Bootstrapping the AdS Virasoro-Shapiro amplitude

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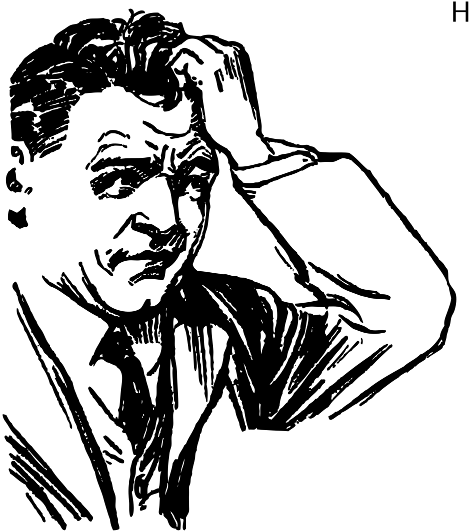
Strings 2024, CERN
June 7, 2024

Based on:

2204.07542, 2209.06223, 2305.03593 with Luis F. Alday, João Silva

2306.12786 with Luis F. Alday

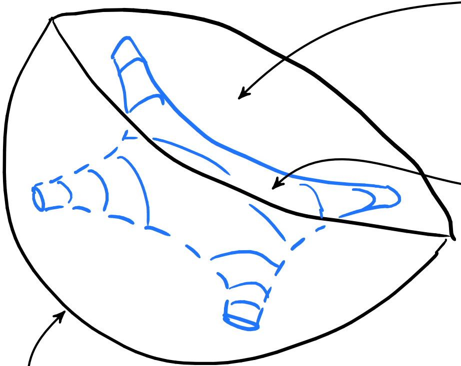


How to formulate string theory on curved spacetime?

At least for AdS_5/CFT_4?

WWVD? – Fix the amplitude first!

1 process - 3 descriptions



5d bulk of AdS:
IIB string theory on $AdS_5 \times S^5$

2d string worldsheet:
2d CFT???

4d boundary of AdS:
$\mathcal{N} = 4$ super Yang Mills theory

This talk:
Find the genus 0 amplitude without quantizing the string.

The AdS amplitude

CFT stress-tensor correlator		AdS graviton amplitude
$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle$	⟷ integral transform	$A(S,T)$

Small curvature expansion:

$$A(S,T) = A^{(0)}(S,T) + \frac{\alpha'}{R^2} A^{(1)}(S,T) + \left(\frac{\alpha'}{R^2}\right)^2 A^{(2)}(S,T) + \dots$$

$$A^{(0)}(S,T) = -\frac{\Gamma(-S)\Gamma(-T)\Gamma(-U)}{\Gamma(S+1)\Gamma(T+1)\Gamma(U+1)}$$

$R =$ AdS radius

$$\frac{\alpha'}{R^2} = \frac{1}{\sqrt{\lambda}}$$

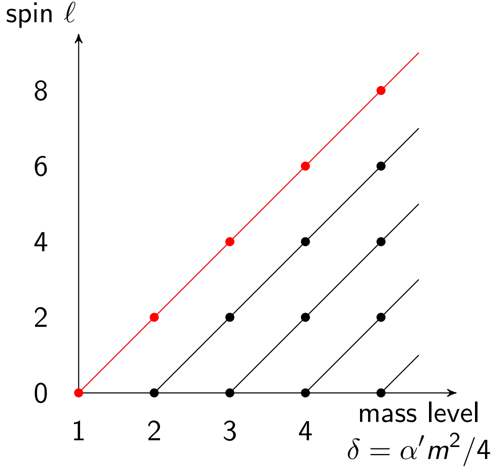
λ ← t'Hooft coupling

Properties of string amplitudes

Partial wave expansion

Flat space:
resonances = massive string modes

$$\lim_{T\to\delta} A^{(0)}(S,T) = \sum_{\ell} \frac{a_{\delta,\ell} P_{\ell}(\cos\theta)}{T-\delta}$$



AdS/CFT:
conformal partial wave expansion (OPE)

$$\Delta = Rm + \ldots = R\sqrt{\frac{4\delta}{\alpha'}}\left(1 + O\left(\frac{\alpha'}{R^2}\right)\right)$$

non-critical string theory
[Gubser,Klebanov,Polyakov;1998]

integrability
[Gromov,Serban,Shenderovich,Volin;2011]

Regge boundedness

String amplitudes have soft UV (Regge) bahaviour

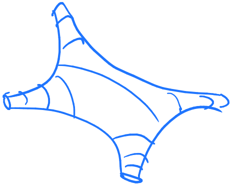
$$\lim_{|S|\to\infty} A^{(0)}(S,T) \sim S^{T+\alpha_0}, \quad \mathrm{Re}(T) < 0$$

Softer than each partial wave $P_\ell(\cos\theta) \propto S^\ell + O(S^{\ell-1})$

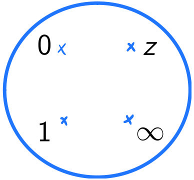
$$A^{(0)}(S,T) = \sum_{\text{spin } \ell} a_\ell(T) P_\ell(\cos\theta)$$

This places strong constraints on the $a_\ell(T)$!

Worldsheet integral



$\rightarrow$



Flat space:

$$A^{(0)}(S,T) = \frac{1}{(S+T)^2} \int d^2z \, |z|^{-2S-2} |1-z|^{-2T-2}$$

Curvature corrections:

$$A^{(k)}(S,T) = \frac{1}{(S+T)^2} \int d^2z \, |z|^{-2S-2} |1-z|^{-2T-2} G^{(k)}_{\text{tot}}(S,T,z)$$

Consider non-linear σ-model for AdS in a small curvature expansion
$\rightarrow$ flat space string amplitudes with extra soft gravitons
$\rightarrow$ $G^{(k)}_{\text{tot}}(S,T,z) = \sum$ single-valued polylogarithmic functions of weight $3k$

Plan of attack

We attack the problem from 2 sides:

CFT dispersion relation

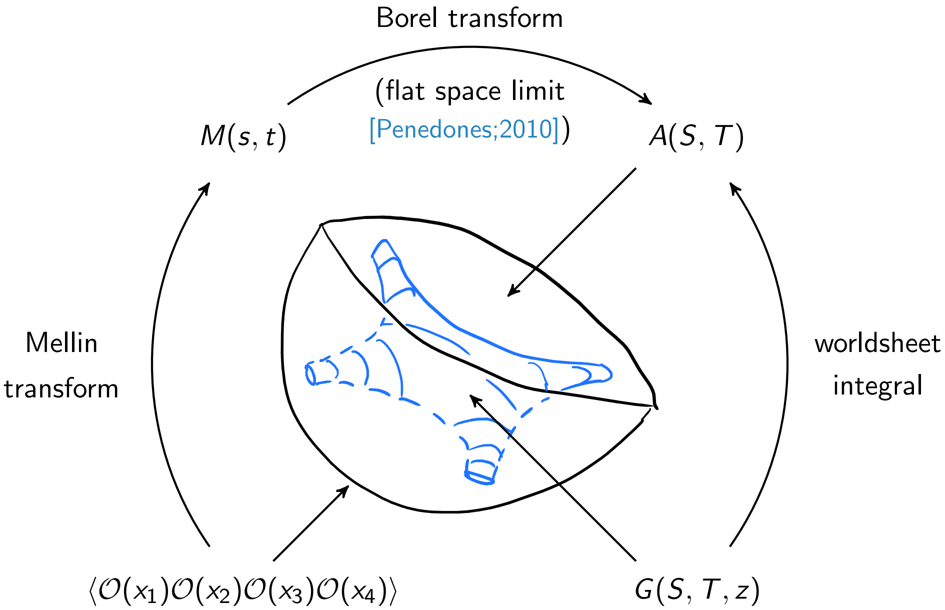
Single-valued worldsheet ansatz

Both have unfixed data.
Equating the two expressions fixes the answer!

Checks:
- Massive string dimensions (vs integrability)
- Low energy expansion (vs localization)
- High energy limit (vs classical scattering)

1. The CFT dispersion relation

Definition of the AdS amplitude



Definition of the AdS amplitude

$$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle \propto \int_{-i\infty}^{i\infty} ds dt\, u^s v^t \Gamma(s,t) M(s,t)$$

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$$

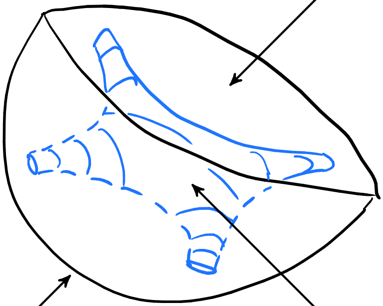
$$v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

$$A(S,T) \propto \int_{-i\infty}^{i\infty} d\alpha \frac{e^\alpha}{\alpha^6} M\left(\frac{\sqrt{\lambda}}{\alpha} S, \frac{\sqrt{\lambda}}{\alpha} T\right)$$

$M(s,t)$ —— Borel transform (flat space limit [Penedones;2010]) ——> $A(S,T)$

$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle$ —— Mellin transform ——> $M(s,t)$

$G(S,T,z)$ —— worldsheet integral ——> $A(S,T)$



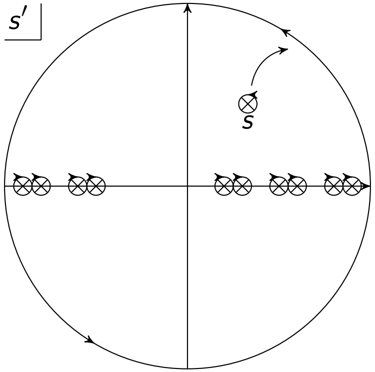
Dispersion relation

$M(s,t)$ has only OPE poles:
[Mack;2009], [Penedones,Silva,Zhiboedov;2019]

$$\text{poles} \sim \frac{C^2_{\Delta,\ell} Q_{\Delta,\ell,m}(t)}{s' - (\Delta - \ell + 2m)}$$

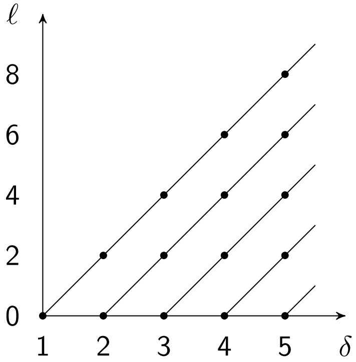
Regge boundedness:

$$\lim_{|s|\to\infty} |M(s,t)| \lesssim |s|^{-2}, \;\; \text{Re}(t) < 2$$



$$M(s,t) = \oint_s \frac{ds'}{2\pi i} \frac{M(s',t)}{(s'-s)} = -\sum_{\text{operators}} \left(\frac{C^2_{\Delta,\ell} Q_{\Delta,\ell,m}(t)}{s - (\Delta - \ell + 2m)} + \frac{C^2_{\Delta,\ell} Q_{\Delta,\ell,m}(t)}{u - (\Delta - \ell + 2m)} \right)$$

Spectrum of massive string operators



massive string modes
= unprotected single-trace operators of $\mathcal{N}=4$ SYM

δ = mass level, ℓ = spin

OPE data expanded in large $\sqrt{\lambda}$:

	$A^{(0)}$ data		$A^{(1)}$ data		$A^{(2)}$ data	
$\Delta_{\delta,\ell} =$	$2\sqrt{\delta}\lambda^{\frac{1}{4}}$	$+$	$\lambda^{-\frac{1}{4}}\Delta^{(1)}_{\delta,\ell}$	$+$	$\lambda^{-\frac{3}{4}}\Delta^{(2)}_{\delta,\ell}$	$+\ldots$
$C^2_{\delta,\ell} =$	$C^{2(0)}_{\delta,\ell}$	$+$	$\lambda^{-\frac{1}{2}}C^{2(1)}_{\delta,\ell}$	$+$	$\lambda^{-1}C^{2(2)}_{\delta,\ell}$	$+\ldots$

Dispersion relation $\to$ Residues

Dispersion relation for $M(s,t) \rightsquigarrow A^{(k)}(S,T)$ expanded around $S=\delta=1,2,\ldots$:

$$A^{(k)}(S,T) = \frac{R^{(k)}_{3k+1}(T,\delta,C^{2(0)}_{\delta,\ell})}{(S-\delta)^{3k+1}} + \ldots + \frac{R^{(k)}_{1}(T,\delta,C^{2(0)}_{\delta,\ell},\ldots,\Delta^{(k)}_{\delta,\ell},C^{2(k)}_{\delta,\ell})}{S-\delta} + \text{reg.}$$

Two lessons:

1. (OPE data)$^{(k-1)}$ fixes most residues of $A^{(k)}(S,T)$!
2. $G^{(k)}_{\text{tot}}(S,T,z)$ should have transcendentality $3k$:

$$\int d^2z\,|z|^{-2S-2}|1-z|^{-2T-2}\log^{3k}|z|^2 \propto \frac{1}{(S-\delta)^{3k+1}} + O\left((S-\delta)^0\right)$$

Next steps (order by order):

- Write worldsheet ansatz for $A^{(k)}(S,T)$.
- Compute its residues and match with the above to fix ansatz.

2. Single-valued functions for the worldsheet

Multiple polylogarithms

Definition ($|z_1 \dots z_r| = r =$ weight) $z_i \in \{0, 1\}$

$$L_{z_1 \dots z_r}(z) = \int_{0 \le t_r \le \dots \le t_1 \le z} \frac{dt_1}{t_1 - z_1} \cdots \frac{dt_r}{t_r - z_r}$$

Properties:

- $\partial_z L_{z_i w}(z) = \dfrac{1}{z - z_i} L_w(z)$
- multi-valued
- holomorphic
- $L_w(1) =$ multiple zeta values

Examples:

- $L_{1^p}(z) = \dfrac{1}{p!} \log^p(1 - z)$
- $L_{0^p 1}(z) = -\mathrm{Li}_{p+1}(z)$

Single-valued multiple polylogarithms

SVMPLs [Brown;2004]

$$\mathcal{L}_w(z) = \sum_{|w_1|+|w_2|=|w|} c_{w_1 w_2} L_{w_1}(z) L_{w_2}(\bar{z})$$

Properties:

- $\partial_z \mathcal{L}_{z_i w}(z) = \frac{1}{z - z_i} \mathcal{L}_w(z)$
- single-valued
- non-holomorphic
- $\mathcal{L}_w(1) \equiv$ single-valued multiple zeta values

Examples:

- $\mathcal{L}_{1^p}(z) = \frac{1}{p!} \log^p |1 - z|^2$
- $\mathcal{L}_{01}(z) = \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) - \log(1 - \bar{z}) \log |z|^2$

Worldsheet correlator (ansatz)

Ansatz:

$$A^{(k)}(S,T) = B^{(k)}(S,T) + B^{(k)}(U,T) + B^{(k)}(S,U)$$

$$B^{(k)}(S,T) = \frac{1}{(S+T)^2} \int d^2z |z|^{-2S-2} |1-z|^{-2T-2} G^{(k)}(S,T,z)$$

Assumed properties of $G^{(k)}(S,T,z)$:

- transcendental weight $3k$ (SVMPLs(z), SVMZVs)
- degree $2k$ polynomial in S, T
- crossing symmetry: $G^{(k)}(S,T,z) = G^{(k)}(T,S,1-z)$

Worldsheet correlator (solution)

First correction: ansatz has 11 rational parameters

Solution

$$G^{(1)}(S,T,z) = (S+T)^2\left(-\frac{1}{6}\mathcal{L}^{+}_{000}(z) + 0\mathcal{L}^{+}_{001}(z) - \frac{1}{4}\mathcal{L}^{+}_{010}(z) + 2\zeta(3)\right) + (S^2-T^2)\left(-\frac{1}{6}\mathcal{L}^{-}_{000}(z) + \frac{1}{3}\mathcal{L}^{-}_{001}(z) + \frac{1}{6}\mathcal{L}^{-}_{010}(z)\right)$$

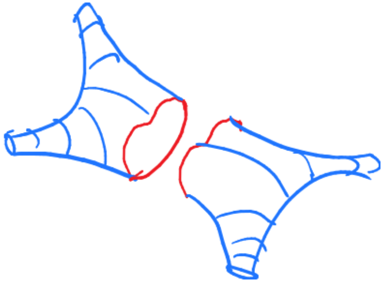
Second correction: ansatz has 115 rational parameters

Solution

$$G^{(2)}(S,T,z) = \frac{1}{18}(S+T)^2(ST-S^2-T^2)\mathcal{L}^{+}_{000000}(z) + \text{ 44 more terms}$$

$$\mathcal{L}^{\pm}_{w}(z) = \mathcal{L}_w(z) \pm \mathcal{L}_w(1-z) + \mathcal{L}_w(\bar{z}) \pm \mathcal{L}_w(1-\bar{z})$$

3. Checks



We extract the OPE data:

		$A^{(0)}$ data		$A^{(1)}$ data		$A^{(2)}$ data	
$\Delta_{\delta,\ell}$	$=$	$2\sqrt{\delta}\lambda^{\frac{1}{4}}$	$+$	$\lambda^{-\frac{1}{4}}\Delta^{(1)}_{\delta,\ell}$	$+$	$\lambda^{-\frac{3}{4}}\Delta^{(2)}_{\delta,\ell}$	$+\dots$
$C^2_{\delta,\ell}$	$=$	$C^{2(0)}_{\delta,\ell}$	$+$	$\lambda^{-\frac{1}{2}}C^{2(1)}_{\delta,\ell}$	$+$	$\lambda^{-1}C^{2(2)}_{\delta,\ell}$	$+\dots$

Leading Regge trajectory ($\delta = 1$ is Konishi):

$$\Delta = 2\sqrt{\delta}\lambda^{\frac{1}{4}}\left(1 + \left(\frac{3\delta}{4} + \frac{1}{2\delta} - \frac{1}{4}\right)\frac{1}{\sqrt{\lambda}} - \left(\frac{21\delta^2}{32} + \frac{1}{8\delta^2} - \frac{(3-12\zeta(3))\delta}{8} - \frac{1}{8\delta} - \frac{17}{32}\right)\frac{1}{\lambda} + \dots\right)$$

Agrees with integrability result!

[Gromov,Serban,Shenderovich,Volin;2011],[Basso;2011],[Gromov,Valatka;2011]

Low energy expansion

Relates to low energy effective action (SUGRA + derivative interactions)

$$A(S,T) = \text{SUGRA} + \sum_{a,b,k=0}^{\infty} \frac{\sigma_2^a \sigma_3^b}{\sqrt{\lambda}^k} \alpha_{a,b}^{(k)}\,, \qquad \sigma_2 = S^2 + T^2 + U^2, \sigma_3 = STU$$

$$= \text{SUGRA} + \underbrace{\alpha_{0,0}^{(0)}}_{R^4} + \underbrace{\frac{\alpha_{0,0}^{(1)}}{\sqrt{\lambda}}}_{D^2R^4} + \underbrace{\sigma_2\, \alpha_{1,0}^{(0)} + \frac{\alpha_{0,0}^{(2)}}{\lambda}}_{D^4R^4} + \underbrace{\sigma_3\, \alpha_{0,1}^{(0)} + \frac{\sigma_2\, \alpha_{1,0}^{(1)}}{\sqrt{\lambda}} + \frac{\alpha_{0,0}^{(3)}}{\sqrt{\lambda}^3}}_{D^6R^6} + \dots$$

$\alpha_{a,b}^{(0)}$ = flat space, we fix all $\alpha_{a,b}^{(1)}$ and $\alpha_{a,b}^{(2)}$, in particular:

$$\alpha_{0,0}^{(1)} = 0\,, \quad \alpha_{1,0}^{(1)} = -\frac{22}{3}\zeta(3)^2\,, \quad \alpha_{0,0}^{(2)} = \frac{49}{4}\zeta(5)\,, \quad \alpha_{1,0}^{(2)} = \frac{4091}{16}\zeta(7)$$

Agrees with localisation result! Altogether we fully fix D^8R^4 and $D^{10}R^4$.

[Binder,Chester,Pufu,Wang;2019],[Chester,Pufu;2020],[Alday,TH,Silva;2022]

High energy limit

Generalizing [Gross,Mende;1987] to AdS [Alday,TH,Nocchi;2023], high energy limit is described by classical solution for

$$\mathcal{S}(X,\Lambda) = \int d^2\zeta \left(\partial X^M \bar{\partial} X_M + \Lambda (X^M X_M + R^2) - i \sum_{j=1}^{4} P_j^M X_M \delta^{(2)}(\zeta - z_j) \right)$$

Matches our result in the high energy limit:

$$\lim_{S,T,R\to\infty} A(S,T) \sim e^{-S(X_{\text{classical}})}$$

Our solution respects exponentiation:

$$\lim_{S,T,R\to\infty} A(S,T) \propto e^{\frac{\alpha'}{R^2} G^{(1)}_{\text{tot}}(S,T,z=\frac{S}{S+T})} \qquad \Rightarrow \qquad G^{(2)}_{\text{tot}}(z=\tfrac{S}{S+T}) = \frac{1}{2}\left(G^{(1)}_{\text{tot}}(z=\tfrac{S}{S+T})\right)^2$$

✓

Checks:

- Massive string dimensions
- Low energy expansion
- High energy limit

AdS Veneziano amplitude

What about open strings?

type IIB string theory in $AdS_5 \times S^5$ with 7-branes $\leftrightarrow$ $4d\ \mathcal{N}=2$ SCFT

We fixed $G^{(1)}_{\text{open}}$ and $G^{(2)}_{\text{open}}$ in the color-ordered gluon amplitude ($G^{(0)}_{\text{open}}=1$):

$$A_{\text{open}}(S,T)=\frac{1}{S+T}\int_0^1 dx\, x^{-S-1}(1-x)^{-T-1}\sum_{k=0}^{\infty}\left(\frac{\alpha'}{R^2}\right)^k G^{(k)}_{\text{open}}(S,T,x)$$

[Alday,Chester,TH,Zhong;2024],[Alday,TH;2024]

Future directions

- Other AdS backgrounds, e.g. type IIA on $AdS_4 \times CP^3$ / ABJM
- Go beyond the small curvature expansion?
- Compute AdS string amplitudes directly from string theory?

Thank you!