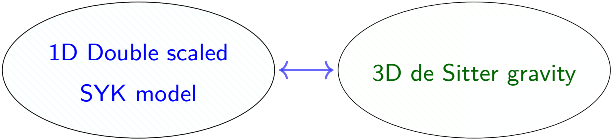
Dual perspectives on double scaled SYK

Herman Verlinde

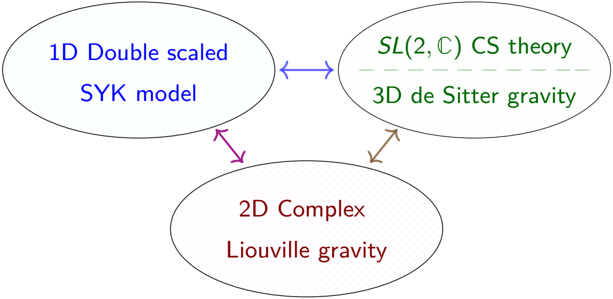
Strings 2025, NYU Abu Dhabi, 01/08/25

Based on: 2310.16991, 2402.00635, 2402.02584, 2409.11551,WIP
with D. Gaiotto, V. Narovlansky, D.Tietto, and M. Zhang

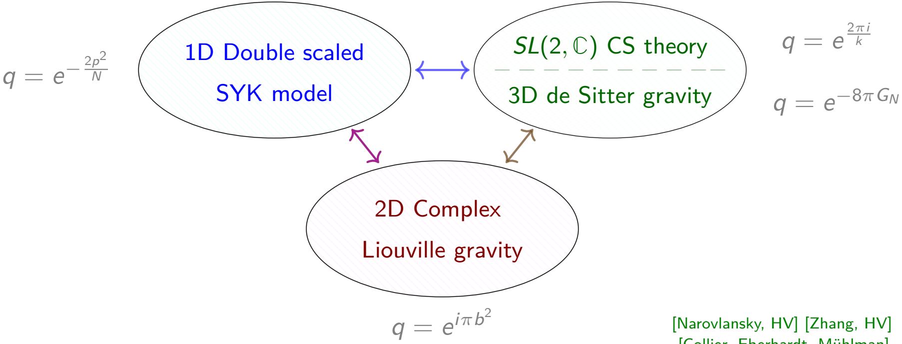


[HV] [Rahman, Susskind]
[Narovlansky, HV] [Zhang, HV]

$$q = e^{-\frac{2p^2}{N}}$$



[Narovlansky, HV] [Zhang, HV]
[Collier, Eberhardt, Mühlman]



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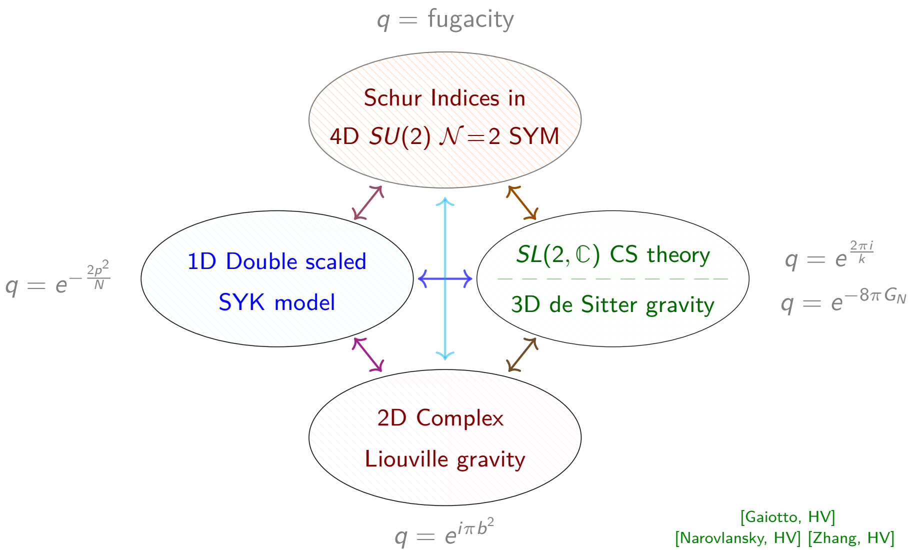

[Gaiotto, HV]
[Narovlansky, HV] [Zhang, HV]
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Double scaled SYK

$$H_{\rm SYK} \;=\; i^{p/2} \sum_{i_1 \ldots i_p} J_{i_1\ldots i_p} \psi_{i_1} \ldots \psi_{i_p} \qquad\qquad \{\psi^i, \psi^j\} = \delta^{ij}$$

random couplings — N majorana variables

In the $N \to \infty$ limit with $\lambda = \frac{p^2}{N}$ finite, the bi-local collective field theory takes the form of a Liouville CFT with complex central charge $c_\pm = 13 \pm i(\frac{6\lambda}{\pi} - \frac{6\pi}{\lambda})$

$$S_{\rm eff} = \frac{N}{8p^2} \int d\tau_1 d\tau_2 \left[\partial_{\tau_1} g \partial_{\tau_2} g - 4\mathcal{J}^2 \exp g(\tau_1, \tau_2)\right]$$

$$G(1,2) = \frac{1}{N} \psi_i(1)\psi_i(2)$$

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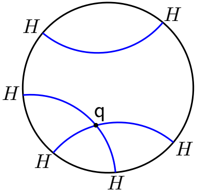
DSSYK is exactly soluble. Its interactions are governed by simple chord rules
. [Berkooz et al]



$$\mathbf{H} | n \rangle = | n+1 \rangle + [n]_{\mathfrak{q}} \, | n-1 \rangle$$

$$[n]_{\mathfrak{q}} = \frac{1-\mathfrak{q}^n}{1-\mathfrak{q}} \qquad \mathfrak{q} \equiv e^{-2\lambda} \equiv e^{-\frac{2p^2}{N}}$$

The Hamiltonian can be expressed in terms of $\mathfrak{q}$-deformed oscillators $\mathfrak{a}$, $\mathfrak{a}^\dagger$ as

$$\boxed{\mathbf{H} = \mathfrak{a}^\dagger + \mathfrak{a}}$$

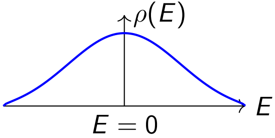
$$\begin{aligned} \mathfrak{a}^\dagger|n\rangle &= |n+1\rangle, \\ \mathfrak{a}|n\rangle &= [n]_{\mathfrak{q}}|n-1\rangle, \\ [\mathfrak{a}, \mathfrak{a}^\dagger]_q &= 1. \end{aligned}$$

$\Longrightarrow$ The energy spectrum and partition function of DSSYK are given by

$$\mathbf{H}|\theta\rangle = \frac{\cos\theta}{\sqrt{\lambda(1-\mathfrak{q})}}|\theta\rangle \qquad \rho(E) = e^{S_0}\,\vartheta_1(2\theta, q)$$

$$\mathcal{Z}_{\rm SYK}(q,\beta) = \mathrm{Tr}[e^{-\beta H}] = e^{S_0}\int_0^\pi \frac{d\theta}{\pi}(q,\, e^{\pm 2i\theta}; q)_\infty\, e^{-\beta E(\theta)}$$

The energy spectrum is bounded and
has a state with maximum entropy.
DSSYK operators span a type II_1 algebra.



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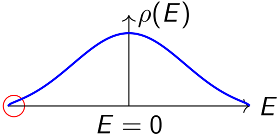
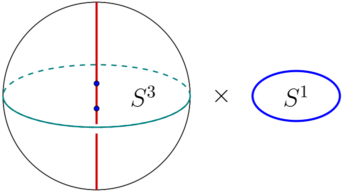


Schur correlators in 4D $\mathcal{N} = 2$ supersymmetric gauge theory

Schur correlation functions in superconformal 4D $\mathcal{N}=2$ gauge theory are defined as the twisted trace over the Hilbert space of states on S^3, or equivalently, as the twisted partition function on $S^3 \times S^1$ decorated by line operators wrapping the S^1:

$$\mathcal{I}(\mathfrak{q}, \beta) = \mathrm{Tr}(-1)^{2R} \mathfrak{q}^{j_3+R} e^{-\beta W_1}$$



Schur correlators are topological: they only depend on operator ordering and do not depend on gauge couplings. Hence they can be computed at zero coupling.[1]

[1] In this talk, we are interested in Schur correlators in pure $SU(2)$ SW theory. SW-theory is not superconformal, so defining the correlators takes some care.

SYK-Schur duality

By introducing a boundary, the Schur index can be generalized to a half-index. Schur-SYK duality is the statement that

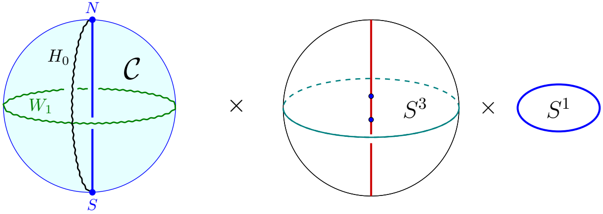
$$\mathcal{Z}_{\rm SYK}(\mathfrak{q},\beta) = \mathcal{I}^{1/2}_{\rm SW}(\mathfrak{q},\beta)$$

[Gaiotto, HV]

What explains this correspondence? What can we learn from it?

Can we generalize it?

SYK-Schur duality is a cousin of AGT duality. Both follow from the class $\mathcal{S}$ description of $SU(2)$ SW theory in terms of 6D (2,0) theory compactified on the sphere $\mathcal{C}$ with two irregular singularities.



The class $\mathcal{S}$ description of SW theory maps Schur correlators to correlators of Wilson line operators of $SL(2,\mathbb{C})$ Chern-Simons theory defined on a 3D manifold B_3 with boundary given by the curve $\mathcal{C}$.

Schur quantization

Gaiotto, Teschner

Pure 2+1 de Sitter gravity can be reformulated as an $SL(2,\mathbb{C})$ CS theory via

$$S_E = \frac{i\kappa}{2\pi}\int \mathrm{tr}(\mathcal{A}d\mathcal{A} + \tfrac{2}{3}\mathcal{A}^2) - \frac{i\kappa}{2\pi}\int \mathrm{tr}(\bar{\mathcal{A}}d\bar{\mathcal{A}} + \tfrac{2}{3}\bar{\mathcal{A}}^3)$$

$$\mathcal{A} = \omega + ie, \qquad \bar{\mathcal{A}} = \omega - ie, \qquad \frac{2\pi}{\kappa} = 8\pi G_N$$

The Hilbert space of $SL(2,\mathbb{C})$ CS theory is spanned by the conformal blocks of a pair of Virasoro-Liouville CFTs with complex central charge

$$S = \frac{i\kappa}{2\pi}\int (\frac{1}{2}\partial\varphi_+\overline{\partial}\varphi_+ + 2e^{\varphi_+}) - \frac{i\kappa}{2\pi}\int (\frac{1}{2}\partial\varphi_-\overline{\partial}\varphi_- + 2e^{\varphi_-})$$

$$c_\pm = 13 \pm 6i(\kappa - \frac{1}{\kappa})$$

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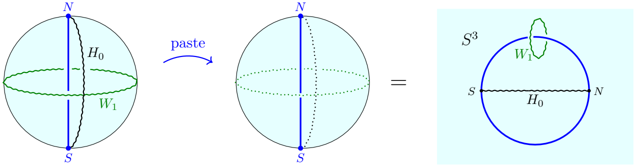
$$c_\pm = 13 \pm 6i(\kappa - \frac{1}{\kappa})$$

The central charges add up to 26 $\implies$ Adding a bc-ghost system produces the worldsheet theory of a soluble string theory, the $\mathbb{C}$omplex Liouville String. Amplitudes of the $\mathbb{C}$LS are invariant under the mapping class group. They evaluate the inner product between states of 3D de Sitter gravity.

$\mathbb{C}$LS amplitudes of a given topology Σ_g compute cosmological correlators in 3D de Sitter gravity in a cosmological spacetime with Cauchy slice Σ_g.

Collier, Eberhardt, Mühlman

Applying this idea to the SW curve $\mathcal{C}$, we are led to identify $\mathcal{C}$ with the spatial section of 3D de Sitter with a localized matter source ("observer") at the podes.



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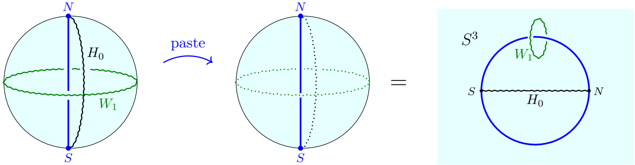
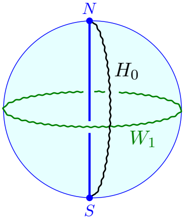


The path-integral over the ball B_3 with boundary $\mathcal{C}$ produces an in-state $|\Psi_{in}\rangle$. Gluing the two spheres amounts to taking an innerproduct $\langle\Psi_{out}|\Psi_{in}\rangle$.

The holonomy operator W_1 measures the deficit angle at the podes. The open line operator H_0 measures the time difference between the podes. In complex Virasoro-Liouville CFT, they correspond to closed and open Verlinde lines.



Classical phase space of $SL(2,\mathbb{C})$ CS theory on $\mathcal{C}$

$$F(\mathcal{A}) = d\mathcal{A} + \mathcal{A}\wedge\mathcal{A} = 0$$

$$(\partial - \mathcal{A})\Psi = 0$$

We parametrize the holonomy M around the equator and the values of the constant section Ψ at the north and south podes via

$$M \;=\; \mathrm{P}\exp\oint_A \mathcal{A} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}$$

$$\theta = \tfrac{\pi}{2}(1-\alpha)$$

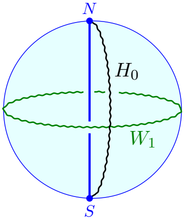
α = deficit angle

$$s \;=\; \Psi_{|\mathrm{S}} = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, \qquad n = \Psi_{|\mathrm{N}} = \begin{pmatrix} s_1\, e^{u} \\ s_2 e^{-u}\end{pmatrix}$$

The gauge invariant holonomies are then given by[2]

$$W_1 = \mathrm{Tr}M = 2\cos\theta \qquad H_k = n\wedge M^k s = \frac{\sinh(u+ik\theta)}{\cos\theta}$$

[2]Here we choose to normalize the section Ψ such that $s\wedge M s = n\wedge M n = 1$.

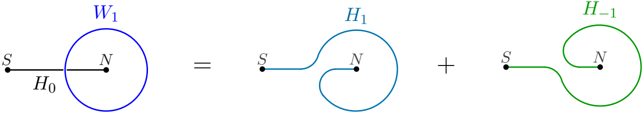


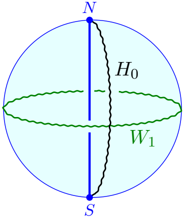
Classical Skein and Ptolemy relations

$$H_0 W_1 = H_{-1} + H_1 \qquad \text{(Skein)}$$

$$H_1 H_{-1} = 1 + H_0^2 \qquad \text{(Ptolemy)}$$

$SL(2,\mathbb{C})$ Wilson line operator in CS theory satisfy the classical skein rule

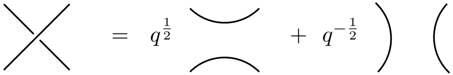




Quantum Skein algebra

$$H_0 W_1 = q^{1/2} H_{-1} + q^{-1/2} H_1$$
$$H_1 H_{-1} = 1 + q\, H_0^2$$
$$H_0\, H_{\pm 1} = q^{\pm 1/2}\, H_{\pm 1} H_0$$

$SL(2,\mathbb{C})$ Wilson line operator in CS theory satisfy the local skein rule



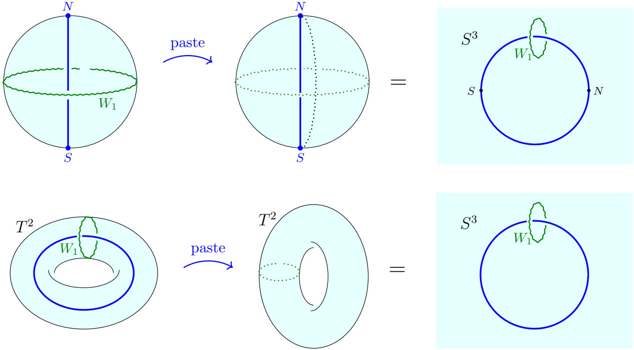
SYK-Schur duality is based on the isomorphism between the Skein algebra of line operators on $\mathcal{C}$ and the q-deformed oscillator algebra via the identifications

$$\mathbf{H} = W_1, \qquad \mathfrak{q}^{-\mathfrak{n}} = H_0, \qquad \mathfrak{a} = \mathfrak{q}^{\mathfrak{n}} H_1, \qquad \mathfrak{a}^\dagger = H_{-1}\mathfrak{q}^{\mathfrak{n}}$$

The hermiticity properties of the q-oscillators match with the $*$-structure and Hilbert space representation of the Skein algebra that follows from Schur quantization of SW theory.

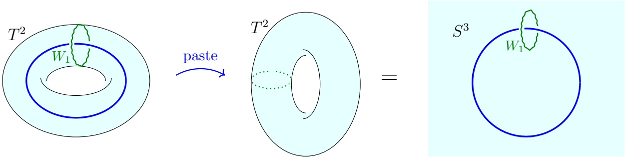
$\Rightarrow$ Note that the vacuum condition $\mathfrak{a}|\mathrm{vac}\rangle = 0$ implies that classically $n = Ms$. $\Leftarrow$

Let us compute the DSSYK partition function from the 3D gravity side! This can be done via the following relatively standard surgery argument:



In general, we should allow for torus bundles with a non-trivial framing. These can included by gluing the tori together via a non-trivial Dehn twist.

We associate wavefunctions $|\Psi_A\rangle$ and $|\Psi_B\rangle$ to the two tori



For $|\Psi_A\rangle$ we take a weighted superposition of W_1 eigenstates

$$|\Psi_A(\beta)\rangle = \int_\gamma dp\, e^{-\beta\mu_b(p)}|\Psi_A(p)\rangle \qquad \mu_b(p) = 2\cos(2\pi b\, p)$$

representing the state for which the blue circle has total geodesic length β.

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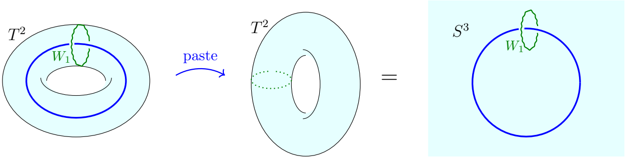

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The innerproduct can be evaluated in terms of $\mathbb{C}$LCFT modular S and T matrix:

$$\mathcal{Z}_n(\beta) = \langle\Psi_B(\beta)|\,\hat{T}^n|\Psi_A(\mathbb{1})\rangle = \int_\gamma dp\, e^{-\beta\mu_b(p)}\,(T_p)^n\, S_{\mathbb{1}p}$$

$$S_{\mathbb{1}p} = \sin(\pi b\, p)\sin(\pi b^{-1} p) \qquad T_p = e^{2\pi i p^2}$$

The special cases $n = 0, 2$ reproduce the $\mathbb{C}$LS and SYK partition functions

$$\mathcal{Z}_0(0) = c_0 \int_0^{\pi} d\theta \, \sin\theta \, \sinh(\frac{2\pi\theta}{\lambda}) = \mathcal{Z}_{\mathbb{C}\mathrm{LS}}(0)$$

$$\mathcal{Z}_2(0) = c_2 \int_0^{\infty} d\theta \, e^{-\frac{2}{\lambda}\theta^2} \sin\theta \, \sinh(\frac{2\pi\theta}{\lambda}) = \mathcal{Z}_{\mathrm{SYK}}(0)$$

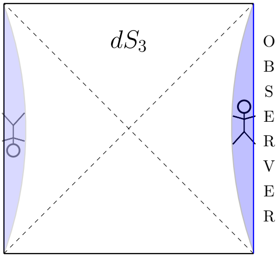
Introducing $\psi = \pi - 2\theta$ and taking the semi-classical limit, we can write

$$\mathcal{Z}_{\mathbb{C}\mathrm{LS}}(0) = \int d\psi \; e^{S_{\mathbb{C}\mathrm{LS}}(\psi)}, \qquad S_{\mathbb{C}\mathrm{LS}} = S_0 - \frac{2\pi\psi}{\lambda}$$

$$\mathcal{Z}_{\mathrm{SYK}}(0) = \int d\psi \; e^{S_{\mathrm{SYK}}(\psi)}, \qquad S_{\mathrm{SYK}} = S_0 - \frac{2\psi^2}{\lambda}$$

Can we interpret these formulas from the gravity side?

The Schwarschild-de Sitter spacetime of an observer with energy E



$$ds^2 = (1-\rho^2)d\tau^2 + \frac{d\rho^2}{1-\rho^2} + \rho^2 d\varphi^2$$

$$\varphi \simeq \varphi + 2\pi - \psi \quad ; \quad \tau \simeq \tau + 2\pi$$

$$2\pi - \psi = 2\pi\sqrt{1-8G_N E}$$

The GH entropy $S_{\rm GH}$ and observer energy E depend on the deficit angle via

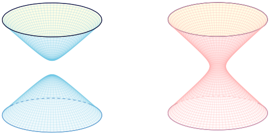
$$S_{\rm GH} = \frac{2\pi - \psi}{4G_N} \qquad ; \qquad \beta_{\rm dS} E = \frac{1}{16\pi G_N}(4\pi\psi - \psi^2)$$

Comparing with the formulas on the previous slide, we see that, after equating $\lambda = 8\pi G_N$ and modulo the overall constant shift by S_0, the entropies are related via

$$S_{\mathbb{C}\rm LS}(\psi) = S_{\rm GH}(\psi) = S_{\rm SYK}(\psi) - \beta_{\rm dS} E(\psi)$$

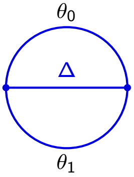
Things I did not have time to talk about:

- q-Schwarzian, Schur TFT
- $\mathbb{C}$LS = Schur $\times$ WP
- dS isometries, $SL(2)_q$
- dS two-point function
- Cosmological correlators
- Possible generalisations
- OTOCs, gravitational interactions
- dS interpretation of the fake disk
- Inflation and dark energy in DSSYK
- Topological minimal string realization of DSSYK



DSSYK 2pt-function = Zamolodchikov's formula = Schur index of trinion SCFT

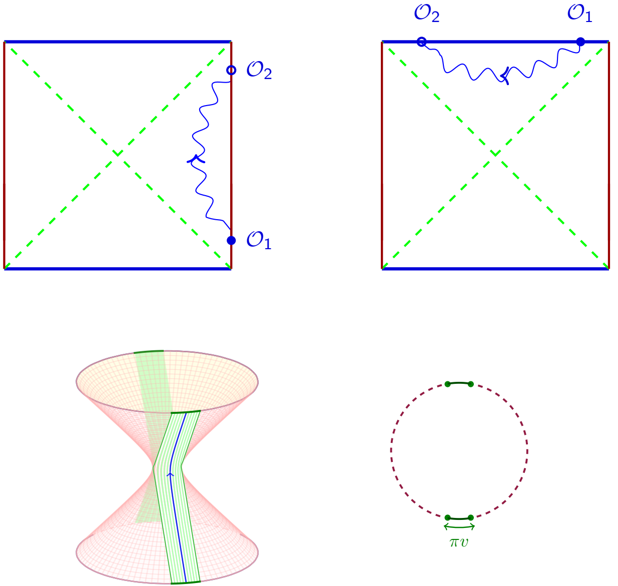
$$\mathcal{A}_3(\Delta, \theta_1, \theta_2) = \frac{\vartheta_1(b^2\Delta, q) \prod_i \vartheta_1(2\theta_1, q)}{\vartheta_1(b^2\Delta \pm \theta_1 \pm \theta_2, q)} =$$



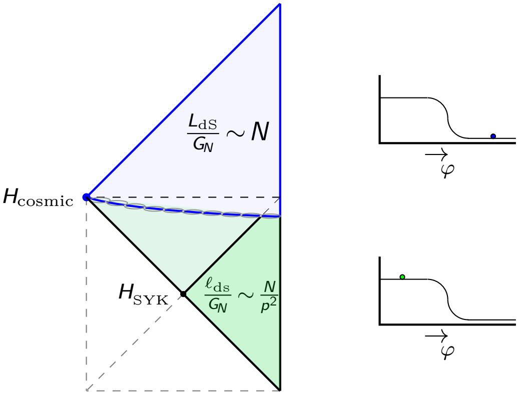
More general *n*-point functions take the form

$$\mathcal{A}_{1,..,n} = \sum_{\Gamma} \frac{1}{|\mathrm{Aut}(\Gamma)|} \prod_e \int dp_e p_e \prod_v \mathcal{A}^{(b)}_{\mathrm{TFT}}(p_v)\, \mathcal{V}^{(b)}_{g_v, n_v}(p_v)$$

Collier et al



$\mathcal{O}_2$
$\mathcal{O}_1$
$\mathcal{O}_2$
$\mathcal{O}_1$
πv



$\frac{L_{\rm dS}}{G_N} \sim N$
$H_{\rm cosmic}$
$H_{\rm SYK}$
$\frac{\ell_{\rm ds}}{G_N} \sim \frac{N}{p^2}$
$\overrightarrow{\varphi}$
$\overrightarrow{\varphi}$