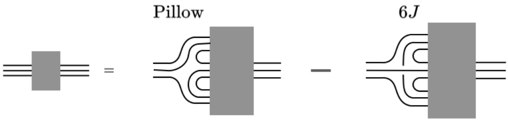
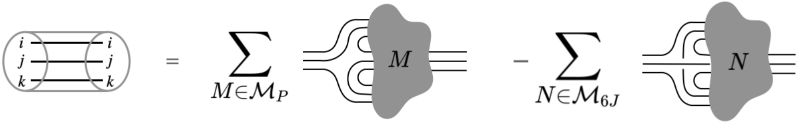
- To match with 3d gravity, we want $f(M, \hbar) \to 1$ as $\hbar \to 0$
- We can check this conjecture using the $\hbar \to 0$ limit of the Schwinger Dyson equation, e.g.



- For a single manifold on the LHS, we have a finite counting problem



Summary and final thoughts

- We showed how the topological expansion of 3d gravity arises from a random ensemble of approximate CFT's.
- The sum over topologies implements the bootstrap constraints order by order in the e^{-c} expansion
- A successful non-perturbative completion of the sum would solve the bootstrap equations exactly, and land us in the space of exact CFT's. We expect this is a $e^{-e^{c}}$ question.
- The matrix model sums over a simpler class of 3 manifolds: perhaps we should start by understanding the non-perturbative completion of this sum.

Outlook

- A matrix model prediction: 3 boundary torus wormhole
- Some loose ends:
  - Solve the Schwinger-Dyson equation : generalization of topological recursion?
  - renormalization /cancellation of S^2 handle divergences
  - Regularization of the $SL(2, \mathbb{Z})$ accumulation point = hyperbolic cusp
- Future work:
  - Random BCFT and EOW branes
  - Adding matter
  - Random ensemble for dS3?
  - Random ensemble of RCFT's