Massless Lifshitz Field Theory for Arbitrary z

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Lifshitz field theory

- LFTs are a class of non-relativistic field theories which are spatially isotropic, homogeneous and admits the scaling symmetry

$$t \to \lambda^z t, \quad x^i \to \lambda x^i, \quad \lambda > 0.$$

- For $z = 2$, Lifshitz scalar field theory in (2+1) dimensions known as Quantum Lifshitz model (QLM) describe the critical point of the well-known Rokhsar-Kivelson Quantum dimer model. [Moessner, Sondhi and Fradkin '01] [Ardonne, Fendley and Fradkin '04]
- Various entanglement measures such as entanglement entropy [Fradkin, Moore, Hsu, Thorlacius....], entanglement negativity [Angel-Ramelli et al. '20], reflected entropy and Markov gap [Berthiere, Chen and Chen '23] have been studied mostly for integer z.
- We employ the notion of fractional derivatives to study the massless Lifshitz theory for arbitrary values of z in any dimensions.

Massless Lifshitz scalar theory and Lifshitz ground state

- Consider the action for the massless Lifshitz scalar field theory in (1+1)-dimensions for arbitrary $z > 1$

$$S = \frac{1}{2} \int dt dx \left[(\partial_t \phi)^2 - \kappa^2 (\nabla_x^z \phi)^2 \right].$$

- In our work, we use the following definition of fractional derivative ∇_x^z

$$\nabla_x^z e^{ikx} \equiv (ik)^z e^{ikx}.$$

- Then, the fractional derivative of any arbitrary function can be obtained using the Fourier analysis with appropriate choice of integral contour

$$\nabla_x^z F(x) = \int_C \mathcal{F}(k) (ik)^z e^{ikx} dk.$$

- The ground state of the Lifshitz theory is given by

$$|\Psi_0\rangle = \frac{1}{\sqrt{\mathcal{Z}}} \int \mathcal{D}\phi \, e^{-S_{\rm cl}[\phi]/2} |\phi\rangle, \quad S_{\rm cl}[\phi] = \kappa \int \left(\nabla_x^{\frac{z}{2}} \phi \right)^2 dx.$$

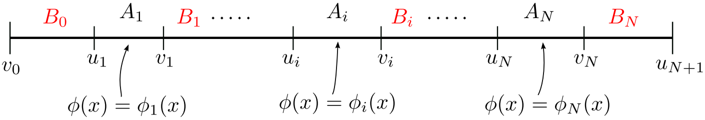
- This ground state takes the form of RK vacuum, it is given by a superposition of quantum states with a quantum mechanical amplitude $c[\phi] \propto e^{-S_{\mathrm{cl}}[\phi]/2}$.
- The propagator of the theory is given by

$$K(\phi_i, \phi_f; x_i, x_f) = \int_{\phi(x_i)=\phi_i}^{\phi(x_f)=\phi_f} \mathcal{D}\phi \exp\left(-\kappa \int_{x_i}^{x_f} \left(\nabla_x^{\frac{z}{2}} \phi\right)^2 dx\right).$$

- Usually the integral can be evaluated by integrating out the fluctuations around the classical solution ϕ_c and expressed as

$$K(\phi_i, \phi_f; l) = \sqrt{\frac{\gamma}{\pi l^{z-1}}} e^{-\gamma(\phi_f - \phi_i)^2 / l^{z-1}}.$$

- Consider a subsystem $A \equiv \bigcup_{i=1}^{N} A_i$



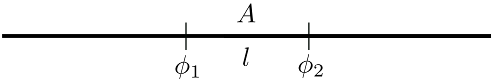
- The trace $\mathcal{Z}_n \equiv \int \mathcal{D}\phi_A (\rho_A^n)_{\phi_A, \phi_A}$ is given by

$$\mathcal{Z}_n = \frac{1}{\mathcal{Z}^n} \int_{-\infty}^{\infty} d\alpha_1 d\beta_1 \cdots d\alpha_N d\beta_N \prod_{i=1}^{N} K^n(u_i, v_i) \prod_{i=1}^{N} K^n(v_i, u_{i+1}).$$

Entanglement entropy

- For a finite subsystem A of length l in an infinite system, the trace $\mathcal{Z}_n$ is given by

$$\mathcal{Z}_n = \mathcal{Z}^{-n} \int d\phi_1 \int d\phi_2 K(\phi_1, \phi_2; l)^n.$$



- Using the form of the propagator, the Rényi entropy may be expressed as

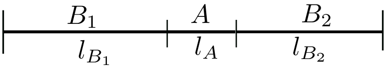
$$S_n(A) = \frac{z-1}{2} \log \frac{l}{\epsilon} + \frac{c_n}{2}.$$

- This is different from the usual case of a conformal vacuum where the UV parts are proportional with a nontrivial n-dependent coefficient

$$[S_n(A)]_{\text{UV}} = \frac{1}{2}(1 + 1/n)[S(A)]_{\text{UV}}.$$

- We observe that the Lifshitz vacuum is different from the vacuum of the CFT.

Mutual information



- The mutual information between B_1 and B_2 is given by

$$I(B_1 : B_2) = \frac{1}{2} \log \frac{\left(l_{B_1}^{z-1} + l_A^{z-1}\right)\left(l_{B_2}^{z-1} + l_A^{z-1}\right)}{l_A^{z-1}\left(l_{B_1}^{z-1} + l_A^{z-1} + l_{B_2}^{z-1}\right)} = \frac{1}{2} \log \frac{1}{1-\tilde{\eta}}.$$

- Here the cross-ratio $\tilde{\eta}(z)$ is given by

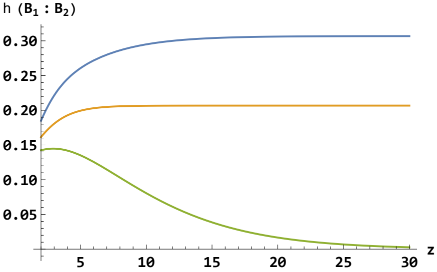
$$\tilde{\eta}(z) := \frac{(l_{B_1} l_{B_2})^{z-1}}{\left(l_{B_1}^{z-1} + l_A^{z-1}\right)\left(l_{B_2}^{z-1} + l_A^{z-1}\right)}.$$

- When $l_A \ll l_{B_i}$, then $\tilde{\eta}(z) \to 1$. It happens same for $l_A < l_{B_i}$ and $z \gg 1$.
- The mutual information maximizes in these cases which is expected since the interactions of the theory have increasing range while the length l_A is small compared to the rest subsystems sizes.

Reflected entropy and Markov gap

- The Markov gap for the configuration of disjoint intervals can be obtained as

$$h(B_1 : B_2) = \frac{1}{\sqrt{1-\tilde{\eta}}} \log\left(\frac{1+\sqrt{1-\tilde{\eta}}}{\sqrt{\tilde{\eta}}}\right) - \log\left(\frac{2(1-\tilde{\eta})}{\sqrt{\tilde{\eta}}}\right).$$



- For $l_A \leq \min\{l_{B_1}, l_{B_2}\}$, $h(B_1 : B_2)$ increases up to a constant value whereas for $l_A > \min\{l_{B_1}, l_{B_2}\}$, $h(B_1 : B_2)$ decays to zero.
- We observe that with increasing degrees of anisotropy of the Lifshitz field theory, the tripartite entanglement can be enhanced or completely destroyed depending on the sizes of the partitions.

THANK YOU!