D-instanton Induced Effective Action and its Gauge Invariance

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The question

Perturbative string theory amplitudes are gauge invariant

– on-shell amplitudes involving pure gauge states are integrals of total derivatives in the moduli space of Riemann surfaces

– vanish after integration over the moduli

String amplitudes also receive non-perturbative contributions from D-instantons

On-shell amplitudes involving pure gauge states are still integrals of total derivatives in the moduli space of Riemann surfaces

However due to IR divergence we need to

– cut-off the integration region near degeneration of Riemann surfaces

– separately integrate over the 'zero modes' after summing over different world-sheet contributions, including disconnected ones

Are the amplitudes still gauge invariant?

Note: Gauge invariance includes diffeomorphism, local supersymmetry etc.

We shall address this question by:

1. Constructing the D-instanton corrected effective action of closed string field theory

2. Showing that this action satisfies the Batalin-Vilkovisky (BV) master equation.

Batalin Vilkovisky (BV) Formalism

Suppose we have a theory with some gauge invariance.

The fields appearing in the gauge invariant action will be called the 'classical fields'

For every gauge symmetry parameter, we introduce a 'ghost field' carrying the same quantum numbers but opposite grassmannality

The classical fields and the ghost fields together will be called fields $\{\psi^\alpha\}$

For every field ψ^α, we introduce an anti-field ψ^*_α carrying opposite grassmannality

Define, for any functional F and G of fields and anti-fields:

$$\{\mathbf{F},\mathbf{G}\} = \frac{\partial_{\mathbf{R}}\mathbf{F}}{\partial\psi^\alpha}\,\frac{\partial_{\mathbf{L}}\mathbf{G}}{\partial\psi^*_\alpha} - \frac{\partial_{\mathbf{R}}\mathbf{F}}{\partial\psi^*_\alpha}\,\frac{\partial_{\mathbf{L}}\mathbf{G}}{\partial\psi^\alpha}, \qquad \Delta\mathbf{F} = (-1)^{\psi_\alpha}\frac{\partial_{\mathbf{L}}}{\partial\psi^\alpha}\left(\frac{\partial_{\mathbf{L}}\mathbf{F}}{\partial\psi^*_\alpha}\right)$$

L,R: left, right derivatives, $(-1)^{\psi_\alpha}$: grassmann parity of ψ_α

The BV master action $S(\psi, \psi^*)$ is defined by demanding the following conditions:

1. It satisfies the BV master equation

$$\frac{1}{2}\{S, S\} + \Delta S = 0 \qquad \Rightarrow \qquad \Delta e^S = 0$$

since, for any grassmann even function F

$$\Delta e^F = \frac{1}{2}\{F, F\} e^F + \Delta F \, e^F$$

2. When we set $\psi^*_\alpha = 0$, S should reduce to the original gauge invariant action involving classical fields.

Path integral is performed weighted by e^S and we integrate over only half of the fields

– e.g. for each α, set $\psi^\alpha = 0$ or $\psi^*_\alpha = 0$ and integrate over ψ^*_α or ψ^α

– equivalent to the Faddeev-Popov procedure but more versatile

String field theory

In the conventional approach, a string amplitude with a given set of external states has one term at every loop order

– given by an integral over the moduli space of a Riemann surface with punctures

– avoids having to sum over large number of Feynman diagrams.

However, often these integrals run into divergences from the boundary of the moduli space

In some cases, these can be tackled by analytic continuation in the external momenta

When it fails we need to take help of string field theory (SFT)

– essential for D-instantons

SFT is a regular quantum field theory (QFT) with infinite number of field modes, one for every state of the world-sheet CFT of matter and ghosts

– naturally formulated in the BV formalism

Perturbative amplitudes: sum of Feynman diagrams

– can be reorganized into integrals over moduli spaces of Riemann surfaces

Each diagram covers part of the integration region

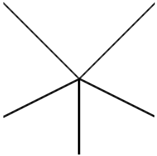
Sum of the diagrams covers the full integration region.

Our theory of interest will be string theory in some background in the presence of r D-branes

– relevant string field theory has open and closed string fields

Riemann surfaces are those with and without boundaries, with the boundaries lying on the D-branes

One particular SFT Feynman diagram is the elementary interaction vertex without propagators



This is given by integral over some regions of the moduli spaces of Riemann surfaces

– give the interaction terms in the BV master action of string field theory

Integrating out fields

$\widetilde{S}_{(r)}$: the BV master action of open-closed string field theory on r D-branes

$S_{(r)}$: the (non-local) BV master action of effective closed string field theory obtained by integrating out the open string fields

$$e^{S_{(r)}} = \int d(\text{open})\, e^{\widetilde{S}_{(r)}}$$

We have

$$(\Delta_{\text{closed}} + \Delta_{\text{open}})\, e^{\widetilde{S}_{(r)}} = 0$$

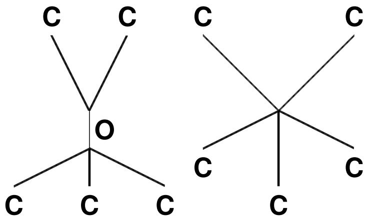
Integrate this over open string fields

$\Rightarrow$ the $\Delta_{\text{open}}\, e^{\widetilde{S}_{(r)}}$ term is a total derivative and integrates to zero.

$$\Rightarrow \qquad \Delta_{\text{closed}}\, e^{S_{(r)}} = 0$$

From now on we denote Δ_{closed} as Δ since $S_{(r)}$ only depends on the closed string fields.

Interaction terms in $S_{(r)}$: sum of all connected Feynman diagrams with closed strings as external states and open strings as internal states



Since each diagram is given by integration over some region of the moduli space, their sum is given by integral over a bigger region of the moduli space of connected Riemann surfaces.

We shall call them ‘EFT world-sheets’

– generate interaction vertices of effective field theory (EFT)

D-instantons

D-instantons are D-branes with Dirichlet boundary condition along all non-compact directions, including Euclidean time

However there are several differences between the way we treat ordinary D-branes and D-instantons

1. Open strings live on 0 dimensional space-time

$\Rightarrow$ path integral over them corresponds to integration over ordinary variables

2. Open strings have zero modes for which the quadratic term vanishes

e.g. modes that describe the position of the instanton in space-time

For such modes the propagator is ill defined

In the world-sheet computation these show up as divergences from the boundaries of the moduli space

Since $S_{(r)}$ were defined as the result of summing over diagrams with internal open string propagators, $S_{(r)}$ cannot be computed using Feynman diagrams

Remedy: Remove the zero mode contribution from the propagators to define $S'_{(r)}$ and do the zero mode integral at the end

$S'_{(r)}$ can be computed from integrals over moduli spaces of EFT Riemann surfaces with the divergences subtracted

We compute $S_{(r)}$ via

$$e^{S_{(r)}} = \int_z e^{S'_{(r)}}$$

Some definitions:

$S_{(0)}$: Contribution from connected EFT world-sheets without boundaries

$\widehat{S}'_{i_1\cdots i_k}$, $k\le r$: contribution to $S'_{(r)}$ from those connected EFT world-sheets which have

– at least one boundary on i_1-th brane, one boundary on i_2-th brane etc.

– and all of whose boundaries lie on branes $i_1,\cdots,i_k$

Then

$$S'_{(1)} = S_{(0)} + \widehat{S}'_1\,, \qquad S'_{(2)} = S_{(0)} + \widehat{S}'_1 + \widehat{S}'_2 + \widehat{S}'_{12}$$

$$e^{S_{(r)}} = \int_z e^{S'_{(r)}} = \int d(\text{open})\, e^{\widetilde{S}_{(r)}}$$

– satisfies $\Delta\, e^{S_{(r)}} = 0$

However $S_{(r)}$ by itself does not give the closed string effective action for r D-instantons

3. Disconnected Riemann surfaces whose boundaries end on the same D-instanton are connected in the target space

$\Rightarrow$ their contribution should be included in the expression for the closed string effective action.

e.g. $\int_z \widehat{S}'_1 \widehat{S}'_2 \widehat{S}'_{12}$ comes from product of three disconnected Riemann surfaces, but is connected in the target space

– should be part of the two instanton contribution to the effective action

$\int_z \widehat{S}'_{12} \widehat{S}'_{34}$ is not a contribution to the effective action but $\int_z \widehat{S}'_{12} \widehat{S}'_{34} \widehat{S}'_{23}$ is.

4. Since D-instantons are finite action solutions we have to sum over different number of D-instantons

$\Rightarrow$ **all of them contribute to the usual closed string amplitudes**

r instanton contribution is weighted by κ^{r} with $\kappa \equiv e^{-\mathcal{T}}$, $-\mathcal{T}$ being the D-instanton action

5. Permutation of r identical instantons gives the same configuration

$\Rightarrow$ **we have a factor of $1/\mathrm{r}!$**

Using all the rules we see that to order κ^2, we get the action:

$$\mathsf{S} = \mathsf{S}_{(0)} + \kappa \int_{\mathsf{z}} \left(e^{\widehat{\mathsf{S}}'_1} - 1 \right) + \frac{1}{2} \kappa^2 \int_{\mathsf{z}} \left(e^{\widehat{\mathsf{S}}'_{12}} - 1 \right) e^{\widehat{\mathsf{S}}'_1 + \widehat{\mathsf{S}}'_2} + \mathcal{O}(\kappa^3)$$

The $\left(e^{\widehat{\mathsf{S}}'_1} - 1 \right)$ factor ensures that we have at least one boundary in the one instanton contribution.

However instead of just $\widehat{\mathsf{S}}'_1$ we also have higher powers of $\widehat{\mathsf{S}}'_1$ since disconnected EFT world-sheets contribute

The $\left(e^{\widehat{\mathsf{S}}'_{12}} - 1 \right)$ factor in the third term ensures that there is at least one factor of $\widehat{\mathsf{S}}'_{12}$

– needed to ensure that the contribution is connected in the target space

After this we can have arbitrary factors of $\widehat{\mathsf{S}}'_{12}$, $\widehat{\mathsf{S}}'_1$ and $\widehat{\mathsf{S}}'_2$

At higher order the analysis gets more complicated

Questions:

1. Can we write a general expression for the effective action S to arbitrary power of κ?

2. Can we verify that this action satisfies the BV master equation?

Answer to this question:

$$S = S_{(0)} + \ln\left(1 + \sum_{r=1}^{\infty} \frac{1}{r!}\,\kappa^{r}\, e^{S_{(r)} - S_{(0)}}\right) - \kappa, \qquad \kappa \equiv e^{-\mathcal{T}}$$

1. There is a combinatoric proof of this statement that we shall skip for now.

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2. Check that it agrees with the earlier result:

$$S = S_{(0)} + \kappa\left(e^{S_{(1)}-S_{(0)}} - 1\right) + \frac{1}{2!}\,\kappa^{2}\left(e^{S_{(2)}-S_{(0)}} - e^{2(S_{(1)}-S_{(0)})}\right) + \mathcal{O}(\kappa^{3})$$

Use

$$e^{S_{(r)}} = \int_{z} e^{S'_{(r)}}$$

$$S'_{(1)} = S_{(0)} + \widehat{S}'_{1}\,, \qquad S'_{(2)} = S_{(0)} + \widehat{S}'_{1} + \widehat{S}'_{2} + \widehat{S}'_{12}$$

to write

$$S = S_{(0)} + \kappa\int_{z}\left(e^{\widehat{S}'_{1}} - 1\right) + \frac{1}{2}\,\kappa^{2}\int_{z}\left(e^{\widehat{S}'_{12}} - 1\right)e^{\widehat{S}'_{1}+\widehat{S}'_{2}} + \mathcal{O}(\kappa^{3})$$

– reproduces the earlier result up to order κ^2

3. Check that it satisfies the BV master equation:

$$\mathbf{S} = \mathbf{S}_{(0)} + \ln\left(1 + \sum_{r=1}^{\infty} \frac{1}{r!}\,\kappa^r\, e^{S_{(r)} - S_{(0)}}\right) - \kappa$$

$$e^{S} = e^{-\kappa} e^{S_{(0)}} \left[1 + \sum_{r=1}^{\infty} \frac{1}{r!}\,\kappa^r\, e^{S_{(r)} - S_{(0)}}\right] = e^{-\kappa}\left[e^{S_{(0)}} + \sum_{r=1}^{\infty} \frac{1}{r!}\,\kappa^r\, e^{S_{(r)}}\right]$$

$\Rightarrow \Delta\, e^{S} = 0$ since $\Delta\, e^{S_{(r)}} = 0$ for $r = 0, 1, 2, \cdots$

$\Rightarrow$ S satisfies the BV master equation

This in turn proves the gauge invariance of D-instanton contribution to on-shell amplitudes in string theory.

4. Proof that S gives the correct effective action:

The complication in analyzing contribution to S

– disconnected world-sheets may represent connected Feynman diagrams in space-time and contribute to S

Strategy: Interaction terms in S and their products should give all connected and disconnected EFT world-sheets

Let S_K be the kinetic term

e^{S-S_K}, expanded in powers of $S-S_K$, generates 1+ product of one or more interaction vertices of the effective theory

$\Rightarrow$ should give 1+ all EFT world-sheets, connected and disconnected, weighted by $\kappa^r/r!$ if its boundaries end on r D-instantons

Proposed formula gives

$$e^{S-S_K} = e^{-\kappa} e^{S_{(0)}-S_K} \left[1 + \sum_{r=1}^{\infty} \frac{1}{r!} \kappa^r e^{S_{(r)}-S_{(0)}} \right]$$

Does it generate 1+ all connected and disconnected EFT world-sheets?

$e^{S_{(0)}-S_K}$: Generates (1+ all connected and disconnected EFT world-sheets without boundaries)

Its coefficient:

$$e^{-\kappa} \left[1 + \sum_{r=1}^{\infty} \frac{1}{r!} \kappa^r \right] = 1$$

– generates correctly EFT world-sheets without boundaries s instanton contribution

– defined as the contribution from connected or disconnected EFT world-sheets for which

• each of the s instantons has at least one boundary lying on it

• each boundary lies on one of the s instantons

Such contributions should be multiplied by $\kappa^s/s!$.

Does our proposal for e^S satisfy this?

Proposed formula gives

$$e^{S-S_K} = e^{-\kappa} e^{S_{(0)}-S_K} \left[1 + \sum_{r=1}^{\infty} \frac{1}{r!} \kappa^r \, e^{S_{(r)}-S_{(0)}} \right]$$

s instanton contribution can appear from the r=s term in the expansion, but also from the r-th term with r $\geq$ s if we only pick terms that use only s of the r instantons

– this can be chosen in $\binom{r}{s}$ ways

$\Rightarrow$ net coefficient of the s-instanton contribution is

$$e^{-\kappa} \sum_{r\geq s} \frac{1}{r!} \kappa^r \binom{r}{s} = e^{-\kappa} \sum_{r\geq s} \frac{1}{s!(r-s)!} \kappa^r = \frac{1}{s!} \kappa^s$$

– gives the correct contribution