# Exploring thermal CFTs

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# Motivation

* Quantum critical points: nonzero temperature in the lab.

* Black Holes through AdS/CFT.

* CFTs on non-trivial manifolds.

# Thermal CFTs

Thermal effects are captured by placing the theory on a **circle**

$$S^1_\beta \times \mathbb{R}^{d-1} \qquad \beta = \frac{1}{T}$$

With periodic boundary conditions for the bosons and anti-periodic for the fermions.

| | |
|---|---|
| Translations | ✓ |
| Spatial rotations | ✓ |
| Boosts | ✗ |
| Dilatations | ✗ |
| Special conformal | ✗ |

# Thermal CFTs

In this talk <u>assume</u> we know the **zero temperature CFT data**:

$$\Delta_{\mathcal{O}} \ , \ f_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}$$

and are interested in computing **<u>new</u> finite temperature data:**

the non-zero ***thermal one-point functions***:

$$\langle \mathcal{O}(x) \rangle_\beta = \frac{b_{\mathcal{O}}}{\beta^{\Delta_{\mathcal{O}}}}$$

for neutral scalar operators.

And more generally for all traceless symmetric tensors:

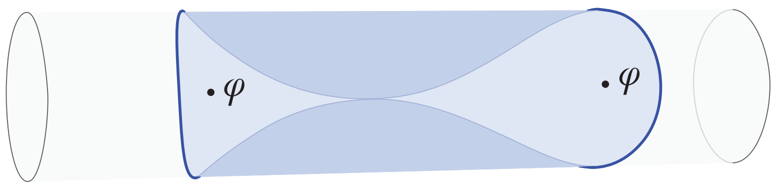
$$\langle \mathcal{O}^{\mu_1 \cdots \mu_J}(x) \rangle_\beta = \frac{b_{\mathcal{O}_J}}{\beta^{\Delta_{\mathcal{O}_J}}} (e^{\mu_1} \cdots e^{\mu_J} - traces)$$

# Thermal CFTs

We can still use the OPE:

$$\mathcal{O}_1(x) \times \mathcal{O}_2(0) = \sum_{\mathcal{O}} f_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}} |x|^{\Delta_{\mathcal{O}} - \Delta_{\mathcal{O}_1} - \Delta_{\mathcal{O}_2} - J} x_{\mu_1} \ldots x_{\mu_J} \, \mathcal{O}^{\mu_1 \ldots \mu_J}(0)$$

But now the radius of convergence is finite: $|x| < \beta$




$$|x| = \sqrt{r^2 + \tau^2}$$

[Iliesiu, Kologlu, Mahajan, Perlmutter, Simmons-Duffin 2018]

# Thermal CFTs

The **two-point function** of identical scalars $\phi$, using the OPE

$$\langle \phi(\tau, r)\phi(0)\rangle_\beta = \sum_{\mathcal{O}} f_{\phi\phi\mathcal{O}} \left(\sqrt{r^2+\tau^2}\right)^{\Delta_\mathcal{O}-2\Delta_\phi-J} x_{\mu_1}\ldots x_{\mu_J} \underline{\langle \mathcal{O}^{\mu_1\cdots\mu_J}\rangle_\beta}$$

and the definition of the Gegenbauer polynomials:

$$\langle \phi(\tau, r)\phi(0)\rangle_\beta = \sum_{\mathcal{O}} \frac{\underline{a_\mathcal{O}}}{\beta^{\Delta_\mathcal{O}}} \left(\sqrt{r^2+\tau^2}\right)^{\Delta_\mathcal{O}-2\Delta_\phi} C_J^{(\nu)}\left(\frac{\tau}{\sqrt{\tau^2+r^2}}\right)$$

$$\underline{a_\mathcal{O} = b_\mathcal{O}} f_{\mathcal{O}\phi\phi}\frac{J!}{2^J(\nu)_J} \qquad (\nu)_J = \frac{\Gamma(\nu+J)}{\Gamma(\nu)} \qquad \nu = \frac{d-2}{2}$$

New Finite Temperature data

# Thermal CFTs

Periodicity of the two-point function is captured by the **KMS condition**:

[Kubo 1957]
[Martin,Schwinger 1959]

$$\langle \phi(\tau, r)\phi(0,0)\rangle_\beta = \langle \phi(\tau + \beta, r)\phi(0,0)\rangle_\beta$$

The OPE expression does not manifestly satisfy KMS, thus imposing it gives a nontrivial **“thermal crossing equation”**.

Variation of KMS:

$$\left\langle \phi\left(\beta/2+\tau\right)\phi(0)\right\rangle_\beta = \left\langle \phi\left(\beta/2-\tau\right)\phi(0)\right\rangle_\beta$$

[El-Showk, Papadodimas 2011]

# Plan of the talk

- ✱ Derive and test thermal Sum Rules.
- ✱ Heavy operators: asymptotic thermal OPE density.
- ✱ Propose a numerical thermal bootstrap method.
- ✱ Temporal line defects (Polyakov loops).

# Sum Rules

# Sum rules from KMS

✱ OPE expand both sides of the El-Showk - Papadodimas formula

$$\left\langle \phi\left(\beta/2+\tau, r\right)\phi(0)\right\rangle_\beta = \left\langle \phi\left(\beta/2-\tau, r\right)\phi(0)\right\rangle_\beta$$

✱ Then further expand the result in powers of $\tau$ and $r$.

✱ Use the definition of Gegenbauer polynomials and the binomial theorem.

$$\sum_{\mathcal{O}\in\phi\times\phi} b_{\mathcal{O}}\, f_{\mathcal{O}\phi\phi}\, F_{\ell,n}(h,J) = 0$$

$n \in \mathbb{N}\,, \ell \in 2\mathbb{N}+1$

$h = \Delta - J$

$$F_{\ell,n}(h,J) = \frac{1}{2^{h+J}}\binom{\frac{h-2\Delta_\phi}{2}}{n}\binom{h+J-2\Delta_\phi-2n}{\ell}\,{}_3F_2\left[\begin{matrix}\frac{1-J}{2}, -\frac{J}{2}, \frac{h}{2}-\Delta_\phi+1\\ \frac{h}{2}-\Delta_\phi-n+1, -J-\nu+1\end{matrix}\middle|\,1\right]$$

# Sum rules from KMS

$$\sum_{\mathcal{O}\in\phi\times\phi} b_{\mathcal{O}}\, f_{\mathcal{O}\phi\phi}\, F_{\ell,n}(h,J) = 0$$

$n \in \mathbb{N}\,,\ell \in 2\mathbb{N}+1$

$h = \Delta - J$

New Finite Temperature ($b_{\mathcal{O}}$)

Known data Zero Temperature ($f_{\mathcal{O}\phi\phi}\, F_{\ell,n}(h,J)$)

We have an **infinite set** of **linear equations** for the combinations $a_{\mathcal{O}} \propto b_{\mathcal{O}} f_{\mathcal{O}\phi\phi}$

Difficulty: **Thermal one-point functions *not*** sign-definite.

Which is crucial for linear programming methods (standard numerical bootstrap).

# Sum rules from KMS

$$\sum_{\mathcal{O}\in\phi\times\phi} b_{\mathcal{O}}\, f_{\mathcal{O}\phi\phi}\, F_{\ell,n}(h,J) = 0$$

$n \in \mathbb{N}\,, \ell \in 2\mathbb{N}+1$

$h = \Delta - J$

New

Finite Temperature

Known data

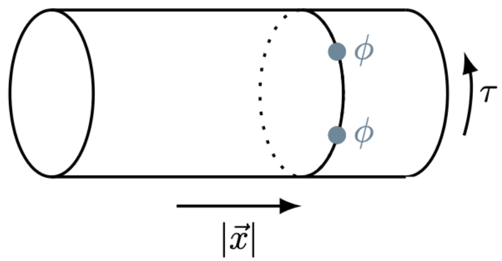
Zero Temperature

Interpretation of the integers:

$$\frac{\partial^{2\ell+1}}{\partial\tau^{2\ell+1}}\frac{\partial^n}{\partial r^n}\langle\phi(\tau,r)\phi(0,0)\rangle_\beta\Big|_{\tau=\frac{\beta}{2},r=0}$$

# Sum rules from KMS

The sum rules for $a_{\mathcal{O}} \propto b_{\mathcal{O}} f_{\mathcal{O}\phi\phi}$




further **simplified for** $r = 0$

$$\frac{\Gamma\left(2\Delta_\phi + \ell\right)}{\Gamma\left(2\Delta_\phi\right)} = \sum_{\Delta \neq 0} \frac{a_\Delta}{2^\Delta} \frac{\Gamma\left(\Delta - 2\Delta_\phi + 1\right)}{\Gamma\left(\Delta - 2\Delta_\phi - \ell + 1\right)} \qquad \ell \in 2\mathbb{N} + 1$$

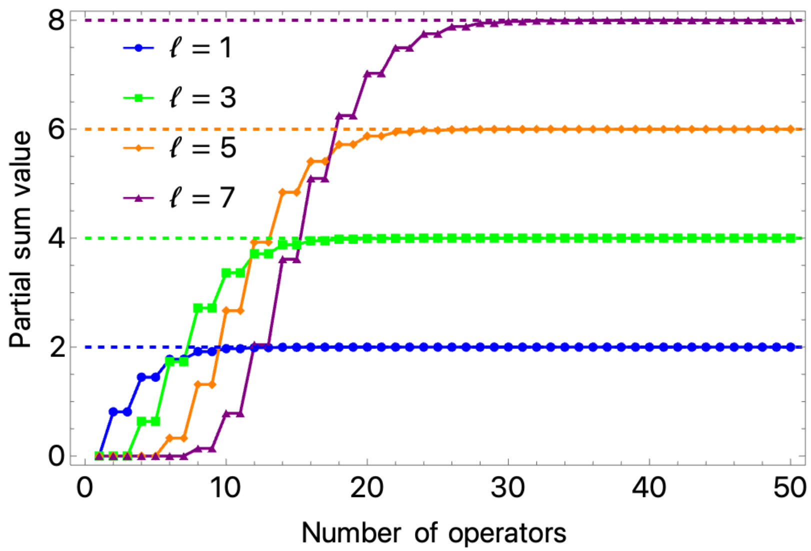
$a_\Delta = \sum_{\mathcal{O} \in \phi \times \phi} a_{\mathcal{O}}\, C_J^{(\nu)}(1)$ for fixed $\Delta$

Operators of same $\Delta$ but different J cannot be distinguished because of $r = 0$.

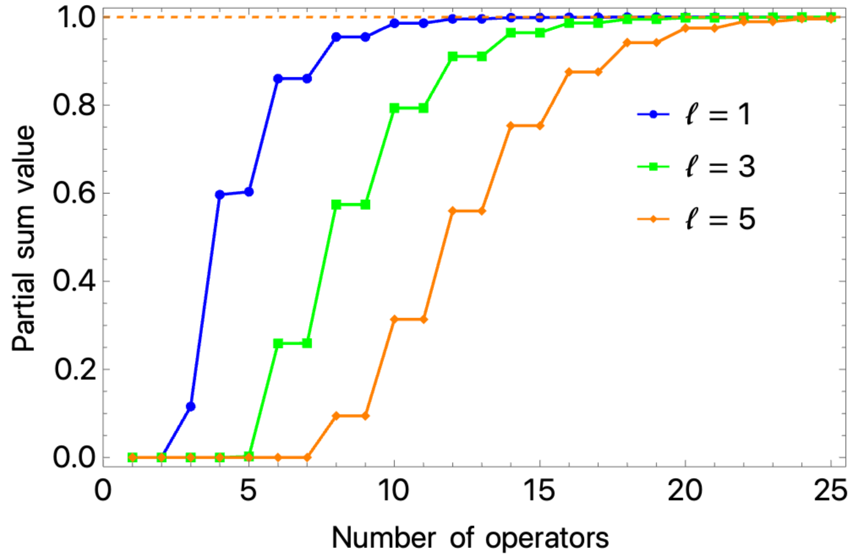
Generically this does not occur, unless there is extra symmetry, like for free theory.

# KMS sum rules: test & learn

### 4-dim free theory




### $\mathrm{O}(N)$ model at large N




Dashed straight lines: the LHS of the sum rule (identity contribution). The RHS plot adding operators.

$$\frac{\Gamma\left(2\Delta_\phi + \ell\right)}{\Gamma\left(2\Delta_\phi\right)} = \sum_{\Delta \neq 0} \frac{a_\Delta}{2^\Delta} \frac{\Gamma\left(\Delta - 2\Delta_\phi + 1\right)}{\Gamma\left(\Delta - 2\Delta_\phi - \ell + 1\right)}$$

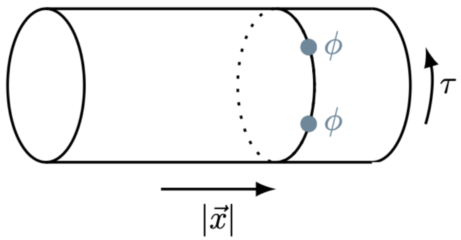
**Observation**: for **small** $\ell$ only **few light operators** contribute.

# Heavy Operators

# Asymptotic thermal OPE density

We want to bound the thermal OPE density for $\Delta \to \infty$. Inspired by [Qiao, Rychkov 2017]

Consider the simplified two-point function at $r = 0$:




$$\langle \phi(\tau)\phi(0) \rangle_\beta = \tau^{-2\Delta_\phi} \sum_{\mathcal{O}\in\phi\times\phi} \frac{a_\mathcal{O}}{\beta^{\Delta_\mathcal{O}}} \tau^{\Delta_\mathcal{O}} = \tau^{-2\Delta_\phi} \int_0^\infty d\Delta\, \rho(\Delta)\, \frac{\tau^\Delta}{\beta^\Delta}$$

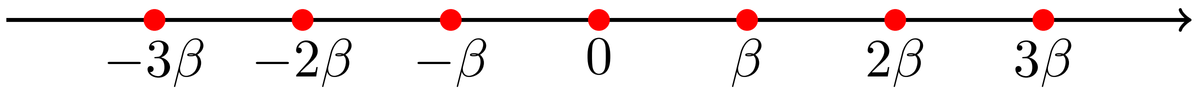
via introducing the spectral density $\rho(\Delta) = \sum_{\Delta'} \delta(\Delta' - \Delta)\, a_\Delta$

$a_\Delta = \sum_{\mathcal{O}\in\phi\times\phi} a_\mathcal{O}\, C_J^{(\nu)}(1)$ for fixed $\Delta$.

# Asymptotic thermal OPE density

We will need:

The locations of the poles of the two-point function on the real axis:




due to periodicity.

$$\langle \phi(\tau)\phi(0)\rangle_\beta \overset{\tau\to k\beta}{\sim} \left(k\beta - \tau\right)^{-2\Delta_\phi}$$

# Asymptotic thermal OPE density

Make an **Ansatz** for the asymptotic behaviour of the density

$$\rho(\Delta) \overset{\Delta\to\infty}{\sim} A\Delta^{-\alpha}$$

Sensible due to Tauberian theory

Doing the integral

$$\langle\phi(\tau)\phi(0)\rangle_\beta \overset{\tau\to\beta}{\sim} A\,\Gamma(2-\alpha)\tau^{-2\Delta_\phi}\left(1-\frac{\tau}{\beta}\right)^{\alpha-2}$$

Comparing with the poles of the two-point function

$$\alpha = -2\Delta_\phi + 2\,, \quad A = \frac{1}{\Gamma(2\Delta_\phi)}$$

$$\rho(\Delta) \overset{\Delta\to\infty}{\sim} \frac{1}{\Gamma(2\Delta_\phi)}\Delta^{2\Delta_\phi-1}$$

# Asymptotic thermal OPE density

Interpretation: the asymptotic thermal density of OPE of Heavy operators.

$$\rho(\Delta) \overset{\Delta\to\infty}{\sim} \frac{1}{\Gamma(2\Delta_\phi)} \Delta^{2\Delta_\phi - 1}$$

Keep in mind, the physical spectrum is discrete $\rho(\Delta) \overset{\Delta\to\infty}{\sim} \sum_{\Delta'} \delta(\Delta' - \Delta)\, a_\Delta$

More correctly: **average density** of OPE **of Heavy operators**

$$\int_0^\Delta \rho(\tilde{\Delta}) d\tilde{\Delta} \overset{\Delta\to\infty}{\sim} \frac{\Delta^{2\Delta_\phi}}{\Gamma(2\Delta_\phi + 1)} \left(1 + \mathcal{O}\left(\frac{1}{\Delta}\right)\right)$$

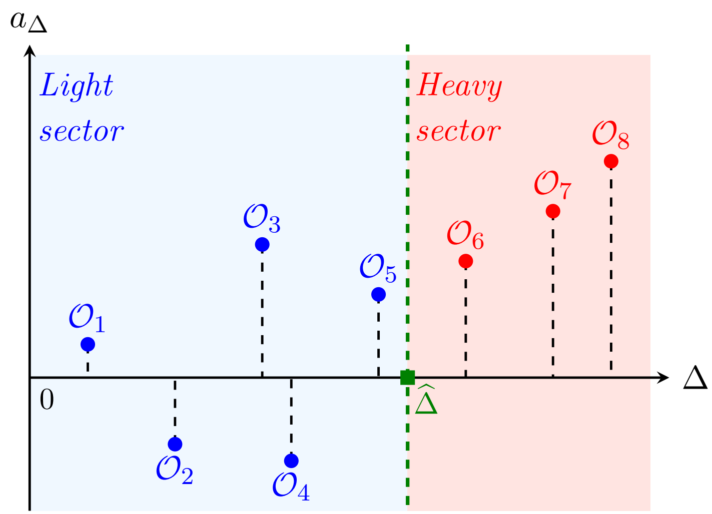
**This result is the correct formal math result (Tauberian theorems) with an error.**

# Asymptotic thermal OPE density

This was a **heuristic derivation** (nonetheless captures the intuition).

In the paper we give a rigorous derivation using **Tauberian theorems** under the assumptions:

- ✱ Unitarity (reality of the thermal OPE coefficients)
- ✱ Boundedness of the thermal OPE from below (we proved)
- ✱ $\Delta_\phi > 1/2$ which is correct because of unitarity ($d \geq 3$)




$$a_\Delta \geq -c$$

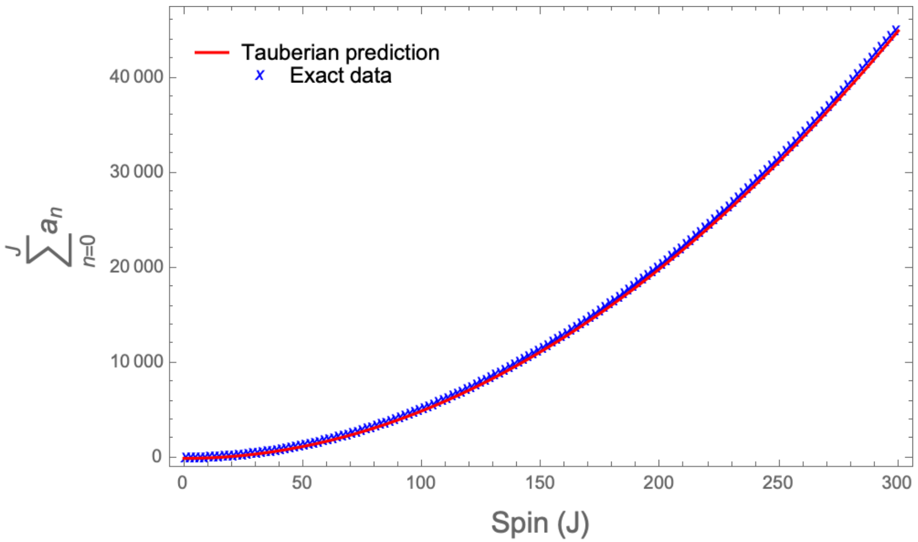
# Free theory in 4d

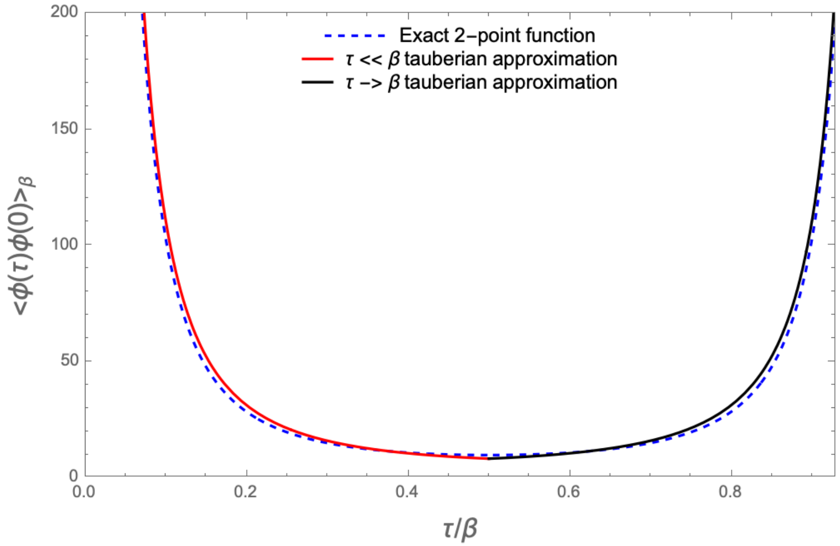
The exact two-point function:

$$\langle\phi(\tau)\phi(0)\rangle_\beta = \left(\frac{\pi}{\beta}\right)^2 \csc^2\left(\frac{\pi}{\beta}\tau\right)$$

Tauberian approximation:

$$\langle\phi(\tau)\phi(0)\rangle_\beta \simeq \begin{cases} \displaystyle\int_0^\infty d\Delta \frac{\Delta^{2\Delta_\phi-1}}{\Gamma(2\Delta_\phi)}\frac{(\beta-\tau)^{\Delta-2\Delta_\phi}}{\beta^\Delta} & \tau/\beta \ll 1 \\ \displaystyle\int_0^\infty d\Delta \frac{\Delta^{2\Delta_\phi-1}}{\Gamma(2\Delta_\phi)}\frac{\tau^{\Delta-2\Delta_\phi}}{\beta^\Delta} & \tau/\beta \sim 1 \end{cases}$$




The Tauberian approximation is very good!

# 3d O(N) model at large N

Hubbard-Stratonovich field

Lagrangian description:

$$\mathcal{L} = \frac{1}{2}(\partial\phi_i)^2 + \frac{1}{2}\sigma\phi_i^2 - \frac{\sigma^2}{4\lambda}$$

Wilson-Fisher expansion $\epsilon = 4 - d \ll 1$

weakly coupled in large N: $\lambda_* = \dfrac{8\pi^2}{N+8}\epsilon$

d=3 non-trivial IR fixed point.

The two-point function:

$$\langle\phi_i(\tau,r)\phi_j(0,0)\rangle_\beta = \delta_{ij}\sum_{m=-\infty}^{\infty}\int\frac{d^2k}{(2\pi)^2}\frac{e^{-i\vec{k}\cdot\vec{x}-i\omega_m\tau}}{\omega_n^2+\vec{k}^2+m_{th}^2} = \delta_{ij}\sum_{m=-\infty}^{\infty}\frac{e^{-m_{th}\sqrt{(\tau+m\beta)^2+r^2}}}{\sqrt{(\tau+m\beta)^2+r^2}}$$

$$\langle\sigma\rangle_\beta = m_{th}^2 = \frac{4}{\beta^2}\log^2\left(\frac{1+\sqrt{5}}{2}\right)$$

[Sachdev, Ye 1992]

[Iliesiu, Kologlu et al. 2018]

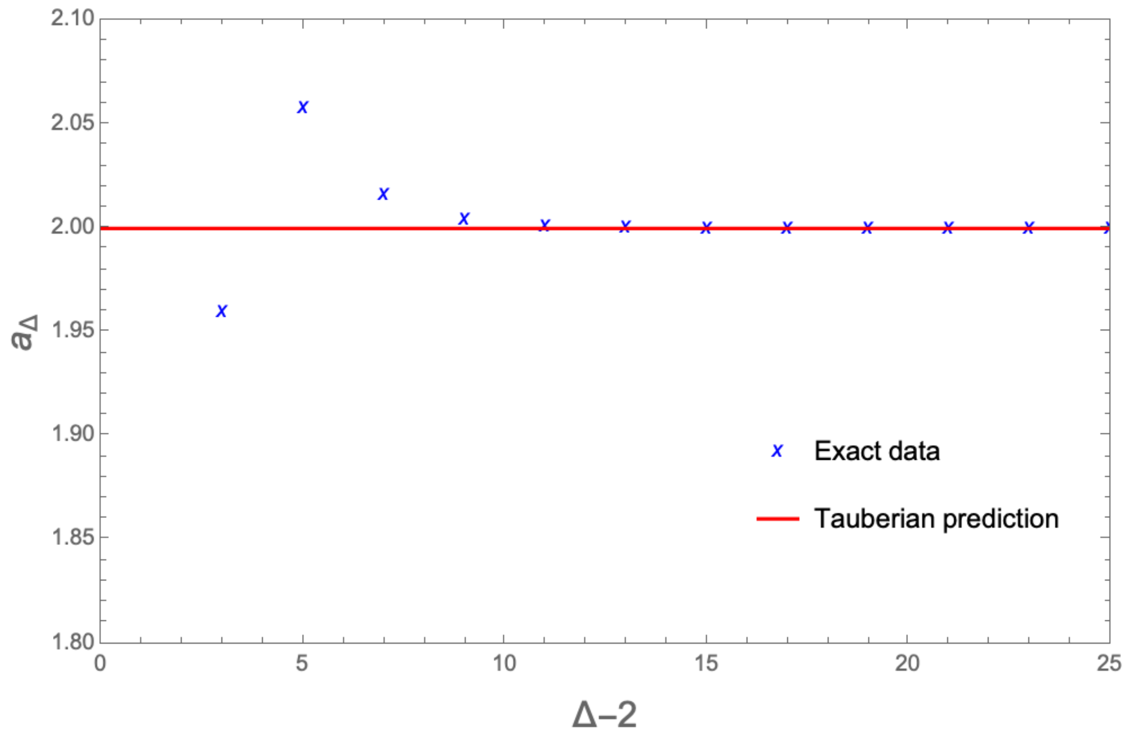
# 3d O(N) model at large N

$$\Delta_{\phi_i} = \frac{1}{2} + \mathcal{O}\left(\frac{1}{N}\right)$$

$$\Delta_{\sigma} = 2 + \mathcal{O}\left(\frac{1}{N}\right)$$

$$a_\Delta \overset{\Delta\to\infty}{\sim} \frac{\Delta^{2\Delta_\phi - 1}}{\Gamma(2\Delta_\phi)}\delta\Delta$$

$$a_\Delta \overset{\Delta\to\infty}{\sim} 2$$




The two-point function for $r = 0$:

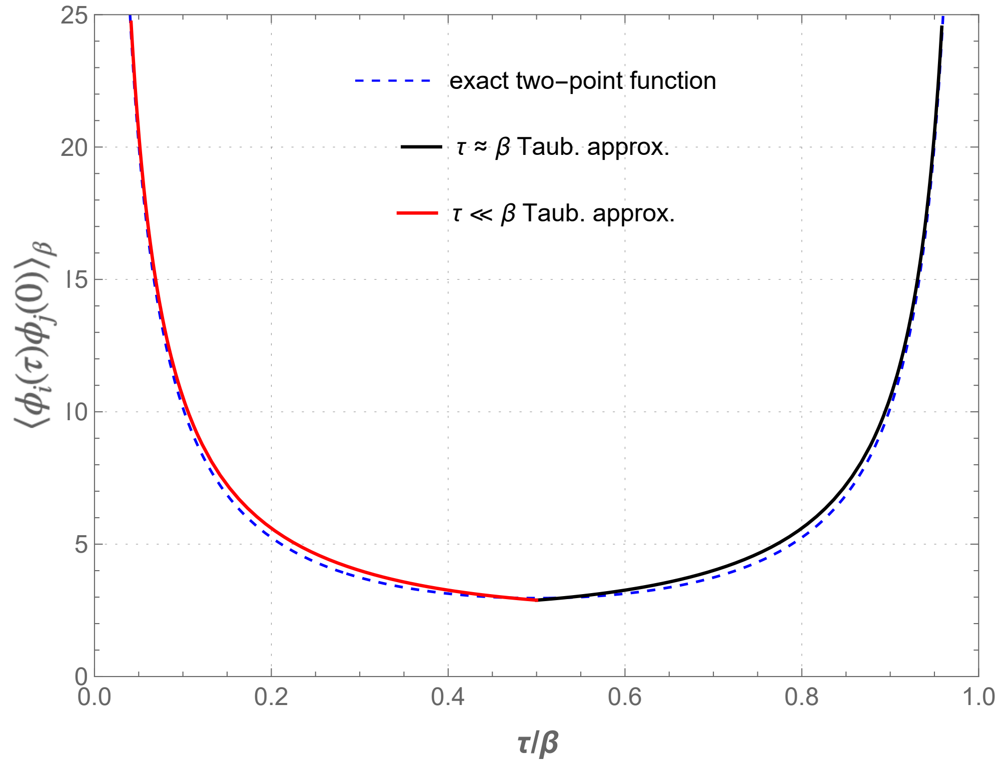
$$\langle \phi_i(\tau)\phi_j(0)\rangle_\beta = \delta_{ij}\left(\frac{e^{m_{th}(\tau-\beta)}}{\beta-\tau}\,{}_2F_1\left(\left\{\begin{matrix}1\ ,\frac{\beta-\tau}{\beta}\\ \frac{2\beta-\tau}{\beta}\end{matrix}\right\}\middle|\ e^{-m_{th}\beta}\right)+\frac{e^{-m_{th}\tau}}{\tau}\,{}_2F_1\left(\left\{\begin{matrix}1\ ,\frac{\tau}{\beta}\\ \frac{\beta+\tau}{\beta}\end{matrix}\right\}\middle|\ e^{-m_{th}\beta}\right)\right)$$

# 3d O(N) model at large N

The Tauberian approximation:

$$\langle \phi(\tau)\phi(0)\rangle_\beta \simeq \begin{cases} \int_0^\infty d\Delta \frac{\Delta^{2\Delta_\phi - 1}}{\Gamma(2\Delta_\phi)} \frac{(\beta-\tau)^{\Delta - 2\Delta_\phi}}{\beta^\Delta} & \tau/\beta \ll 1 \\ \int_0^\infty d\Delta \frac{\Delta^{2\Delta_\phi - 1}}{\Gamma(2\Delta_\phi)} \frac{\tau^{\Delta - 2\Delta_\phi}}{\beta^\Delta} & \tau/\beta \sim 1 \end{cases}$$

Gives the two-point function with **less than 10% error!**




The Tauberian approximation is very good!

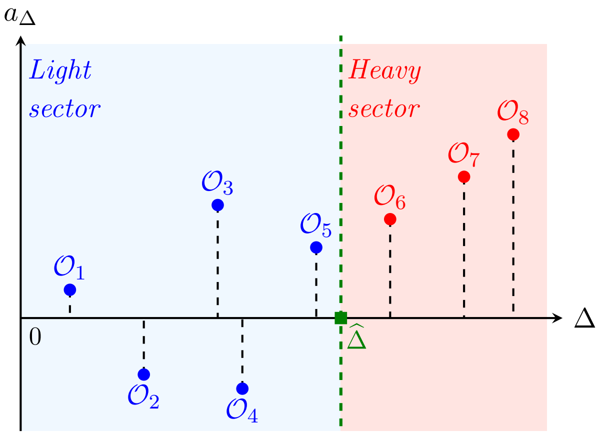
# Setting up the Numerics

# Numerical method

Inspired by [Gliozzi 2013] [Poland, Prilepina, Tadić' 2023] [W. Li 2023]

1. **Input**: zero Temperature spectrum and **Output**: $a_\Delta$ & $c_i$ .

2. Truncate the sum + *improved* Tauberian asymptotic:




$$f(\ell) = \sum_{\Delta < \hat{\Delta}} a_\Delta \, F(\Delta, \ell) + \sum_{\Delta > \hat{\Delta}} a^T_\Delta \, F(\Delta, \ell)$$

Light operators

Heavy operators: Tauberian

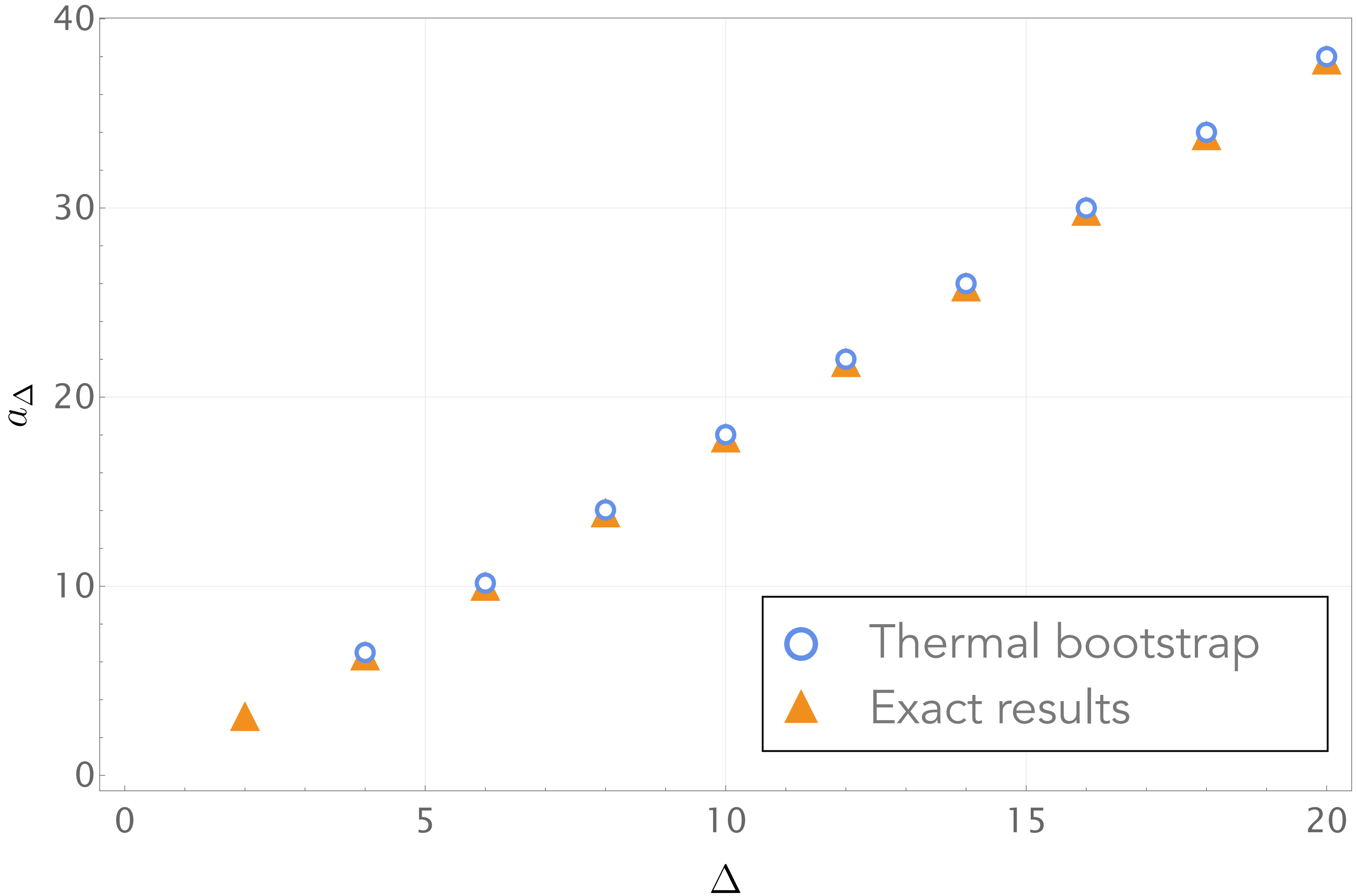
$$a^T_\Delta \sim \frac{\Delta^{2\Delta_\phi - 1}}{\Gamma(2\Delta_\phi)} \delta\Delta \left(1 + \frac{c}{\Delta} + \dots\right)$$

3. Numerically minimize with "random" coefficients the square of the sum rules.

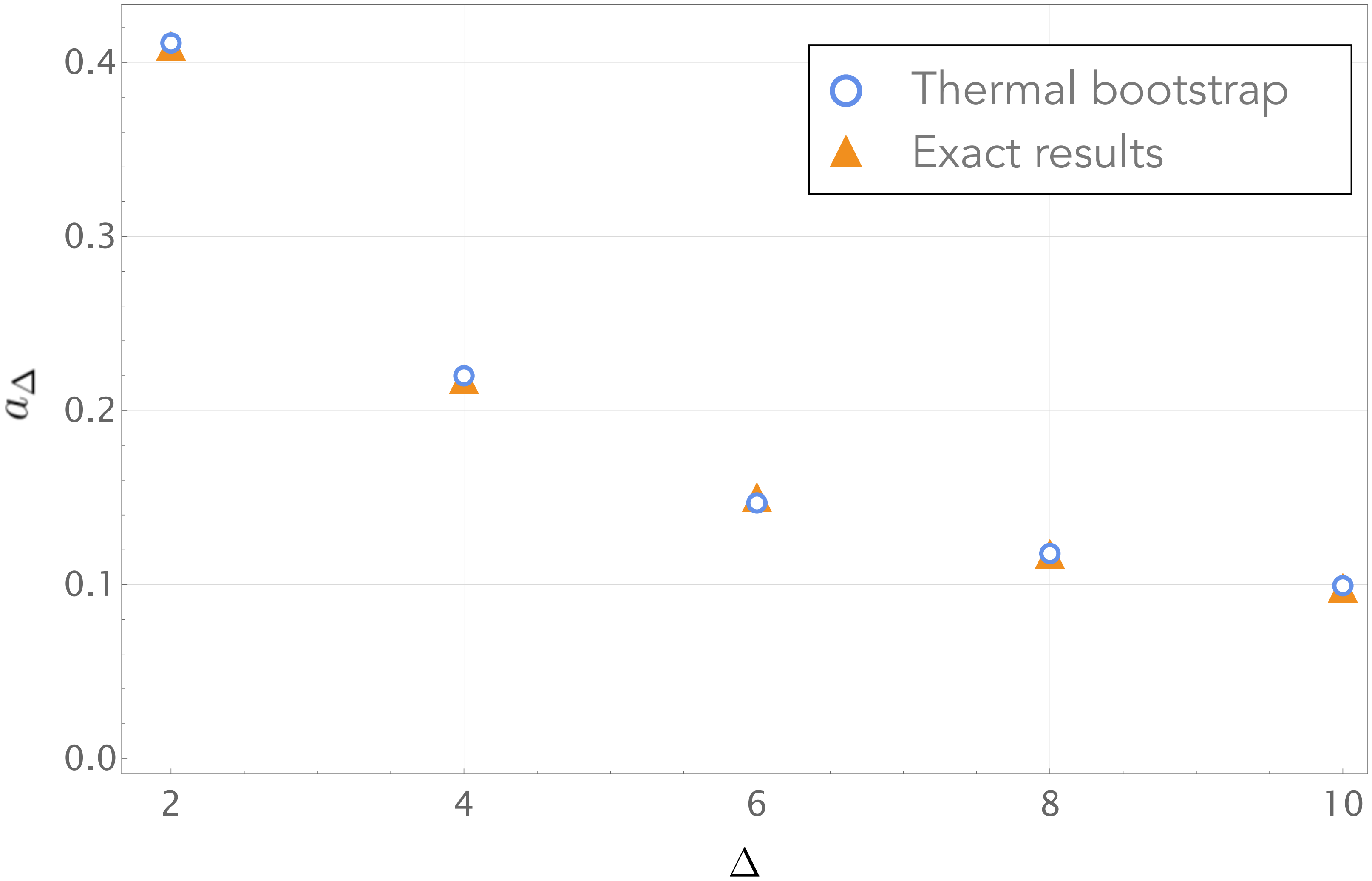
$$\min \left[ \sum_{\ell \leq \ell_{max}} r_\ell \, f^2(\ell) \right]$$

Random coefficients

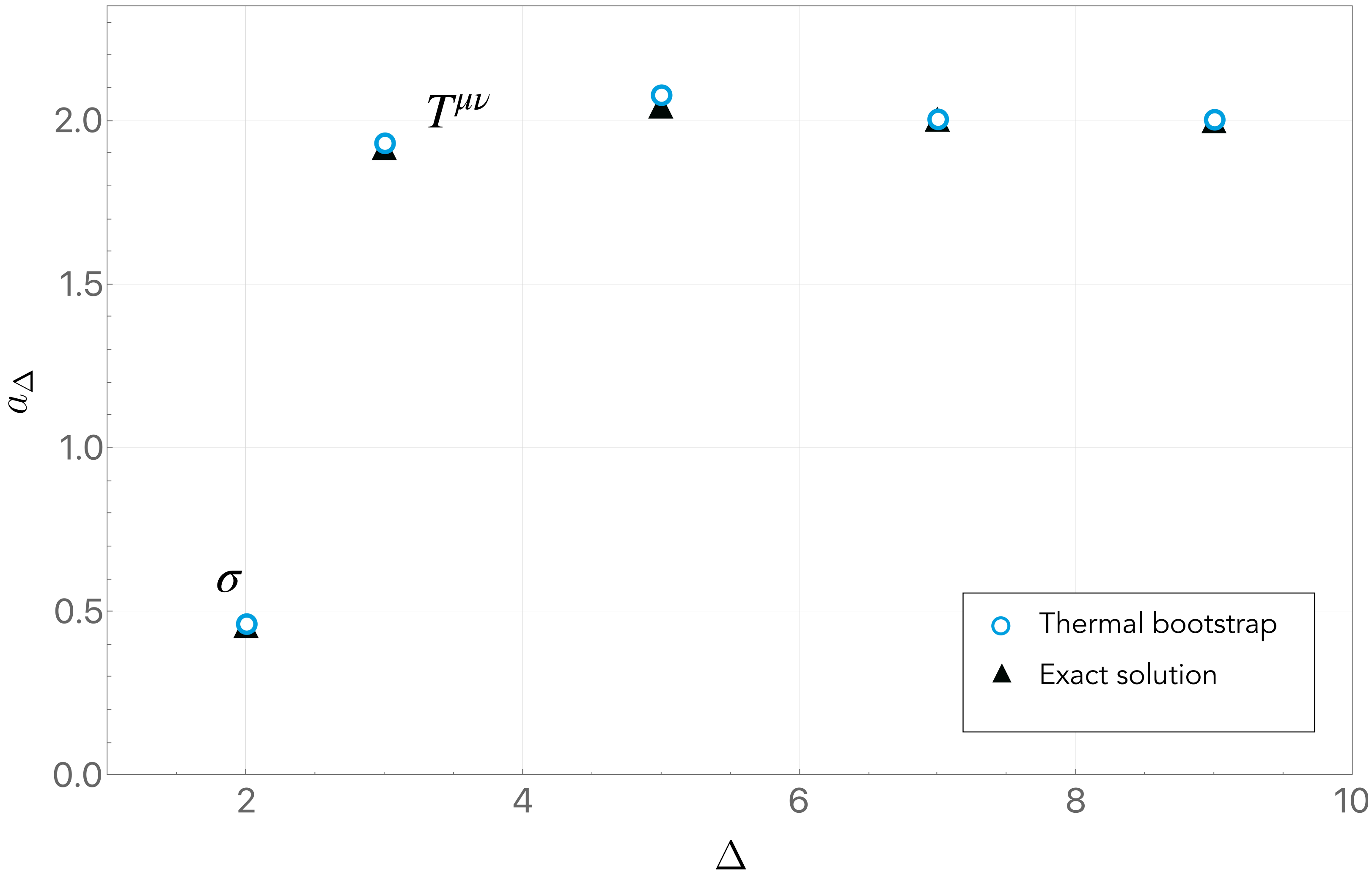
# Free theory in 4d



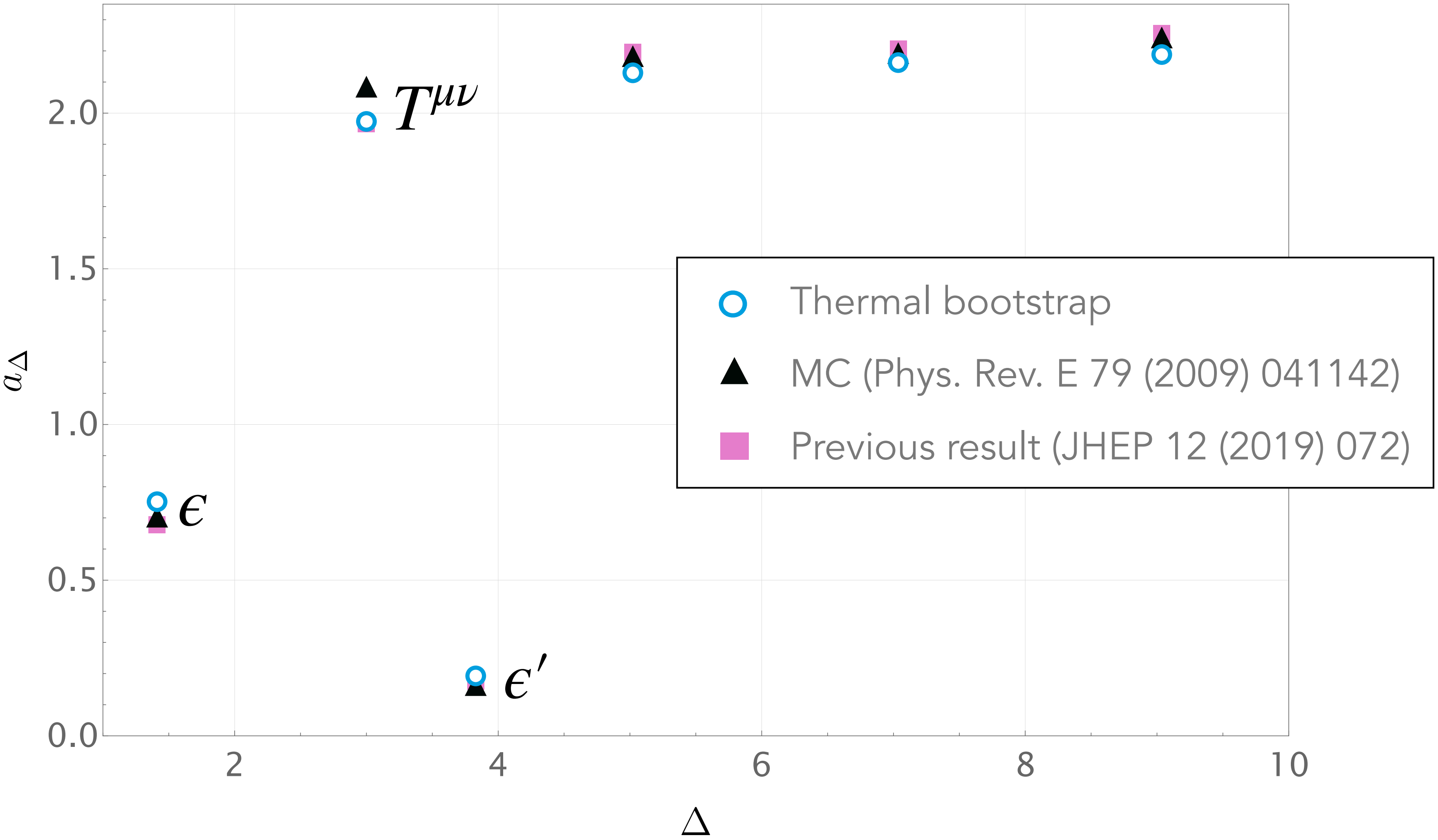
# 2d Ising $\langle \sigma\sigma \rangle_\beta$



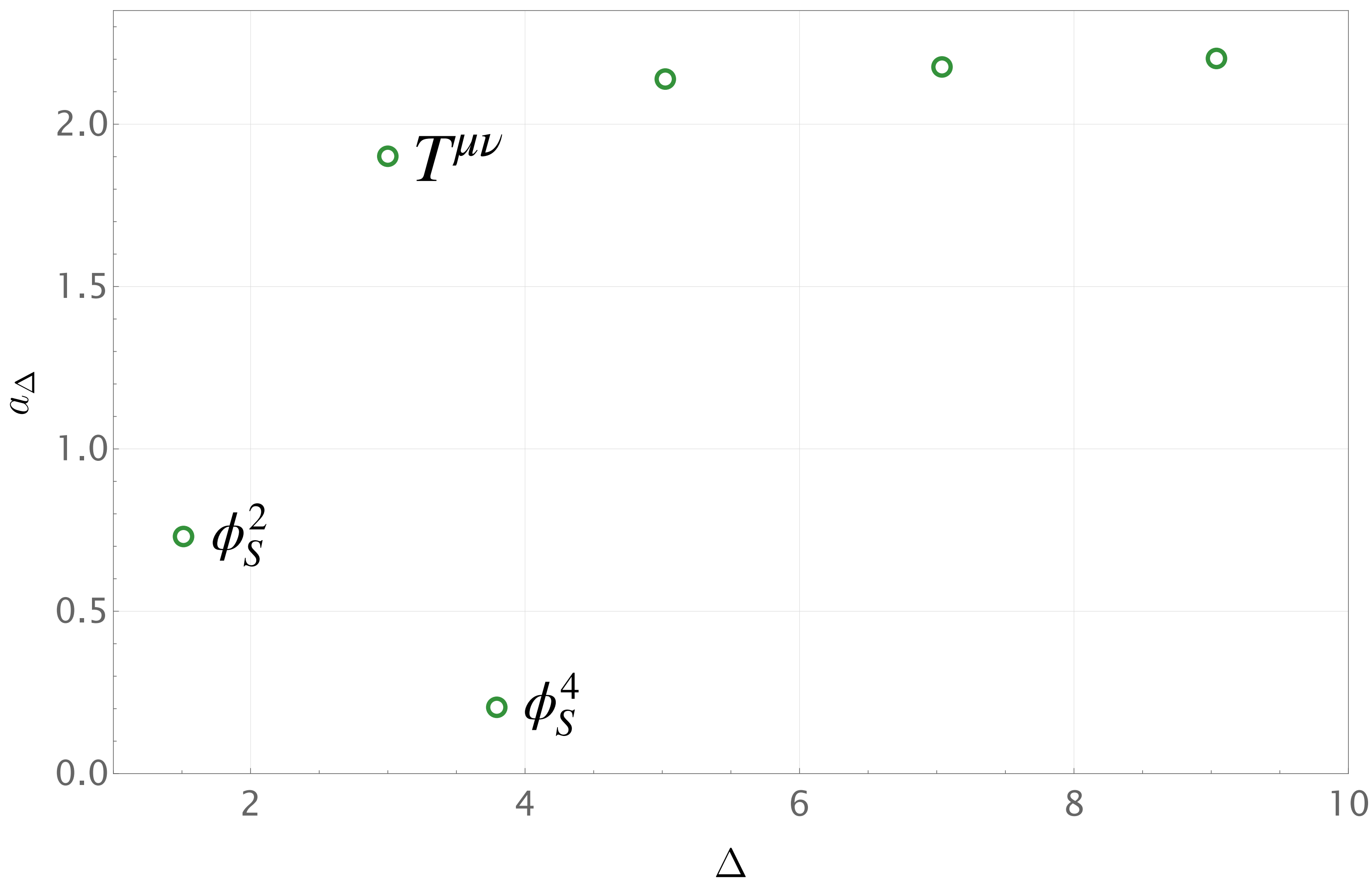
# 3d O(N) model at large N



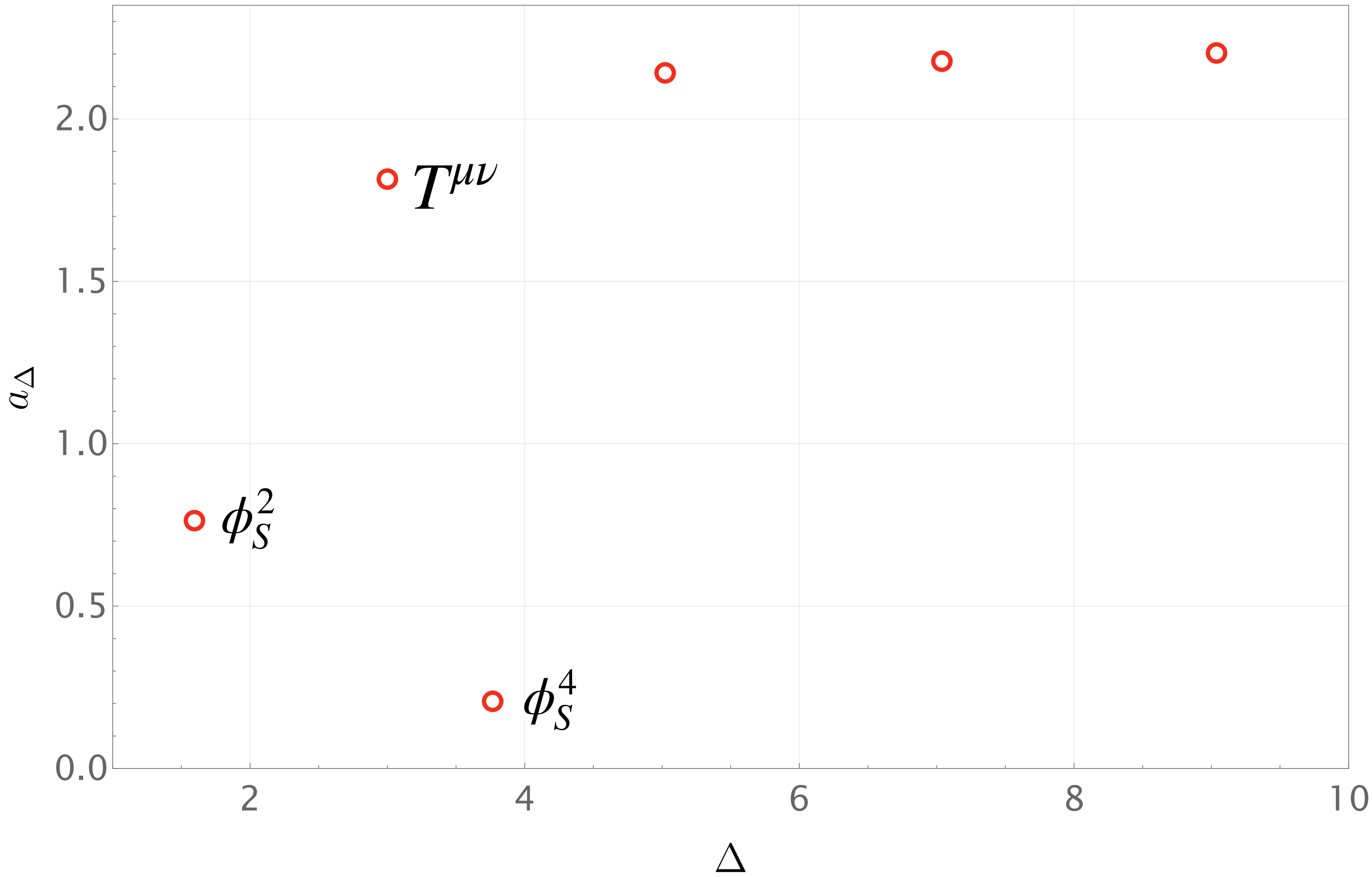
# The 3d Ising model



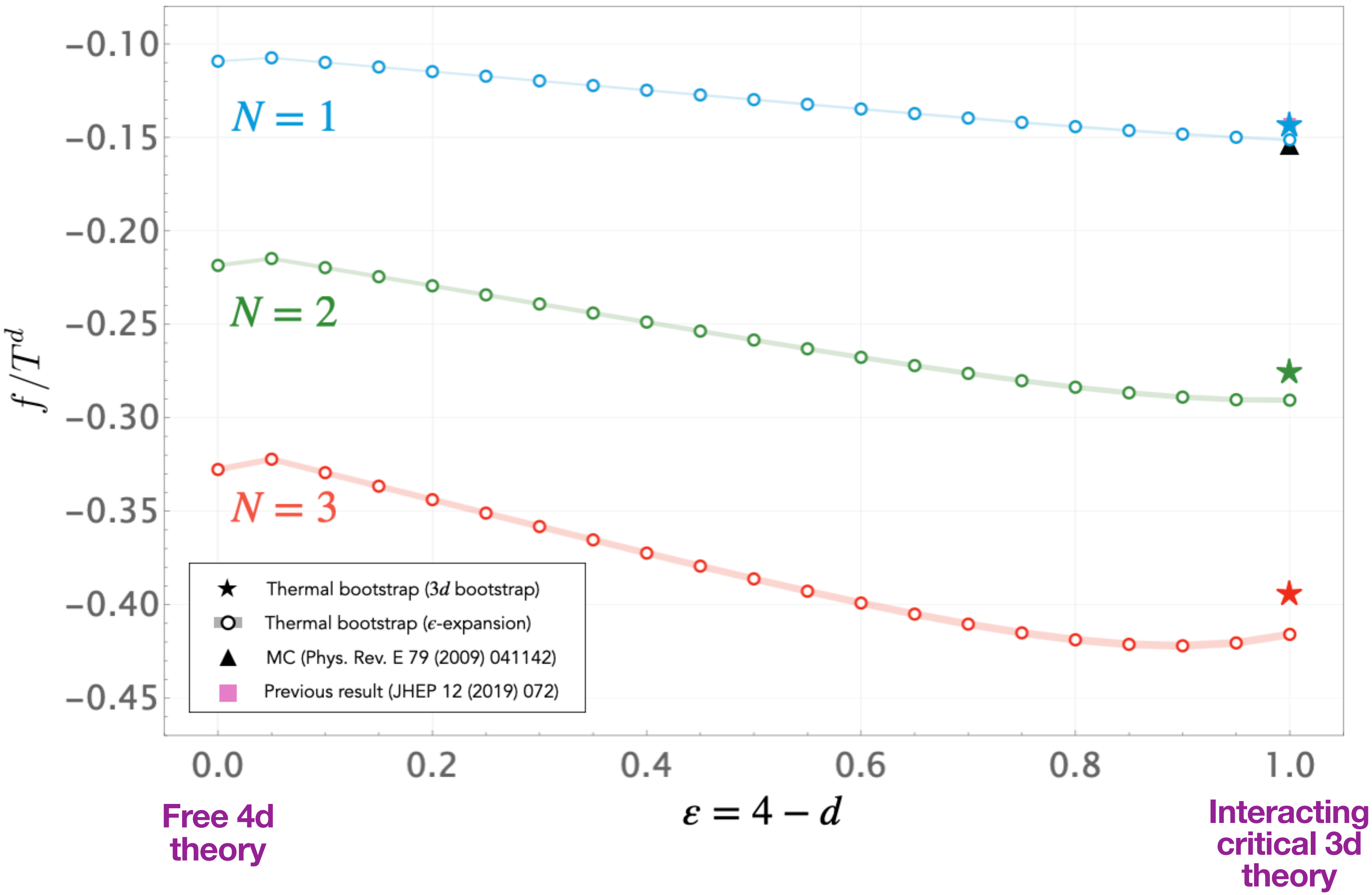
# The 3d $\mathrm{O}(2)$ XY model



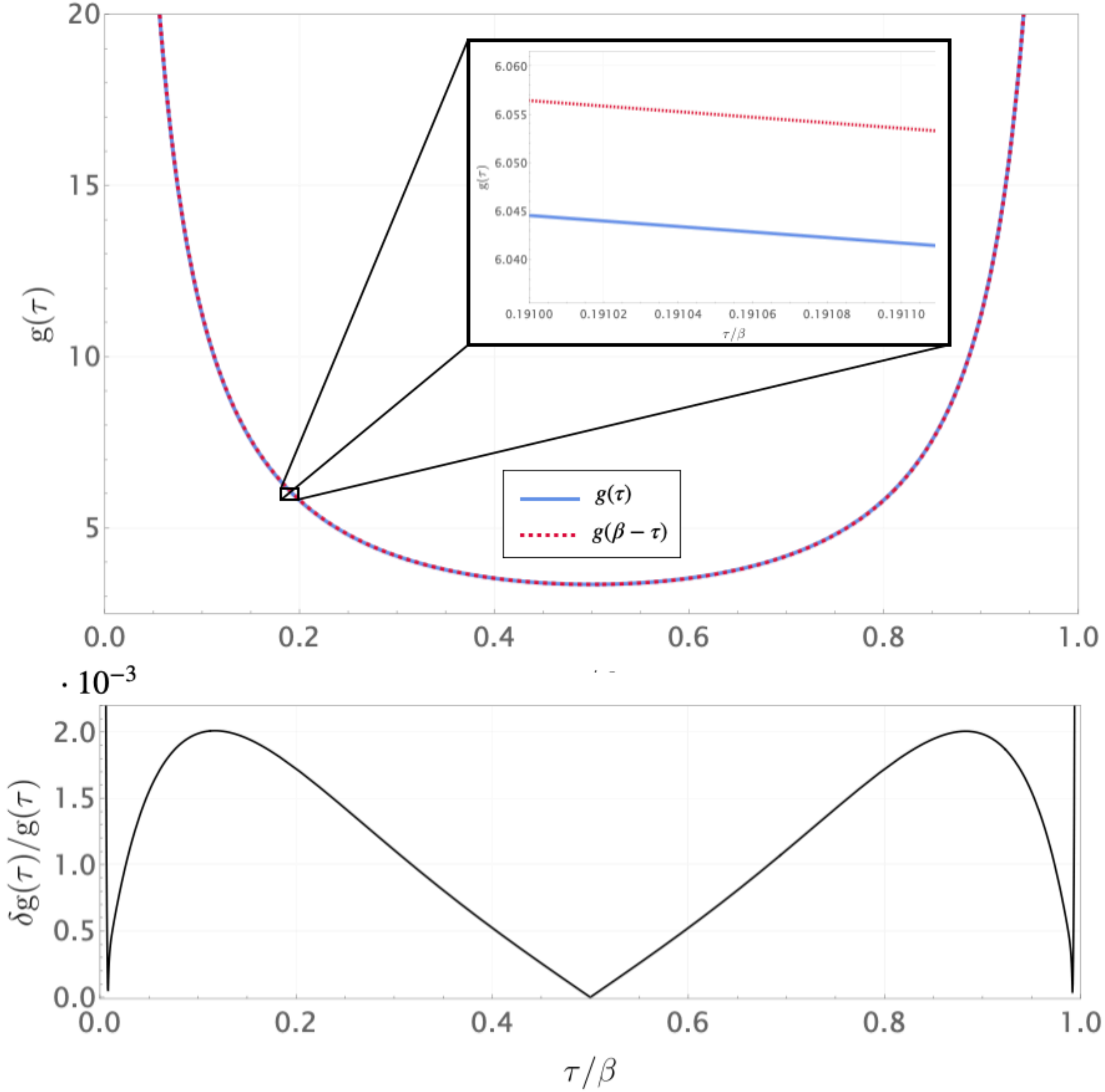
# The 3d $\mathrm{O}(3)$ Heisenberg model



# Free energy across dimensions



# Two-point function of 3d Ising




$$g(\tau) = \tau^{-2\Delta_\phi} \sum_{\Delta} \frac{a_\Delta}{\beta^\Delta} \tau^\Delta$$

$$\frac{\partial^\ell}{\partial \tau^\ell} \left[ g\left(\frac{\beta}{2} + \tau\right) - g\left(\frac{\beta}{2} - \tau\right) \right]_{\tau=0} = 0$$

The failure of KMS is only of order $10^{-3}$!

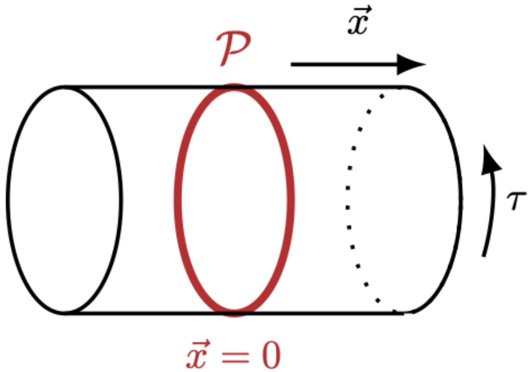
Each plot takes a couple of minutes on a laptop!

Keeping only the first 3 light operators and seven derivatives

# Temporal line defects

# Polyakov loops

Polyakov loops are temporal Wilson loops wrapping the thermal circle




They were introduced as a criterion for confinement. [Polyakov 1978]

* In the weak coupling using Feynman diagrams.
* In the strong coupling using holography.

Can we use the thermal bootstrap to compute them?

# Temporal line defects

The line defects can be studied using “**thermal**” **1d defect CFT** methods.

**OPE from bulk to defect**
Zero temperature data

$$\mathcal{O}(\tau, \vec{x}) = \sum_{\hat{\mathcal{O}}^{i_1\ldots i_s}} \mu_{\mathcal{O}\hat{\mathcal{O}}} \, |\vec{x}|^{\hat{\Delta}-\Delta-s} x_{i_1}\ldots x_{i_s} \hat{\mathcal{O}}^{i_1\ldots i_s}(\tau)$$

“Thermal” 1d defect CFT, non-perturbative exact result:

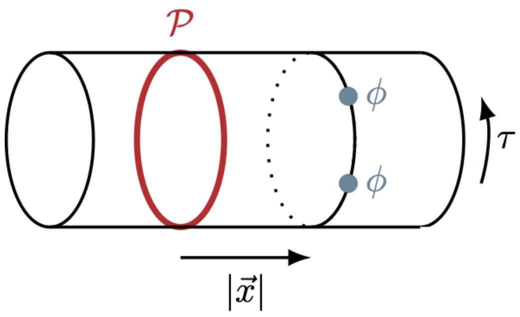
$$\langle \mathcal{O}(\tau, \vec{x}) \mathcal{P} \rangle_\beta = \sum_{\hat{\mathcal{O}}^{i_1\ldots i_s}} b_{\hat{\mathcal{O}}} \mu_{\mathcal{O}\hat{\mathcal{O}}} \frac{|x|^{\hat{\Delta}-\Delta}}{\beta^{\hat{\Delta}}}$$

New finite Temperature vevs of 1d defect CFT

Zero temperature data

# Sum rules from KMS

$$\langle \phi(\beta/2-\tau)\phi(0)\mathscr{P}\rangle_\beta = \langle \phi(\beta/2+\tau)\phi(0)\mathscr{P}\rangle_\beta$$




The sum rules obtained are of the form:

$$\sum_{\Delta,\widehat{\Delta}} \hat{b}_{\widehat{\mathcal{O}}}\, f_{\phi\phi\mathcal{O}}\lambda_{\mathcal{O}\widehat{\mathcal{O}}} \binom{\Delta-2\Delta_\phi}{n} \frac{z^{\widehat{\Delta}-\Delta}}{2^\Delta} = 0$$

New finite Temperature vevs of 1d defect CFT

Zero temperature data

Defect thermal sum rules are very similar to their bulk counterparts.

# Where we currently are

* Derived and tested thermal sum rules.

* Heavy operators: asymptotic thermal OPE density.

* Proposed an efficient numerical thermal bootstrap.

* Temporal line defects (Polyakov loops).

# Where are we going?

- Analytical approach using a new, thermal dispersion relation.
- More theories $\mathcal{N}$=4 SYM, ABJM, …
- Numerical approach for Polyakov loops.
- Study the $S^1 \times S^{d-1}$ geometry.
- Black holes, hydro and CFT data.

Thank you!