Recent Developments in Generalized Symmetries

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Generalized Symmetry "Revolution": **10 years**

Generalized Global Symmetries

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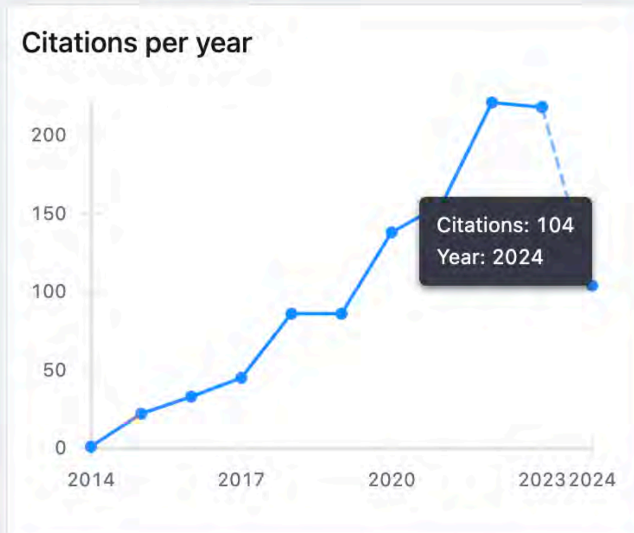


Plan

1. Overview of recent developments
2. Symmetry TFT and Generalized Charges
3. Application: Classification of Symmetric Phases

1. Overview of recent developments

Generalized Symmetry "Revolution"

The identification that "Topological defects in a QFT are global symmetry generators" has hugely enlarged our horizon of what a global symmetry can be:

1. **Higher Symmetry Groups**:
   # Symmetries acting on extended operators
   [Gaiotto, Kapustin, Seiberg, Willett, '14]
   # Higher-groups, such as extensions of higher-form symmetries
   [Kapustin, Thorngren][Benini, Cordova, Hsin][Cordova, Dumitrescu, Intrilligator]

2. **Non-invertible** or **Categorical Symmetries**:
   A non-invertible symmetry $\mathcal{S}$ satisfies composition "fusion": $a, b \in \mathcal{S}$

$$a \otimes b = N_1 c_1 \oplus \cdots \oplus N_k c_k \,, \quad c_i \in \mathcal{S} \,, \; N_i \in \mathbb{N}$$

   $\otimes$ and $\oplus$ makes this akin to an **algebra**. There is extra data that makes it into a **fusion category**.
   # Long-history in 2d QFTs: [Fuchs, Runkel, Schweigert][Bhardwaj, Tachikawa][$\cdots$]
   # Many constructions in $d > 2$ QFTs [Starting in '21]

This has led to an avalanche of new results and exciting cross-fertilization between hep-th, hep-ph, cond-mat, and math.

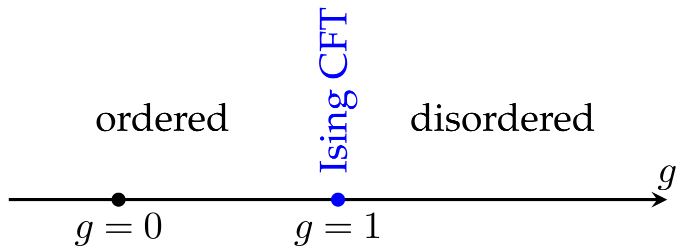
Two simple examples of Non-Invertible Symmetries in 2d

Transverse field **Ising model**: $\mathcal{H} = (\mathbb{C}^2)^L$ with nearest neighbor Hamiltonian

$$H = -\sum_j Z_j Z_{j+1} - g \sum_j X_j \,.$$

There is a $\mathbb{Z}_2$ spin flip symmetry $\eta = \prod_j X_j$.

- $g = 0$: two ground states, $|\uparrow^L\rangle$ and $|\downarrow^L\rangle$: "ordered phase"
- $g \gg 1$: ground state preserves the $\mathbb{Z}_2$: "disordered phase"
- $g = 1$: critical Ising CFT at $c = 1/2$.



Kramers-Wannier duality:
$X_i \to Z_j Z_{j+1}$ and $Z_j Z_{j+1} \to X_{j+1}$, corresponds to $g \to g^{-1}$.
At $g = 1$: symmetry of the critical Ising CFT, which realizes the non-invertible defect:

$$N^2 = 1 + \eta$$

Two simple examples of Non-Invertible Symmetries in 2d:

Ising fusion category:
Generators are 1, η, N, where $\eta \otimes \eta = 1$ is a $\mathbb{Z}_2$ group, and $N \otimes \eta = \eta \otimes N = N$, but N is non-invertible

$$N \otimes N = 1 \oplus \eta\,.$$

N is the Kramers-Wannier self-duality of the critical Ising model.

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- **Representations of a finite non-abelian group G**:
  e.g. permutation group on 3 elements S_3:

  $\text{Rep}(S_3) =$ representations of S_3 with the tensor product form a fusion category .

  The generators are the irreducible representations:
  the trivial (1), sign (U) and 2d representation E, respectively, with tensor product (fusion):

$$U \otimes U = 1\,, \qquad E \otimes U = U \otimes E = E\,, \qquad E \otimes E = 1 \oplus U \oplus E\,.$$

Non-Invertible Symmetries in $d = 4$

- 4d Kramers-Wannier duality defects:
  [Kaidi, Ohmori, Zheng][Choi, Cordova, Hsin, Lam, Shao]

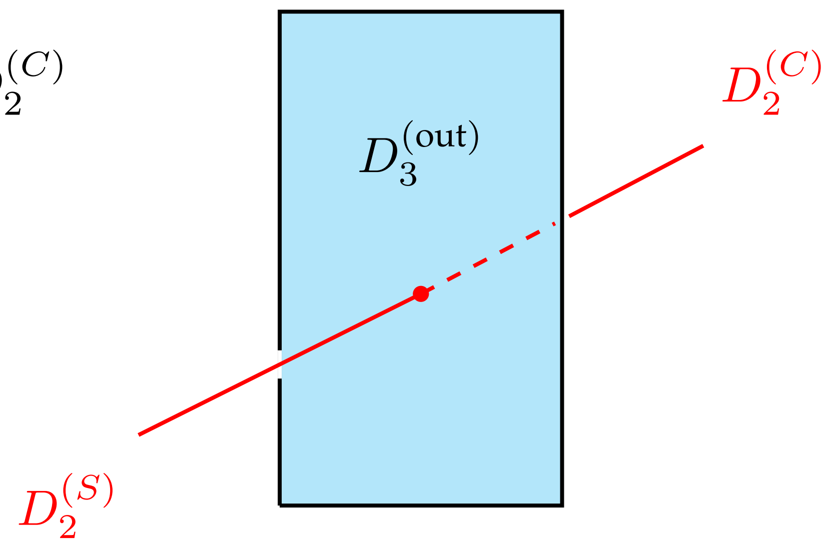
$$\text{QFT} \cong \text{QFT}/D \qquad \Rightarrow \qquad \text{non-invertible 0-form symmetry}$$

- Condensation defects from higher-gauging : [Roumpedakis, Seifnashri, Shao]

$$\mathcal{C}_d \sim \sum_{\Sigma \in H_q(M_d, \mathbb{Z}_N)} e^{i \int_\Sigma b}$$

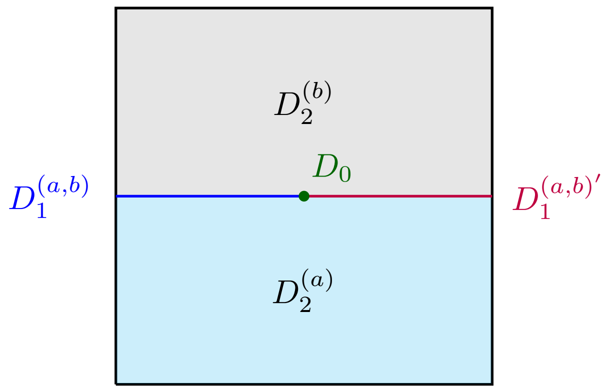
- Gauging outer automorphisms [Bhardwaj, Bottini, SSN, Tiwari]:
  E.g. 1-form symmetry $\mathbb{Z}_2^{(S)} \times \mathbb{Z}_2^{(C)}$ of Spin$(4n)$ exchanged by outer automorphism

$$D_2^{\text{inv}} = D_2^{(S)} \oplus D_2^{(C)}$$

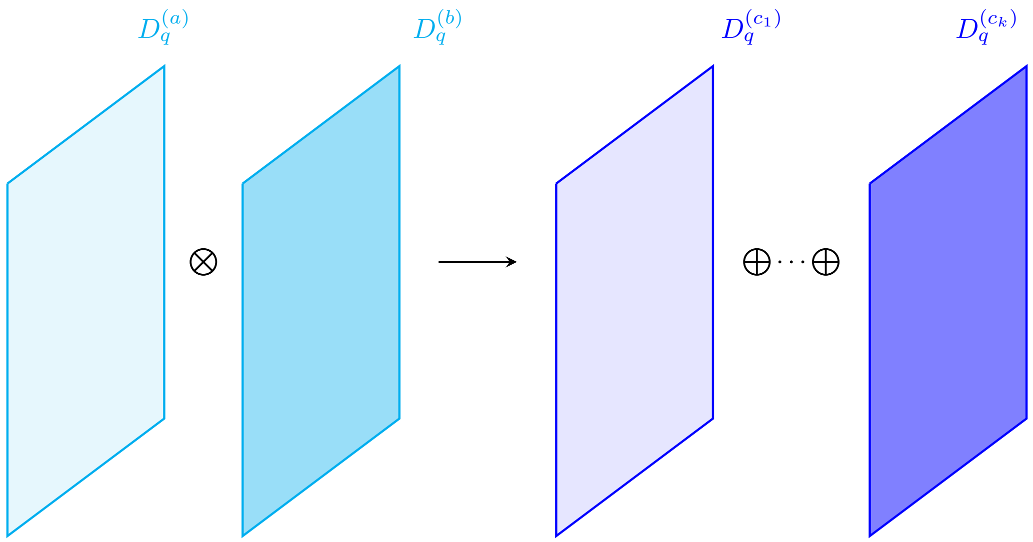


Fusion Higher-Categories

In $d > 2$ topological defects of dimension $0, 1, \cdots, d-1$ exist



Topological defects of fixed dimension have a fusion:



$\Rightarrow$ Fusion higher-category. In d dimensions: $(d-1)$ category.

Ubiquity of Non-Invertible Symmetries

NI 1. String Theory – World-sheet

NI 2. String Theory – Geometric engineering

NI 3. String Theory – Holography

NI 4. String Theory – Branes

NI 5. QFT: Standard Model

NI 6. Lattice models

NI 7. **QFT: Constraining Symmetric Phases**

Many, many references. Since last spring we finally have some reviews:

[SSN: 2305.18296] [Brennan,Hong: 2306.00912] [Bhardwaj et al: 2307.07547]

[Luo, Wang^2: 2307.09215] [Shao: 2308.00747]

NI 1. String Theory – World-sheet

2d CFTs that are rational with respect to a chiral algebra have so-called Verlinde topological lines [Verlinde][Petkova, Zuber][Chang, Lin, Shao, Yifan Wang,Yin].

Virasoro minimal models have lines labeled by the conformal weights λ

$$L_\lambda \otimes L_\mu = \bigoplus_\nu N^\nu_{\lambda,\mu} L_\nu$$

and $N^\nu_{\lambda,\mu} \in \mathbb{N}$ are the fusion coefficients.

Example: $c = 1/2$ Ising CFT has Ising fusion category symmetry $N^2 = 1 \oplus \eta$, where $\eta = L_{1/2}$ and $N = L_{1/16}$.

Most recent application:
Using integrable flows that commute with the Verlinde lines, consistency with the symmetry implies modified crossing for the S-matrix [Copetti, Lucia Cordova, Komatsu].

NI 1. String Theory – World-sheet

Gepner Models.

Calabi-Yau n-fold sigma-models can have Gepner points on the conformal manifold, which are orbifolded tensor products of $N = 2$ Minimal Models

$$(\otimes_i \mathrm{MM}_{k_i})/\Gamma\,,$$

where

$$\mathrm{MM}_k = \frac{\mathfrak{su}(2)_k \oplus \mathfrak{u}(1)_2}{\mathfrak{u}(1)_{k+2}}$$

and Γ is a finite group action $\mathbb{Z}_{\mathrm{lcm}(k_i+2)}$.

The Gepner model inherits non-invertible symmetries from the Verlinde lines of the MMs [Cordova, Rizi][Angius, Giaccari, Volpato].

What is it good for? Non-invertible symmetries commuting with exactly marginal deformations, can constrain spectra, correlators.

Example: the Quintic 3-fold $(\mathrm{MM}_{k=3})^5/\mathbb{Z}_5$ has a subspace of deformations that preserve a Fibonacci fusion category $W^2 = 1 \oplus W$.

NI 2. String Theory and Geometric Engineering

[Morrison, SSN, Willet][Albertini, del Zotto, Garcia-Extebarria, Hosseini][Cvetic, Heckman, Hubner, Thorres], [Penn, Durham, Uppsala, Oxford...][**Garcia-Extebarria Strings '22**]

String theory/M-theory on non-compact (special holonomy) space X engineers QFTs, whose generalized symmetries are encoded in the topology of X:

$$\frac{H_{p+1}(X, \partial X, \mathbb{Z})}{H_{p+1}(X, \mathbb{Z})} = \text{Infinitely heavy probes, modulo screening}$$

Example:
In M-theory, the 1-form symmetry is obtained by considering M2-branes on relative 2-cycles (line operators), modulo screening by M2-branes wrapping compact cycles (local operators).

't Hooft anomalies for generalized symmetries:
Reduction on the link ∂X [Apruzzi, Bonetti, Garcia-Extebarria, Hosseini, SSN] gives the $d+1$ dimensional topological theory (Symmetry TFT – more on that later)

Compact models/Swampland: How are global, non-invertible symmetries in theories of quantum gravity broken/gauged?
⇒ [Miguel Montero's talk]

NI 3. String Theory and Holography

AdS_3/CFT_2: Non-invertible symmetries from S_N orbifolds
$\mathfrak{T}^N/S_N$ symmetric orbifold has non-invertible defects (recall $\text{Rep}(S_3)$), where $\mathfrak{T} = U(1)^4$. These can be mapped to the tensionless limit of AdS_3 string theory [Gutperle, Li, Rathroe, Roumpedakis][Knighton,Sriprachyakul, Vosmera]

- AdS_5/CFT_4 [Witten '98]
- AdS_4/CFT_3 [Berman, Tachikawa, Zafrir]
- Klebanov-Strassler $N = 1$ SYM confinement [Apruzzi, van Beest, Gould, SSN]
- Duality Defects [Antinucci, Benini, Copetti, Galati, Rizi][Aguilera Damia,Argurio, Benini, Benvenuti, Copetti, Tizzano]

The relation between (finite) global symmetries and the bulk holography is:

Topological couplings in the (consistent truncations of) supergravity actions correspond to the SymTFT for the global symmetries of the boundary FT.

Invertible 1-form symmetries: [Witten '98]

$\mathrm{AdS}_5 \times S^5$ dual to 4d $\mathcal{N} = 4$ $\mathfrak{su}(N)$ Super-Yang-Mills: leading terms in the derivative expansion are

$$\frac{1}{2\pi} \int_{\mathrm{AdS}_5} N b_2 dc_2$$

where b_2 and c_2 are the backgrounds for the electric and magnetic $\mathbb{Z}_N \times \widehat{\mathbb{Z}}_N$ 1-form symmetries, whose generators are

$$D_2^{(b)} = e^{i \int b_2} \,, \qquad D_2^{(c)} = e^{i \int c_2} \,,$$

which are mutually non-local

$$D_2^{(b)}(M) D_2^{(c)}(M') = \exp\left(\frac{2\pi i L(M, M')}{N}\right) D_2^{(c)}(M') D_2^{(b)}(M) \,.$$

L is the linking of the two 2d surfaces in 5d. A global form of the gauge group is chosen by fixing b.c. for b_2 and c_2, e.g.

b_2 : Dirichlet, i.e. $D_2^{(b)}$ gives rise to line operators (charges)

c_2 : Neumann, i.e. $D_2^{(c)}$ gives rise to symmetry generators

Fixing a global form of the gauge group, corresponds to selecting a maximal set of mutually local such defects.

NI 3. String Theory and Holography

Non-invertible Symmetries: [Apruzzi, Bah, Bonetti, SSN]
For the holographic dual to 4d $\mathcal{N} = 1$ $\mathfrak{su}(N)$ SYM, the topological couplings of the Klebanov-Strassler solution are

$$\frac{1}{2\pi} \int_{M_5} N b_2 \wedge dc_2 + 2N a_1 \wedge dc_3 + \frac{1}{2\pi} a_1 b_2^2 \,,$$

where c_3 is the background for the chiral 0-form symmetry $\mathbb{Z}_{2N}$ with mixed anomaly with the 1-form symmetry.

For $PSU(N)$ gauge group, the symmetry generators are

$$\mathcal{N}_3^{(1)}(M_3) = \int [Da] e^{2\pi i \int_{M_3} \left(c_3 + \frac{1}{2} N a da + a b_2 \right)}$$

with non-invertible fusion

$$\mathcal{N}_3^{(1)} \otimes \mathcal{N}_3^{(1)\dagger} = \mathcal{C}_{\mathbb{Z}_N} = \text{condensation defect for the 1-form symmetry}$$

NI 4. String Theory and Branes

[Apruzzi, Bah, Bonetti, SSN][Garcia-Extebarria] [Heckman, Hubner, Thorres, Zhang][X. Yu]

Proposal: in the near horizon limit, branes inserted in a holographic setup furnish symmetry generators. Close to the boundary $r \to \infty$: $T_{\rm Dp} \sim r^p$, $p > 0$, only topological WZ terms on brane remain.

Holographic dual to $\mathcal{N} = 1$ SYM:
D5-branes on $S^3 \times M_3 \subset T^{1,1} \times M_4$ have topological couplings in the near horizon limit $r \to r_0 \to \infty$

$$S_{\rm D5} = 2\pi \int_{M_3} \left(c_3 + \frac{N}{2} a_1 da_1 + a_1 db_1 \right) ,$$

where $C_6 = c_3 \wedge \omega_{S^3}$, $C_4 = b_1 \wedge \omega_{S^3}$ and a_1 is the gauge field on the brane. For $PSU(N)$ gauge group (b_1 Dirichlet), this is the non-invertible defect $\mathcal{N}_3^{(1)}$.

Non-Invertible fusion from tachyon condensation:
The $\mathcal{N}_3^{(1)} \otimes (\mathcal{N}_3^{(1)})^\dagger$ = condensation defect for the 1-form symmetry on M_3, is D5-$\overline{\rm D5}$ via tachyon condensation (with a remnant D3-charge) [Sen] resulting in the condensation defect [Apruzzi, Bah, Bonetti, SSN][Bah, Leung, Waddleton]

Generalized Charges from Hanany-Witten Transition

[Apruzzi, Bah, Bonetti, SSN[Apruzzi, Gould, Bonetti, SSN]

How do the non-invertible symmetry generators act on ’t Hooft lines in $PSU(N)$ SYM?

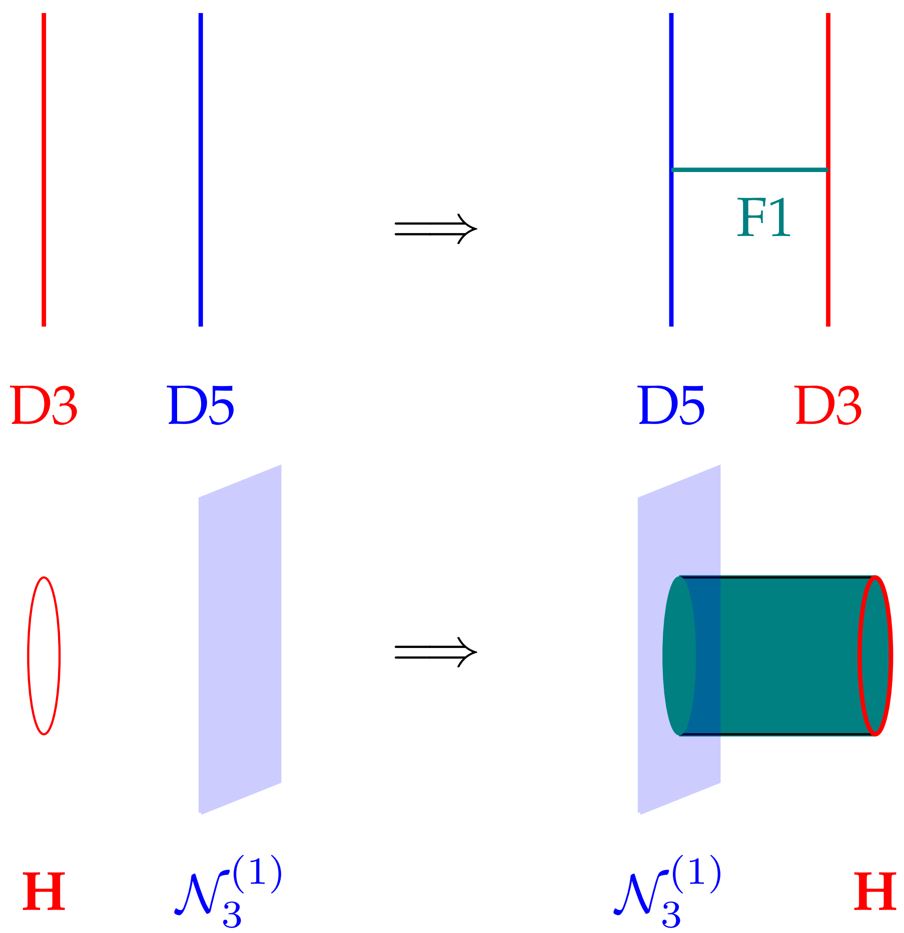
- Charged line operators:
  D3s stretching along the radial direction and wrapped on $S^2 \times S^1$ give rise to ’t Hooft lines.

- Topological defects:
  D5s on $S^3 \times M_3$ generate the non-invertible codim 1 topological defects.

Brane	x_0	x_1	x_2	x_3	r	z_1	z_2	w_1	w_2	w_3
D3	X				X	X	X			
D5	X	X	X					X	X	X

Charge conservation implies that the total linking of the branes is conserved – in particular when we exchange the position of the D3 and D5.

Preserving the linking requires the creation of an F1:



’t Hooft loop gets flux attachment when it crosses the non-invertible defect – similar to disorder operator in Kramers-Wannier duality.

General Feature of Non-Invertible Symmetries: maps genuine to non-genuine (attached to topological defects) operators

String Theory lessons for Generalized Symmetries

Symmetry	String Theory
Symmetry TFT	Topological subsector of sugra
Symmetry generators	Branes in topological limit
Generalized charges	Branes wrapping relative cycles
Fusion	Tachyon condensation
Linking action of symmetry on charges	Hanany-Witten moves

NI 5. QFT – Standard Model

[Ohmori Strings '22, Cordova Strings '23]

In QED with massless charge 1 Dirac fermion, the axial current $j_\mu = \frac{1}{2}\bar{\Psi}\gamma_5\gamma_\mu\Psi$ is not conserved due to the ABJ anomaly

$$d \star j = \frac{1}{8\pi^2} F \wedge F \,.$$

Define instead a defect dressed by 3d TQFT that has opposite anomaly

$$\mathcal{D}_{\frac{1}{N}}(M_3) = \int [Da] \exp\left(\int_{M_3} \frac{2\pi i}{N} \star j + \frac{iN}{4\pi} ada + \frac{i}{2\pi} adA \right) \,.$$

It is topological, but satisfies non-invertible fusion

$$\mathcal{D}_{\frac{1}{N}}(M) \times \overline{\mathcal{D}_{\frac{1}{N}}(M)} = \mathcal{C}_N(M).$$

- [Choi, Lam, Shao][Cordova, Ohmori] application to pion decay
- [Cordova, Hong, Koren, Ohmori] Z′ with non-invertible chiral symmetry, gives breaking by exponentially small amount, application to neutrino masses
- [Cordova, Hong, Koren] Non-Invertible Peccei-Quinn Symmetry and the Massless Quark Solution to the Strong CP Problem

NI 6. Lattice Models

Huge progress on studying lattice models with non-invertible symmetries – hep meets cond-mat.

UV lattice models, which have non-invertible symmetries:

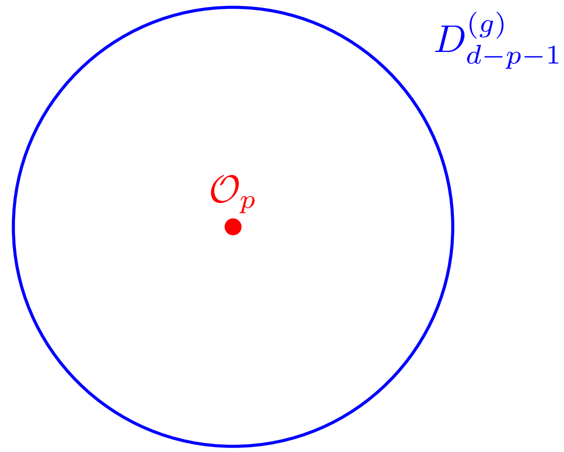
- Anyon chain: 1+1d model, that realizes any fusion category symmetry [Feiguin et al][Aasen, Mong, Fendley][Lootens et al]
- Tensor product Hilbert space realizations: anomalies and LSM theorems [Cheng, Seiberg], Ising symmetry [Seiberg, Shao][Seiberg, Shao, Seifnashri], Rep(S_3) and Rep(D_8) phases [Eck, Fendley][Bhardwaj, Bottini, SSN, Tiwari][Chatterjee, Aksoy, Wen][Shao, Seifnashri]
- Gapped/gapless phases in (1+1)d lattice models with any fusion category symmetry [Bhardwaj, Bottini, SSN, Tiwari].
- 2-fusion category symmetries on the honey-comb lattice [Inamura, Ohmori]

In Summary:
There is an abundance of non-invertible symmetries.

What do non-invertible symmetries do for you?

Question 1: Generalized Charges

For group-like symmetries, the charged operators **"charges"** transform in representations. Charges are detected by linking with the symmetry generators:



The charge under the symmetry group element $g \in G$ is measured by determining the linking of $\mathcal{O}_p$ with symmetry generator $D^{(g)}_{d-p-1}$.

(Genuine) Operators will form representations or multiplets.

Question 1: What replaces this for non-invertible symmetries?
I.e. what are generalized charges.

Question 2: Symmetric Phases

Landau paradigm:
A continuous (2nd order) phase transition is a symmetry breaking transition.

G is a symmetry group, which is spontaneously broken to a subgroup H, resulting in $|G/H|$ vacua, which are acted upon by the broken symmetry (+SPT phases).

Example: $G = \mathbb{Z}_2$**.**
There are two gapped $\mathbb{Z}_2$-symmetric phases:

- Trivial phase ($H = \mathbb{Z}_2$): $\mathbb{Z}_2$ symmetric single vacuum
- Spontaneously Symmetry Broken (SSB) Phase ($H = 1$): charged operator $\mathcal{O}_-$ gets a vacuum expectation value, two vacua, and the broken $\mathbb{Z}_2$ exchanges them

Between these there is a 2nd order phase transition: in (1+1)d, the critical Ising CFT

$$\boxed{\mathbb{Z}_2 \text{ SSB Phase}} \longleftarrow \boxed{\text{Ising CFT}} \longrightarrow \boxed{\text{Trivial Phase}}$$

Question 2: What replaces this for non-invertible symmetries?
I.e. what are implications of a Categorical Landau Paradigm.

Q1. Generalized Charges and Symmetry TFT

Generalized Charges for Non-Invertible Symmetries

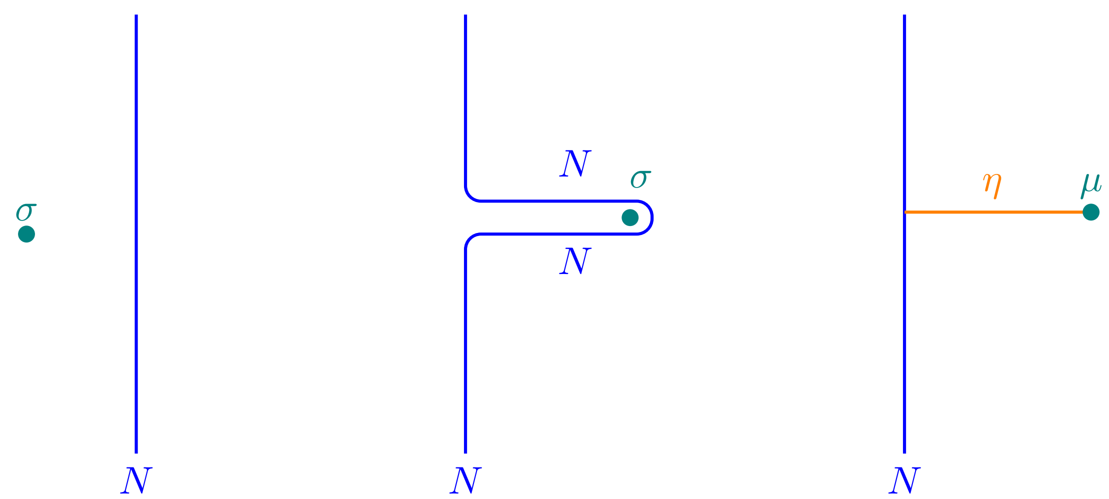
Generalized q-charge
= q-dim defect in a "Representations of a Non-Invertible Symmetry".

In 2d: tube algebra and lasso-action [Fröhlich, Fuchs, Runkel, Schweigert][Lin, Okada, Seifnashri, Tachikawa][Bhardwaj, SSN][Bartsch, Bullimore, Ferrari, Pearson]

Example: Ising fusion symmetry of the critical Ising model

$$\eta^2 = 1\,, \qquad N\eta = \eta N = N\,, \qquad N^2 = 1 \oplus \eta\,.$$

We can act on the spin operator σ (1/16 primary):



The non-invertible symmetry maps σ to μ, which is attached to an η-line ("twisted sectors, non-genuine" operators).

From Generalized Charges to Linking $d+1$ dims

A complete characterization of generalized charges can be given by going to $d+1$ dimensions [Bhardwaj, SSN]:

Generalized charges = Linking of topological defects in $d+1$ dims

The $d+1$ dimensional theory is the

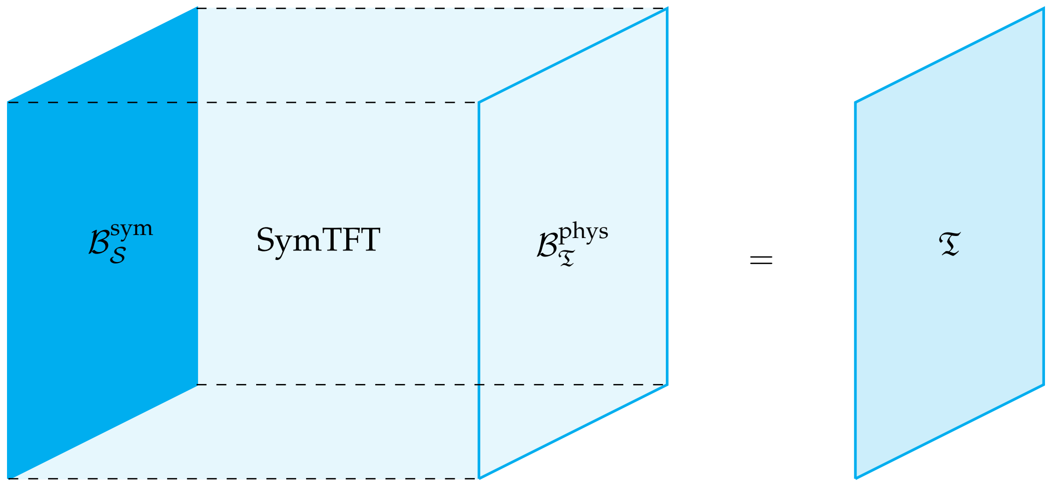
Symmetry Topological Field Theory (SymTFT) of $\mathcal{S}$.

Examples:
BF-theory for abelian p-form symmetries (with anomalies).

SymTFT ("Sandwich")

[Gaiotto, Kulp][Apruzzi, Bonetti, Garcia-Extebarria, Hosseini, SSN] [Freed, Moore, Teleman]

Given a physical QFT $\mathfrak{T}$ with (finite) symmetry $\mathcal{S}$ in d dimensions. The SymTFT is a $d+1$ dimensional TQFT $\mathfrak{Z}(\mathcal{S})$ by gauging $\mathcal{S}$ in $(d+1)$ dims:

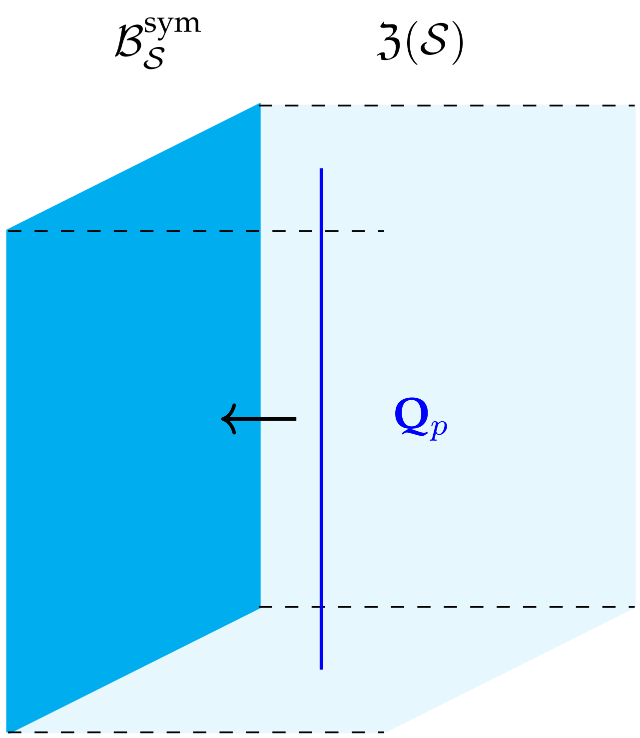


- $\mathcal{B}^{\text{sym}}_{\mathcal{S}}$ = Symmetry boundary:
  condense a maximal number of mutually local topological defects. The remaining defects generate $\mathcal{S}$.
- $\mathcal{B}^{\text{phys}}_{\mathfrak{T}}$ = Physical boundary:
  condense a subset of mutually local defects (braiding trivially with each other, but not necessarily maximal)

The interval compactification gives $\mathfrak{T}$ with symmetry $\mathcal{S}$.

The topological defects $\mathbf{Q}_p$ of the SymTFT form the Drinfeld Center of $\mathcal{S}$.

SymTFT: Recovering $\mathcal{S}$

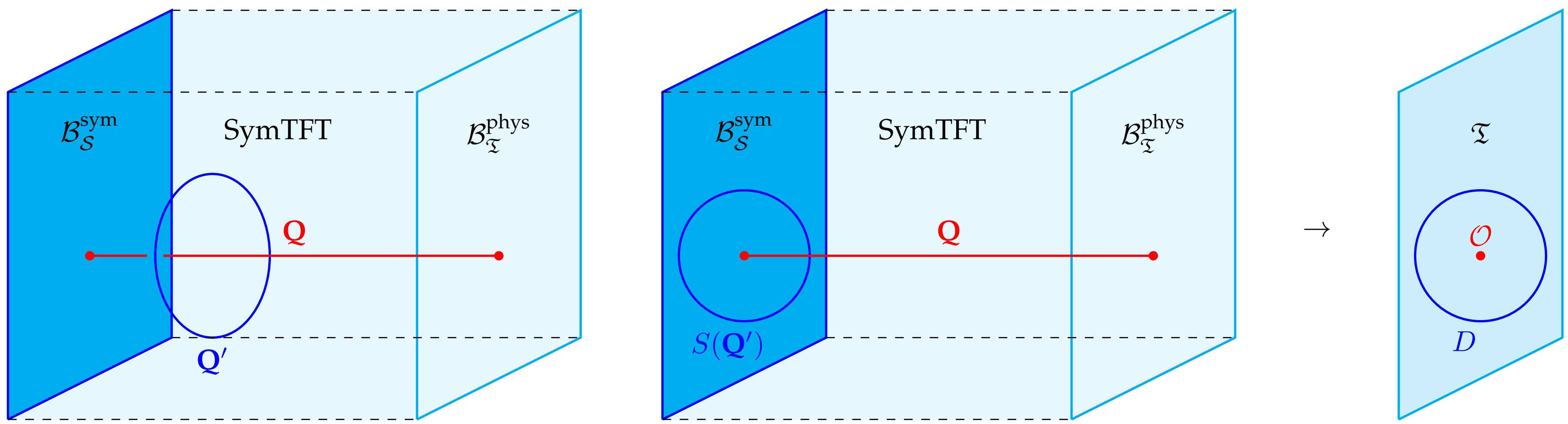


$\mathcal{B}^{\text{sym}}_{\mathcal{S}}$: gapped (topological) boundary conditions of the SymTFT:

$\Rightarrow$ Determined by a maximal set of mutually local topological defects, which form a Lagrangian algebra

$\mathbf{Q}_p$ with Neumann b.c.s give rise to symmetry generators $\mathcal{S}$.

Linking of Topological Defects is Action of Symmetry

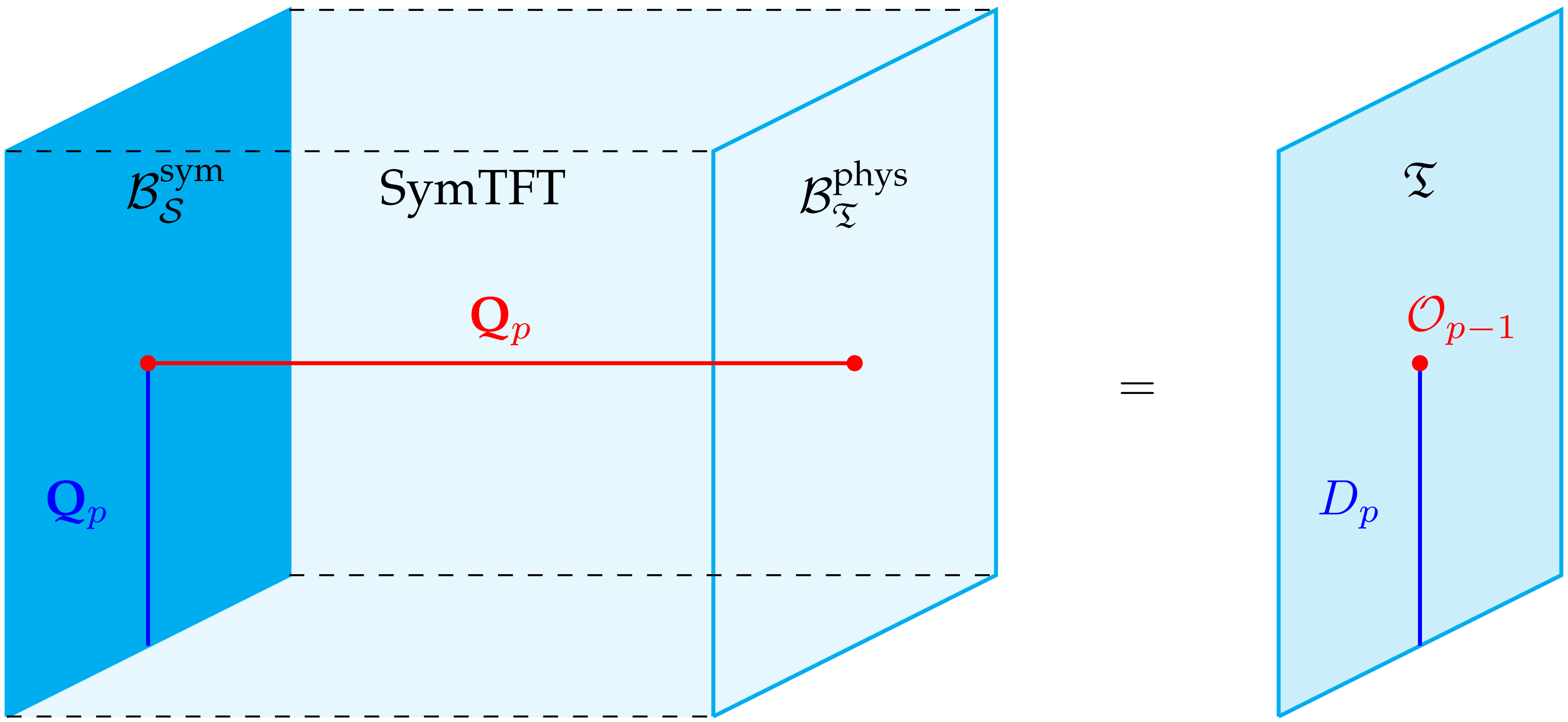


[Bhardwaj, SSN '23]:
The generalized charges are the topological defects **Q** of the SymTFT, which condense on both boundaries.

Q′ that have Neumann b.c. on the $\mathcal{B}^{\text{sym}}_{\mathcal{S}}$ boundary are the generators of the symmetry $\mathcal{S}$.

Linking of **Q** and **Q**′ determines the charge under the symmetry.

SymTFT: Non-genuine Operators



$\mathcal{O}$ is attached to a topological line, i.e. a non-genuine operator.

SymTFT for 0-form symmetry groups in d dims

For (non-abelian) $G^{(0)}$, the topological defects of the SymTFT are labeled by

$$\mathbf{Q}^{[g],\mathbf{R}},$$

- conjugacy classes $[g]$
- representations of the stabilizer group H_g of $g \in [g]$.

Example: S_3 in 2d:
$S_3 = \mathbb{Z}_3 \rtimes \mathbb{Z}_2 = \{\mathrm{id}, a, a^2, b, ab, a^2 b\}$
Irreps: $+$ (trivial), $-$ (sign), E (2d representations).
Conjugacy classes:

$$\begin{aligned} [\mathrm{id}]\,, &\quad H_{\mathrm{id}} = S_3 \\ [a]\,, &\quad H_a = \{\mathrm{id}, a, a^2\} = \mathbb{Z}_3 \\ [b]\,, &\quad H_b = \{\mathrm{id}, b\} = \mathbb{Z}_2\,. \end{aligned}$$

$H_a = \mathbb{Z}_3$ reps: 1d and characterized by $1, \omega = e^{2\pi i/3}, \omega^2$.
$H_b = \mathbb{Z}_2$ reps: labelled by $\pm$.

SymTFT for S_3 and $\text{Rep}(S_3)$

The topological lines of the SymTFT $\mathfrak{Z}(S_3)$:

$$\begin{aligned}
\mathbf{Q}_1^{([\text{id}],\mathbf{R})} &: \qquad \mathbf{R} = 1, 1_-, E \\
\mathbf{Q}_1^{([a],\mathbf{R})} &: \qquad \mathbf{R} = 1, \omega, \omega^2 \\
\mathbf{Q}_1^{([b],\mathbf{R})} &: \qquad \mathbf{R} = \pm\,.
\end{aligned}$$

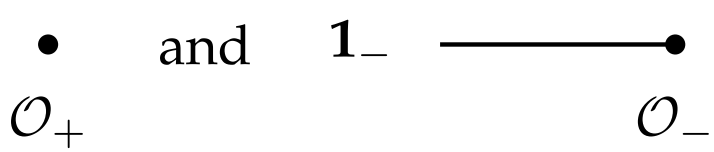
The topological b.c.s (Lagrangian algebras) are

$$\mathcal{L}_{S_3} = ([\text{id}],1) \oplus ([\text{id}],1_-) \oplus 2([\text{id}],E) \qquad \mathcal{L}_{\text{Rep}(S_3)'} = ([\text{id}],1) \oplus ([\text{id}],1_-) \oplus 2([a],1)$$

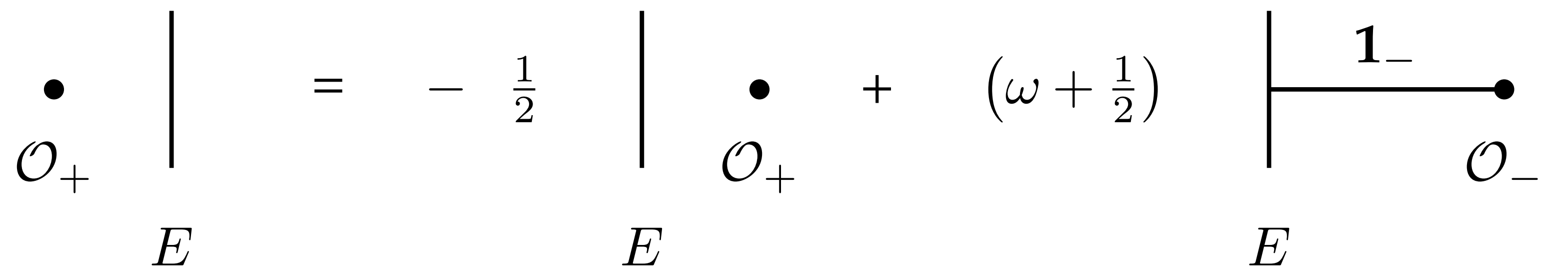
$$\mathcal{L}_{S_3'} = ([\text{id}],1) \oplus ([\text{id}],E) \oplus ([b],1) \qquad \mathcal{L}_{\text{Rep}(S_3)} = ([\text{id}],1) \oplus ([a],1) \oplus ([b],1)$$

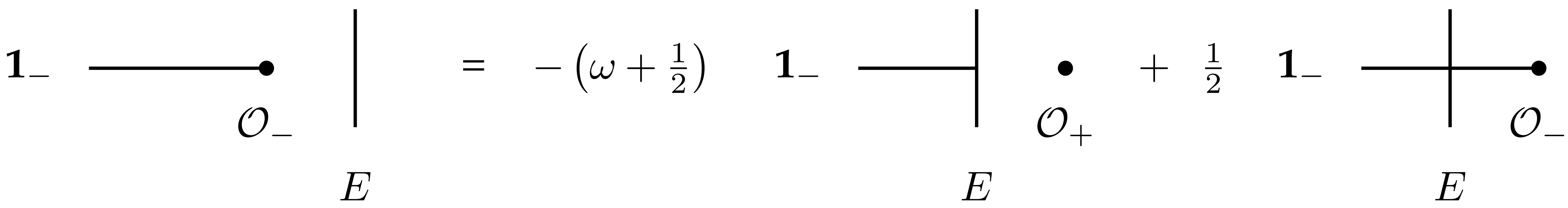
Multiplet structure:

- S_3: $\mathbf{Q}_1^{([\text{id}],\mathbf{R})}$ are untwisted; $\mathbf{Q}_1^{([a])}$ and $\mathbf{Q}_1^{([b])}$ are twisted sector reps
- $\text{Rep}(S_3)$: $\mathbf{Q}_1^{([\text{id}],\mathbf{R})}$ are twisted (attached to $\mathbf{R}$ lines).
  $\mathbf{Q}_1^{([a],1)}$ contains two operators:



This can be derived from the action of the symmetry on defects in the SymTFT:





SymTFT

- Topological defects are the generalized charges
- Gauging a symmetry = change symmetry b.c.
- If $\mathcal{S}$ and $\mathcal{S}'$ that are related by gauging, they have the same SymTFT
- The SymTFT captures 't Hooft anomalies ($\to$ recall examples in holography)

Remarks:

- SymTFT exists for any higher-fusion category: the topological defects are the so-called Drinfeld Center. For 2-fusion categories see [Kong et al][Bhardwaj, SSN]

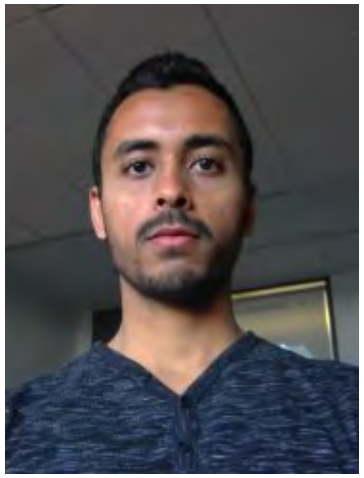
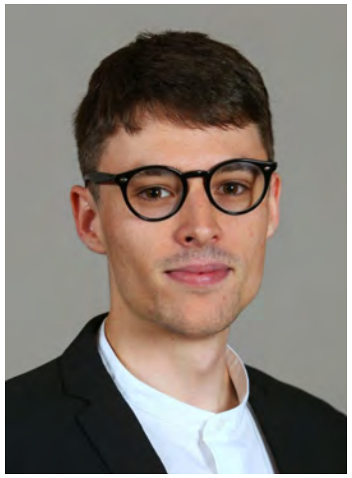
$$\mathfrak{Z}(2\text{Vec}_G^{\omega}) = \bigoplus_{[g]} 2\text{Rep}^{\omega_g}(H_g)$$

- Recently: SymTFT or SymT for continuous abelian and non-abelian symmetries [Antinucci, Benini][Apruzzi, Bedogna, Dondi][Bonetti, del Zotto, Minasian][Brennan, Sun]. This can be important for higher-group symmetries which mix continuous and finite symmetries.

Q2. Application: Classification of Symmetric Phases

Based on work in collaboration with:

Lakshya Bhardwaj (Oxford)
Lea Bottini (Oxford)
Daniel Pajer (Oxford)
Alison Warman (Oxford)
Apoorv Tiwari (NBI, Copenhagen)

`2310.03786`: Categorical Landau Paradigm and SymTFT
`2310.03784`: Gapped Phases with Non-Invertible Symmetries
`2312.17322`: The Club Sandwich: Gapless Phases and Phase Transitions with Non-Invertible Symmetries
`2403.00905`: Hasse Diagrams for Gapless SPT and SSB Phases with Non-Invertible Symmetries
`2405.05964`: Phases of Lattice Models with Non-Invertible Symmetries

Related works – many in cond-mat:

Gapped phases:

[Thorngren, Wang][Inamura][Huang, Lin, Seifnashri][Cordova, Zhang][S. Huang, Meng Cheng]

Gapless phases:

[Chatterjee, Aksoy, Wen][Wen, Potter]

With fermions:

[S. Huang][Bhardwaj, Inamura, Tiwari]

Non-Invertible SPT phases from lattice models:

[Fechisin, Tantivasadakarn, Albert][Seifnashri, Shao][Jia]

Non-Invertible symmetries and phase transitions in lattice models (RepS_3):

[Eck, Fendley][Bhardwaj, Bottini, SSN, Tiwari][Chatterjee, Aksoy, Wen]

Categorical Landau Paradigm

Conjecture/Hope: **Generalized (Categorical) Landau Paradigm:**
Explain phase transitions using a suitably generalized notion of symmetry.

Let $\mathcal{S}$ be a non-invertible symmetry. We develop a framework that determines:

- All $\mathcal{S}$-symmetric gapped phases including the order parameters, i.e. generalized charges that acquiring vevs
- Gapless phase transitions between $\mathcal{S}$-symmetric gapped phases:

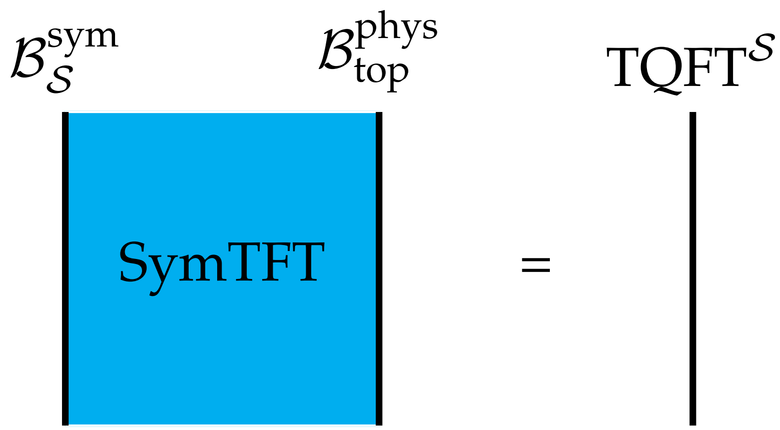
$$\boxed{\mathcal{S}\text{ Gapped Phase}} \quad \longleftarrow \quad \boxed{\text{CFT}} \quad \longrightarrow \quad \boxed{\mathcal{S}\text{ Gapped Phase'}}$$

Generalizes the Landau paradigm to $\mathcal{S}$ a categorical symmetry

$\Rightarrow$ Categorical Landau Paradigm [Bhardwaj, Bottini, Pajer, SSN]

Classification of gapped $\mathcal{S}$-symmetric phases

Choose $\mathcal{B}^{\text{phys}}$ to be also a topological (gapped) boundary condition.



A gapped boundary condition of the SymTFT is a Lagrangian algebra, i.e.

mutually local and maximal subset of defects.

Fix symmetry boundary to be $\mathcal{L}_{\mathcal{S}}$:
A gapped $\mathcal{S}$-symmetric phase is given by a Lagrangian algebra $\mathcal{L}$:

- SPT (symmetry protected topological phase): $\mathcal{L} \cap \mathcal{L}_{\mathcal{S}} = 1$
  Cannot deform to the trivial theory without breaking symmetry
- SSB (spontaneous symmetry breaking): $\mathcal{L} \cap \mathcal{L}_{\mathcal{S}} \supsetneq 1$

# of vacua = # of topological defects that condense on both boundaries, which are also the order parameters.

Gapped Phases with Group-Symmetry in 2d

Landau type classification: $\mathcal{S} = G$ then

- $H < G$ the unbroken symmetry
- $\omega \in H^2(H, U(1))$ cocycle/SPT phase.

Example: $G = \mathbb{Z}_4$

The SymTFT is a 3d topological order ($\mathbb{Z}_4$ Dijkgraaf-Witten-theory) $\int b_1 \cup \delta c_1$, with anyons $e = e^{i \int b_1}$ and $m = e^{i \int c_1}$:

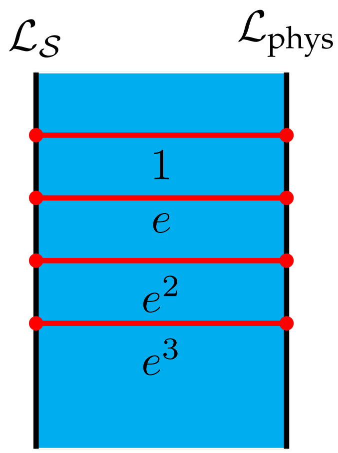
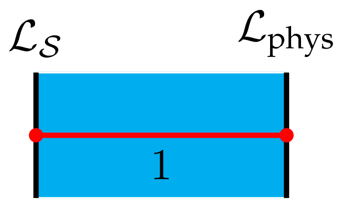
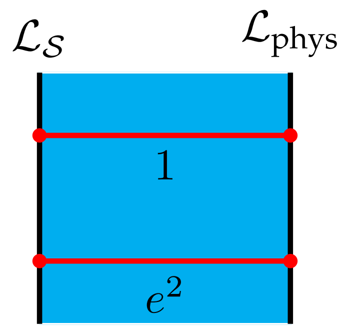
$$\text{Topological defects (anyons):} \qquad (e^i, m^j), \quad e^4 = 1, \; m^4 = 1 \, .$$

e and m braid non-trivially. The Lagrangian, i.e. maximal, trivially braiding subsets of anyons are:

1. $\mathcal{L}_{\text{Dir}} = 1 \oplus e \oplus e^2 \oplus e^3$
2. $\mathcal{L}_{\text{Neu}} = 1 \oplus m \oplus m^2 \oplus m^3$
3. $\mathcal{L}_{\text{Neu}(\mathbb{Z}_2)} = 1 \oplus e^2 \oplus m^2 \oplus e^2 m^2$

The symmetry boundary is $\mathcal{B}^{\text{sym}}_{\mathcal{S}=\mathbb{Z}_4} = \mathcal{L}_{\text{Dir}}$.

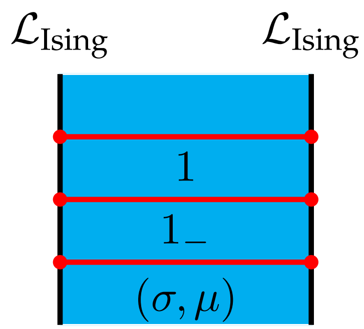
Gapped Phases with $\mathbb{Z}_4$ Symmetry via the SymTFT

<table>
<tr><th>$\mathcal{L}_{\text{phys}} = \mathcal{L}_{\mathcal{S}} = \mathcal{L}_{\text{Dir}}$</th><th>$\mathcal{L}_{\text{phys}} = \mathcal{L}_{\text{Neu}}$</th><th>$\mathcal{L}_{\text{phys}} = \mathcal{L}_{\text{Neu}(\mathbb{Z}_2)}$</th></tr>
<tr><td></td><td></td><td></td></tr>
<tr><td>$\mathbb{Z}_4$ SSB:
4 identical vacua, permuted by $\mathbb{Z}_4$</td><td>$\mathbb{Z}_4$ Trivial Phase:
single vacuum with $\mathbb{Z}_4$ acting trivially</td><td>$\mathbb{Z}_2$ SSB:
2 identical vacua, permuted by $\mathbb{Z}_2$</td></tr>
</table>

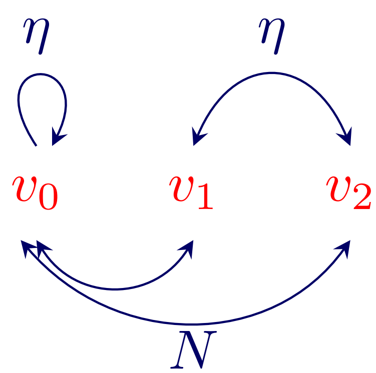
Gapped Phases with Non-Invertible Symmetry: Ising Category

The SymTFT is Ising $\boxtimes \overline{\text{Ising}}$ and there is a unique subset of mutually local anyons (gapped b.c./Lagrangian algebra):

$$\mathcal{L}_{\text{Ising}} = 1 \oplus 1_- \oplus (\sigma, \mu)$$



Resulting in 3=2+1 vacua, with the symmetry acting as

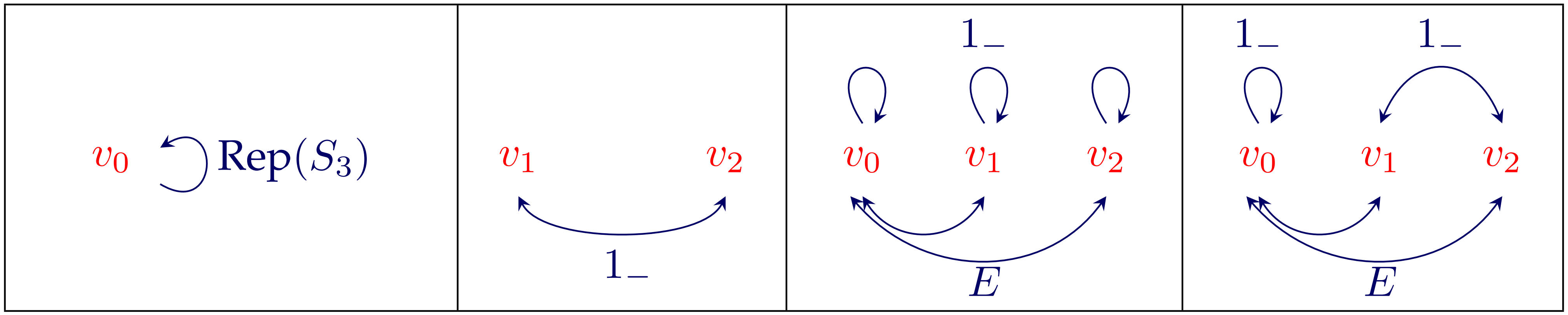


Unique Ising symmetric gapped phase: SSB phase with 3 vacua.

Gapped Phases with Non-Invertible Symmetry: Rep(S_3)

Repeating a similar SymTFT analysis now for the non-invertible symmetry Rep(S_3) (1, 1_-, E irreps) we find four gapped phases:

Rep(S_3) trivial phase	$\mathbb{Z}_2$ SSB	Rep$(S_3)/\mathbb{Z}_2$ SSB	Rep(S_3) SSB



Phase Transitions

Consider two gapped $\mathcal{S}$-symmetric phases, how do we determine the $\mathcal{S}$-symmetric phase transitions?

$$\boxed{\mathcal{S}\text{ gapped}} \quad \longleftarrow \quad \boxed{\mathcal{S}\text{ gaplesss}} \quad \longrightarrow \quad \boxed{\mathcal{S}\text{ gapped}'}$$

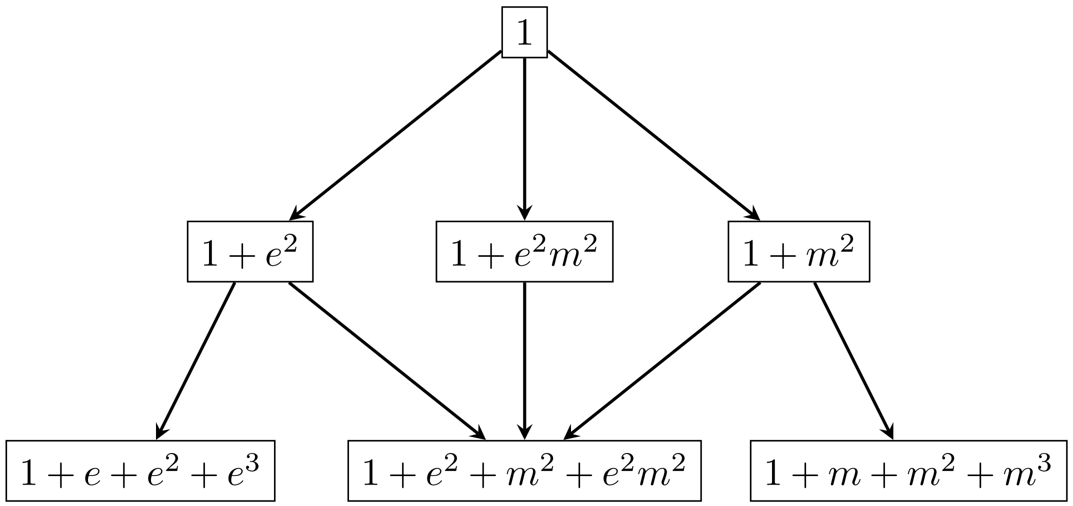
- Gapped phase: determined by Lagrangians $\mathcal{L}_i$
- Gapless phase transition between $\mathcal{L}_1$ and $\mathcal{L}_2$ is characterized by

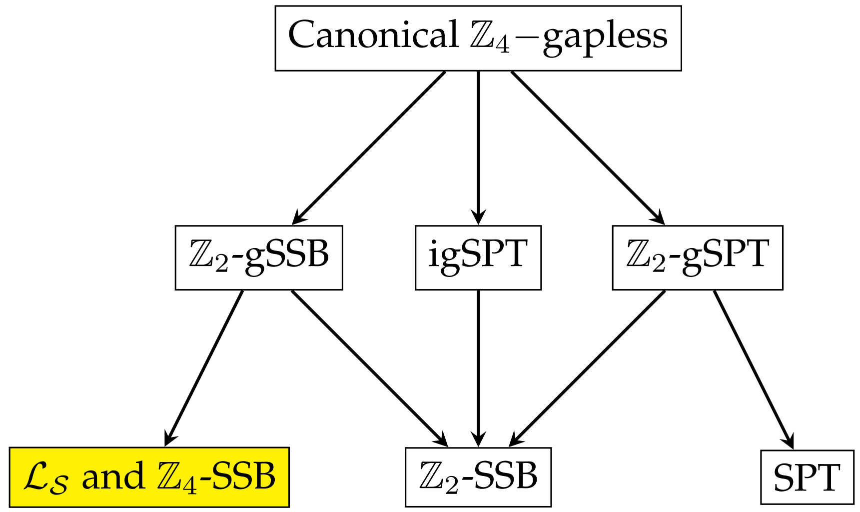
  $$\mathcal{A}_{12} = \mathcal{L}_1 \cap \mathcal{L}_2$$

  i.e., a non-maximal set of mutually local topological defects.
- One can also tune and consider $\cap_i \mathcal{L}_i$ for any subset of Lagrangian algebras.

Condensable algebras have a partial order, and thus a Hasse diagram.

Gapless Phases and Phase Transitions for $\mathbb{Z}_4$



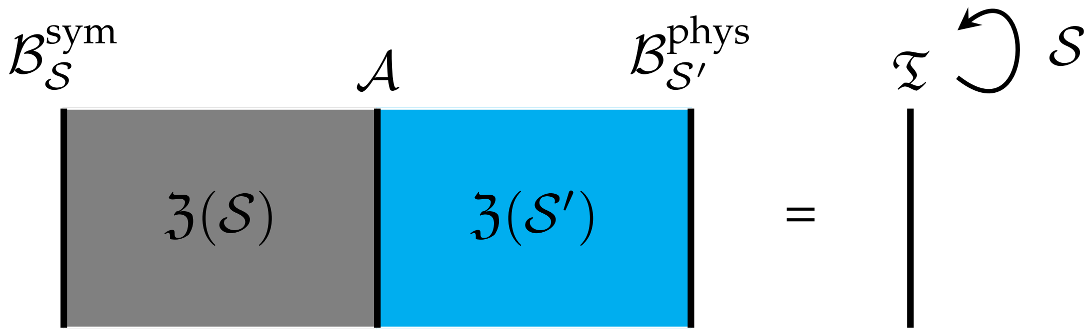


- gSPT (gapless SPT): $\mathcal{A} \cap \mathcal{L}_\mathcal{S} = 1$
- igSPT (intrinsically gapless SPT): gSPT that cannot be deformed to an SPT
- gSSB (gapless SSB): $\mathcal{A} \cap \mathcal{L}_\mathcal{S} \supsetneq 1$
- igSSB (intrinsically gapless SSB): gSSB with n universes, that cannot be deformed to an SSB with n vacua

For $\mathbb{Z}_4$: igSPT was found in [Wen, Potter].
First non-invertible igSPT: Rep(D_{8n}) [Bhardwaj, Pajer, SSN, Warman].

Club Sandwich and Phase Transitions

Non-maximal ("Non-Lagrangian") condensable algebras define interfaces between topological orders $\mathfrak{Z}(\mathcal{S})$ and $\mathfrak{Z}(\mathcal{S}')$, where the latter is a reduced topological order:



Concretely this can be used to make new phase transitions out of old:

$\Rightarrow$ Kennedy-Tasaki-transformations: $\mathcal{S}'$-symmetric to $\mathcal{S}$-symmetric theories

New Phase Transitions from Old

Consider an input phase transition between $\mathcal{S}'$-symmetric gapped phases

$$\mathfrak{T}_1^{\mathcal{S}'} \longleftarrow \mathcal{C}_{12}^{\mathcal{S}'} \longrightarrow \mathfrak{T}_2^{\mathcal{S}'}$$

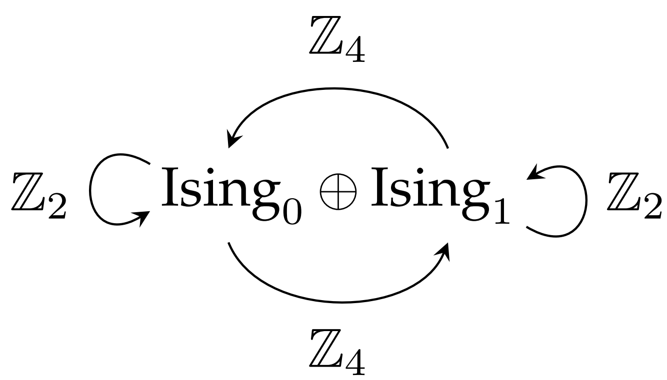
The club sandwich produces a phase transition for the symmetry $\mathcal{S}$, which is the KT transformation of the initial input phase transition:

$$\mathfrak{T}_1^{\mathcal{S}} \longleftarrow \mathcal{C}_{12}^{\mathcal{S}} \longrightarrow \mathfrak{T}_2^{\mathcal{S}}$$

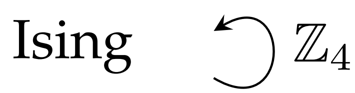
Gapless Phases from SymTFT for $\mathbb{Z}_4$

Gapless phase between $\mathbb{Z}_4$-SSB to $\mathbb{Z}_2$-SSB:

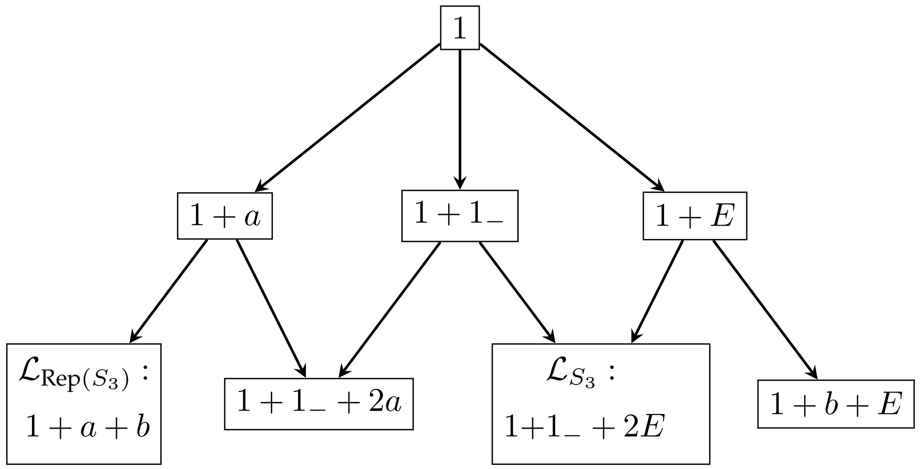
$\mathcal{S} = \mathbb{Z}_4$ with algebra $\mathcal{A} = 1 \oplus e^2$, implies $\mathcal{S}' = \mathbb{Z}_2$, so the input transition is an Ising transition:

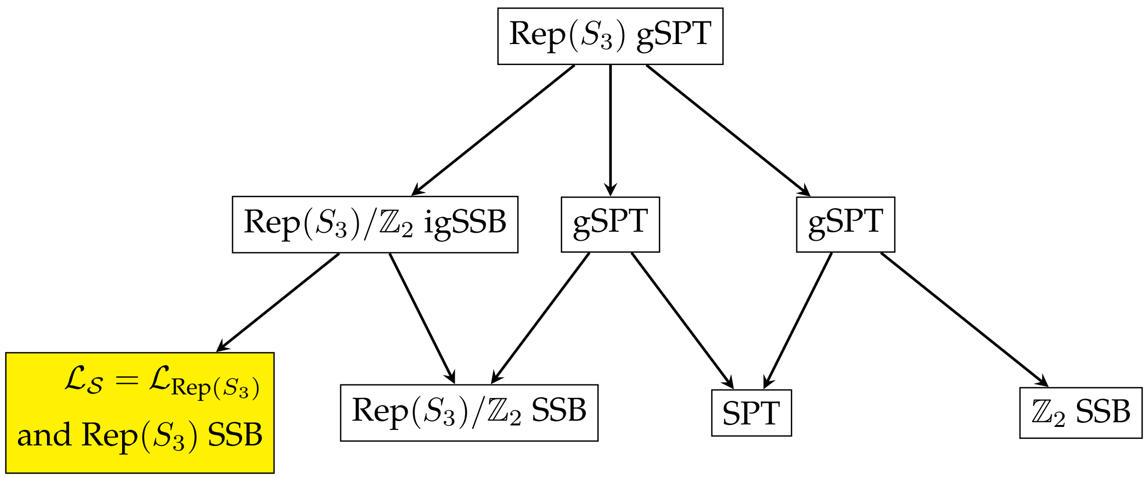


Gapless phase between $\mathbb{Z}_4$ trivial and $\mathbb{Z}_2$ SSB:



Hasse Diagram for Rep(S_3)

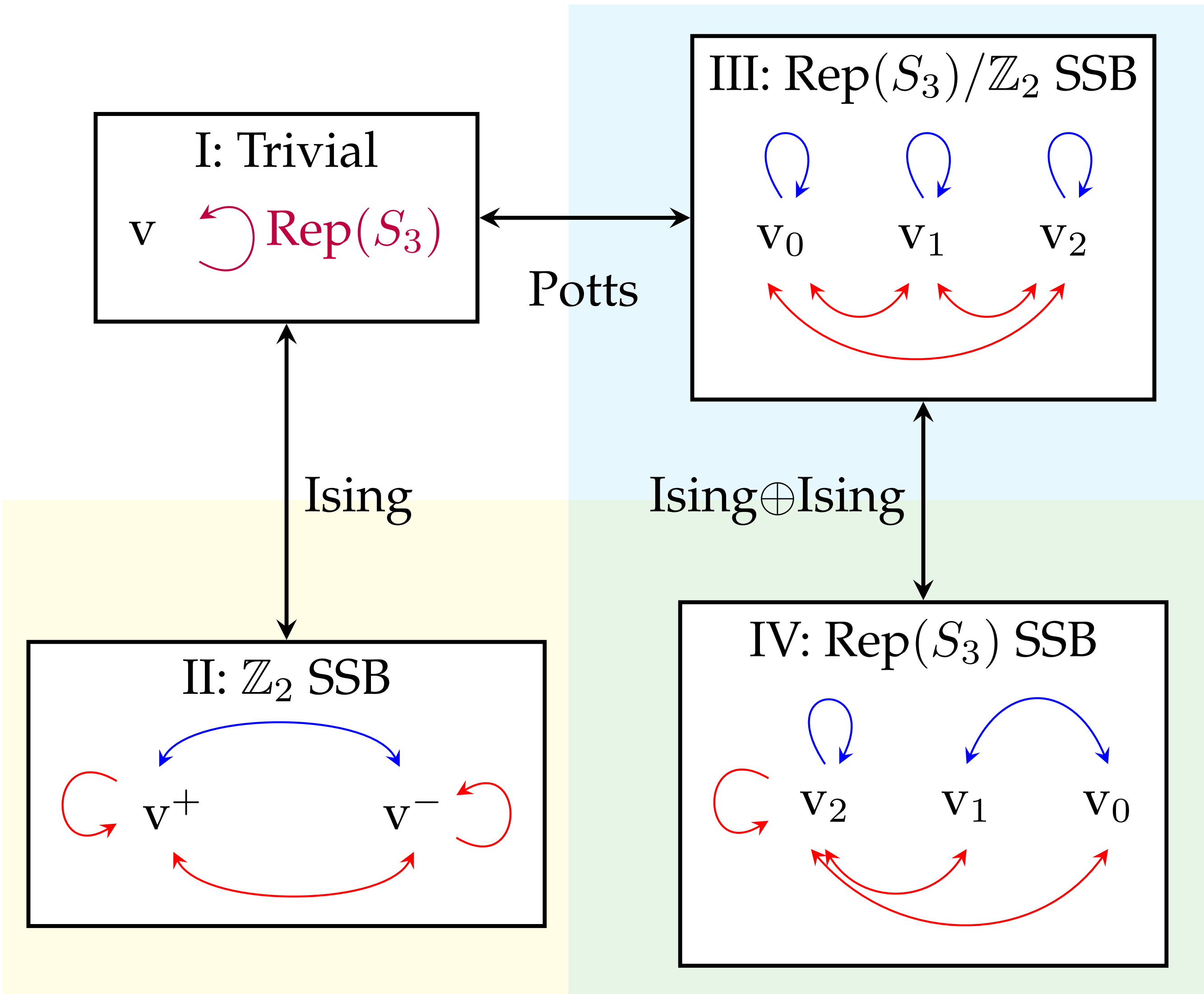




Phase diagram for Rep(S_3) in 2d

Rep(S_3)= $\{1, \sigma, E\}$. Both from continuum and from spin-chain models

[Bhardwaj, Pajer, SSN, Warman][Bhardwaj, Bottini, SSN, Tiwari][Chatterjee, Aksoy, Wen]



A Roadmap of Phases with Symmetry $\mathcal{S}$

- Construct the SymTFT and its topological defects.
- Determine all condensable algebras of topological defects.
- In particular: $\mathcal{L}_1$ and $\mathcal{L}_2$ are Lagrangians, that give rise to gapped phases, then the gapless phase between these is given by $\mathcal{A}_{12} = \mathcal{L}_1 \cap \mathcal{L}_2$.
- SymTFT encodes the order parameters and symmetry implementation.

Results in new phases with non-invertible symmetries, e.g. found non-invertible SPTs and igSPTs for $\text{Rep}(D_{8n})$.

Crucially, this is applicable to any fusion category symmetry.

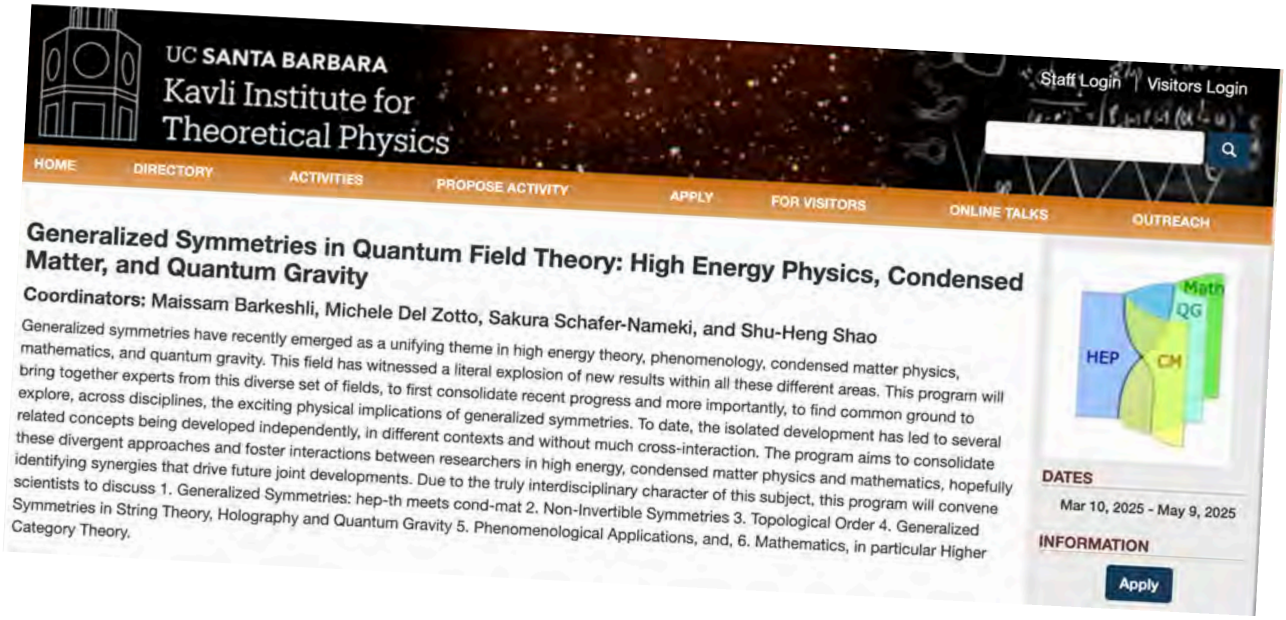
Conclusions and Open Questions

Non-Invertible/Categorical symmetries are everywhere. Don't think about escaping them (or: stop worrying and learn about categories).
Open Questions:

1. Classification of symmetric phases: 3d and 4d where the full structure of higher fusion categories will need to be tapped in [wip Oxford]
2. gSPT, igSPT, gSSB, igSSB phases in higher dimensions: QFT examples? [Antinucci, Copetti, SSN, wip]
   $\rightarrow$ gSPTs in 4d [Thomas Dumitrescu's talk.]
3. Categorical Landau: Is there a LG model for non-invertible symmetries?
4. SymTFT: Inclusion of spacetime symmetries (in lattice models: [Seiberg, Seifnashri, Shao]) and continuous symmetries into the SymTFT.
5. How is the full structure of higher-fusion categories encoded in string theory/holography? Condensation completion, higher associators, etc.
6. Develop constraints on RG-flows from non-invertible symmetries beyond 2d.
7. Lattice model realizations for the above: lattice realizations of non-invertible symmetries in 3d, 4d.

https://sites.google.com/view/symmetries2024/home

https://www.kitp.ucsb.edu/activities/gensym25



Supplementary Material

What is a categorical symmetry?

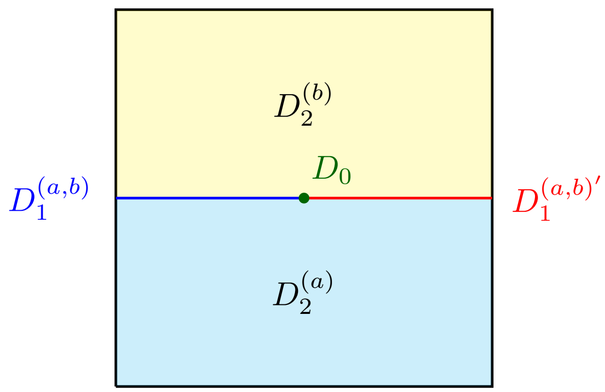
Topologiocal defects = symmetries.

$(d-p-1)$-dimensional defect links in d dimensions with a p-dimensional charged operator. ”p-form symmetry”.

A non-invertible symmetry in d-spacetime dimensions is a $(d-1)$-fusion category:

- Topological defects of dimension $(d-1)$, up to 0: $(d-1)$ objects, $(d-2)$ morphisms, $(d-3)$ 2-morphisms, etc.
- Fusion of defects in each dimension
- Compatibility/associativity conditions

$d=3$:

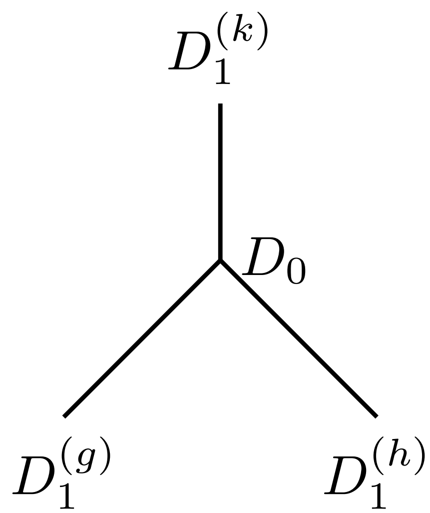


Fusion Categories

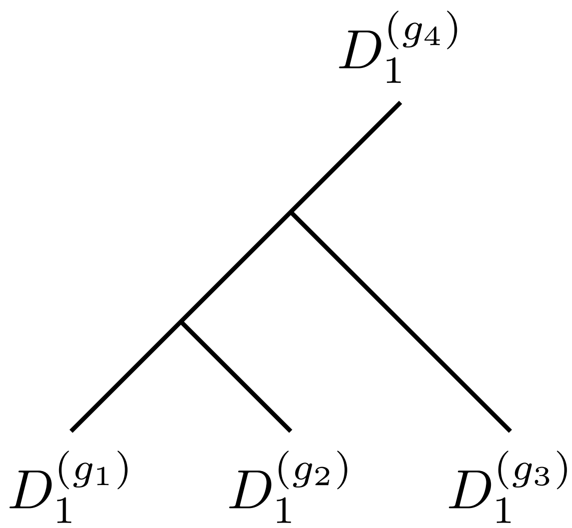
In (1+1)d:

- Objects: topological lines $D_1^{(g)}$,
- Morphisms: topological point operators $D_0 \in \mathrm{Hom}(D_1^{(g)}, D_1^{(h)})$.
- Fusion:

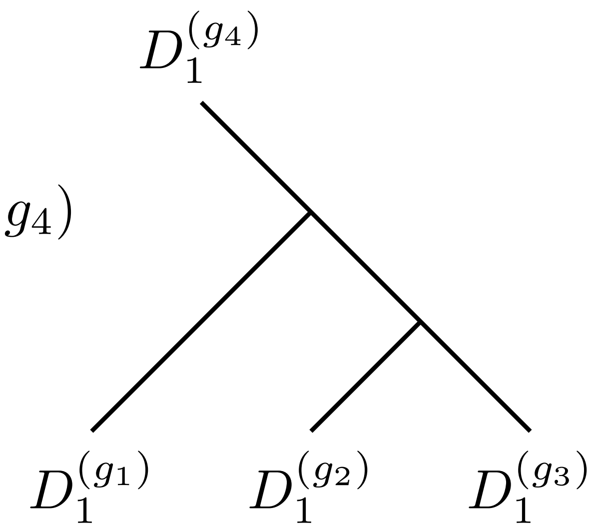
$$D_1^{(g)} \otimes D_1^{(h)} = \bigoplus_k N_k^{g,h} D_1^{(k)} \,.$$



- Associativity:



$$= \omega(g_1, g_2, g_3, g_4)$$



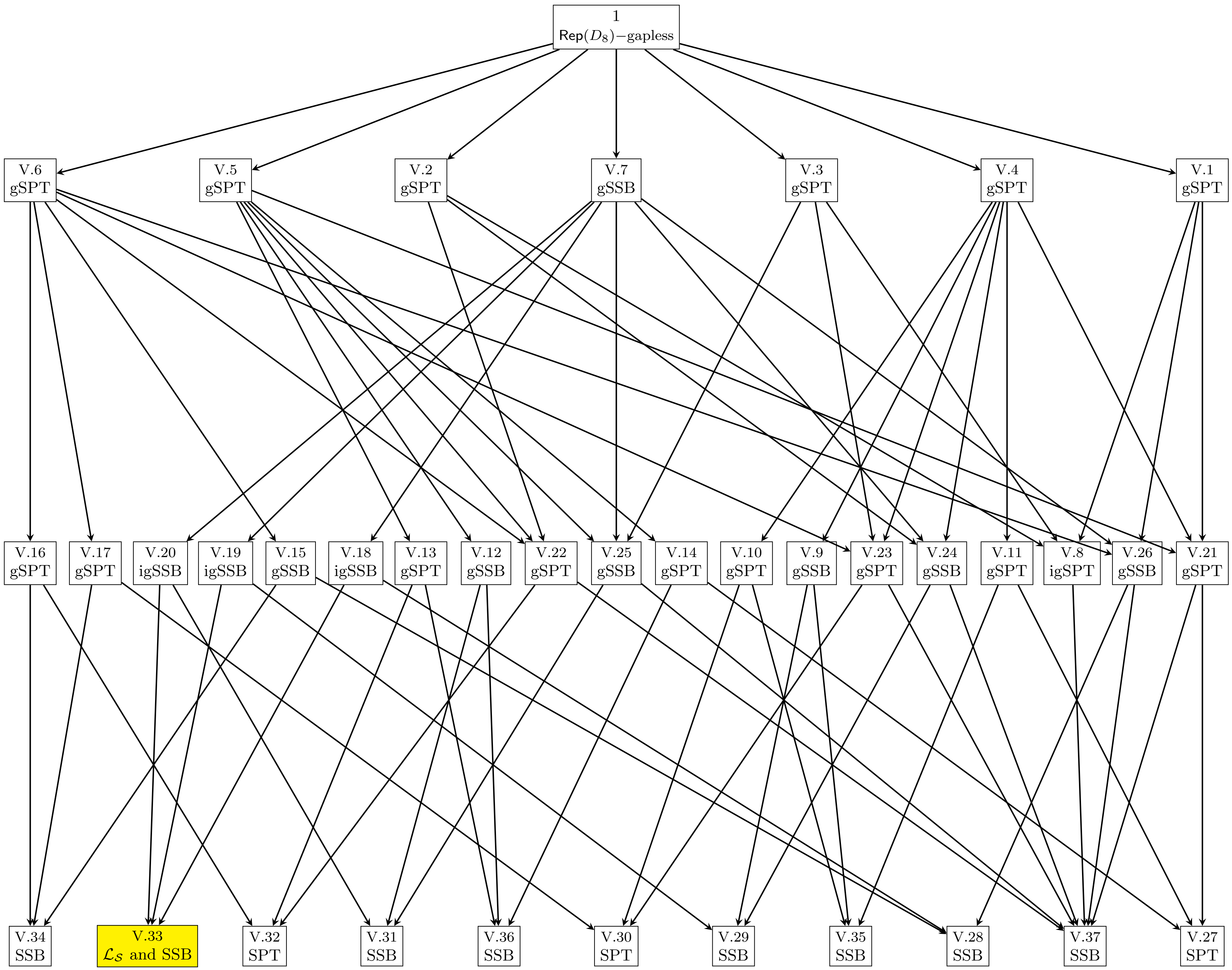
Classification of Phases

Phase	Physical characterization	Energy gap Δ Symmetry gap $\Delta_{\mathcal{S}}$	Condition on $\mathcal{A}$ in (1+1)d	n
SPT	Gapped system with energy gap $\Delta > 0$. IR: trivial TQFT. $\mathcal{S}$-charges confined in IR appear at an energy scale (symmetry gap) $\Delta_{\mathcal{S}} \geq \Delta > 0$. Order parameters (OPs) are all of string type (i.e. in twisted-sectors for $\mathcal{S}$).	$\Delta > 0$ $\Delta_{\mathcal{S}} > 0$	$\mathcal{A} = \mathcal{L}$ $\mathcal{A} \cap \mathcal{L}_{\mathcal{S}} = 1$	1
gSPT	Gapless system with $\Delta = 0$ and a unique ground state on circle. Not all charges of $\mathcal{S}$ appear in IR. The confined charges appear at a symmetry gap $\Delta_{\mathcal{S}} > 0$. OPs are all of string type.	$\Delta = 0$ $\Delta_{\mathcal{S}} > 0$	$\mathcal{A} \neq \mathcal{L}$ $\mathcal{A} \cap \mathcal{L}_{\mathcal{S}} = 1$	1
igSPT	A gSPT phase that cannot be deformed to a gapped SPT phase, because it has confined charges not exhibited by any of the gapped SPTs.	$\Delta = 0$ $\Delta_{\mathcal{S}} > 0$	$\mathcal{A} \neq \mathcal{L}$ $\mathcal{A} \cap \mathcal{L}_{\mathcal{S}} = 1$	1
SSB	Gapped system with n degenerate vacua (labeled by i) permuted by $\mathcal{S}$ action. Each vacuum i has energy gap $\Delta^{(i)} > 0$. Going from i to j costs $\Delta^{(ij)} > 0$. Not all charges realized in IR $\Longrightarrow$ symmetry gap $\Delta_{\mathcal{S}} > 0$. OPs are multiplets with string and non-string type.	$\Delta^{(i)} > 0$ $\Delta^{(ij)} > 0$ $\Delta_{\mathcal{S}} > 0$	$\mathcal{A} = \mathcal{L}$ $\mathcal{A} \cap \mathcal{L}_{\mathcal{S}} \supsetneq 1$	> 1
gSSB	Gapless system with n degenerate gapless universes labeled by i. Each universe has a unique ground state on a circle. Going from i and j costs $\Delta^{(ij)} > 0$. Not all charges realized in IR $\Longrightarrow$ symmetry gap $\Delta_{\mathcal{S}} > 0$. OPs string and non-string type	$\Delta^{(i)} = 0$ $\Delta^{(ij)} > 0$ $\Delta_{\mathcal{S}} > 0$	$\mathcal{A} \neq \mathcal{L}$ $\mathcal{A} \cap \mathcal{L}_{\mathcal{S}} \supsetneq 1$	> 1
igSSB	A gSSB phase with n universes that cannot be deformed to a gapped SSB phase with n vacua.	$\Delta^{(i)} = 0$ $\Delta^{(ij)} > 0$ $\Delta_{\mathcal{S}} > 0$	$\mathcal{A} \neq \mathcal{L}$ $\mathcal{A} \cap \mathcal{L}_{\mathcal{S}} \supsetneq 1$	> 1

Δ is the energy gap. $\Delta_{\mathcal{S}}$ the symmetry gap: not all $\mathcal{S}$-charges are realized in the IR. The missing/confined charges are realized by excited states. The symmetry gap $\Delta_{\mathcal{S}}$, is the energy of the first excited state carrying one of the confined charges. The symmetry becomes less faithful going downwards.

Hasse Diagram for $\mathsf{Rep}(D_8)$

Dim	Condensable Algebra of $\mathcal{Z}(\mathsf{Rep}(D_8))$ (with label)		Reduced TO $\mathcal{S}'$	Phase for $\mathcal{S} = \mathsf{Rep}(D_8)$	n
1	1	(V.0)	$\mathcal{S}$	$\mathsf{Rep}(D_8)-$gapless	1
2	$1 \oplus e_{RG}$	(V.1)	$\mathbb{Z}_4$	gSPT	1
2	$1 \oplus e_{GB}$	(V.2)	$\mathbb{Z}_4$	gSPT	1
2	$1 \oplus e_{RB}$	(V.3)	$\mathbb{Z}_4$	gSPT	1
2	$1 \oplus e_{R}$	(V.4)	$\mathbb{Z}_2 \times \mathbb{Z}_2$	gSPT	1
2	$1 \oplus e_{G}$	(V.5)	$\mathbb{Z}_2 \times \mathbb{Z}_2$	gSPT	1
2	$1 \oplus e_{B}$	(V.6)	$\mathbb{Z}_2 \times \mathbb{Z}_2$	gSPT	1
2	$1 \oplus e_{RGB}$	(V.7)	$\mathbb{Z}_2 \times \mathbb{Z}_2$	gSSB	2
4	$1 \oplus e_{GB} \oplus e_{RB} \oplus e_{RG}$	(V.8)	$\mathbb{Z}_2^{\omega}$	igSPT	1
4	$1 \oplus e_R \oplus m_{GB}$	(V.9)	$\mathbb{Z}_2$	gSSB	2
4	$1 \oplus e_R \oplus m_G$	(V.10)	$\mathbb{Z}_2$	gSPT	1
4	$1 \oplus e_R \oplus m_B$	(V.11)	$\mathbb{Z}_2$	gSPT	1
4	$1 \oplus e_G \oplus m_{RB}$	(V.12)	$\mathbb{Z}_2$	gSSB	2
4	$1 \oplus e_G \oplus m_R$	(V.13)	$\mathbb{Z}_2$	gSPT	1
4	$1 \oplus e_G \oplus m_B$	(V.14)	$\mathbb{Z}_2$	gSPT	1
4	$1 \oplus e_B \oplus m_{RG}$	(V.15)	$\mathbb{Z}_2$	gSSB	2
4	$1 \oplus e_B \oplus m_R$	(V.16)	$\mathbb{Z}_2$	gSPT	1
4	$1 \oplus e_B \oplus m_G$	(V.17)	$\mathbb{Z}_2$	gSPT	1
4	$1 \oplus e_{RGB} \oplus m_{RG}$	(V.18)	$\mathbb{Z}_2$	igSSB	3
4	$1 \oplus e_{RGB} \oplus m_{GB}$	(V.19)	$\mathbb{Z}_2$	igSSB	3
4	$1 \oplus e_{RGB} \oplus m_{RB}$	(V.20)	$\mathbb{Z}_2$	igSSB	3
4	$1 \oplus e_G \oplus e_R \oplus e_{RG}$	(V.21)	$\mathbb{Z}_2$	gSPT	1
4	$1 \oplus e_B \oplus e_G \oplus e_{GB}$	(V.22)	$\mathbb{Z}_2$	gSPT	1
4	$1 \oplus e_B \oplus e_R \oplus e_{RB}$	(V.23)	$\mathbb{Z}_2$	gSPT	1
4	$1 \oplus e_{GB} \oplus e_R \oplus e_{RGB}$	(V.24)	$\mathbb{Z}_2$	gSSB	2
4	$1 \oplus e_G \oplus e_{RB} \oplus e_{RGB}$	(V.25)	$\mathbb{Z}_2$	gSSB	2
4	$1 \oplus e_B \oplus e_{RG} \oplus e_{RGB}$	(V.26)	$\mathbb{Z}_2$	gSSB	2
8	$1 \oplus e_G \oplus e_R \oplus e_{RG} \oplus 2m_B$	(V.27)	trivial	SPT	1
8	$1 \oplus e_B \oplus e_{RG} \oplus e_{RGB} \oplus 2m_{RG}$	(V.28)	trivial	SSB	4
8	$1 \oplus e_{GB} \oplus e_R \oplus e_{RGB} \oplus 2m_{GB}$	(V.29)	trivial	SSB	4
8	$1 \oplus e_B \oplus e_R \oplus e_{RB} \oplus 2m_G$	(V.30)	trivial	SPT	1
8	$1 \oplus e_G \oplus e_{RB} \oplus e_{RGB} \oplus 2m_{RB}$	(V.31)	trivial	SSB	4
8	$1 \oplus e_B \oplus e_G \oplus e_{GB} \oplus 2m_R$	(V.32)	trivial	SPT	1
8	$1 \oplus e_{RGB} \oplus m_{GB} \oplus m_{RB} \oplus m_{RG}$	(V.33)	trivial	$\mathcal{L}_{\mathcal{S}}$ and SSB	5
8	$1 \oplus e_B \oplus m_G \oplus m_R \oplus m_{RG}$	(V.34)	trivial	SSB	2
8	$1 \oplus e_R \oplus m_B \oplus m_G \oplus m_{GB}$	(V.35)	trivial	SSB	2
8	$1 \oplus e_G \oplus m_B \oplus m_R \oplus m_{RB}$	(V.36)	trivial	SSB	2
8	$1 \oplus e_B \oplus e_G \oplus e_{GB} \oplus e_R \oplus e_{RB} \oplus e_{RG} \oplus e_{RGB}$	(V.37)	trivial	SSB	2


1
Rep(D_8)−gapless
V.6
gSPT
V.5
gSPT
V.2
gSPT
V.7
gSSB
V.3
gSPT
V.4
gSPT
V.1
gSPT
V.16
gSPT
V.17
gSPT
V.20
igSSB
V.19
igSSB
V.15
gSSB
V.18
igSSB
V.13
gSPT
V.12
gSSB
V.22
gSPT
V.25
gSSB
V.14
gSPT
V.10
gSPT
V.9
gSSB
V.23
gSPT
V.24
gSSB
V.11
gSPT
V.8
igSPT
V.26
gSSB
V.21
gSPT
V.34
SSB
V.33
$\mathcal{L}_{\mathcal{S}}$ and SSB
V.32
SPT
V.31
SSB
V.36
SSB
V.30
SPT
V.29
SSB
V.35
SSB
V.28
SSB
V.37
SSB
V.27
SPT