Representation Theory of Solitons

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References and Collaborators

Particle-Soliton Degeneracies from Spontaneously Broken Non-Invertible Symmetry [arxiv:2403.08883]

Representation Theory of Solitons [arxiv:2408.11045]

Topological Cosets via Anyon Condensation and Applications to Gapped QCD_2 [arxiv:2412.01877]

Particle-Soliton Degeneracy in 2d Quantum Chromodynamcs [arxiv:2412.21153]

Nick Holfester

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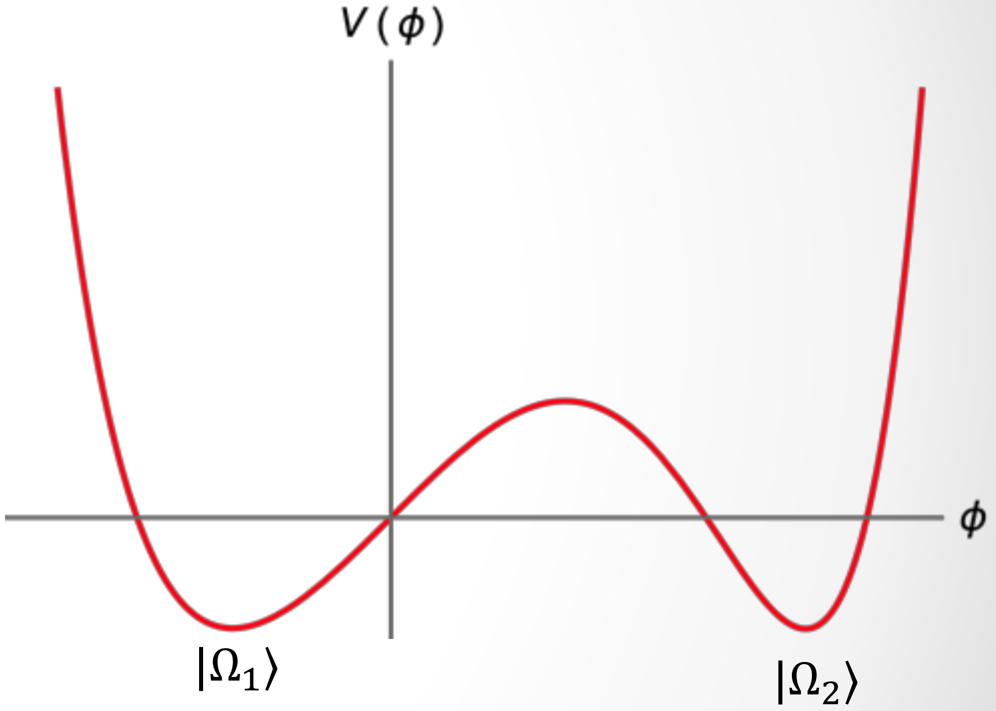
Motivation

Renormalization Group Flows

Consider a scalar field theory in 2d. Let the potential be $V(\phi)$ e.g.:

$$V(\phi) = \phi^6 - 10\,\lambda^3\phi^3 + 12\,\lambda^5\phi$$



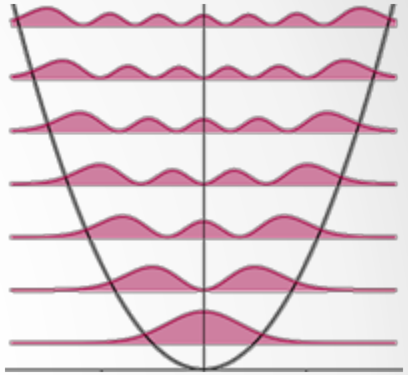
no reflection symmetry

At $\lambda \to 0$ this model describes a tricritical point

Non-zero λ, turns on a relevant deformation

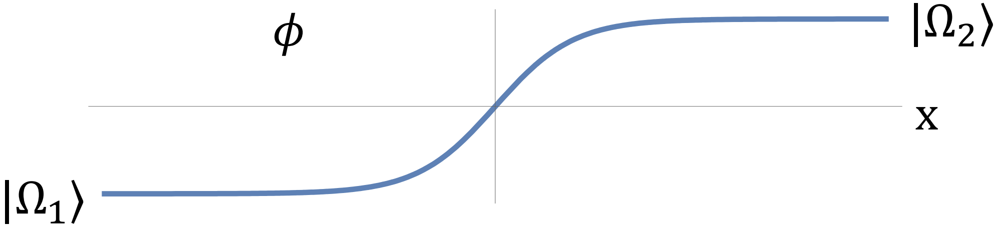
Low energy physics gapped. Two minima/ground states $|\Omega_1\rangle$ and $|\Omega_2\rangle$

Particles and Solitons



Two qualitatively different types of excitations:

- **Particles:** e.g. small excitations of ϕ above each vacuum
- **Solitons:** where the field ϕ interpolates between the vacua



The minimum energy soliton is stable: cannot deform to particle excitation

Ordinary global symmetry organizes the energy eigenstates into multiplets.

Particles Solitons and Global Symmetry

Decompose Hilbert space into sectors labelled by the boundary conditions

$$\mathcal{H} \cong \mathcal{H}_{1,1} \oplus \mathcal{H}_{2,2} \oplus \mathcal{H}_{1,2} \oplus \mathcal{H}_{2,1}$$

In $\mathcal{H}_{i,j}$ the field is at $|\Omega_i\rangle$ at $x = -\infty$, and $|\Omega_j\rangle$ at $x = +\infty$

These are superselection sectors: local operators preserve sector

Symmetry is not spontaneously broken: acts on a fixed sector above

Symmetry spontaneously broken (acts on vacua): permutes the sectors

Standard symmetries map particles to particles and solitons to solitons

Main Question and Results

Suppose the QFT has a non-invertible/generalized global symmetry $\mathcal{C}$

What is the implication on the massive spectrum? What are the multiplets?

What selection rules does $\mathcal{C}$ imply on the scattering matrix of these states?

[Copetti-L. Córdova-Komatsu]

Results:

- Describe the mathematical setup to attack this question
- Given non-invertible symmetry, obtain novel degeneracies e.g.:

$$m_{\text{particle}} = m_{\text{soliton}}$$

Reflects intrinsically-quantum/strong-coupling nature of these symmetries!

Non-Invertible Symmetry in 2d

Non-Invertible Symmetry in 2d

Symmetries defined by operators commuting with Hamiltonian (conserved)

Locality $\Longrightarrow$ topological lines $a, b, c, \cdots$. Mathematically a **fusion category $\mathcal{C}$**

Finite range of simple lines $a, b, c, \cdots$. Analog of finite symmetry group

Multiplication/fusion rule for symmetries $a, b, c, \cdots$ generally non-invertible:

$$a \times b = \sum_c N_{ab}^{c}\, c$$

If extended in time, each operator defines a defect Hilbert space: $\mathcal{H} \rightarrow \mathcal{H}_a$

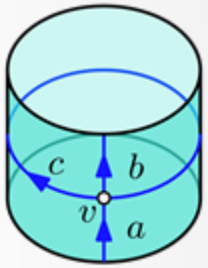
State operator map: $\mathcal{H}_a$ describes twisted sector operators

Action of Symmetry on Operators

Fusion category $\mathcal{C}$ acts in different ways depending on spatial topology

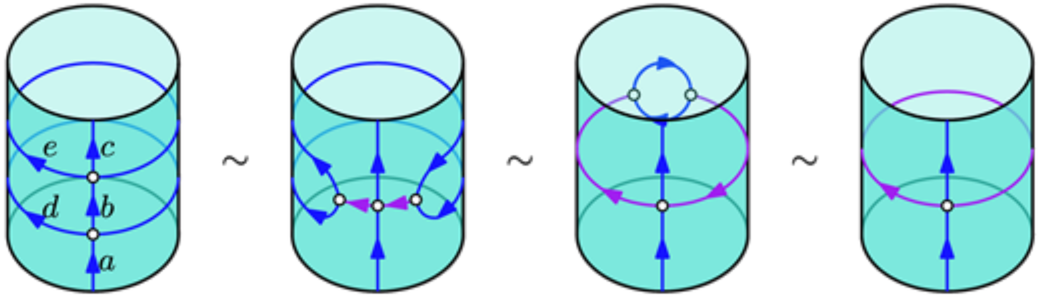
[Ocneanu]

For spatial circle, symmetry action via the tube algebra, $\mathbf{Tube}(\mathcal{C})$



Elements are decorated cylinders (c transforming $\mathcal{H}_a$ to $\mathcal{H}_b$):

Composition in $\mathbf{Tube}(\mathcal{C})$ via stacking and simplifying with associator:



Action of $\mathbf{Tube}(\mathcal{C})$ on direct sum $\bigoplus_{c\in\mathcal{C}} \mathcal{H}_c$. Representation category is center:

$$\mathbf{Rep}\big(\mathbf{Tube}(\mathcal{C})\big) \cong \mathcal{Z}(\mathcal{C})$$

Open Spatial Manifolds and the Strip Algebra

[related recent work: Choi-Rayhaun-Zheng, Bhadwaj-Copetti-Pajer-Schafer-Nameki, Heymann-Quella, Das-Molina-Vilaplana-Saura-Bastida, Dimofte-Niu]

Physics with Boundaries

Natural to study the implications of symmetry on manifolds with boundary

One spatial dimension $\Longrightarrow$ only option spatial interval. Various interpretations:

- The interval is finite in length. For instance, a CFT on an interval with conformal boundary conditions is a natural condensed matter problem.

- The interval is infinite in length, e.g. space is $\mathbb{R}$. Boundary conditions at infinity are required to define finite energy states. (particles/solitons)

The setup we will describe applies to both gapless and gapped scenarios

In gapped situation, IR is a TQFT with multiple vacua acted on by $\mathcal{C}$

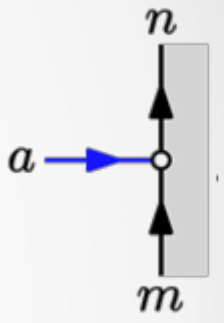
Boundary Conditions and Modules

Boundaries necessitate additional algebraic structure: **module category $\mathcal{M}$**

$\mathcal{M}$ has simple boundaries $m, n, \cdots$, and bulk-boundary fusion rules:

$$a \times m = \sum_n \tilde{N}_{am}^{n}\, n \quad ,$$

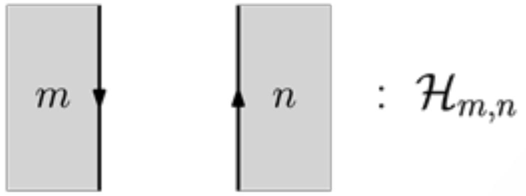
Graphically a acts on boundary:



If the bulk line a changes the boundary conditions it is spontaneously broken

The total Hilbert space is split into sectors labelled by pairs of objects in $\mathcal{M}$:

$$\mathcal{H} \cong \bigoplus_{m,n \in \mathcal{M}} \mathcal{H}_{m,n}$$

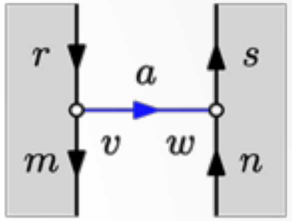


Boundary changing operators or particle/soliton depending on context

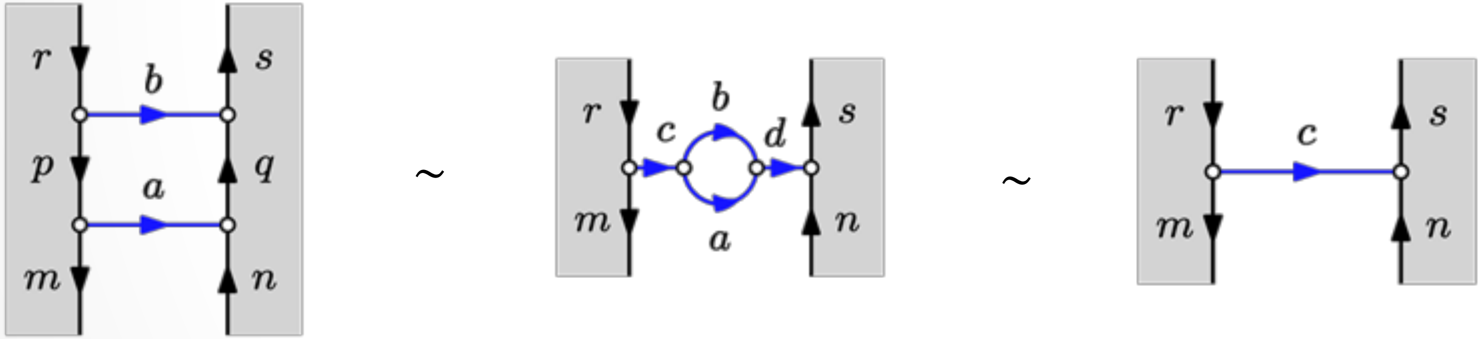
The Strip Algebra

On spatial interval with boundary module $\mathcal{M}$, $\mathcal{C}$ acts by strip algebra: $\mathbf{Str}_{\mathcal{C}}(\mathcal{M})$

Elements are decorated strips (a transforming $\mathcal{H}_{mn}$ to $\mathcal{H}_{rs}$):



Composition in $\mathbf{Str}_{\mathcal{C}}(\mathcal{M})$ via stacking and simplifying:



[Similar Algebras: Barter-Bridgeman-Jones, Bridgeman-Lootens-Verstraete, Kitaev-Kong, Jia-Tan-Kaszlikowski, Choi-Rayhaun-Zheng]

Action of $\mathbf{Str}_{\mathcal{C}}(\mathcal{M})$ on direct sum $\mathcal{H} \cong \bigoplus_{m,n\in\mathcal{M}} \mathcal{H}_{m,n}$

Goal: Understand representations of $\mathbf{Str}_{\mathcal{C}}(\mathcal{M})$ and physical consequences

Representations of the Strip Algebra

$\mathbf{Str}_{\mathcal{C}}(\mathcal{M})$ is an example of a C^* weak Hopf algebra. Generalizes group rings

Duals and tensor products can be defined (antiparticle and multiparticle)

[Böhm-Nill-Szlachányi]

Category of representations is known abstractly:

$$\mathbf{Rep}\big(\mathbf{Str}_{\mathcal{C}}(\mathcal{M})\big) \cong \mathcal{C}^*_{\mathcal{M}} = \mathrm{Hom}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$$

Unpack degeneracies by associating each representation to quiver/graph:

- Nodes are elements in $\mathcal{M}$. Interpreted as distinct vacua of the theory
- Morphisms are arrows from m to n. If $m \neq n$, soliton. If $m = n$, particle

Quiver diagram yields explicit degeneracies implied by symmetry $\mathcal{C}$

[Similar: Cecotti-Vafa]

An Integrable Example

Flows from Rational CFTs

Consider the tricritical Ising model CFT (central charge $c = 7/10$):

Tricritical Ising Model M_4			
Kac label	Conformal Weight	Verlinde Line	Quantum Dimension
$(1,1)$ or $(3,4)$	$h_{1,1} = 0$	1	$d_{1,1} = 1$
$(1,2)$ or $(3,3)$	$h_{1,2} = 1/10$	$W \otimes \eta$	$d_{1,2} = \frac{1+\sqrt{5}}{2}$
$(1,3)$ or $(3,2)$	$h_{1,3} = 3/5$	W	$d_{1,3} = \frac{1+\sqrt{5}}{2}$
$(1,4)$ or $(3,1)$	$h_{1,4} = 3/2$	η	$d_{1,4} = 1$
$(2,2)$ or $(2,3)$	$h_{2,2} = 3/80$	$W \otimes N$	$d_{2,2} = \sqrt{2}(\frac{1+\sqrt{5}}{2})$
$(2,4)$ or $(2,1)$	$h_{2,1} = 7/16$	N	$d_{2,1} = \sqrt{2}$

There are six primaries/Verlinde lines to consider:

[Chang-Lin-Shao-Wang-Yin]

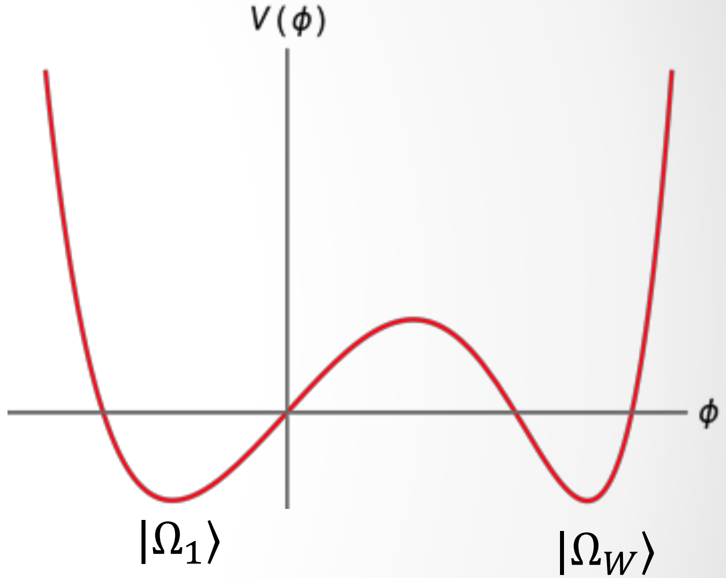
- Fibonacci line: $W \times W = 1 + W$. Preserved by RG flow generated by $\mathcal{O}_{2,1}$
- Tambara-Yamagami lines: $\eta \times \eta = 1,\ N \times N = 1 + \eta$. Preserved by $\mathcal{O}_{1,3}$ flow

The $\mathcal{O}_{2,1}$ RG Flow

The flow triggered by $\mathcal{O}_{2,1}$ results in a gapped theory with two vacua

In Landau Ginzburg realization, sextic potential with degenerate minima and no $\mathbb{Z}_2$ symmetry of the scalar

$$V(\phi) = \phi^6 - 10\,\lambda^3\phi^3 + 12\,\lambda^5\phi$$



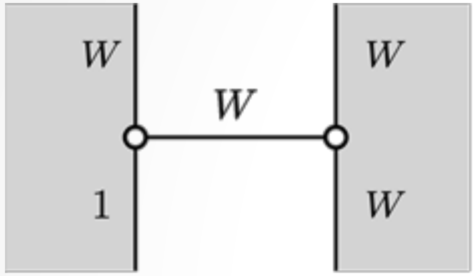
$\lambda \to 0$ describes tricritical Ising, $\lambda \neq 0$ the deformation

The IR TQFT is has therefore two local operators and Fibonacci symmetry $\mathcal{C}$

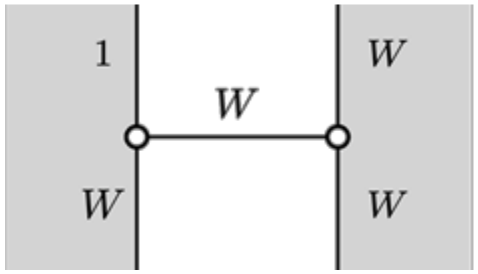
This is a regular module category $\Rightarrow$ label boundary conditions by $\mathcal{C}$

Degeneracies from Fibonacci

Two interesting elements of the strip algebra (maps are not isomorphisms!):



$: \mathcal{H}_{1,W} \to \mathcal{H}_{W,W}$



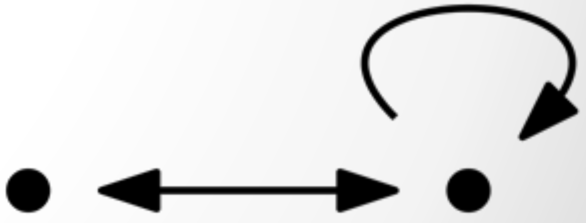
$: \mathcal{H}_{W,W} \to \mathcal{H}_{1,W}$

Minimum energy state $|s\rangle \in \mathcal{H}_{1,W}$ is a soliton. Has CPT conjugate, $|s^*\rangle \in \mathcal{H}_{W,1}$

Fibonacci symmetry maps these onto a non-zero state in $|p\rangle \in \mathcal{H}_{W,W}$

The states $\{|s\rangle, |s^*\rangle, |p\rangle\}$ form a triplet of equal energy $E > 0$

Visualize in quiver. Nodes vacua, arrows excitations:



[Lassig-Mussardo-Cardy, Zamolodchikov]

Symmetry does not predict number of triplets. Known to be one (integrability)

Quantum Chromodynamics

2d Gapped QCD

Gauge theory with gauge group G and massless fermions in representation $\boldsymbol{R}$

At short distances free fermions, at long distances a coset model:

$$\frac{\mathrm{Spin}\big(\dim(\boldsymbol{R})\big)_1}{G_{I(\boldsymbol{R})}}$$

We have bosonized. $I(\boldsymbol{R})$ is index of G in $\mathrm{Spin}\big(\dim(\boldsymbol{R})\big)$ (symmetry of fermions)

If central charge of coset is zero, the IR has a mass gap $\Longrightarrow$ topological coset

[Classification: Delmastro-Gomis]

Solution consists of describing massive particles, solitons, and their scattering

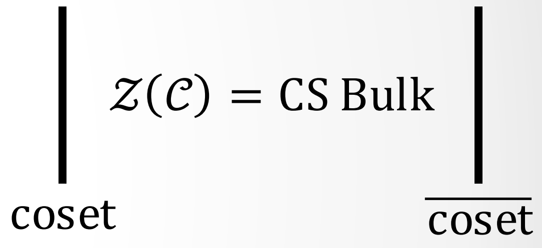
2d QCD and Categorical Symmetry

Gapped QCD always has an interesting fusion category symmetry $\mathcal{C}$

[Komargodski-Ohmori-Roumpedakis-Seifnashri]

Visualize via 3d: in IR Chern-Simons theory with coset boundary conditions

Defines the topological lines and local operators in 2d



Mathematically, the Chern-Simons theory is the Drinfeld center of $\mathcal{C}$:

$$\mathcal{Z}(\mathcal{C}) = \text{Spin}\big(\dim(\boldsymbol{R})\big)_1 \times G_{-I(\boldsymbol{R})} \qquad \text{(simply-connected } G\text{)}$$

This symmetry $\mathcal{C}$ is spontaneously broken in IR, leading to multiple vacua

Particle-Soliton Degeneracy in 2d QCD

We can apply the fusion category symmetry $\mathcal{C}$ to constrain the spectrum!!

Example: SO(3) gauge theory, with matter in a symmetric tensor $\boldsymbol{R} = \mathbf{5}$

Relevant bulk TQFT is: $\mathcal{Z}(\mathcal{C}) = (\mathrm{Spin}(5)_1 \times SU(2)_{-10})/\mathbb{Z}_2$

(Quotient by $\mathbb{Z}_2$ changes gauge group to SO(3). Eliminates 1-form symmetry)

Coset has vanishing central charge. Symmetry $\mathcal{C}$ is three simple lines $\{1, v, A\}$:

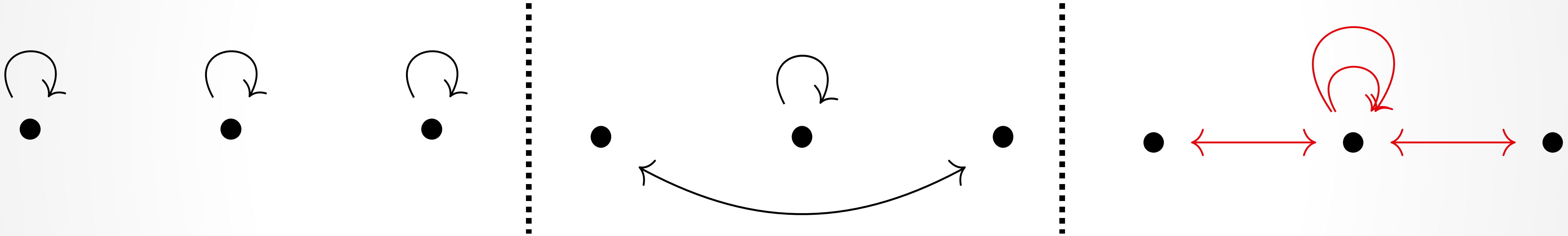
$$v \times v = 1, \qquad v \times A = A \times v = A, \qquad A \times A = 1 + v + 2A$$

In the IR the symmetry $\mathcal{C}$ is spontaneously broken leading to three vacua

Particle-Soliton Degeneracy in 2d QCD

This spontaneous symmetry breaking pattern allows three possible irreps.

Visualize in a quiver: nodes as vacua, arrows stable states (particles/solitons)



A priori, symmetry alone does not tell us how many of each multiplet exists

However, we know each superselection sector of Hilbert space is non-empty

Implies stable states furnish a connected quiver. **Thus at least red states exist!**

Conclusions

Non-invertible symmetries can imply novel degeneracies of massive spectra.
They can relate particles (same vacua) and solitons (distinct vacua)

Key role played by module categories induced from boundary/IR. Used to define the strip algebra $\mathbf{Str}_{\mathcal{C}}(\mathcal{M})$ which controls the representation theory.

't Hooft solved the large color limit of QCD_2. Half a century later,
non-invertible symmetry has given us tools to make progress at finite N_C

Future topics:

How do we generalize these lessons to higher spacetime dimensions?