Entanglement dynamics from universal low-lying modes

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work in progress with Sanjay Moudgalya and Tibor Rakovszky

Entanglement entropy and thermalization

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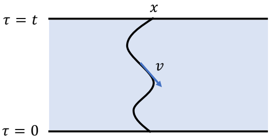
- A more fine-grained probe of the growth of correlations is the entanglement entropy of a subsystem.
- We expect that the strong interactions in chaotic systems cause them to thermalize: any initial state starts to resemble a thermal state at sufficiently late times.
- This leads to the universal expectation that irrespective of the initial state, entanglement entropy saturates to an extensive value at late times.

Universality in approach to equilibrium

- Evolution of entanglement entropy in generic chaotic time-evolutions is very difficult to study analytically.
- But the few analytically tractable examples we can study suggest surprising universality.
- In both random circuits and holographic CFTs, the evolution of entanglement entropy at late times can be expressed in terms of a membrane formula.
- Conjectured to hold universally in Jonay, Huse, Nahum.

Membrane picture for entanglement growth

- In one spatial dimension, suppose we want to find the entanglement entropy of the left half-line at time t.
- Extend the system in time direction from $\tau = 0$ to $\tau = t$, and consider all possible curves:



- We can integrate a function $\mathcal{E}(v)$ along the curve, and minimize over all possible curves.

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  - Mezei showed that for large system size and time, we can get rid of the radial direction in the bulk, and reduce the HRT formula to a minimization problem in the boundary.
  - The resulting membrane tension satisfies non-trivial constraints from Jonay, Huse, Nahum.

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- But we would like to have a more precise understanding of the following questions:

1. What is the source of the velocity-dependent function $\mathcal{E}(v)$?
2. Is there an underlying structure in terms of low-lying modes, which we could look for in a continuum theory such as a holographic CFT?

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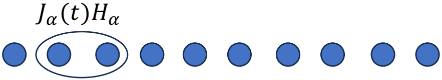
$$H(t) = \sum_{\alpha} J_{\alpha}(t) H_{\alpha},$$

where the H_{α} are local operators, and $J_{\alpha}(t)$ are random numbers, uncorrelated for different times and different α.



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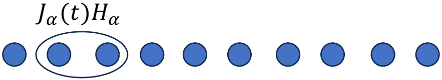

- Previously, such models have allowed a derivation of diffusion in two-point functions Moudgalya and Motrunich; Ogunnaike, Feldmeier, Lee.

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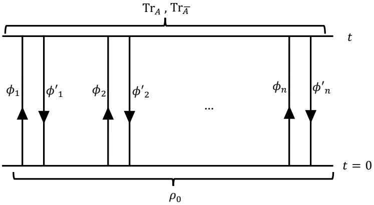

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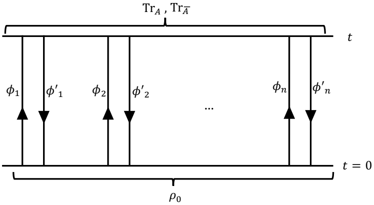

- The key simplification in Brownian models is that the Lorentzian evolution on $2n$ copies can be replaced with a Euclidean evolution:

$$\overline{(U(t) \otimes U(t)^*)^{\otimes n}} = e^{-P_{2n} t}$$

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$$\approx (1 + iH_a(t)\epsilon - \frac{1}{2} H_a(t)^2 \epsilon^2 + ...) \otimes (1 - iH_b(t)^T \epsilon - \frac{1}{2} H_b(t)^{T\,2} \epsilon^2 + ...)$$

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$$= 1 - \epsilon P_2 + O(\epsilon^2) \quad \approx \quad e^{-\epsilon P}$$

where

$$P_2 = \sum_{\alpha} (H_{a,\alpha} - H^T_{b,\alpha})^2$$

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- The equilibrium saturation value of the n-th Renyi entropy is determined by the zero energy states of P_{2n}.

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- Approach to equilibrium is determined by low energy eigenstates, which have a universal structure.

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  P_4 has two degenerate ground states:

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  The low-energy excitations include a "one-particle" band approximately given by:

$$|\psi_k\rangle = \Sigma_x \, e^{i\,k\,x} \; |\downarrow\rangle|\downarrow\rangle|\downarrow\rangle|\downarrow\rangle|\downarrow\rangle|\downarrow\rangle|\downarrow\rangle_x|\phi_{x+1,\dots,x+d}\rangle|\uparrow\rangle_{x+d+1}|\uparrow\rangle|\uparrow\rangle|\uparrow\rangle|\uparrow\rangle|\uparrow\rangle|\uparrow\rangle$$

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- Dispersion relation at $O(1)$ values of k is physically important for satisfying certain constraints on $\mathcal{E}(v)$.

Plan

- Introduce expression for the second Renyi entropy as a transition amplitude, and the definition of $|\uparrow\rangle$ and $|\downarrow\rangle$.
- Derive the low-energy excitations in a simplifying limit.
- Discuss how the structure remains robust more generally.

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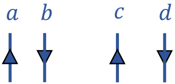

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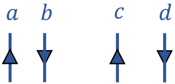

- Let us introduce the following "spins" on four copies of a single site in particular, Zhou and Nahum

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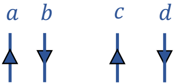

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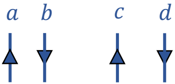

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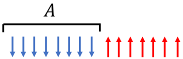


Equilibrium value in models without conserved quantities

- P_4 generally has exactly two zero energy eigenstates:

$$|\uparrow \dots \uparrow\rangle \,, \quad |\downarrow \dots \downarrow\rangle$$

- This gives the Page value for the entropy of pure state at late times:

$$\lim_{t\to\infty} S_{2,A}(t) = \min(\log d_A, \log d_{\bar{A}})$$

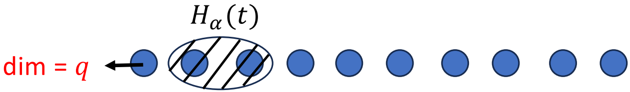
- We would now like to understand the approach to this value using the low-energy modes of P_4.

Low energy excitations: GUE model

GUE model

- Take each $H_\alpha(t)$ to be an i.i.d. random Hermitian matrix on adjacent sites drawn from the GUE ensemble:

$$H_\alpha(t) = H_{i,i+1}^{(\mathrm{GUE})}(t)$$



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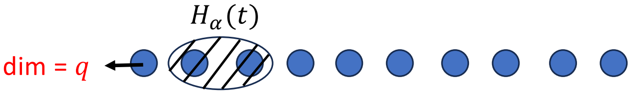


- Using the average over these random matrices, P_4 can be expressed entirely in terms of $|\uparrow\rangle\,,|\downarrow\rangle$.

Analytically solvable large q limit

- In the large q limit, P_4 is exactly solvable, and has a very simple action on a single domain wall

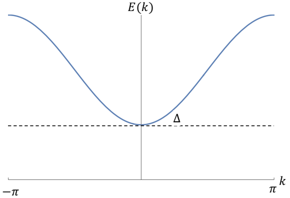
$$\langle D_x| \equiv \langle \downarrow\downarrow \dots \downarrow_x \uparrow_{x+1} \uparrow \dots \uparrow|$$

$$\langle D_x| P_4 = \langle D_x| - \frac{1}{q}(\langle D_{x-1}| + \langle D_{x+1}|)$$

- This leads to the following band of lowest excited states:

$$\langle \psi_k| = \sum_x e^{ikx} \langle D_x|$$

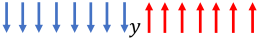
$$E(k) = 1 - \frac{2}{q} \cos k$$



Second Renyi entropy for half-line region

- Let us return to the second Renyi entropy of a half-line region:

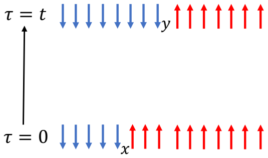
$$e^{-S_2(y,t)} = \langle D_y | e^{-P_4 t} | \rho_0, e \rangle$$



- Since $\langle D_y |$ only evolves to a superposition of $\langle D_x |$ at other locations,

$$e^{-S_2(y,t)} = \sum_x \langle D_y | e^{-P_4 t} | \bar{D}_x \rangle \langle D_x | \rho_0, e \rangle$$

$$= \sum_x \langle D_y | e^{-P_4 t} | \bar{D}_x \rangle \, e^{-S_2(x,t=0)}$$



Membrane picture from one domain wall band

- Using one-particle eigenstates in domain wall propagator:

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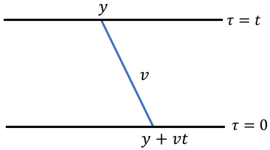
- At late times: using saddle-point approximation for the propagator,

$$S_2(y,t) = \min_v \left[\, \mathcal{E}(v) \; t + S_2(y+vt, t=0) \, \right]$$

where

$$\mathcal{E}(v) = E(k_v) - ik_v v, \quad k_v \text{ is solution to } E'(k_v) = iv \, .$$



One-particle band at finite q

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- From numerical diagonalization of P_4:

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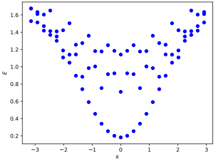
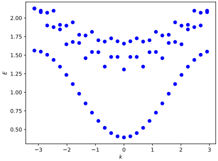
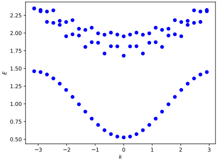


$q = 3$



$q = 4$



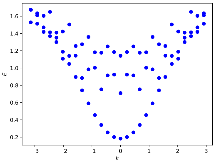
Gapped spectrum in all cases.

One-particle band at finite q

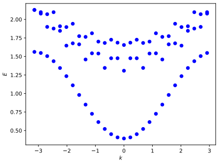
- Away from large q limit, interactions can cause domain walls to split, so the eigenstates and eigenvalues are modified.

- From numerical diagonalization of P_4:

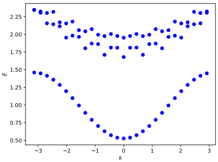
$q = 2$



$q = 3$



$q = 4$



Gapped spectrum in all cases.

- Is the structure of the eigenstates robust?

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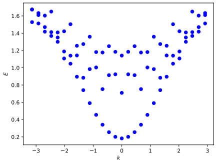
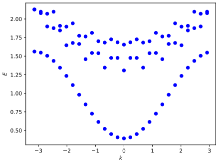
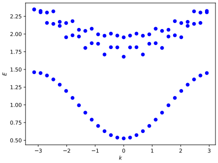


$q = 3$



$q = 4$



Gapped spectrum in all cases.

- Is the structure of the eigenstates robust?

- Is there still a well-defined one-particle band within the continuum for $q = 2$?

Structure of eigenstates at finite q

- Let us consider a variational ansatz for the eigenstates:

$$|\psi_k\rangle = \sum_x e^{ikx} |\downarrow \ldots . \downarrow_x\rangle |\phi_{x+1,\ldots,x+d}\rangle |\uparrow_{x+d+1} \ldots \uparrow\rangle$$

- We can increase the value of d, and at each d, minimize

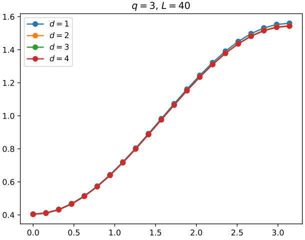
$$E_{\rm var}(k) = \langle \psi_k | P_4 | \psi_k \rangle$$

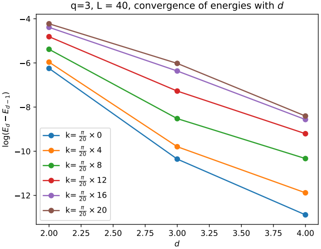
over all choices of $|\phi\rangle$.

- Rapid convergence of $E_{\rm var}(k)$ with d would tell us that the eigenstates are well-approximated by $|\psi_k\rangle$. Haegeman, Spyridon, Michalakis, Nachtergaele, Osborne, Schuch, Verstraete

Variational results for $q = 3$

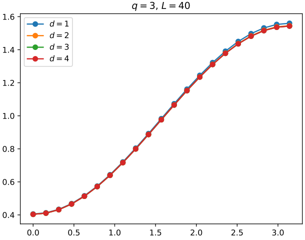
From minimizing $\langle \psi_k | A | \psi_k \rangle$ over all choices of $|\phi\rangle$ for various d:

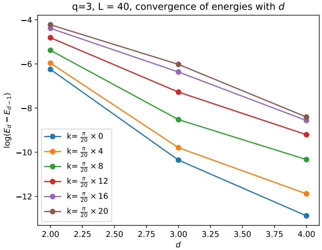




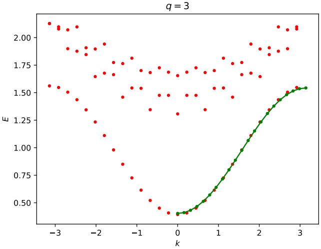
Variational results for $q = 3$

From minimizing $\langle\psi_k|A|\psi_k\rangle$ over all choices of $|\phi\rangle$ for various d:



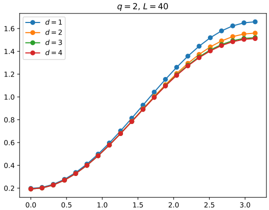


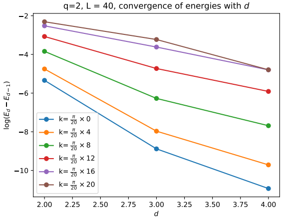
Good agreement with exact diagonalization results:



Variational results for $q = 2$

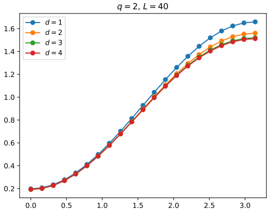
Still very good convergence with d:

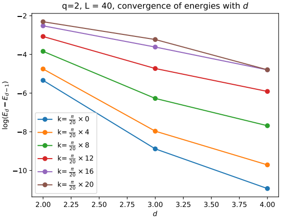




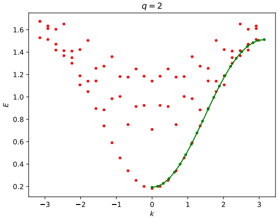
Variational results for $q = 2$

Still very good convergence with d:





- Confirms that there is still a well-defined domain wall band within the continuum. ($\langle D_x|$ will only have significant overlap with this band.)



Constraints on membrane tension

- Overall, we still get the membrane formula at finite q, with $\mathcal{E}(v)$ given by Legendre transform of exact dispersion relation.

- Using the numerically obtained dispersion relations, we can find $\mathcal{E}(v)$, and check that the general constraints Jonay, Huse, Nahum are satisfied

$$\mathcal{E}(v_B) = v_B, \quad \mathcal{E}'(v_B) = 1, \quad \mathcal{E}(v) \geq v, \quad \mathcal{E}''(v) > 0\,.$$

- First two constraints are needed to ensure entropy of equilibrium state does not increase. They translate to the following condition on the dispersion relation:

$$E(i \log q) = 0$$

  which cannot be satisfied by small k expansion.

Generalizations

- In cases where coupling operators are fixed, P_4 is no longer expressed entirely in terms of $|\uparrow\rangle, |\downarrow\rangle$. But similar variational calculation shows that low energy eigenstates still have a dressed domain wall structure.
- Can be seen as explicit realization of ideas of Zhou and Nahum emphasizing importance of $|\uparrow\rangle, |\downarrow\rangle$ in general chaotic systems.
- We find a similar dynamics of domain walls in higher dimensions. In the large q, small k limit, we obtain the same membrane tension as in (1+1)D.

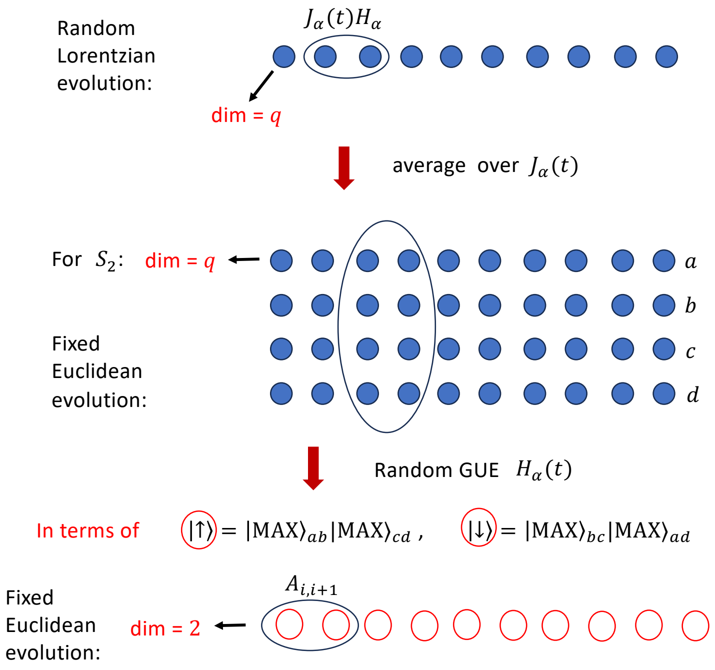
Summary and further questions

- In Brownian models without conserved quantities, the membrane picture is a result of gapped low-energy modes that resemble plane waves of domain walls between permutations.

Questions:

- How does this picture generalize to finite temperature?
  - How does the picture change in Brownian circuits with conserved quantities?
  - How can a similar set of modes emerge in systems without random averaging, including holographic CFTs?
- Can we quantitatively analyse the higher-dimensional case?
- Can these modes be used to formulate an effective field theory of hydrodynamics for entanglement?

Thank you!


Random Lorentzian evolution:
$J_\alpha(t)H_\alpha$
dim = q
average over $J_\alpha(t)$
For S_2: dim = q
a
b
c
d
Fixed Euclidean evolution:
Random GUE $H_\alpha(t)$
In terms of $|\uparrow\rangle = |\mathrm{MAX}\rangle_{ab}|\mathrm{MAX}\rangle_{cd}$, $|\downarrow\rangle = |\mathrm{MAX}\rangle_{bc}|\mathrm{MAX}\rangle_{ad}$
$A_{i,i+1}$
Fixed Euclidean evolution:
dim = 2