RESURGENCE AND NON-PERTURBATIVE TOPOLOGICAL STRINGS

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Introduction and motivation

String theory is only defined perturbatively, and it has been known for some time that its genus expansion diverges factorially, like *(2g)!*

String Perturbation Theory Diverges

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This is a signal of exponentially small, non-perturbative (NP) effects in the string coupling constant, coming from sectors of the theory which are invisible in a perturbative framework.

We expect these NP effects to be closely related to D-branes [Polchinski]. This expectation has been verified in detail in non-critical strings, where we have a good control of both perturbative and non-perturbative sectors [Martinec, Alexandrov-Kazakov-Kutasov, …] [see Raghu Mahajan's talk].

In this talk I will summarize recent progress in understanding NP effects for **topological string theory** on general Calabi-Yau (CY) manifolds. In line with the above expectations, we will see that the perturbative topological string free energies know secretly about the spectrum of BPS D-branes on the CY.

The tool

To describe the NP effects in the topological string, I will use the **theory of resurgence**.

The physics insight behind this theory is that the large order behavior of the perturbative series gives access to NP sectors [Bender, Wu, 't Hooft, Parisi, Lipatov, Brezin, Zinn-Justin,…]. This insight is routinely used in QCD to obtain information on non-perturbative effects (renormalon analysis).

The reconstruction of NP sectors in the theory of resurgence is a well-defined mathematical problem, but in general hard to solve explicitly.

A crash course on resurgence

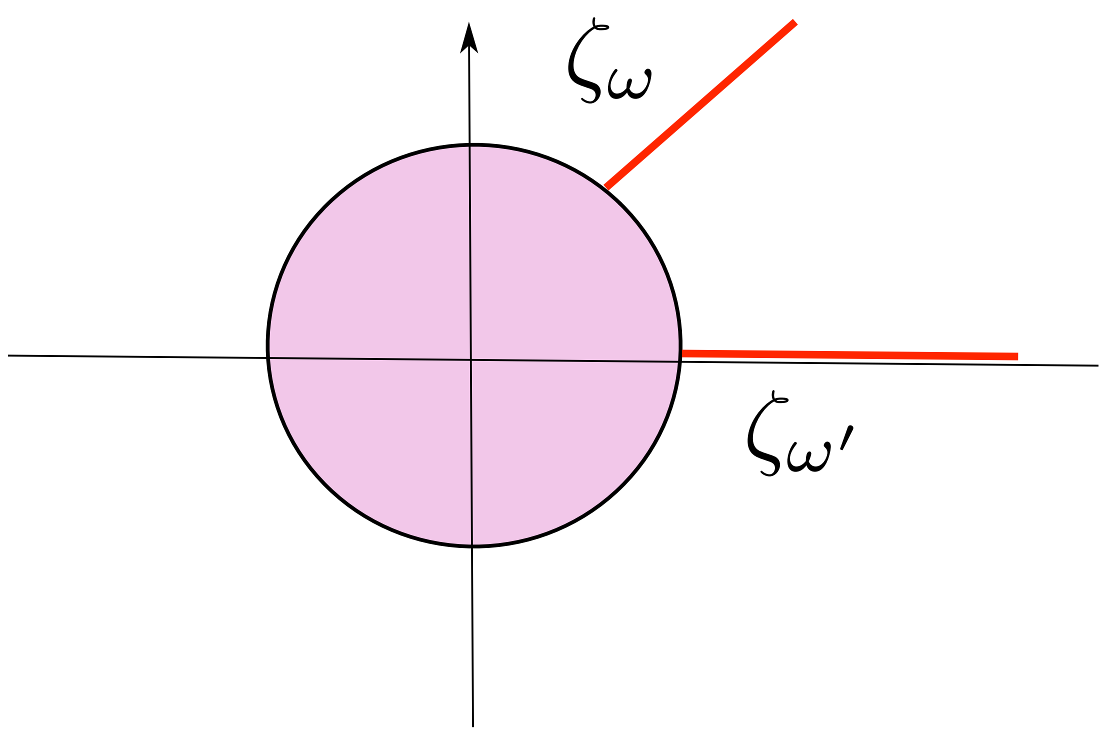
$$\varphi(z) = \sum_{n\geq 0} a_n z^n$$

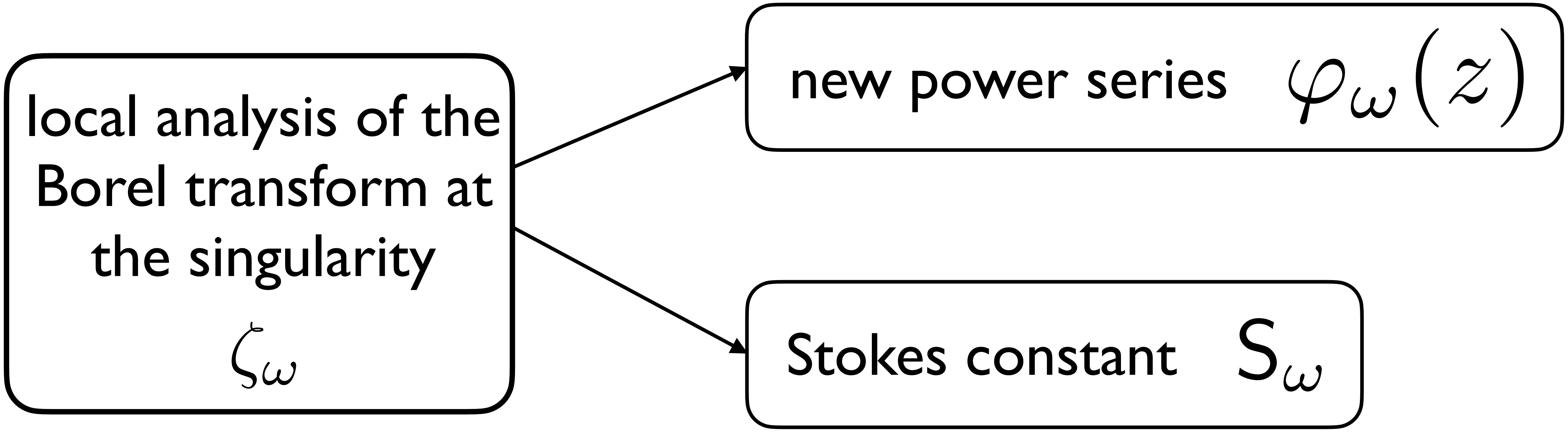
Borel transform

$$\widehat{\varphi}(\zeta) = \sum_{n\geq 0} \frac{a_n}{n!}\zeta^n$$

$$a_n \sim n!$$

The Borel transform of a factorially divergent series is analytic at the origin, and has **singularities** in the complex plane (poles, branch cuts).





With this, we build up a non-perturbative amplitude or **trans-series**

$$\Phi_\omega(z) = \mathsf{S}_\omega \, \mathrm{e}^{-\zeta_\omega / z} \varphi_\omega(z)$$

The resulting collection of trans-series, for all singularities, is called the **resurgent structure** of the original perturbative series $\varphi(z)$. It can be also obtained from the large order behaviour of this series.

Enter topological strings

Let *M* be a Calabi-Yau (CY) threefold. At each genus *g* one can consider the topological string free energy $F_g(X)$, which depends on the moduli of the CY, given by "flat coordinates" X (I will often consider one-modulus CYs for simplicity, and I use the A-model picture)

At large *X* ("large radius") this has an expansion encoding Gromov-Witten invariants of *M*, which "count" holomorphic curves of genus *g* and degree *m*:

$$F_g(X) = \sum_{m \geq 1} N_{g,m} \, \mathrm{e}^{-mX}$$

The flat coordinates and genus zero free energy can be computed in the mirror manifold $M^\star$, as **periods** of the holomorphic 3-form over a symplectic basis of 3-cycles

[Candelas-de la Ossa-Green-Parkes]

$$X^I = \int_{\alpha^I} \Omega \qquad \mathcal{F}_I = \int_{\beta_I} \Omega = \frac{\partial F_0}{\partial X^I}$$

The **total free energy** is given by a factorially divergent expansion in the string coupling constant

$$F(X, g_s) = \sum_{g \geq 0} F_g(X) g_s^{2g-2} \qquad F_g(X) \sim (2g)!$$

What is the resurgent structure of the topological string perturbative series?

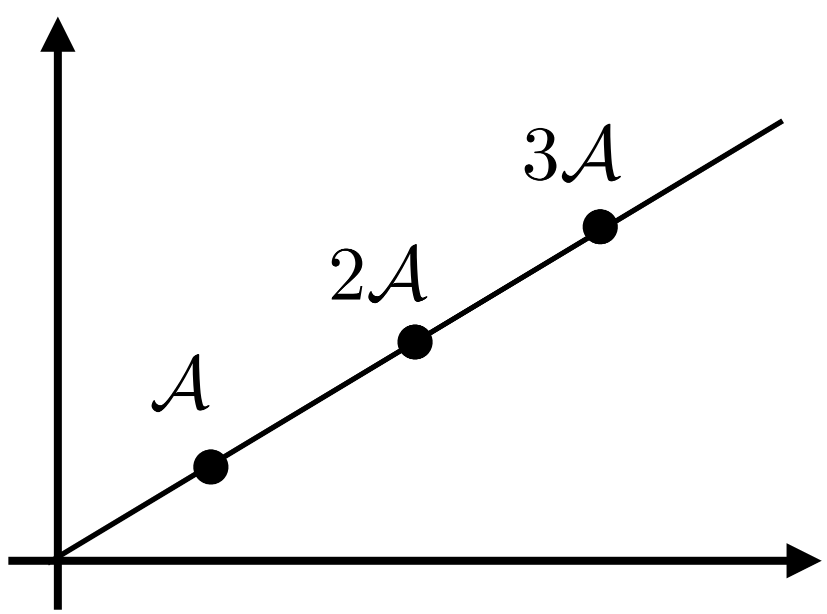
We have to determine first the location of Borel singularities. For each of them, we have to determine the corresponding trans-series (including the Stokes constant)

Borel singularities and D-branes

The Borel singularities are (conjecturally) of the form $\ell\mathcal{A}$, where ℓ is a non-zero integer and $\mathcal{A}$ is the central charge of a BPS D-brane with charges (c^I, d_I)

$$\mathcal{A} = c^I \mathcal{F}_I + d_I X^I$$

Therefore, the Borel singularities give us the spectrum of stable D-branes! Note the "multi-covering" structure:



Non-perturbative amplitudes

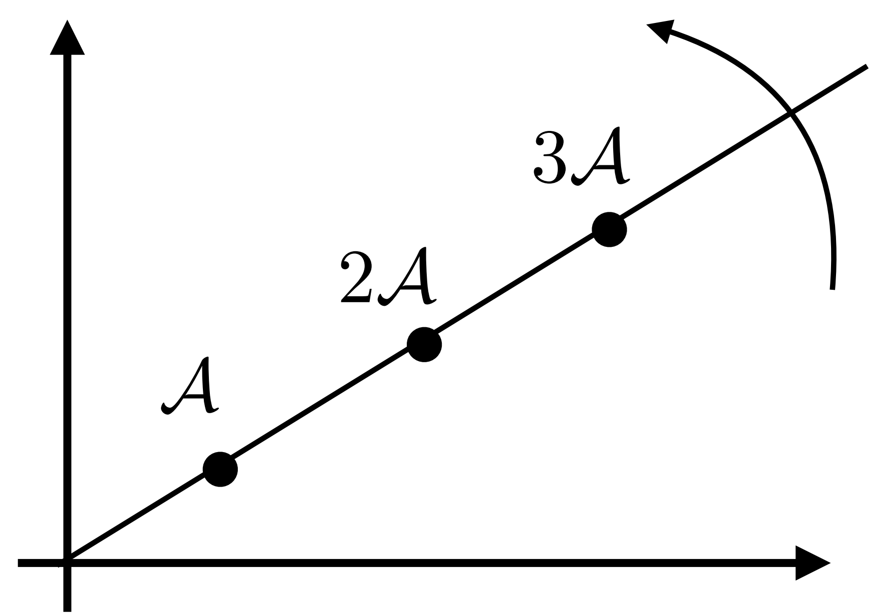
The trans-series can be derived in closed form from the BCOV holomorphic anomaly equations [Gu-KashaniPoor-Klemm-M.M.], as first suggested by [Couso-Edelstein-Schiappa-Vonk]. Except for the Stokes constant, they only involve perturbative data!

For $\ell = 1$ one finds the "one-instanton" amplitude (when $d_I = 0$)

$$\Phi_{\mathcal{A}} = \mathsf{S}_{\mathcal{A}} \left(1 + g_s c^J \partial_J F(X^I - g_s c^I)\right) \mathrm{e}^{F(X^I - g_s c^I) - F(X^I)}$$

This is similar to **eigenvalue tunneling** in matrix models, suggesting that the the CY periods X^I are **quantized** in units of the string coupling constant, as in large *N* dualities

The "multi-instanton" trans-series can be also obtained in closed form [Iwaki-M.M., Alexandrov-M.M.-Pioline]. It can be regarded as Ecalle's "Stokes automorphism" through the ray of singularities.



It turns out to be closely related to Kontsevich-Soibelman automorphisms through BPS rays, and to the geometry of hypermultiplet moduli space [Alexandrov, Persson, Pioline, Coman, Longhi, Teschner, …]

Stokes constants as BPS invariants

Stokes constants are conjecturally given by Donaldson-Thomas (DT) invariants, counting the multiplicity of BPS states

$$\mathsf{S}_{\mathcal{A}} = \frac{\Omega(c^I, d_I)}{2\pi} \qquad \mathcal{A} = c^I \mathcal{F}_I + d_I X^I$$

There is direct and indirect evidence for these conjectures. For example, the Gopakumar-Vafa formula implies that near the large radius point, D2-D0 bound states lead to Borel singularities with the above Stokes constant [Pasquetti-Schiappa, Alim-Tulli-Teschner, Gu-KashaniPoor-Klemm-M.M.].

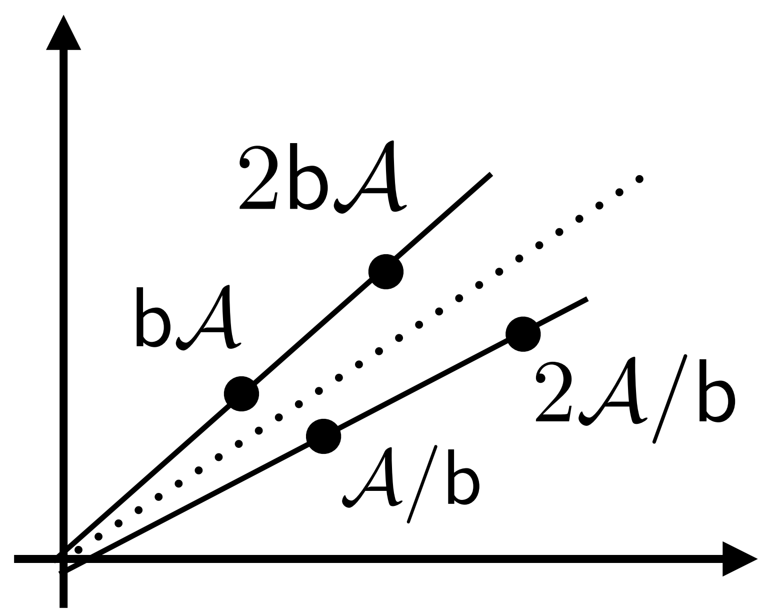
Applications

One can associate topological string free energies to Seiberg-Witten curves of ***N*=2 theories**. Their resurgent structure leads to a new approach to obtain the BPS spectra of these theories [M.M.-Schwick]

The results for the non-perturbative amplitudes apply also to multi-cut Hermitian **matrix models**. They lead to **new results for large *N* instantons** in these models, beyond conventional eigenvalue tunneling [M.M.-Miravitllas]

Generalizations

The **refined** topological string can be regarded as a deformation of the conventional one by a parameter $\mathrm{b} \neq 1$, and all the results above generalize naturally [Alexandrov-M.M.-Pioline, Grassi-Hao-Neitzke]



One can also obtain results for the resurgent structure of Walcher's **real** topological string. BPS invariants counting disks arise as Stokes constants [M.M.-Schwick]

Conclusions

Non-perturbative aspects of the topological string can be explored with the theory of resurgence. The large order behavior of the string perturbative series contains information about the D-brane spectrum.

This physical insight leads to precise mathematical conjectures connecting the analytic properties of the topological string, to BPS counting through Donaldson-Thomas invariants.

Can we prove some of these conjectures? Can we exploit ideas and techniques of resurgence to actually compute BPS degeneracies? (so far most of the non-trivial computations have been numerical).

Trans-series, once Borel-resummed, give proposals for non-perturbative definitions of the theory. Can this be done here? Are we missing additional NP sectors?

Can these ideas be extended to more general string theories and/or large *N* theories?

Thank you for your attention!