# An Analytic Language for 2D CFT

Eric Perlmutter, IPhT Saclay & IHES

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Strings 2025

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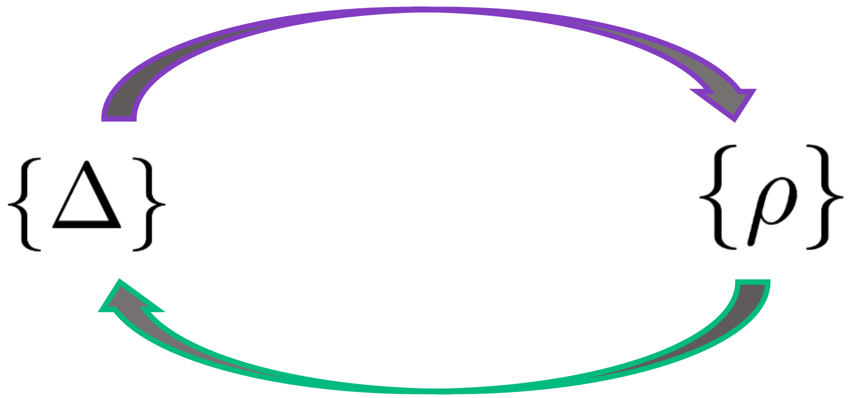
One motivation: to probe *substructure* of black hole microstate spectra in $AdS_3$ quantum gravity.

In 2D CFT, this means finding and solving constraints that act solely on high-energy states.

We propose that the analytic language of L-functions is the right one to address this.



# Outline

1) L-functions for 2D CFT

2) Random Matrix Universality as Subconvexity

3) New Modular Bootstrap: Proof of $\frac{c}{12}$

4) Hints of a Lattice

# L-functions

An $L$-function is a Dirichlet series that meromorphically continues to the whole complex plane under a certain type of functional equation, and perhaps possessing other special properties.

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad a_n \in \mathbb{C}$$

The *completed* L-function $\Lambda_L(s) := q^{s/2}\gamma(s)L(s)\,, \qquad \gamma(s) := \pi^{-ds/2}\prod_{i=1}^{d}\Gamma\left(\frac{s+\kappa_i}{2}\right)$ *Gamma factor*

has poles at most at s=0,1, and obeys $\Lambda_L(s) = \epsilon\overline{\Lambda}_L(1-s)$ ("analytic normalization").

*e.g. Riemann zeta*: self-dual $L$-function of degree-one with unit conductor, root number +1, and $\kappa_1 = 0$.

$$\xi(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \xi(1-s)$$

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*Conductor* $q \in \mathbb{Z}_+$

*Degree-d*

*Spectral parameters* $\kappa_i \in \mathbb{C}$

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*Root number* $|\epsilon| = 1$

*Dual L-function* (*series coefficients* $\overline{a_n}$)

*e.g. Riemann zeta*: self-dual *L*-function of degree-one with unit conductor, root number +1, and $\kappa_1 = 0$.

$$\xi(s) = \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s) = \xi(1-s)$$

# L-functions

- L-functions are often attached to **automorphic forms**, both holomorphic and non-holomorphic. (e.g. Maass cusp forms for **SL(2,ℤ)** + congruence subgroups)

$$f(\tau) = \sum_n a_n e^{2\pi i \tau n} \quad \mapsto \quad L_f(s) = \sum_n \frac{a_n}{n^s}$$

$$f(\tau) = f(\gamma\tau) \quad \mapsto \quad \Lambda_f(s) = \epsilon\overline{\Lambda}_f(1-s)$$

  Langlands: *all* L-functions (suitably defined) are attached to automorphic forms.

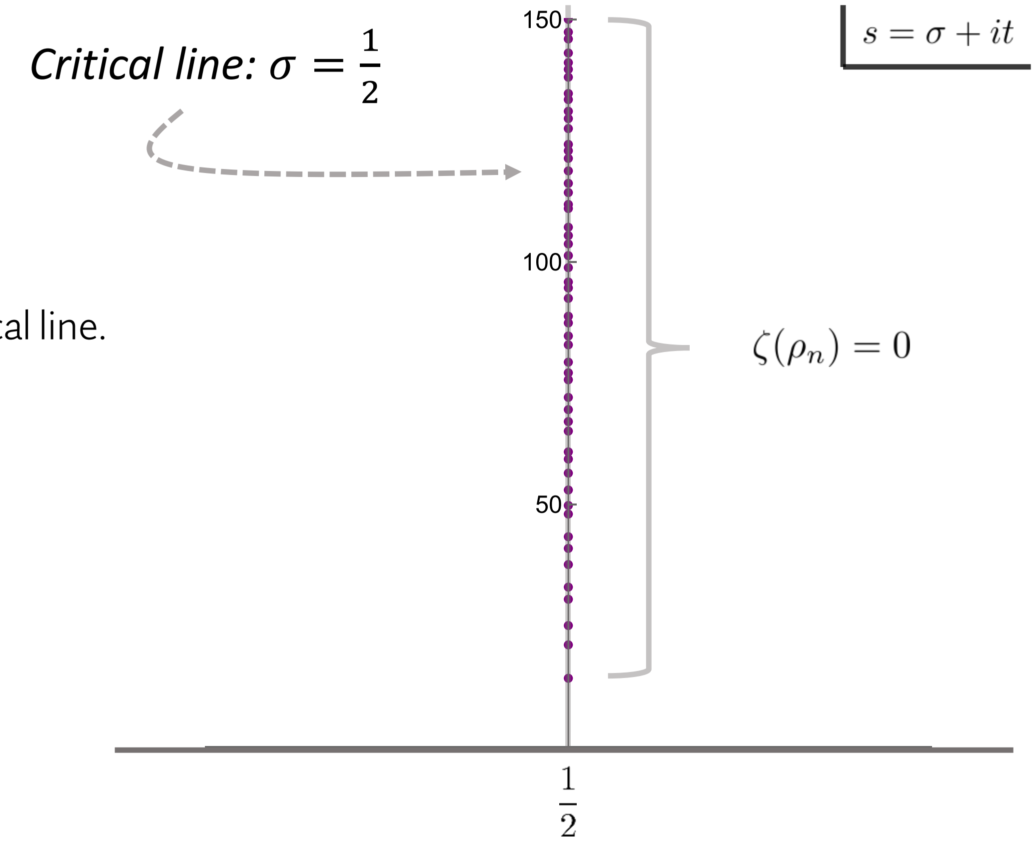
- Standard L-functions also admit an **Euler product**,

$$L(s) = \prod_{i=1}^{d}\prod_p (1-\alpha_i(p)p^{-s})^{-1}$$

  which guarantees **zero-free regions** and connects to the primes (more generally, a set of **primitives**)

- Most well-studied, narrowest class of L-functions are the “Selberg class”.
  On the other hand, natural generalizations lack an Euler product and/or violate Riemann (e.g. Hurwitz zeta)
  Generally speaking, mathematicians’ definition of L-functions is broadening over time.

# L-functions




## Main open problems in the land of L-functions

1) *Where are the zeros?*
   - **Grand Riemann Hypothesis:** all non-trivial zeros lie on the critical line.
   - Zero-free regions/density estimates (**Density Hypothesis)**
   - Random matrix statistics
   - Highest lowest zero

[Montgomery; Odlyzko; Katz, Sarnak]

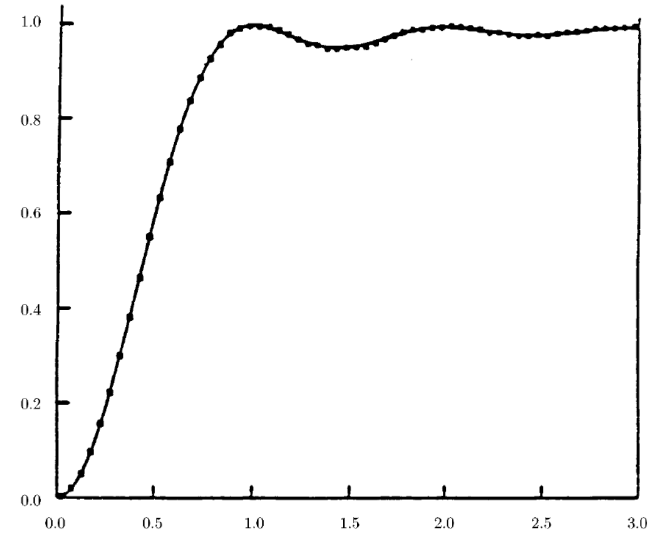
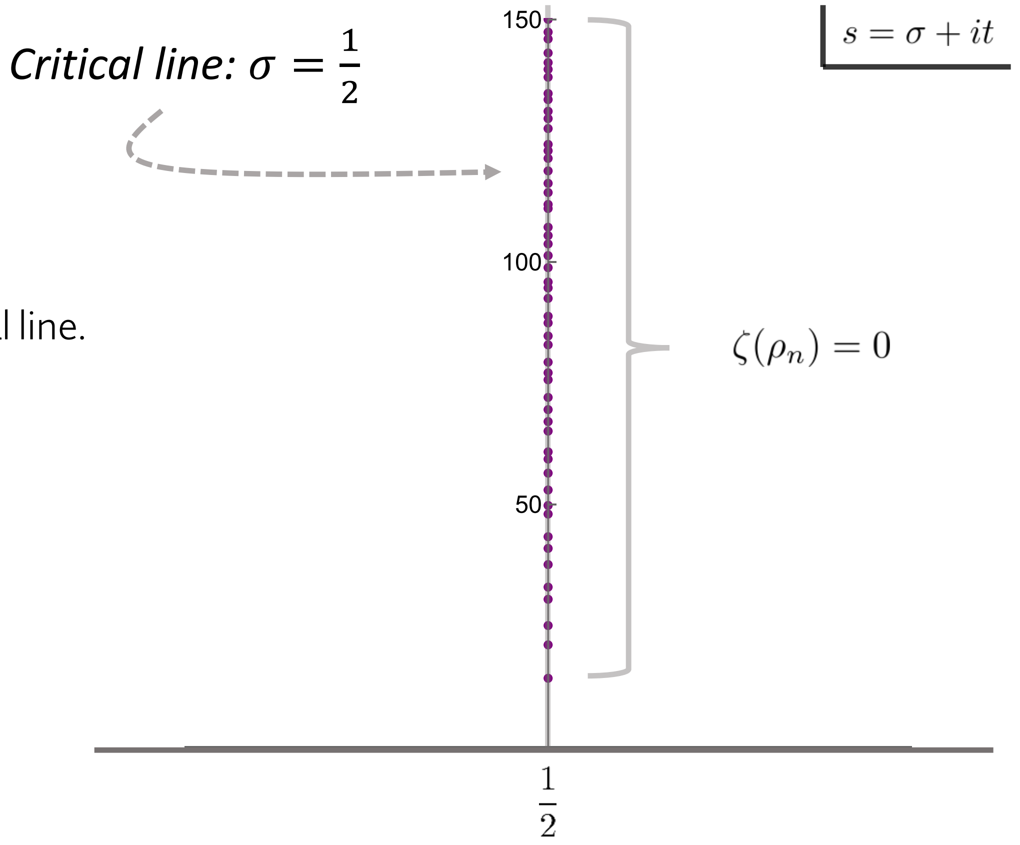



FIGURE 2. Pair correlation for zeros of zeta based on $8 \times 10^6$ zeros near the $10^{20}$-th zero, versus the GUE conjectured density $1 - \left(\frac{\sin \pi x}{\pi x}\right)^2$.

[da la Vallee Poussin; Littlewood; Siegel; Iwaniec; Iwaniec, Sarnak; Rudnick, Sarnak; Miller; Tang, Miller; Conrey; Kowalski; Ivic; Hiary; ...]

# L-functions

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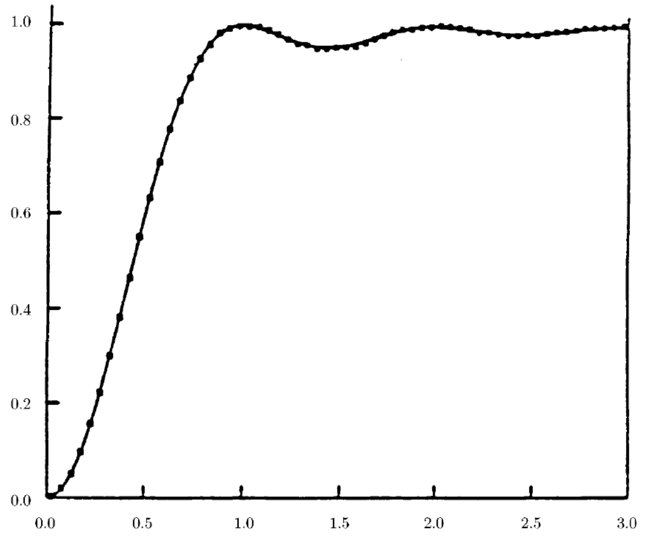


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[Montgomery; Odlyzko; Katz, Sarnak]

REMARKS. Needless to say we believe that every $L$-function (subject to our definition in Section 5.1) satisfies the Grand Riemann Hypothesis. Yet, proving this even for one $L$-function would be an achievement on a historical scale for human beings. Note that an $L$-function may have zeros on the line $\mathrm{Re}\,(s) = 0$ (certainly some trivial zeros, probably not the genuine ones), but as we already

[Iwaniec-Kowalski, Ch. 5]

[da la Vallee Poussin; Littlewood; Siegel; Iwaniec; Iwaniec, Sarnak; Rudnick, Sarnak; Miller; Tang, Miller; Conrey; Kowalski; Ivic; Hiary; ...]

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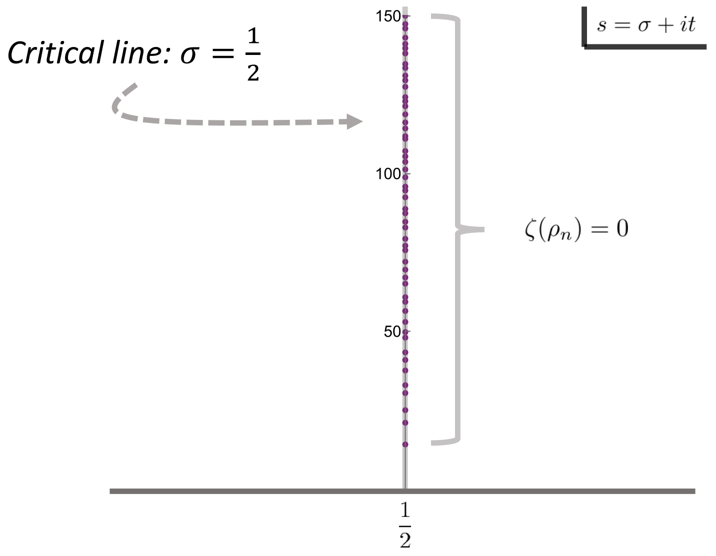
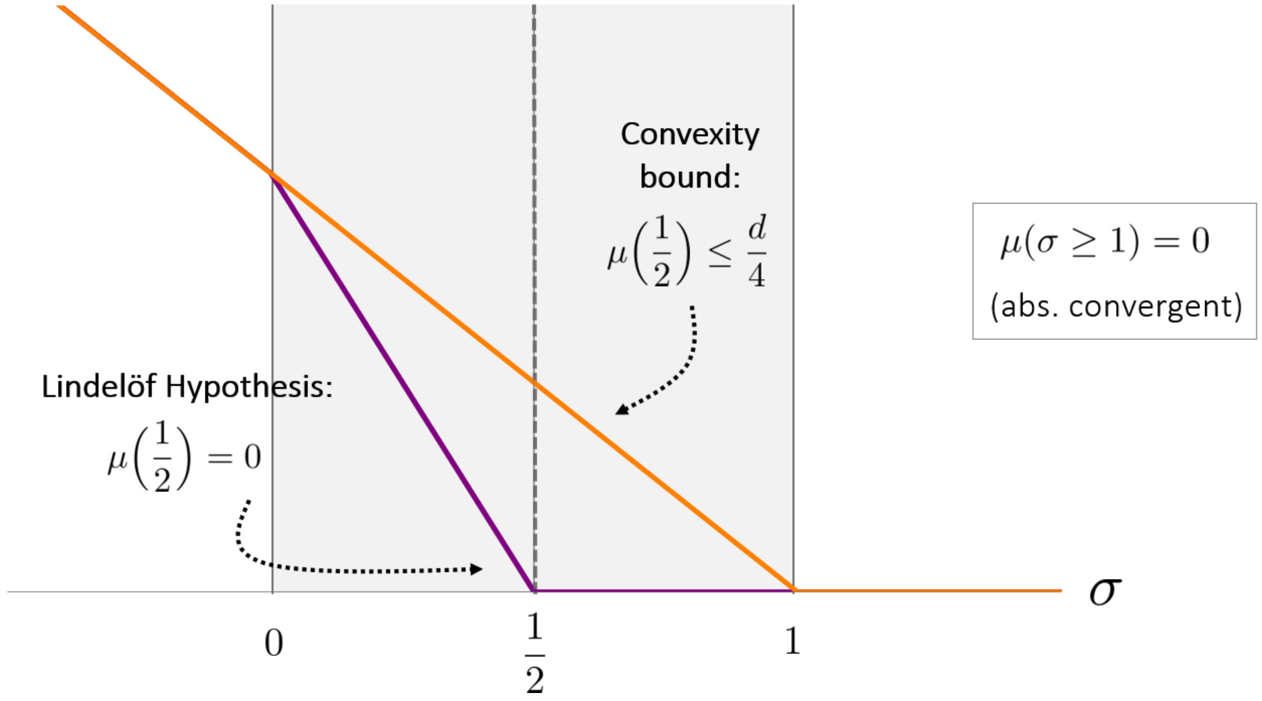



2) *Subconvexity: how do L-functions grow high on the critical line?*
   - **Lindelöf Hypothesis:** slower than any power of *t*.

$$\mu(\sigma) := \inf\left(x \,\middle|\, L(\sigma+it) = O(t^{x+\varepsilon})\right)$$



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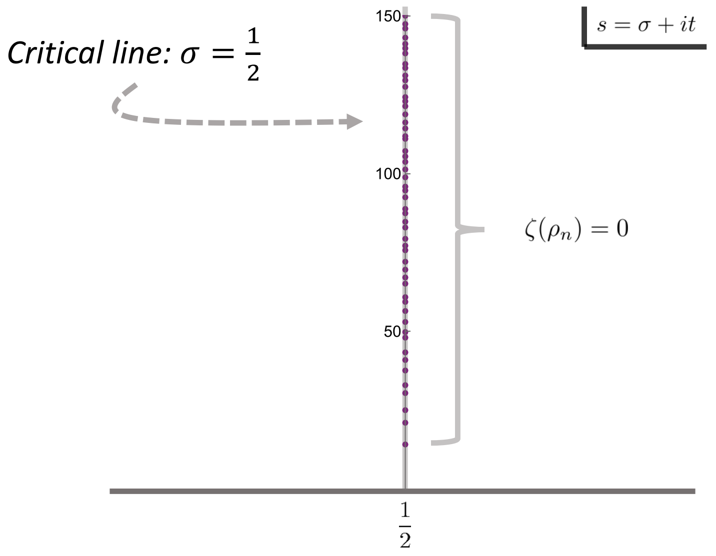


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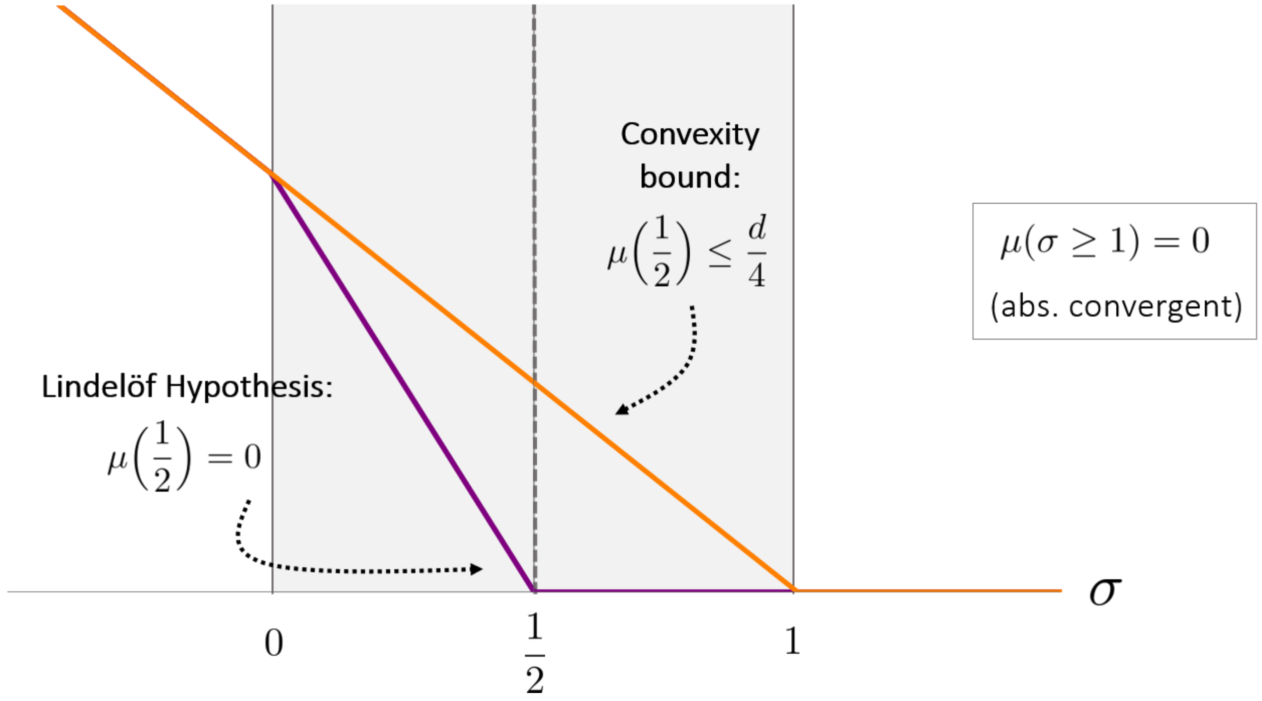


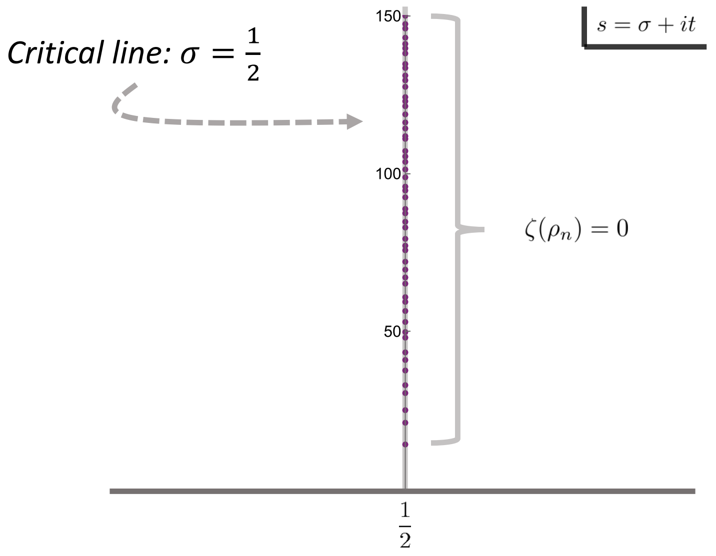

*Note*: RH ⇒ Lindelöf ⇒ Density

*Note*: Best Riemann zeta bound is

$$\mu_\zeta\left(\frac{1}{2}\right) \leq \frac{13}{84} \approx .155$$

[Bourgain ’14]

# L-functions

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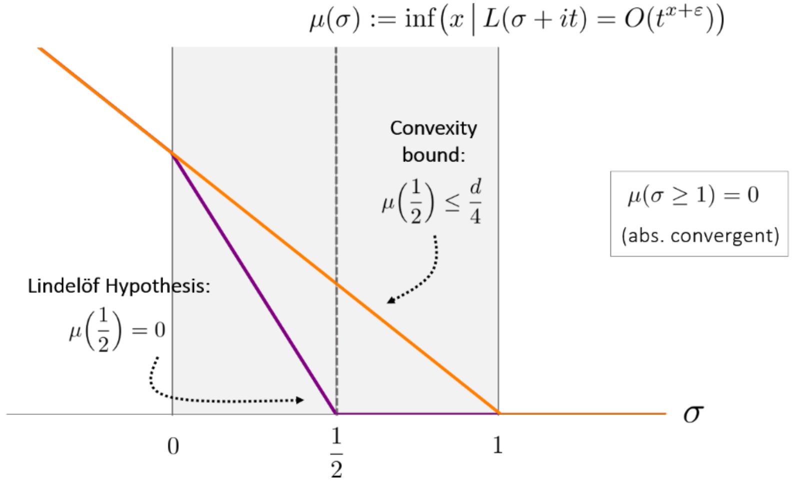
   - **Lindelöf Hypothesis:** slower than any power of *t*.
   - Different "aspects" (height *t*, conductor *q*, spectral parameters, families, …)
   - Moments of L-functions ("Lindelöf on average"), random matrix heuristics thereof
   - Progress from bounding exponential sums (cf. approximate functional eqs.)

   [Weyl; Ingham; Hardy, Littlewood; Bourgain; Nelson; Michel, Venkatesh; Duke, Friedlander, Iwaniec; Soundararajan; Heath-Brown; Keating, Snaith; Conrey, Farmer, Keating, Rubenstein, Snaith; Blomer; Radziwill; Jutila, Motohashi; Harper; …]



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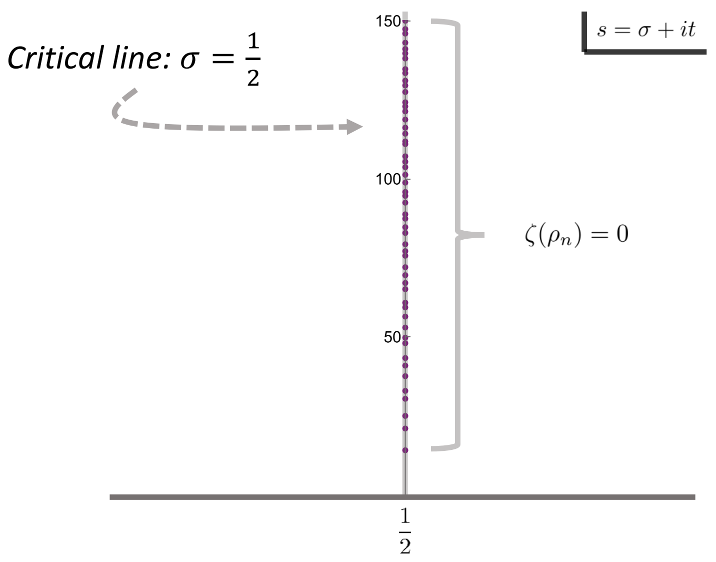


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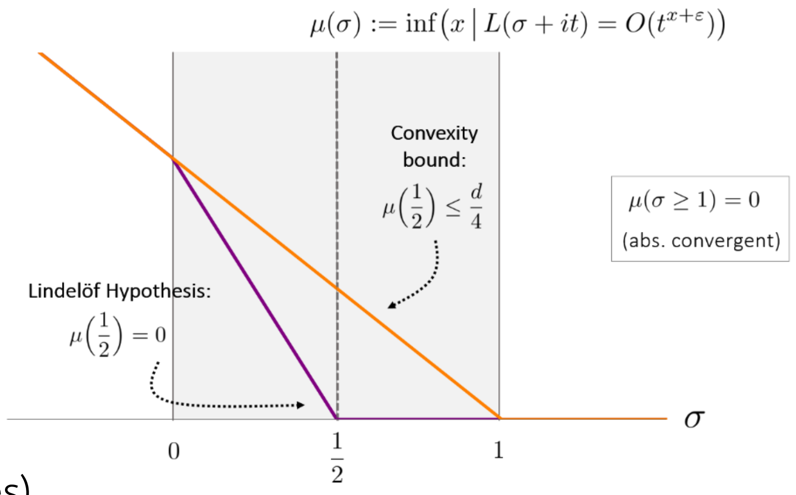
3) *Central values*
   - Distribution of zeros near central point ($s = 1/2$) for families of L-functions
   - Deep connections to arithmetic (e.g. Birch-Swinnerton-Dyer conjecture, elliptic curves)

4) *Distributions of Fourier coefficients $a_n$ (e.g. Sato-Tate/Ramanujan conjectures)*

5) *Proving conjectured analytic properties of automorphic L-functions*

6) *Applications to equidistribution on modular domains (e.g. Quantum Unique Ergodicity)*




[Artin; Langlands; Sarnak; Katz, Sarnak; Iwaniec, Sarnak; Miller; Blomer, Thorner; Humphries, Khan; Ichino; Watson; Lindenstrauss; Soundararajan; ...]

# Introducing L-functions for 2D CFT

Now we construct L-functions – rather, a generalization thereof – for CFTs (Virasoro, compact, $c > 1$).

Taking $\tau = x + iy$, the Virasoro primary partition function is

$$Z_p(\tau) := \sqrt{y}|\eta(\tau)|^2 Z(\tau) = \sqrt{y} \sum_{j \in \mathbb{Z}} e^{2\pi i j x} {\sum_{\Delta}}' d_{\Delta}^{(j)} e^{-2\pi \Delta y}$$

**Where are the L-functions?**

Whereas Hecke-Maass cusp forms have a single constant $a_n$ characterizing the $n^{\text{th}}$ Fourier mode, things must work differently here: each mode is labeled by an infinite set of data $\{\Delta, d_{\Delta}^{(j)}\}$ …

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To start, we could construct a *spectral zeta function* for the primaries: $\zeta_Z(s) = {\sum_{\Delta}}' \frac{d_\Delta}{\Delta^s}$

But this is badly divergent for all *s*: $\Delta_{n+1} - \Delta_n \sim e^{-S_{\text{Cardy}}(\Delta_n)}\,, \quad n \to \infty$

# Introducing L-functions for 2D CFT

Instead, take a difference of two primary partition functions with the *same light spectra*, $\Delta \leq \frac{c-1}{12}$

$$\mathcal{Z}(\tau) := Z_p(\tau) - \widetilde{Z}_p(\tau) \stackrel{!}{=} Z_H(\tau) - \widetilde{Z}_H(\tau)$$

Then form its *scalar zeta function*

$$\zeta_{\mathcal{Z}}(s) = \sum_{\{\lambda\}} \frac{a_\lambda}{\lambda^{s-\frac{1}{2}}}$$

$$a_\lambda \mapsto d_\lambda - \widetilde{d}_\lambda$$

$$\lambda \mapsto \Delta^{(0)} - \frac{c-1}{12}$$

with frequencies $\{\lambda \mid 0 < \lambda_1 < \lambda_2 < \ldots, \quad \lambda_{n\to\infty} \to \infty\}$ supported on the CFT spectra.

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**Claim I**: $\zeta_{\mathcal{Z}}(s)$ converges for $\sigma > 1$, and admits a meromorphic continuation to the whole complex plane.

**Claim II**: Multiplying by a zeta factor – $L_{\mathcal{Z}}(s) := \zeta(2s)\zeta_{\mathcal{Z}}(s)$ – this transforms as an L-function:

$$\Lambda_{\mathcal{Z}}(s) := 2^s \gamma(s) L_{\mathcal{Z}}(s)$$

$$\Lambda_{\mathcal{Z}}(s) = \Lambda_{\mathcal{Z}}(1-s)$$

$$\gamma(s) = \pi^{-2s} \prod_{i=1}^{4} \Gamma\left(\frac{s+\kappa_i}{2}\right), \quad \kappa_i = 0, 1, \pm\frac{1}{2}$$

# Introducing L-functions for 2D CFT

[Hardy, Riesz; Titchmarsh]
[Iwaniec, Terras]

We can prove this by appeal to $\mathrm{SL}(2,\mathbb{Z})$ spectral resolution and the theory of general Dirichlet series.

**I.** $\mathcal{Z}(\tau) \in L^2(\mathcal{F})$. In particular, *unitarity* can be shown to imply that $Z_p(\tau) < \infty \quad \forall \quad \tau \in \mathcal{F} \backslash \{i\infty\}$

($\mathcal{F}$= SL(2,$\mathbb{Z}$) fundamental domain)

It thus resolves into a complete SL(2,$\mathbb{Z}$)-invariant eigenbasis (1 + Eisensteins + Maass cusp forms), w/ scalar mode

$$\mathcal{Z}_0(y) = \langle \mathcal{Z} \rangle + \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=\frac{1}{2}} ds \, (\mathcal{Z}, E_{1-s}) \, y^{1-s}$$
$$\stackrel{!}{=} \sqrt{y} \sum_{\lambda} a_\lambda e^{-2\pi \lambda y}$$

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$$(\mathcal{Z}, E_{1-s}) = (2\pi)^{\frac{1}{2}-s}\,\Gamma\left(s-\frac{1}{2}\right)\zeta_{\mathcal{Z}}(s)$$

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**II.** The inner product must factorize like *this* (cf. "Converse Mapping Theorem") to yield *this*.

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**II.** The inner product must factorize like *this* (cf. "Converse Mapping Theorem") to yield *this*.

**III.** By modularity/analyticity of the Eisenstein series, the inner product is regular for $\mathrm{Re}(s) > 1$ (hence the zeta function converges there) and obeys a functional equation $\Rightarrow \quad \Lambda_{\mathcal{Z}}(s) = \Lambda_{\mathcal{Z}}(1-s)$

# J-subtraction

Given a partition function, we can construct the corresponding L-function in a canonical way.

We call this ***J-subtraction***: in the $q$-expansion of $Z(\tau)$, replace every light state with the unique linear combination of modular J-functions which introduces no further light states nor singularities on $\mathcal{F}$.

$$\mathcal{Z}(\tau) := Z_p(\tau) - Z_p^{(J)}(\tau) = \sqrt{y}\, O(|q|^{\varepsilon})$$

$$\left[\begin{aligned} J(\tau) &= J(\gamma\tau)\,, \ \gamma \in SL(2,\mathbb{Z}) \\ &\approx q^{-1} + 196884q + \dots \end{aligned}\right]$$

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That is, define a non-holomorphic (Hecke-like) operator

$$T_{\alpha,\beta}(\tau) = J(\tau)^{\alpha} J(\bar{\tau})^{\beta} \sum_{m,n=0}^{m+n \leq \alpha+\beta-\frac{1}{12}} \gamma_{mn} J(\tau)^{-m} J(\bar{\tau})^{-n} \qquad (\alpha, \beta \in \mathbb{R}_+)$$

$$\stackrel{!}{=} q^{-\alpha}\bar{q}^{-\beta} + O(|q|^{-\frac{1}{12}+\varepsilon})$$

Defines constants $\gamma_{mn}$ uniquely.

Then the fiducial partition function $Z^{(J)}(\tau) = \sum_{\Delta \leq \frac{c-1}{12}} d_{h,\bar{h}} T_{\frac{c}{24}-h,\frac{c}{24}-\bar{h}}(\tau)$ enables the desired subtraction.

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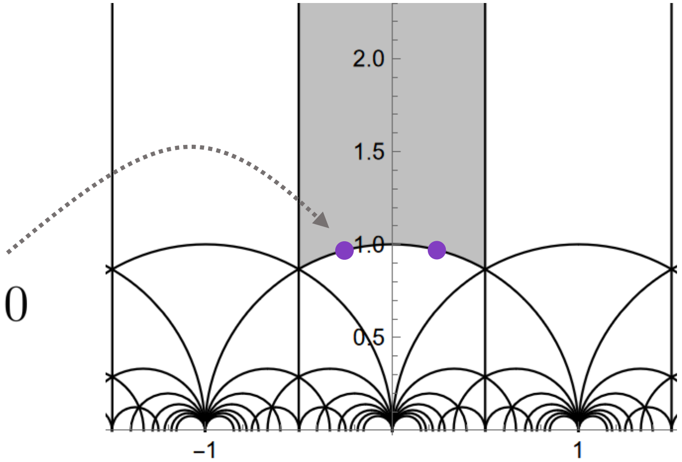
$\mathcal{Z}(\tau)$ is

- ✓ Modular-invariant (cf. spin quantization)
- ✓ Discrete
- ✓ $\in L^2(\mathcal{F})$: J-subtraction does *not* add new poles: $J(\tau_*) = J(\bar\tau_*) = 0$



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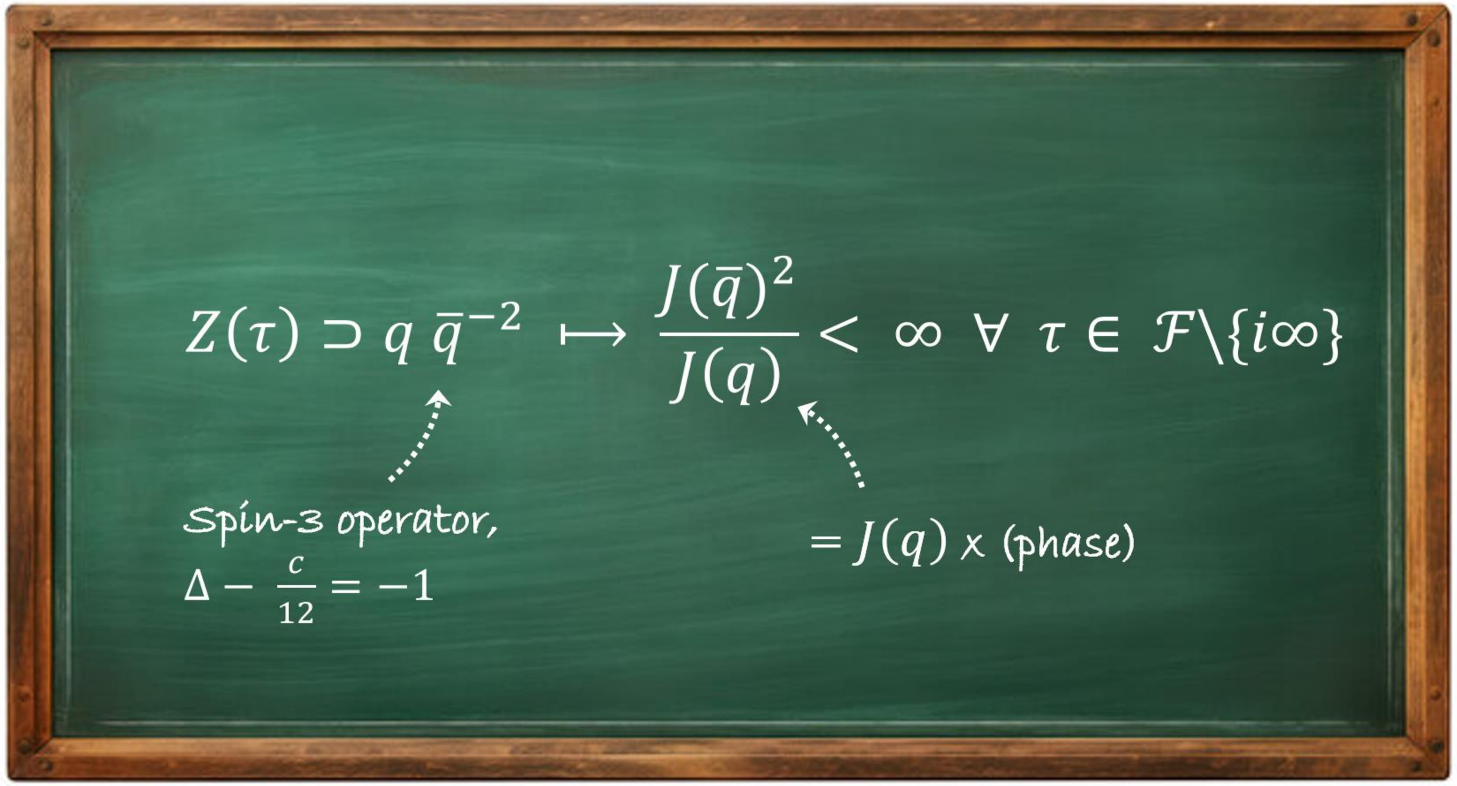



$$\begin{bmatrix} J(\tau) = J(\gamma\tau)\,,\ \gamma \in SL(2,\mathbb{Z}) \\ \approx q^{-1} + 196884q + \ldots \end{bmatrix}$$

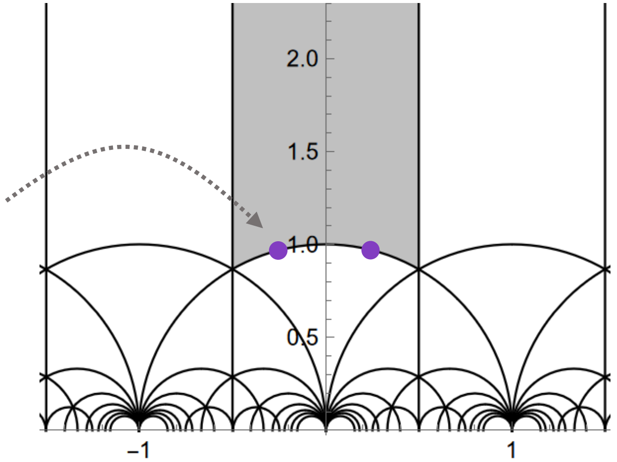
$^{-n}$ $(\alpha, \beta \in \mathbb{R}_+)$

$\mathcal{Z}(\tau)$ is

✓

✓

✓



# L-functions for 2D CFT

Attached to every compact, unitary Virasoro CFT is a self-dual, degree-four L-function.

It has a universal gamma factor with spectral parameters $\kappa_i = 0, 1, \pm\frac{1}{2}$,

root number $+1$, and a simple zero at the central point.

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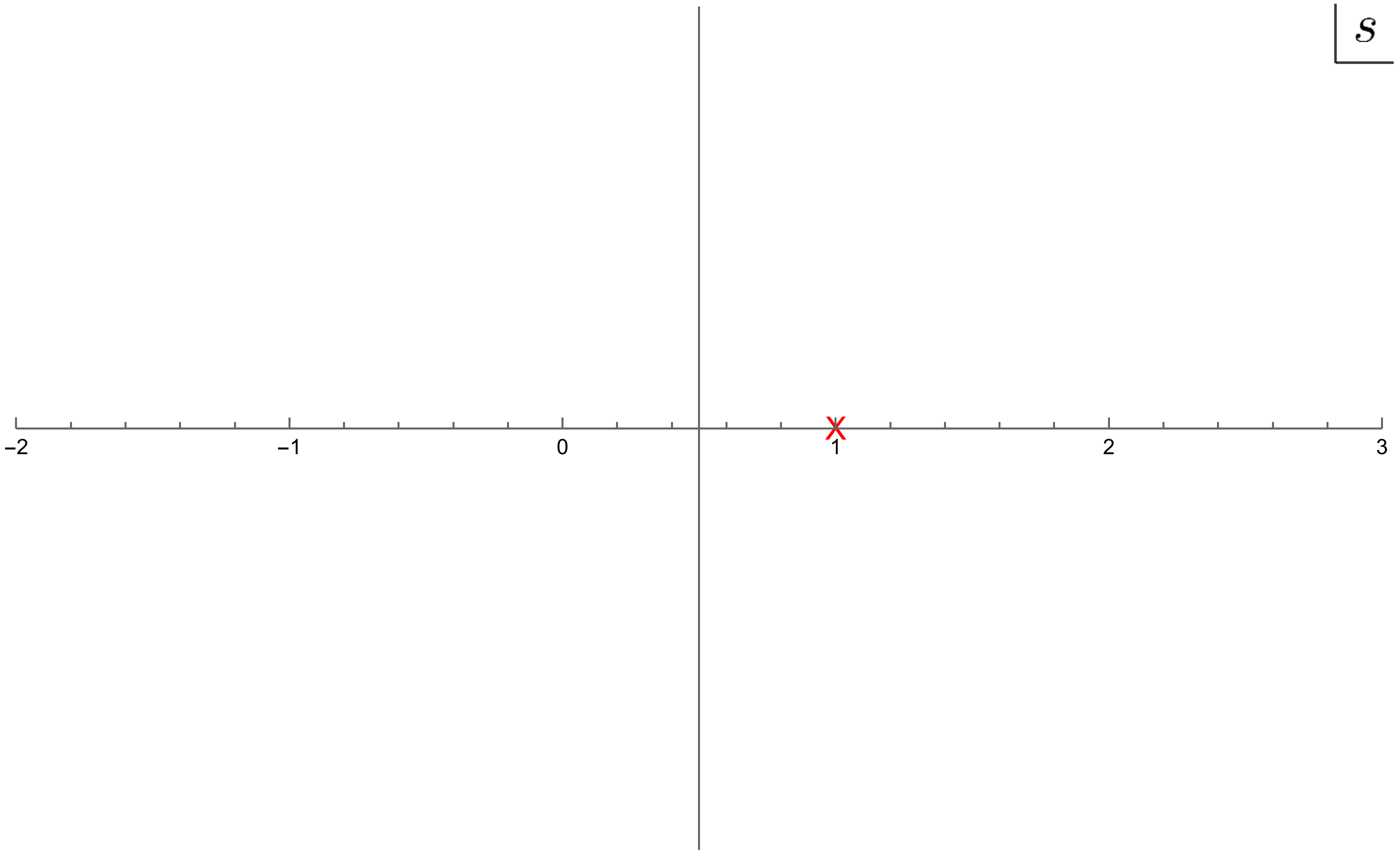
(These are not your father's L-functions.)

# Analytic Structure

The analytic structure of $L_{\mathcal{Z}}(s)$ is determined by discreteness & the functional equation.




$\operatorname{Res}_{s=1} L_{\mathcal{Z}}(s) \propto \langle \mathcal{Z} \rangle$

**Poles**

X $s=1$: due to modular average

**Zeros**

● *Trivial zeros*: $s = \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \ldots$

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$$s \in \left\{ \frac{\rho_n}{2}, 1 - \frac{\rho_n}{2} \right\} \quad (\zeta(\rho_n) = 0)$$

● ***Non-trivial zeros*** $\{\rho_{\mathrm{NT}}\}$

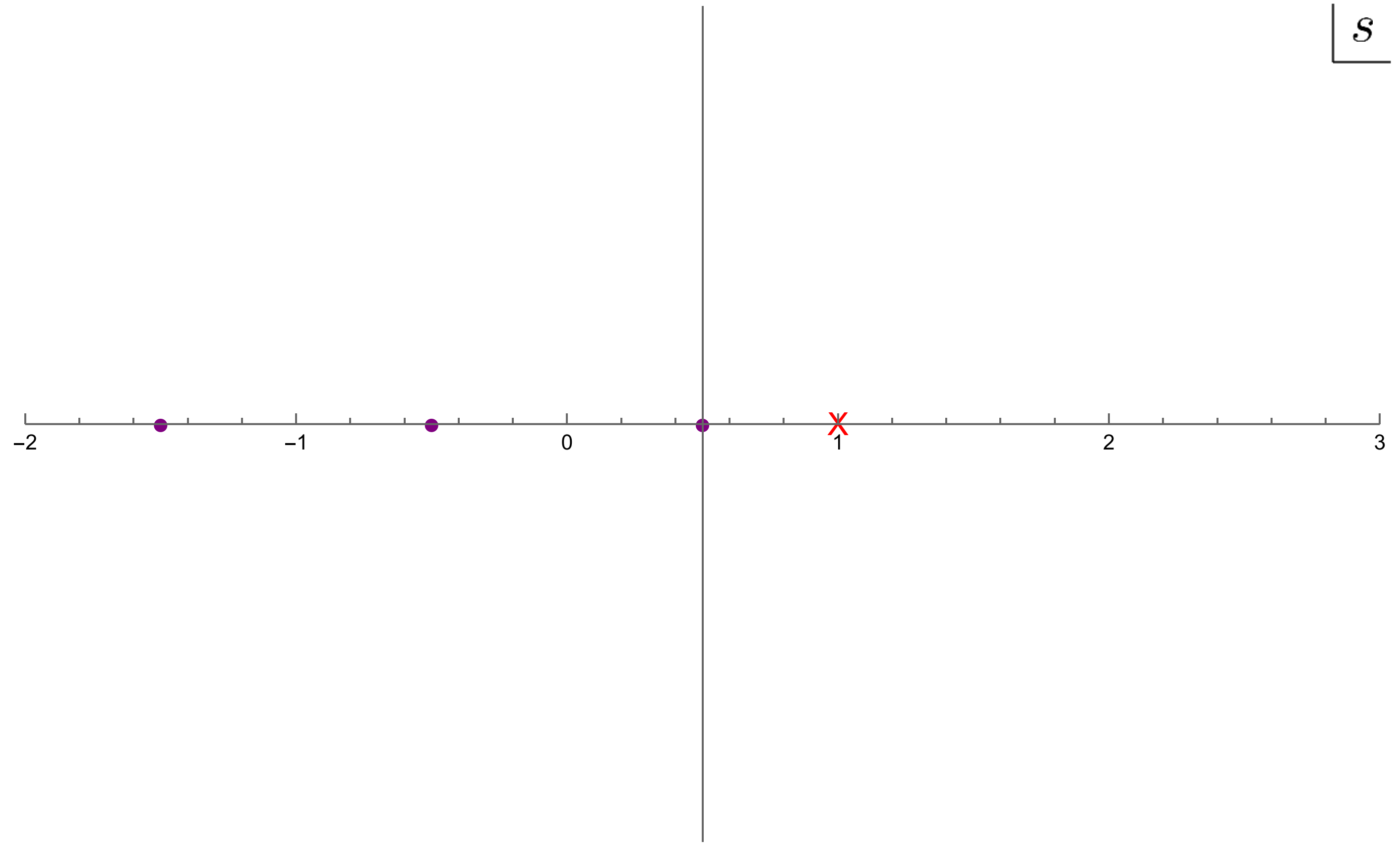
Infinitely many, $\mathcal{Z}$-dependent.

Self-duality ⇒ $\mathbb{Z}_4$ symmetry.

All poles/zeros are simple.

# Analytic Structure

The analytic structure of $L_{\mathcal{Z}}(s)$ is determined by discreteness & the functional equation.




$\operatorname{Res}_{s=1} L_{\mathcal{Z}}(s) \propto \langle \mathcal{Z} \rangle$

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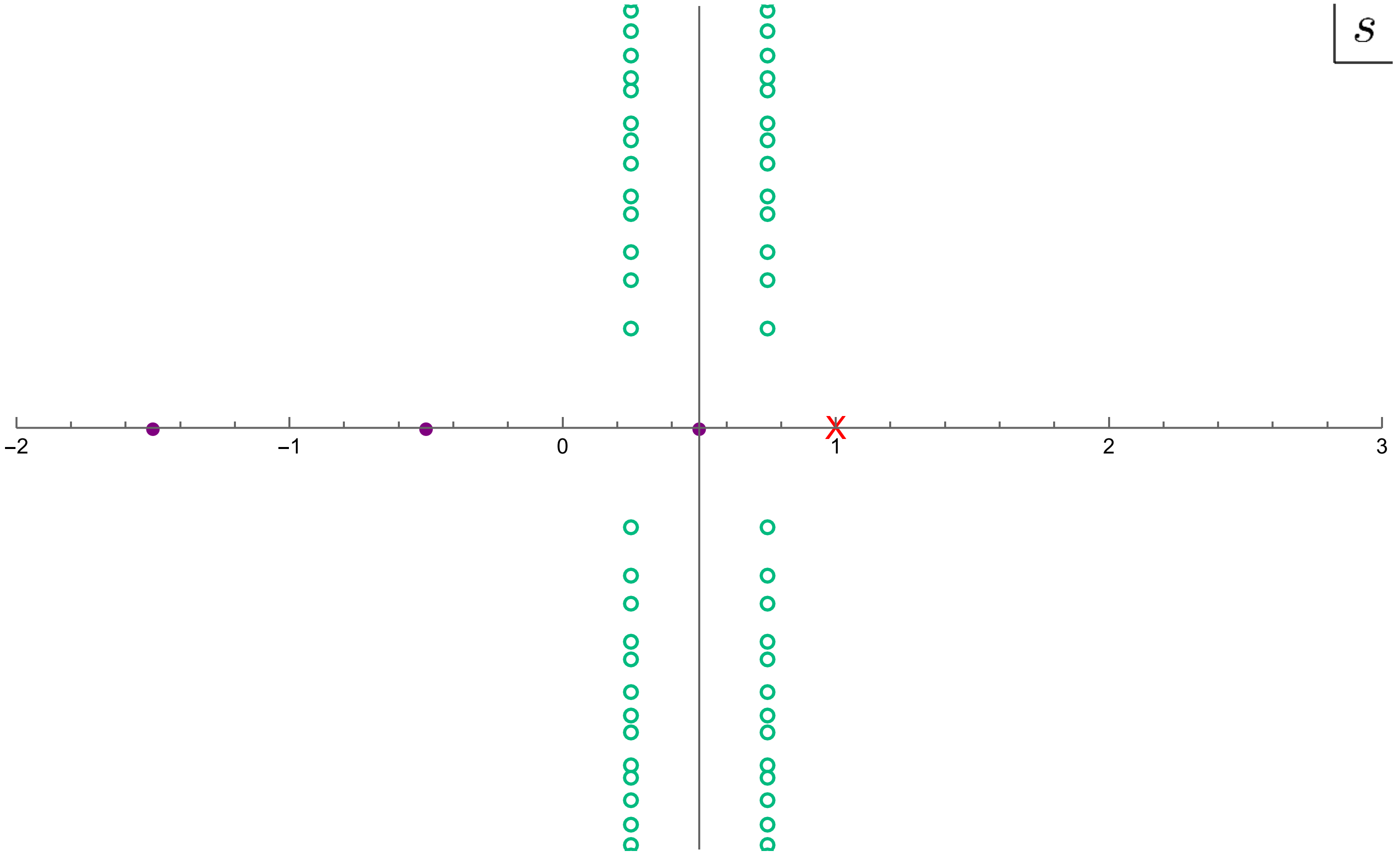


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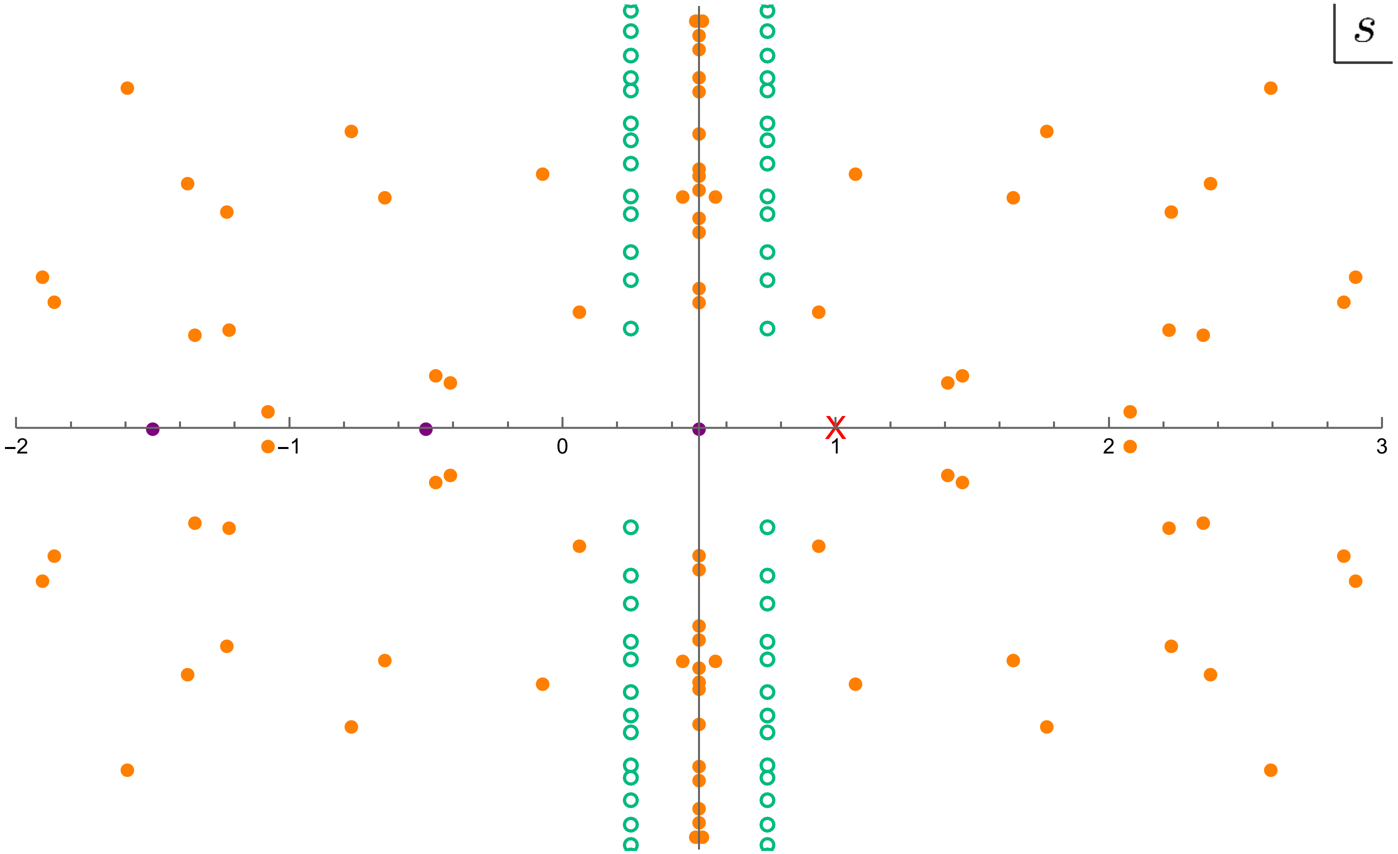


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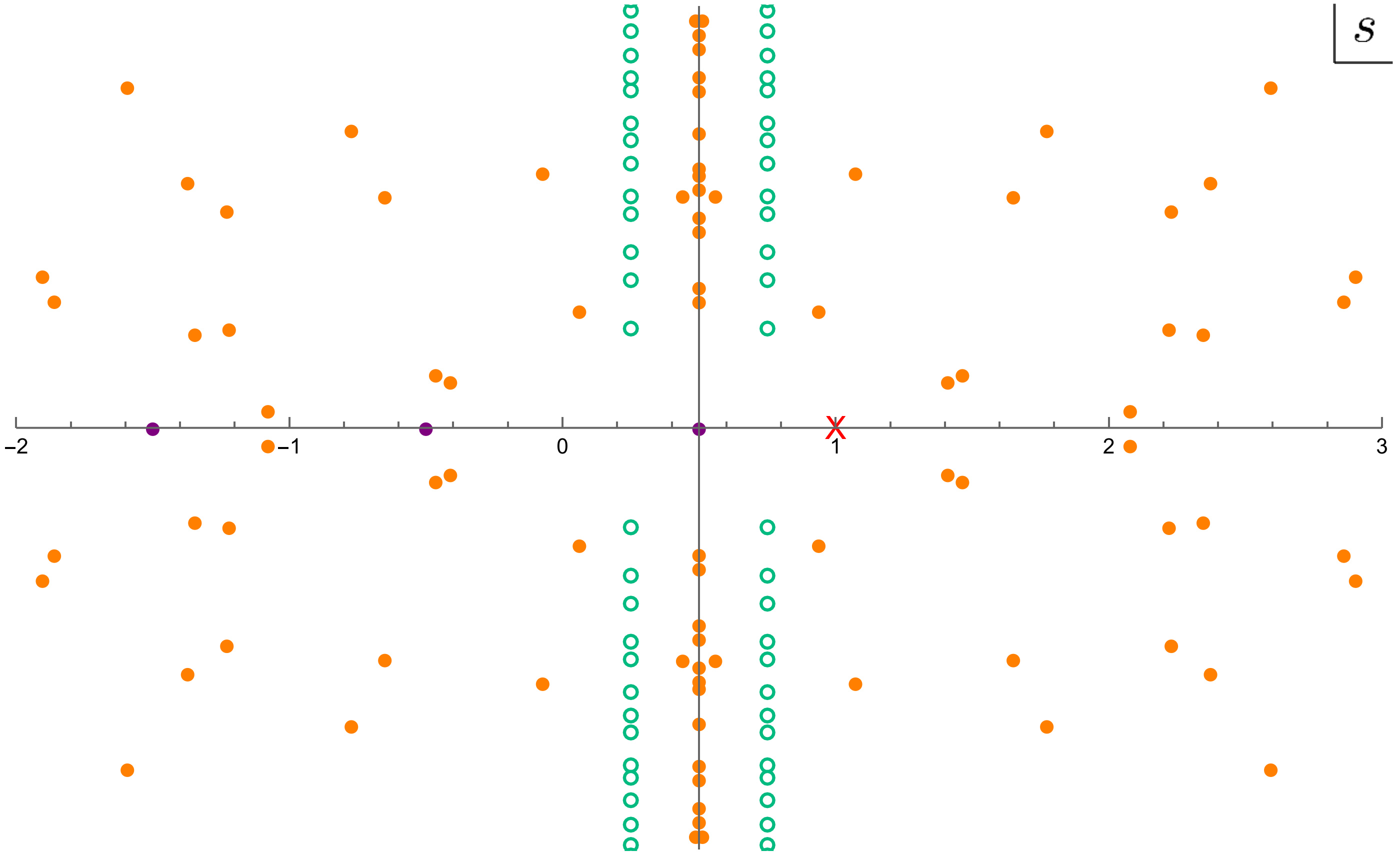


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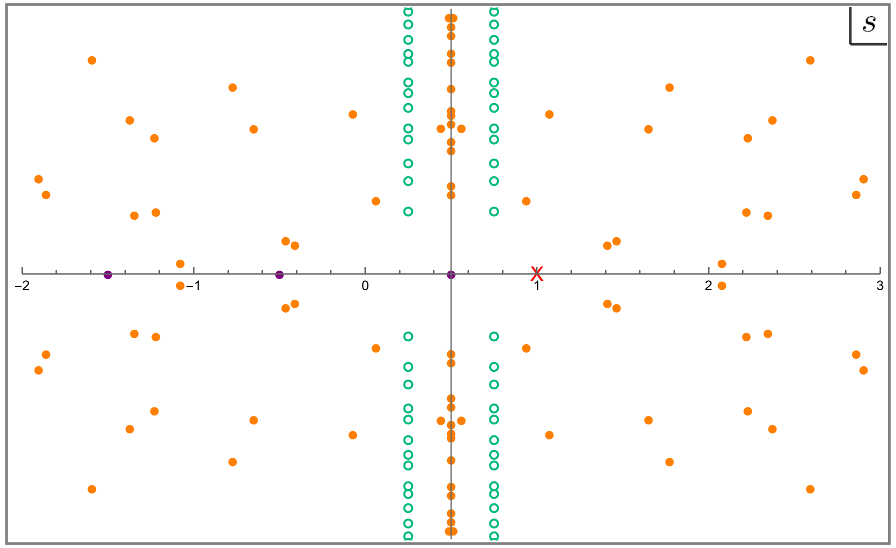
$$-\sum_{\rho} \frac{1}{\rho} = 1 - 2\gamma_E - \log(8\pi^2) + \frac{L'_{\mathcal{Z}}}{L_{\mathcal{Z}}}(0) < \infty$$

All poles/zeros are simple.

# The high-energy spectrum (what's left in the UV?)




Two features of $a_\lambda$, the subtracted high-energy density of states:

1) It vanishes "on average":

$$\zeta_{\mathcal{Z}}\left(\frac{1}{2}-n\right) = 0 = \sum_\lambda a_\lambda \lambda^n \Big|_{\text{regularized}} \quad \forall\ n \in \mathbb{Z}_{\geq 0}$$

Note: $\operatorname*{ord}_{s=\frac{1}{2}} L_{\mathcal{Z}}(s) = 1$

Interesting in view of Birch-Swinnerton-Dyer conjecture

2) Square root cancellation:

$$\sum_\lambda \frac{a_\lambda}{\lambda^{\frac{1}{2}+\varepsilon}} < \infty \qquad \sum_{\lambda \leq \lambda_N} a_\lambda \sim \sqrt{\lambda_N}$$

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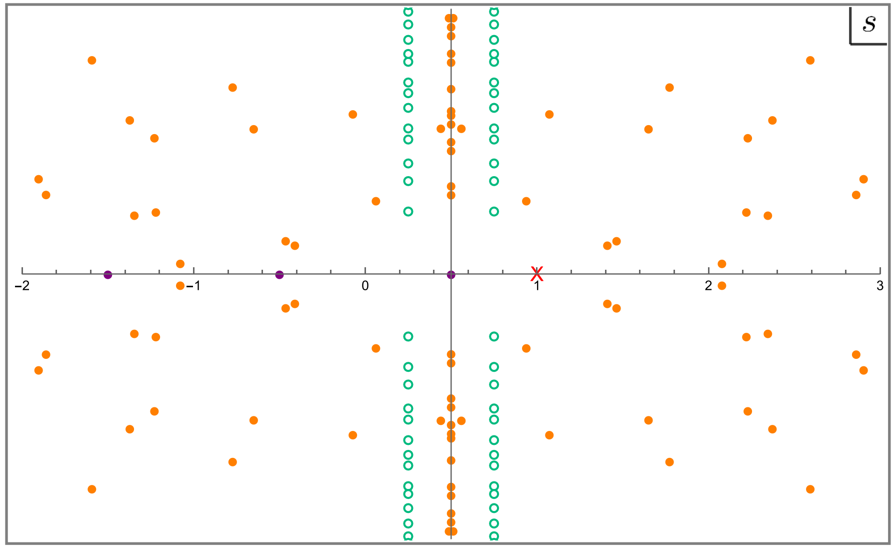


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$\implies$ Very strong UV-IR relations in 2D CFT.

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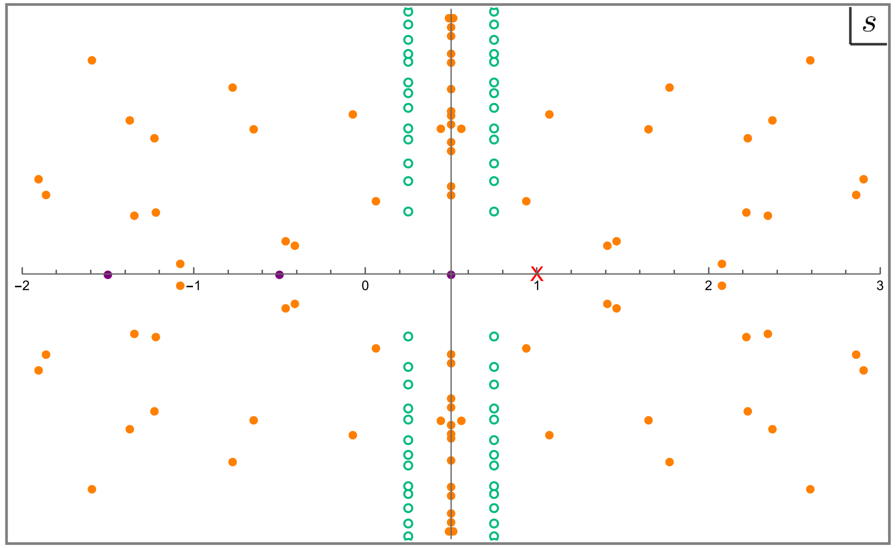


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This is a central concept in analytic number theory; Lindelöf and Riemann are equivalent to it.

$$\sum_{n \lesssim \sqrt{t}} n^{-\frac{1}{2}-it} = O(t^\varepsilon) \qquad \sum_{n \leq N} \mu(n) = O(N^{\frac{1}{2}+\varepsilon})$$

$\Longrightarrow$ Existence of compact irrational CFTs is a "miracle" of precisely the same type.

# Comments (CFT)

- Consistency constraints on $L_{\mathcal{Z}}(s)$ act on high-energy states "after crossing".

- $L_{\mathcal{Z}}(s)$ knows about the central charge *c*, but only via asymptotic level spacings… this is suggestive.

- We are heading towards a bootstrap based on *modularity*, *discreteness*, and *square-integrability*… and a little bit of unitarity:

  i) $\mathcal{Z}(\tau) \in L^2(\mathcal{F})$ follows from (but does not require) unitarity

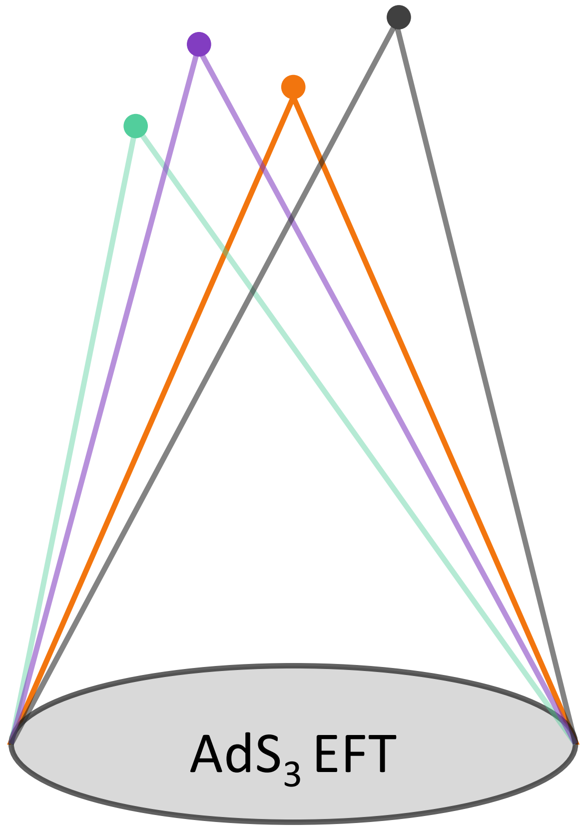
  *ii)* *Reality* of the degeneracies $\Rightarrow$ *Self-duality* of the L-function

- (∃ L-functions for other torus observables, e.g. $\langle\mathcal{O}\rangle_{T^2}$ , $\langle\mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle_{\mathbb{R}^2}$ )

# Comments (gravity)

High-energy 2D CFT primaries are dual to black hole microstates in $AdS_3$ quantum gravity.

Space of UV completions
=
Space of L-functions



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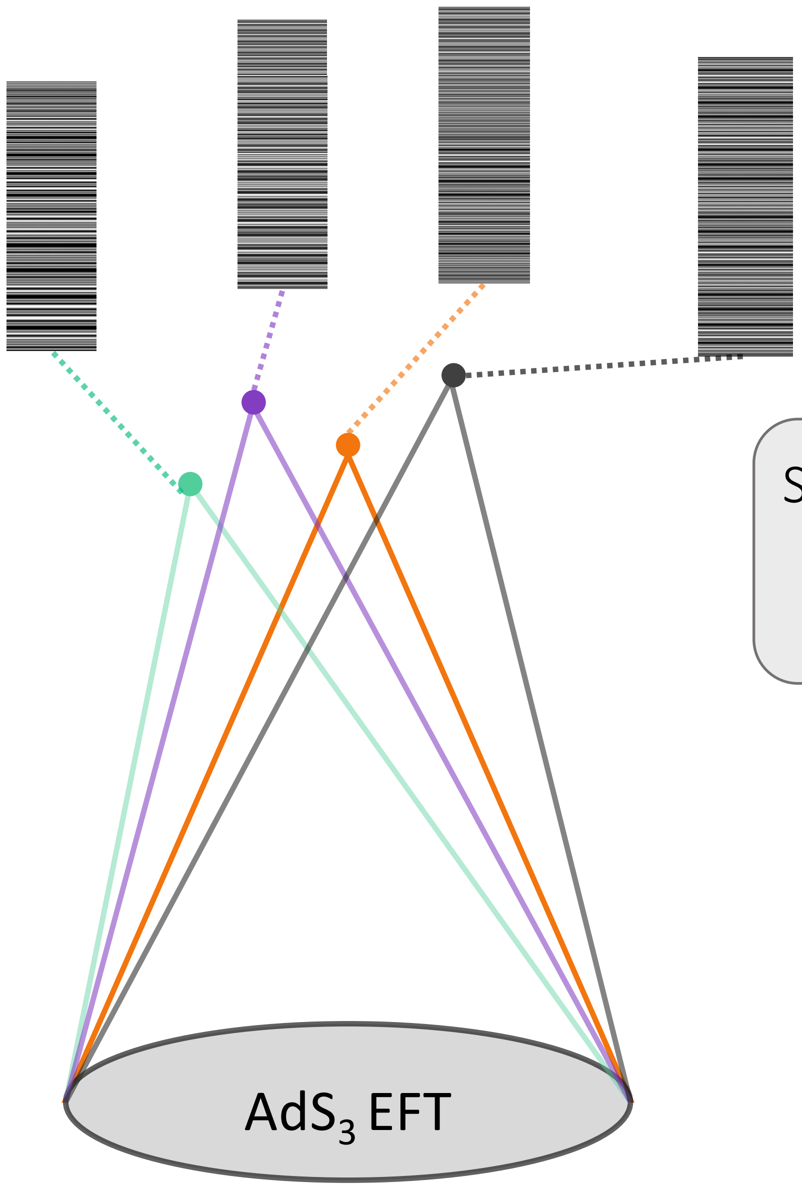


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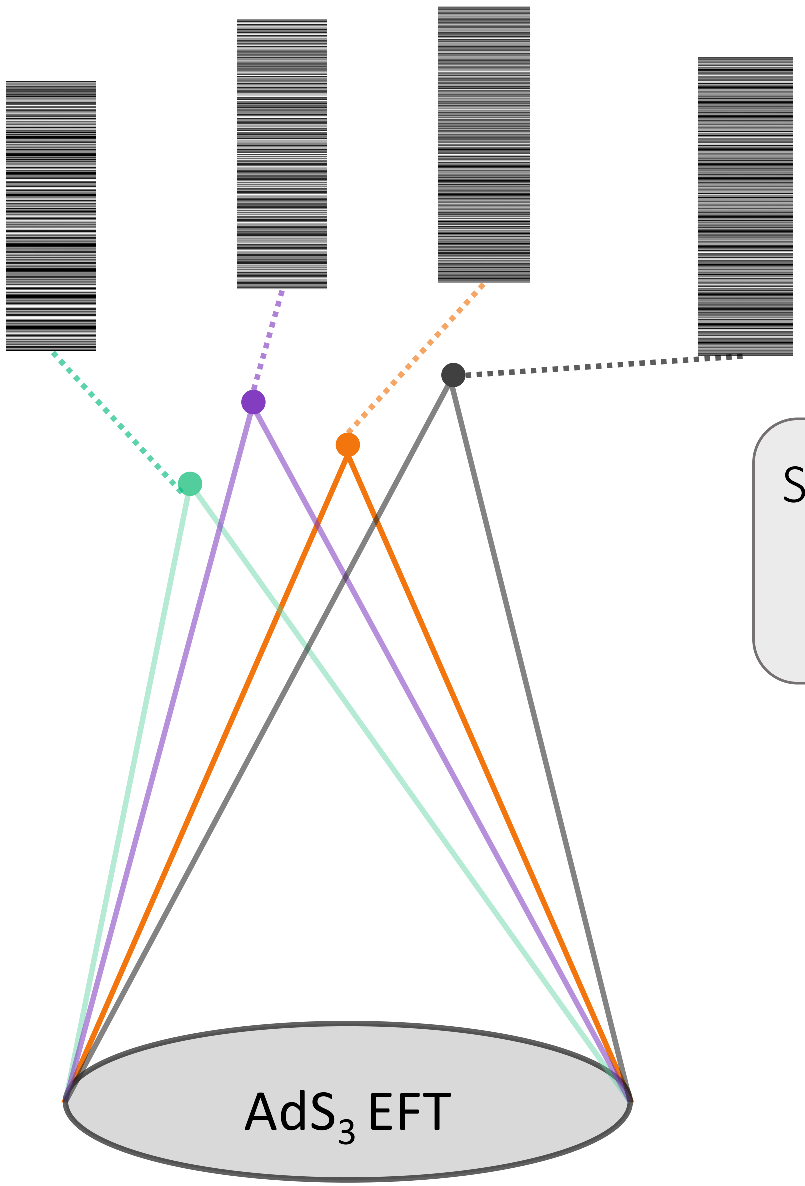


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Space of BH microstate spectra
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Space of L-functions ★

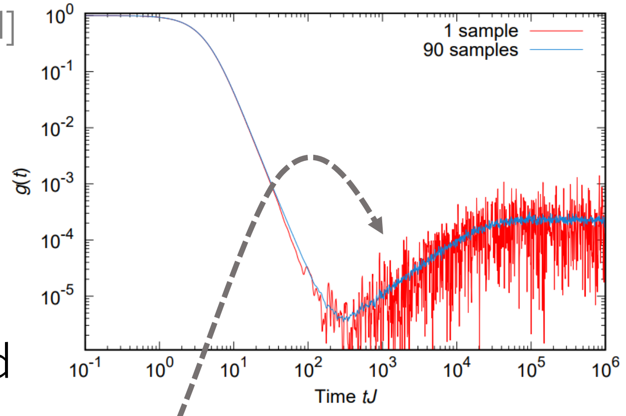
★ Can trade the spectrum of black hole masses $\{\Delta\}$ for *black hole zeros* $\{\rho\}$.

How do properties of 2D CFTs look in the language of L-functions?

Here is a nice example:

# Random Matrix Universality as Subconvexity

[Cotler et al]




$\langle |Z(\beta + iT)|^2 \rangle \sim T$

Quantum chaotic systems exhibit random matrix universality (RMU). [Bohigas Giannoni Schmit]

A conformally- and modular-invariant definition of RMU for 2D CFTs was formulated in [Di Ubaldo, EP '23], + a necessary/sufficient condition on high-energy states for a ramp.

That condition implies

$$\left| L_{\mathcal{Z}}\left(\frac{1}{2} + it\right) \right| \ll |t|^{\frac{1}{2}+\varepsilon} \quad \longleftrightarrow \quad \mu_L\left(\frac{1}{2}\right) \leq \frac{1}{2} \qquad \text{(RMU)}$$

On the other hand,

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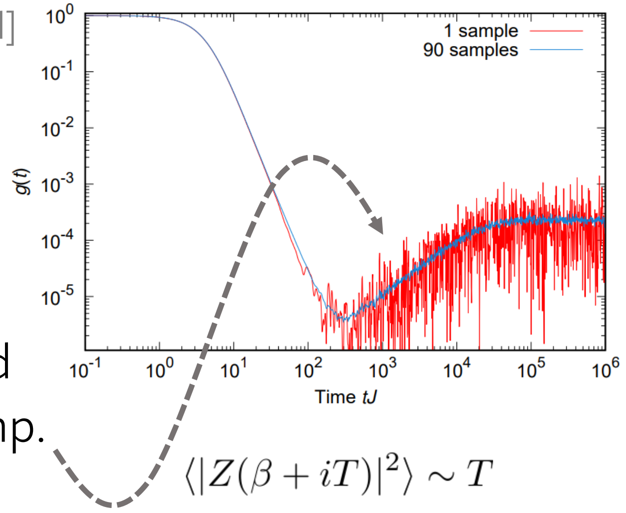


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On the other hand,

$$\left|L_{\mathcal{Z}}\left(\frac{1}{2}+it\right)\right| \ll |t|^{1+\varepsilon} \quad \longleftrightarrow \quad \mu_L\left(\frac{1}{2}\right) \le 1 \qquad \text{(Convexity bound for degree-4)}$$

RMU ⇒ Subconvexity

- Why? Phase incoherence of chaotic phases $\sum_\lambda e^{-it\log\lambda}$ (cf. "approximate functional equation")
- Random matrices are used to build heuristic models for computing moments of L-functions [Conrey et al]. This is different: we are *inputting* RMU as a physical property of *frequencies*, not *observing* RMU of *zeros*.

Reformulating 2D CFT with L-functions offers new proof strategies for old problems.

# New Modular Bootstrap

This setup is well-suited for bootstrapping the spin-0 spectral gap, $\Delta_1^{(0)}$ :

*Consider the L-function attached to a difference of primary partition functions with gaps to at least* $\frac{c-1}{12}$ .

*If* $\lambda_1 \leq \lambda_* \quad \Rightarrow \quad \Delta_1^{(0)} \leq \frac{c-1}{12} + \lambda_*$ *with a single possible exceptional spectrum.*

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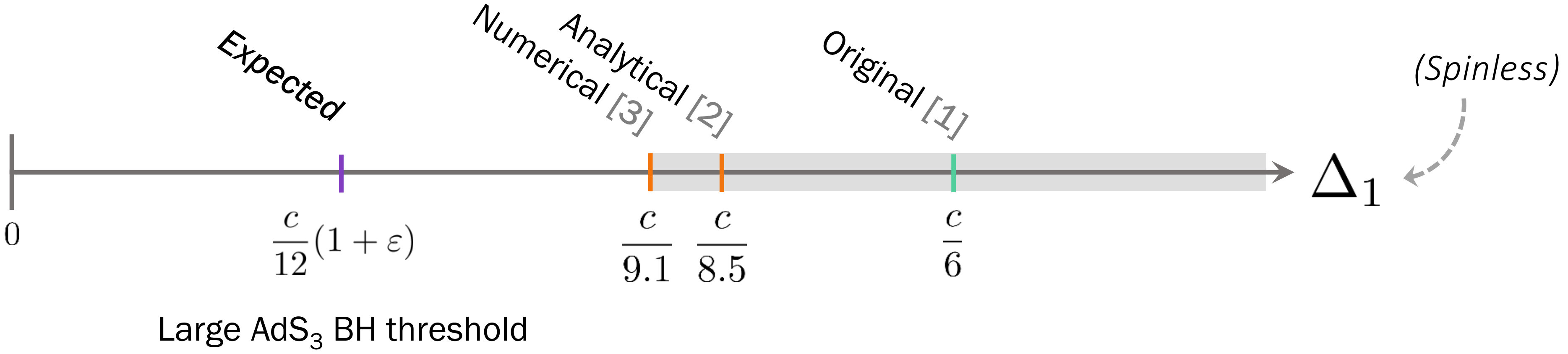
<u>State-of-the-art at $c \to \infty$:</u>




$\Delta_1 \lesssim \frac{c}{12}$ has long been the horizon for this problem.

L-functions can get us there. .


[1: Hellerman '09]

[2: Hartman, Mazac, Rastelli '19]

[3: Afkhami-Jeddi, Hartman, Tajdini '19]

# Proof

The idea of the proof is to play two representations of the L-function off of each other:

[Hadamard] $$\Lambda_{\mathcal{Z}}(s) = \frac{\#}{s(1-s)} \prod_{\rho} \left(1 - \frac{s}{\rho}\right)$$

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Taking the log derivative,

$$\boxed{\sum_{\rho} \frac{1}{s-\rho} = -\log \lambda_1 + \frac{\hat{\zeta}'_{\mathcal{Z}}}{\hat{\zeta}_{\mathcal{Z}}}(s) + F(s)}$$

$$= \frac{1}{s} - \frac{1}{1-s} - \log(8\pi^2) + 2\psi(2s-1) + 2\frac{\zeta'}{\zeta}(2s)$$

Suppose for a moment that the last two terms were absent at large $\lambda_1$:

$$\sum_{\rho} \frac{1}{s-\rho} \approx -\log \lambda_1 \quad (\lambda_1 \to \infty)$$

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*Strategy*: validate the approximation (thus bounding $\Delta_1^{(0)}$) when $\lambda_1 = \frac{c}{12}\varepsilon\,,\ c \to \infty\,,\ \varepsilon > 0$ fixed

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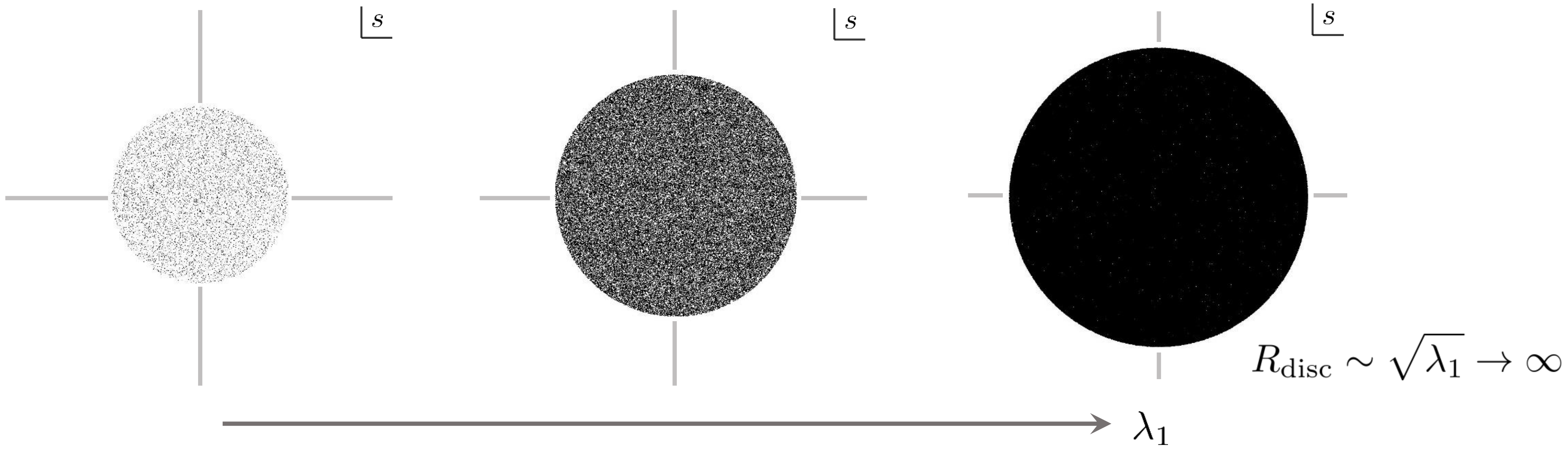
**Next**, in order to drop *F(s)*, *s* must not be "pushed out" too far: $F(s) \sim \log s^2 \quad \text{as} \quad |s| \to \infty$

Therefore,

$$\boxed{\sum_\rho \frac{1}{s-\rho} \approx -\log\lambda_1 \quad \forall \quad s \in \mathcal{R}_\infty \,, \; |s| \ll \sqrt{\lambda_1}}$$

# Proof

So, a bootstrap bound follows as long as the L-function isn't *dense with zeros* inside a large disc, like this:



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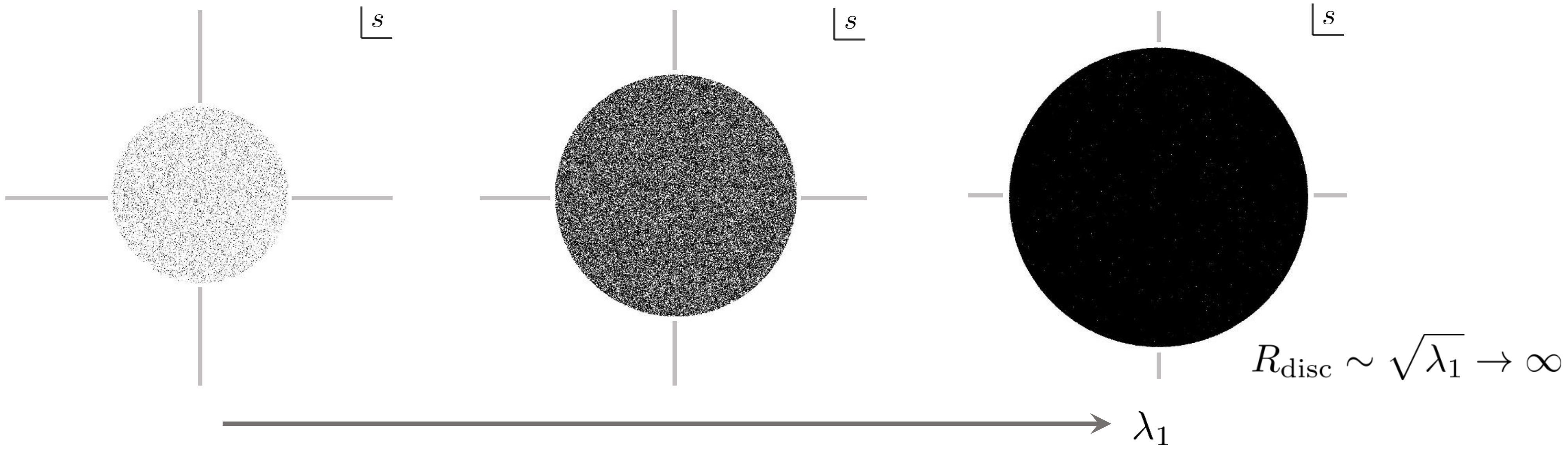


But the dense disc is impossible: Hadamard correlates the *number of NT zeros* with the *asymptotic falloff,*

$$\lambda_1 > \frac{1}{2} \quad \Rightarrow \quad N_{\text{NT}}(L_{\mathcal{Z}}) \ll N_{\text{NT}}(\zeta)$$

Famously, Riemann zeta zeros are logarithmically dense *on a line* – not in a disc.

Even without assuming RH, rigorous density estimates for $\zeta(s)$ zeros imply $\rho_{\text{disc}}(\zeta) \ll \lambda_1^{-\frac{11}{26}+\varepsilon}$. [Guth, Maynard '24]

# Proof

In conclusion,

$$\Delta_1^{(0)} \leq \frac{c-1}{12} + o(c) \quad \text{as} \quad c \to \infty$$

(modulo a single possible exceptional spectrum)

Our proof strategy passes a reassuring check:

CFTs with $N$ generating currents host L-functions with frequencies $\lambda = \Delta^{(0)} - \frac{c-N}{12}$

In Narain CFT,

$$\Delta_1^{U(1)^N} \lesssim \frac{N}{2\pi e} \quad \text{as} \quad N \to \infty$$

Since $c = N$, our proof should break down.

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(*Speculation*: Is $\Delta_1^{(0)} \leq \frac{c-1}{12} + \frac{1}{2}$ at *finite* c? )

# Hints of a lattice?

After a little massaging, one notices the following: $L_{\mathcal{Z}}(s/2)$ obeys precisely the same functional equation as L-functions attached to elliptic curves (the *Hasse-Weil L-function*).

In this relation, the sum over integers in Hasse-Weil maps to the sum over Liouville momenta.

Is this just a coincidence? Is there an "elliptic curve for irrational CFTs"?

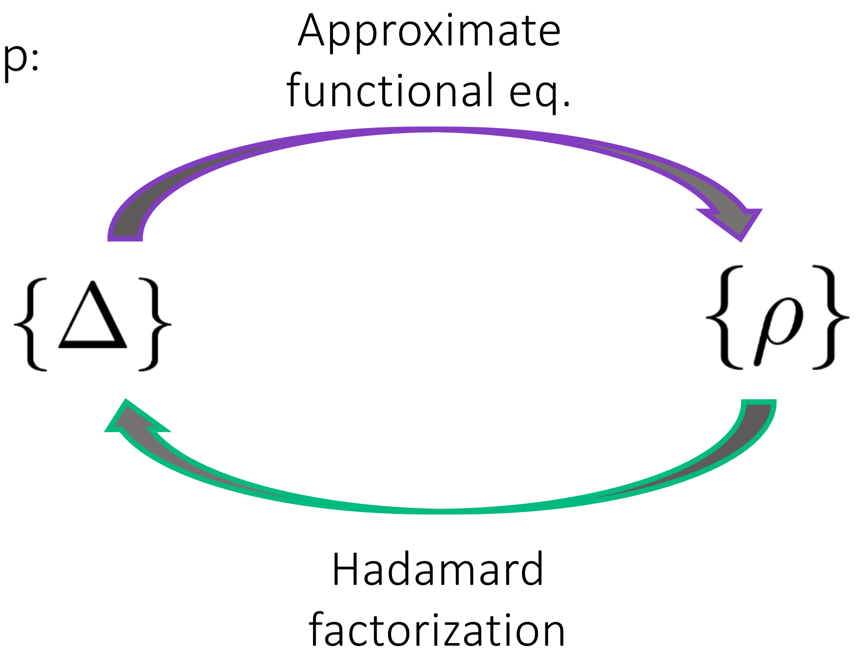
Irrational CFTs are chaotic at high energies. This breeds the familiar intuition that, upon properly accounting for symmetries, the high-energy spectrum of an irrational CFT is totally irregular: that is, structureless apart from the requirement of random matrix statistics.

We submit that this intuition is missing something.

# Final thoughts

We have initiated a structural formulation of 2D CFT in the language of analytic number theory.

Imagine a new kind of bootstrap:




$$L_{\mathcal{Z}}(s) = \zeta(2s)\zeta_{\mathcal{Z}}(s)$$
$$\Lambda_{\mathcal{Z}}(s) = \Lambda_{\mathcal{Z}}(1-s)$$

It seems promising to think more about *black hole zeros*, and the L-functions of extremal CFTs.

What is the L-function of $AdS_3$ pure gravity?

# Amazing backup slide

Explicit L-functions for Narain CFT:

$$L^{(c)}(s) = 2^{s_c}\zeta(2s) \sum_{(n,w)\,\in\,\mathbb{Z}^c\times\mathbb{Z}^c\setminus\{0\}} \frac{\delta_{n\cdot w,0}}{(2\Delta_{n,w}(m))^s} \qquad s_c = s + \frac{c-2}{2}$$