Rational Vertex Operator Algebras from Supersymmetric Field Theories

Heeyeon Kim (KAIST)

Based on various collaborations with
D. Gang, S. Stubbs, N. Garner, A. Ferrari, T. Creutzig, D. Gaiotto,
B. Park, J. Song, B. Go, S. Kim, Q. Jia

Related works by A. Ardehali, M. Dedushenko, M. Litvinov

Rational Vertex Operator Algebras

A rational vertex operator algebra describes a chiral half of a 2d rational conformal field theory, which is characterized by the fact that the Hilbert space decomposes into a finite sum:

$$\mathcal{H} = \bigoplus_{\alpha,\bar{\alpha}} N_{\alpha,\bar{\alpha}} V_\alpha \otimes V_{\bar{\alpha}}$$

Its representation theory gives rise to the modular tensor category (MTC). [Moore-Seiberg 89]

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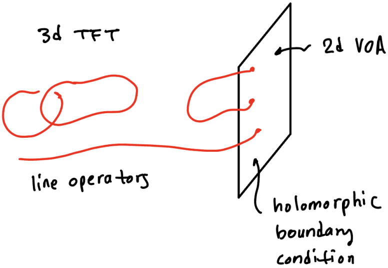
To every rational VOA, we can associate a 3d semi-simple topological field theory (TFT).

3d Semi-Simple TFTs



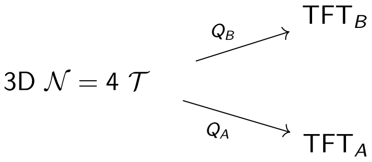
The space of conformal blocks can be identified with the Hilbert space of the 3d TFT. The only local operator in a semi-simple TFT is the identity:

$$Z_{S^2 \times S^1} = \dim(\mathcal{H}_{\mathrm{TFT}, S^2}) = 1 \ .$$

Explicit TFT construction for non-unitary rational VOA is not understood very well. (e.g. Virasoro minimal models with $|p-q| \neq 1$)

Twisted 3d $\mathcal{N} = 4$ SUSY QFTs

A large class of non-unitary 3d TFT can be obtained from topologically twisted 3d $\mathcal{N} = 4$ theories.



They correspond to the Rozansky-Witten twist (B) and the dimensional reduction of the Donaldson twist (A) respectively.

The twisted theories can be put on a half-space with 2d boundary (e.g. $\mathbb{C} \times \mathbb{R}_-$) with a holomorphic boundary condition compatible with Q_A/Q_B.

The algebra of local operators on the boundary forms a VOA.

Rational VOAs from twisted 3d $\mathcal{N} = 4$

However, generic VOAs from this construction are *not* rational.

This is due to the existence of local operators in $TFT_{A/B}$, originated from the Coulomb/Higgs branch operators of 3d $\mathcal{N} = 4$ theory.

Exception: 3d SCFTs with zero-dimensional Coulomb and Higgs branch

$$\rightsquigarrow \text{“Rank-zero” SCFTs.}$$

They are non-Lagrangian, but often admits a simple UV gauge theory description with $\mathcal{N} = 2$ symmetry, which is expected to flow to an $\mathcal{N} = 4$ rank-zero fixed point.

Example: $\mathcal{T}_{\text{min}}$

"Minimal rank-zero SCFT", $\mathcal{T}_{\text{min}}$, with an $\mathcal{N} = 2$ description:

$$U(1)_{3/2} + \text{a chiral multiplet}$$

[Gang-Yamazaki 2018]

There are various indirect evidences that this theory flows to a rank-zero fixed point with supersymmetry enhancement to $\mathcal{N} = 4$.

Being an $\mathcal{N} = 4$ theory, $\mathcal{T}_{\text{min}}$ can be topologically twisted to produce a pair of TFTs, which supports a rational VOA on their boundary.

It was originally conjectured in [Gang-Kim-Lee-Shim-Yamazaki 2021] that $\mathcal{T}_{\text{min}}$ supports the Virasoro minimal model $M(2, 5)$ on the boundary.

OPE calculations

OPEs of boundary operators can be calculated, via "two-step twisting"

(i) Perform holomorphic-topological twist of the $\mathcal{N} = 2$ theory

(ii) Deformation to the full A/B-twist

The boundary operators survives the Q_B cohomology (Dirichlet b.c.):

$$J\ , V_+\ , V_-\ , \theta_+, \theta_-$$

Their OPEs give [Ferrari-Garner-HK 23]

$\rightsquigarrow$ Affine Lie superalgebra $\mathfrak{osp}(1|2)$ at level 1.

This is related to $M(2, 5)$ by level-rank duality. [Creutzig-Garner-HK 24]

Characters of Rational VOAs

Q. Is there a more systematic way to construct rank-zero SCFTs?

Insight from the 2d RCFT classification program: Modular invariance.
The characters of irreducible modules satisfy

$$\chi_\alpha(\tau+1) = T_{\alpha\beta}\chi_\beta(\tau) \ , \qquad \chi_\alpha\left(-1/\tau\right) = S_{\alpha\beta}\chi_\beta(\tau) \ .$$

- Modular linear differential equation (MLDE) [Mathur-Mukhi-Sen 88]
- Nahm's conjecture [Nahm],[Terhoevan],[Zagier],...

$$\chi_{(A,B,C)}(\tau) = \sum_{n\in\mathbb{Z}_+^r} \frac{q^{n^t A n + Bn + C}}{(q)_n} \ , \qquad q = e^{2\pi i \tau} \ ,$$

For which (A, B, C) does this become a modular function? $\rightsquigarrow$ Classification via properties of dilogarithm evaluated at certain algebraic numbers.

A class of $\mathcal{N} = 2$ abelian CSM theories

Notice that the Nahm sum formula also appears as the half-index of an abelian 3d $\mathcal{N} = 2$ theory with a specific boundary condition,

$$\chi_{(A,B,C)}(q) = Z_{D^2 \times S^1}[\mathcal{T}_{3d}](q)$$

which coincides with the vacuum character of the boundary algebra. This motivates us to classify the IR phases of abelian CSM theories:

Consider $\mathcal{N} = 2$ abelian Chern-Simons matter theories labeled by the level matrix K, matter representation Q and superpotential W. Classify all such theories that flow to

- a $\mathcal{N} = 4$ rank-zero fixed point, or
- a unitary TFT

A new class of rank-zero SCFTs

In [Gang-HK-Park-Stubbs, 24] we scanned over $\mathcal{N} = 2$ CSM theories

$$U(1)^r_K + r \text{ chiral multiplets.}$$

and search for K that admits a superpotential deformation so that the theory flows to a rank-zero fixed point. We find 28 sporadic K-matrices for $r = 1, 2, 3$, along with several infinite families.

This reproduces and slightly generalizes the analogous search for the modular Nahm sums classified in [Zagier 07]. These theories (with appropriate b.c.) are expected to support the following rational VOAs:

- Virasoro minimal models $M(2, 2r + 3)$
- $\mathcal{N} = 1$ super-virasoro minimal models $SM(2, 4r)$
- Affine Lie superalgebra $osp(1|2n)_k$
- Some minimal W-algebras

Relation to 4D SCFT/2D VOA correspondence

Many of these rational VOAs from 3d rank-zero SCFTs appear in the 4d $\mathcal{N}=2$ SCFT/VOA correspondence.

$$4\text{d } \mathcal{N}=2 \text{ SCFT} \quad \xrightarrow{\chi} \quad 2\text{d VOA}$$

[Beem-Lemos-Liendo-Peelaers Rastelli-van Rees, 13]

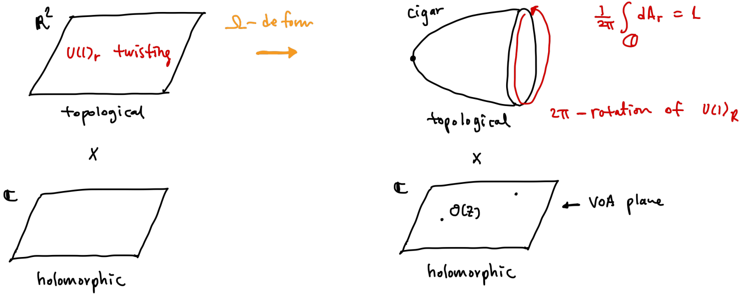
For example, this map gives

$$4\text{D } \mathcal{N}=2 \ (A_1, A_{2r}) \text{ Argyres-Douglas theories} \quad \xrightarrow{\chi} \quad M(2, 2r+3) \ .$$

What is the precise relation between the two constructions?

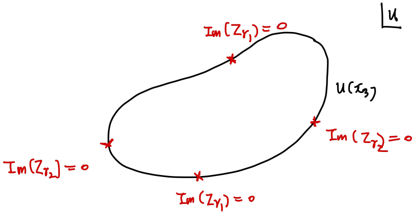
Cigar compactification

Holomorphic-topological twist of 4d $\mathcal{N} = 2$ SCFTs on a Melvin cigar [Dedushenko 23] [Ardehali-Dedushenko-GangLitvinov 24] [Dedushenko-Gukov -Nakajima-Pei-Ye 18] [Cecotti-Cordova-Vafa 13] [Cecotti-Neitzke-Vafa 10]



Compactification along the cigar circle S^1 gives rise to our 3d-2d system. This involves a 2π rotation of $U(1)_R$ symmetry along S^1, which is non-trivial for Argyres-Douglas type theories.

Janus loop



Consider a closed path in the Coulomb branch parametrized by the coordinate of S^1, along which the central charges make continuous 2π rotation:

$$Z_\gamma(u_*) \to e^{2\pi i} Z_\gamma(u_*)$$

The path can be chosen to be homotopic to a Janus configuration of effective theory parametrized by $u(x_3)$, which preserves a 3d $\mathcal{N} = 2$ subalgebra.

The 3d BPS particles "trapped" at various points on the path with $|Z_{4d}| = |Z_{3d}| = M$, which are described by 3d chiral multiplets. Inserting duality domain walls in appropriate loci on the path, we obtain an effective 3d abelian CSM theory. [Cecotti-Neitzke-Vafa 10] [Cecotti-Cordova-Vafa 13] [HK-Gaiotto 24]

The trace formula

This gives rise to the IR trace formula for the Schur index of 4d SCFT, which computes the vacuum character of corresponding VOAs. [Cordova-Shao, 15] [Cecotti-Neitzke-Vafa, 10]

For a given BPS particle with electromagnetic charge γ, we assign X_γ, a quantum torus algebra generator satisfying $X_\gamma X_{\gamma'} = q^{\langle \gamma, \gamma' \rangle} X_{\gamma'} X_\gamma$. Then

$$I_{\mathrm{Schur}}(q) = \mathrm{tr}\ (-1)^F q^{\Delta - R} = \mathrm{Tr}\ \prod_{i=1}^{2N} E_q(X_{\gamma_i})$$

where

$$E_q(z) = \frac{1}{(-q^{1/2}z; q)_\infty} = \sum_{n=0}^{\infty} \frac{(-q^{1/2}z)^n}{(q)_n} \ .$$

I_{Schur} is a wall-crossing invariant quantity.

3d gauge theories from Schur indices

Evaluating the trace gives the character in a Nahm sum formula:

$$I_{\text{Schur}}(q) = (q)_\infty^{\text{rk}(\Gamma/\Gamma_f)} \sum_{\sum n_i \gamma_i \in \Gamma_f} \frac{q^{\frac{1}{2}\sum_i n_i}(-q)^{\frac{1}{2}\sum_{i<j}\langle \gamma_i, \gamma_j \rangle n_i n_j}}{\prod_i (q)_{n_i}}$$

from which we can construct a candidate 3d abelian CSM theory whose half-index coincides with I_{Schur}.

Example: (A_1, A_2) AD theory: $\langle \gamma_1, \gamma_2 \rangle = 1$. The IR formula gives

$$I_{\text{Schur}}(q) = (q)_\infty^2 \sum_{n_i=0}^{\infty} \frac{q^{n_1 n_2 + n_1 + n_2}}{(q)_{n_1}^2 (q)_{n_2}^2} .$$

This gives an 3d $\mathcal{N} = 2$ $U(1)^2$ CSM theory with four chiral multiplets. By various elementary dualities, one can argue that it flows to $\mathcal{T}_{\text{min}}$.

Wall-crossing invariants to 3d partition functions

This analysis motivates us to propose a universal formula that computes the ellipsoid partition function of the twisted rank-zero theory:

$$S_b = \mathrm{Tr}_{\mathcal{H}} \prod_i \Phi_b(x_{\gamma_i})$$

where now the function $\Phi_b(x)$ is the Fadeev quantum dilogarithm, which acts on an auxiliary Hilbert space $L^2(\mathbb{R}^{\mathrm{rk}(\Gamma/\Gamma_f)/2})$. The variables now satisfy the Weyl algebra

$$[x_\gamma, x_{\gamma'}] = \frac{1}{2\pi i}\langle\gamma, \gamma'\rangle \ .$$

The pentagon identity of Φ_b insures that S_b is a wall-crossing invariant quantity. We check this proposal for a large class of Argyres-Douglas theories. [HK-Gaiotto 24]

Higher powers of monodromy operators

It is natural to consider multiple wrappings of the Janus circle, which lead to a family of 3d theories with (see also [Cecotti-Song-Vafa-Yan 15])

$$S_b^{(n)} = \mathrm{Tr}\left[\prod_i \Phi_b(x_{\gamma_i})\right]^n , \quad I_q^{(n)} = \mathrm{Tr}\left[\prod_i E_q(X_{\gamma_i})\right]^n$$

If the R-charges of the Coulomb branch operators are fractional, there exists an integer N such that N-th wrapping trivializes the $U(1)_R$ twisting. Indeed one can explicitly check that

$$\left[\prod_i \Phi_b(x_{\gamma_i})\right]^N = \mathbf{1} \quad \in L^2(\mathbb{R}^{\mathrm{rk}(\Gamma/\Gamma_f)/2}) ,$$

for (A_1, G) type AD theories. [Go-Jia-HK-S.Kim, to appear]

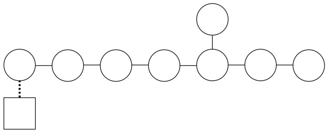
Intermediate VOAs?

This gives rise to a family of VOAs from a 4d SCFTs of AD type. For (A_1, A_2) theory, we obtain four VOAs whose modular data are related by Galois conjugations:

$$M(2,5)\ ,(\hat{G}_2)_1\ ,(\hat{F}_4)_1\ ,osp(1|2)_1$$

An interesting observation: For $n = -4$, the monodromy traces gives the modular invariant characters of mysterious intermediate algebra called $(E_{7\frac{1}{2}})_1$, which was a "missing hole" in the classification of RCFTs with two characters. [Mathur-Mukhi-Sen 88]

From this, we propose a simple UV description of a 3d TFT which support a VOA with these characters. [HK-Song, 24]



Summary and future directions

- The twisted compactification of 4d SCFTs can be understood in the Coulomb branch effective theory. Dynamics of BPS particles are encoded in the Janus loop configuration, which gives a large class of 3D $\mathcal{N}=2$ abelian CSM theories.
- They are expected to flow to 3d $\mathcal{N}=4$ rank-zero fixed point which can be twisted to produce TFT that support various non-unitary rational VOAs.
- 4d interpretation of the family of VOAs?
- Explicit computations of boundary OPEs, especially those related to intermediate VOAs?
- Relation to holomorphic modular bootstrap program?