Modular Invariance, Completeness and selection rules in 2d CFTs

Javier M. Magán
Instituto Balseiro

Based on

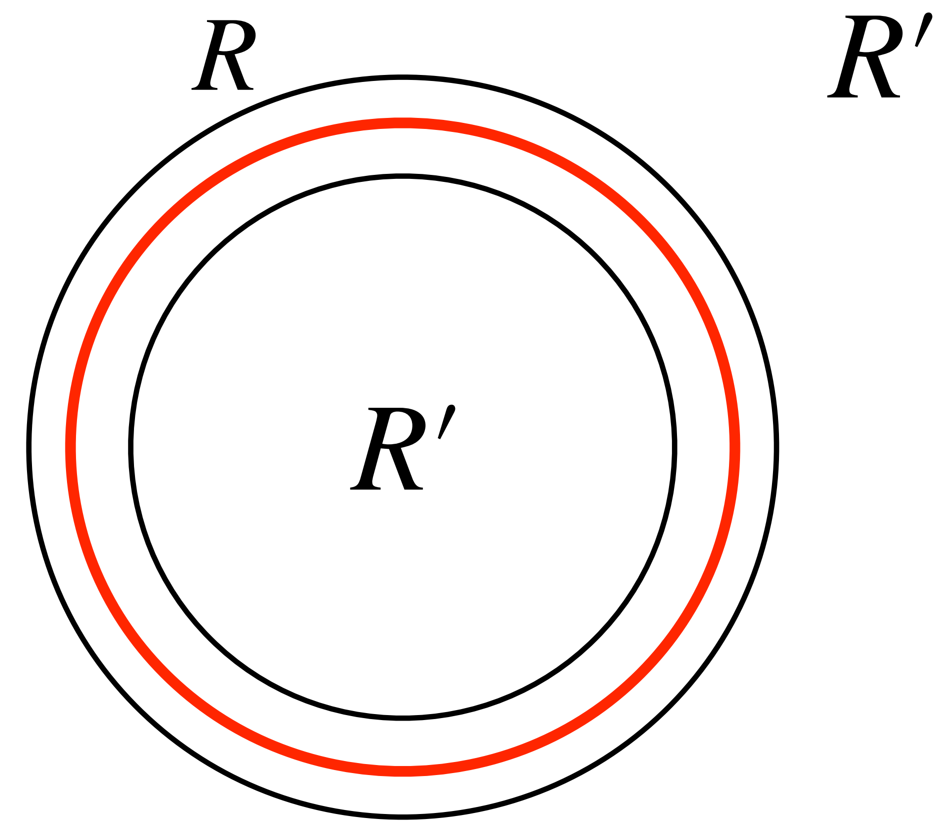
“Modular invariance as completeness”, V. Benedetti, H. Casini, Y. Kawahigashi, R. Longo and J.M.M

”Selection rules for RG flows of minimal models”, V. Benedetti, H. Casini, and J.M.M

Brief review and motivating question

Intrinsic notion of non-local operators

It is "localized" in R: it commutes with local operators in R'



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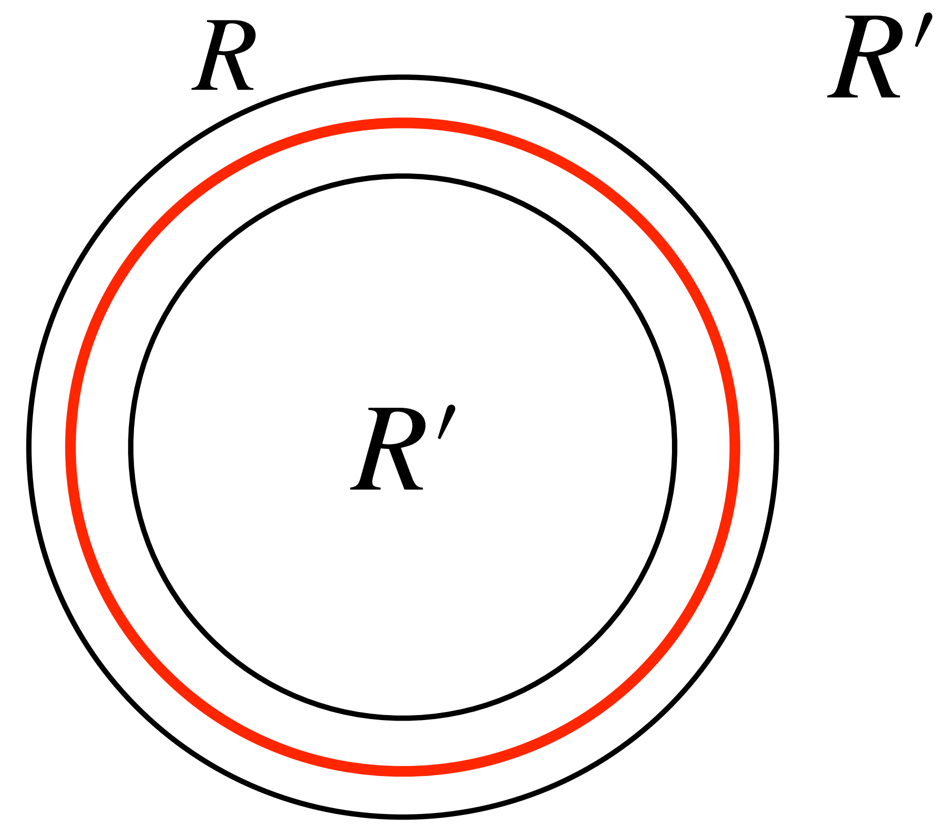

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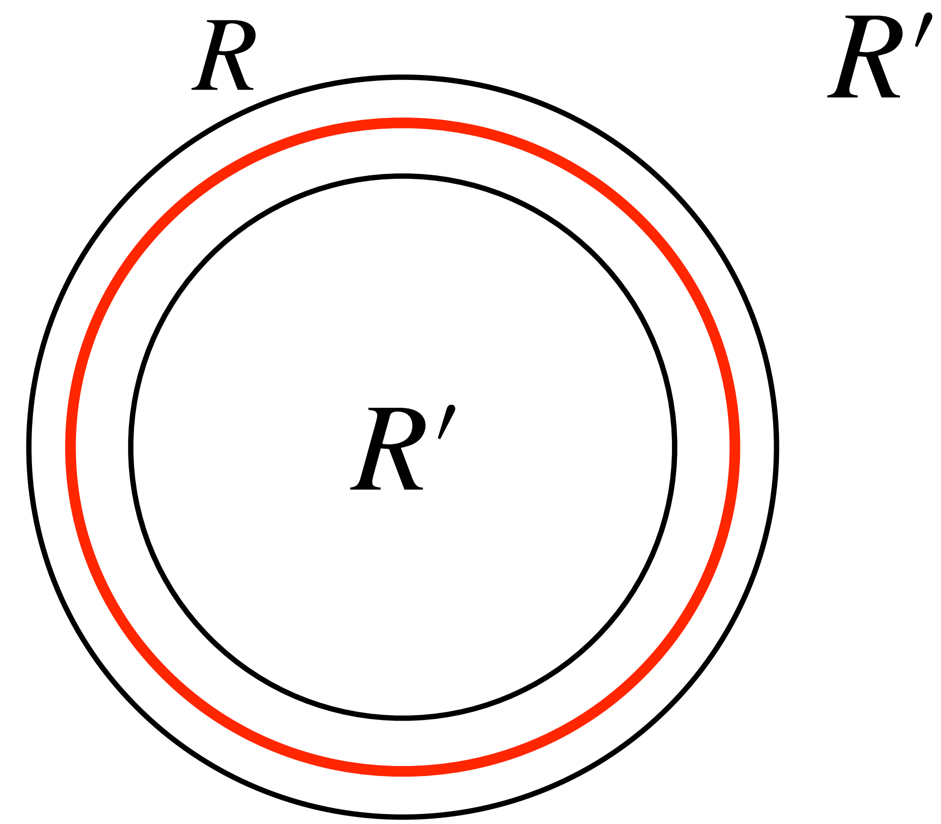

Existence of two algebras for the same region R

Additive algebra: All operators localized in R that can be locally generated in R $\mathscr{A}_R$

Maximal algebra Additive algebra plus all non local operators $\mathscr{A}_R^{max} \equiv \mathscr{A}'_{R'}$

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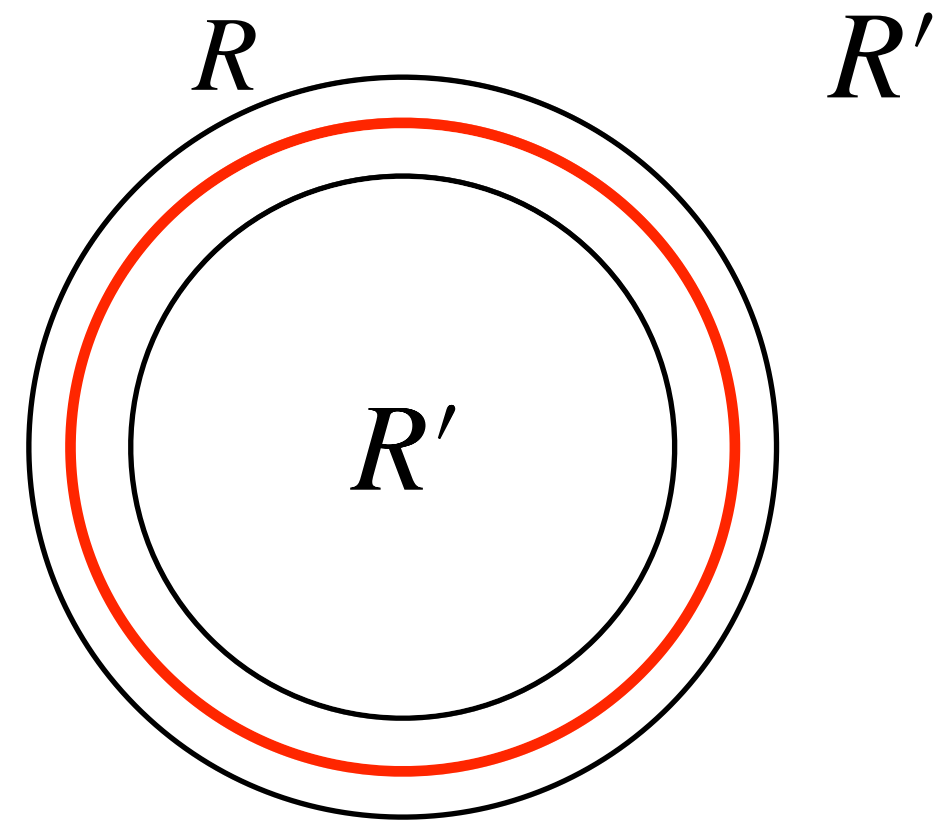

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This suggests an intrinsic definition of completeness in QFT

[Casini, Huerta, J.M.M, Pontello 2020]

[Casini, J.M.M 2021]

Maximal algebra equals additive algebra for all regions

There are no non-local operators in the QFT

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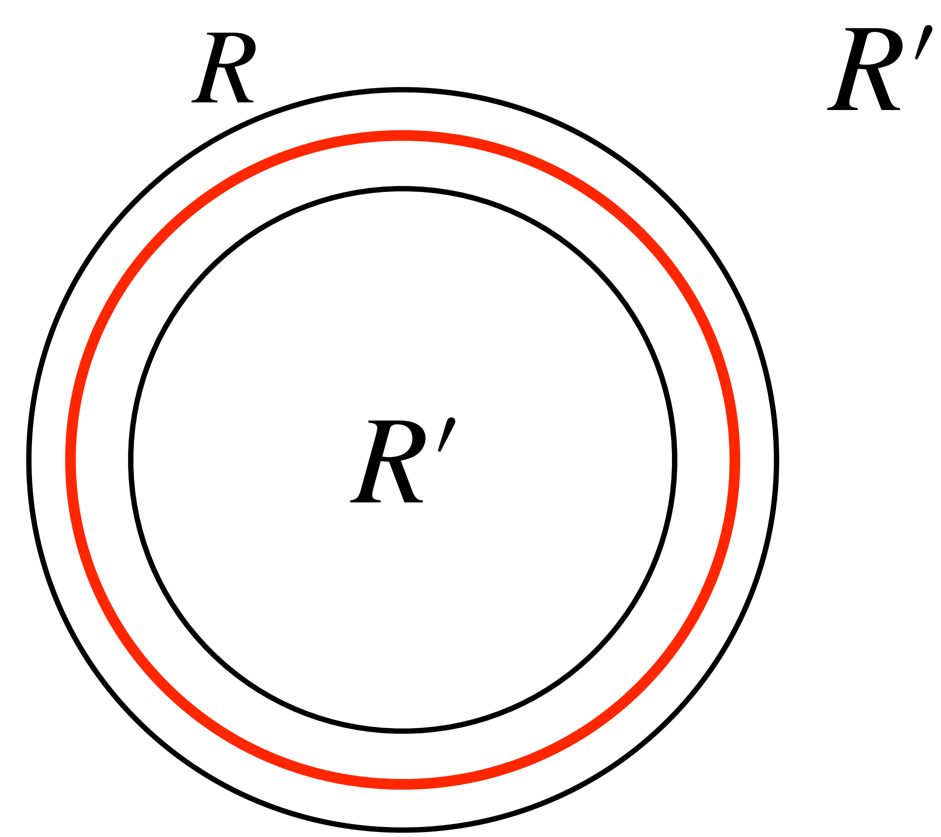

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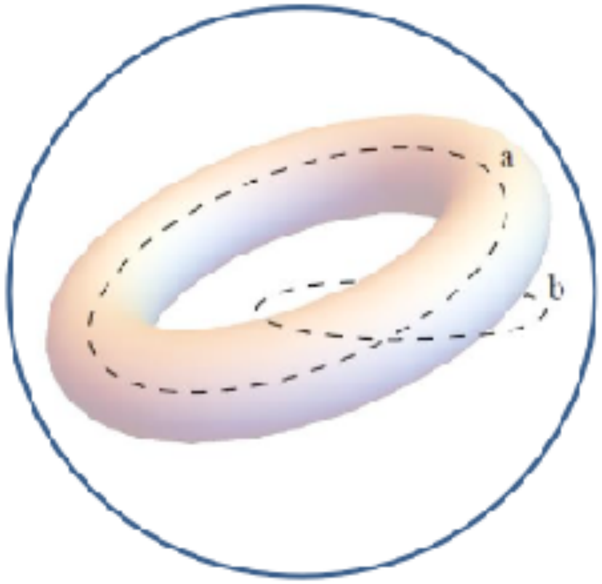
Maximal algebra equals additive algebra for all regions

There are no non-local operators in the QFT. Haag duality holds in any region

Coincides with completeness of electric/magnetic spectrum in gauge theories

Brief review and motivating question

But the notion of non-local operator is a relative one!!



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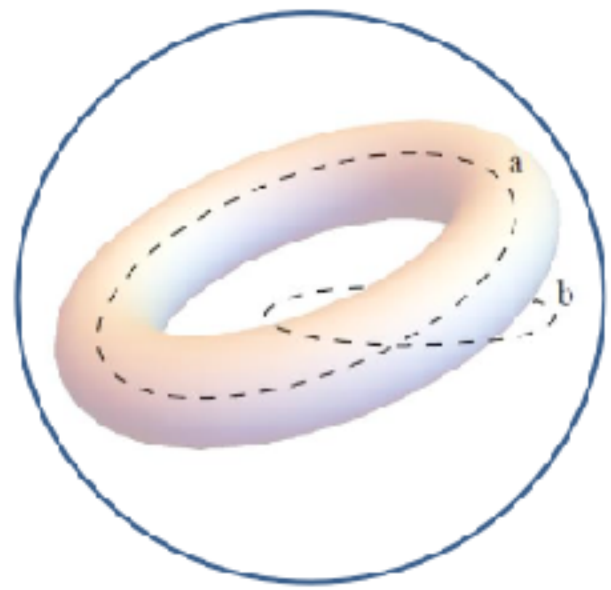

All non-local operators in Relativistic QFT can be locally generated in a ball that contains them: Fundamental Haag Duality (valid in CFT in balls through Bisognano-Wichmann)

Wilson or 't Hooft loops in both abelian and non-abelian theories can be locally generated in a ball containing them

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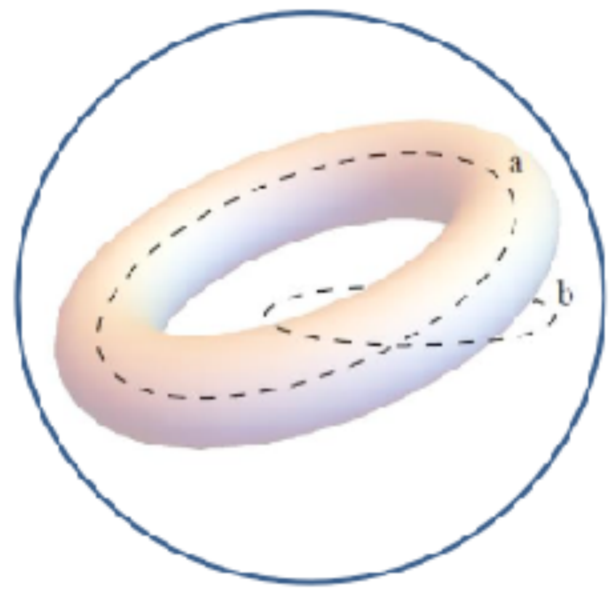

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The description of a CFT in terms of bootstrap data (spectrum and OPE) does not need any addition to describe non-local operators (or Haag duality).

Guiding basic question: what is the imprint of non-local operators in the bootstrap data?

Plan of the talk or takeaway messages

- Modular invariance as completeness
- Finer classification of minimal models
- Selection rules for RG flows

Modular invariance as completeness

Modular invariance is understood as a property of Euclidean CFT on the torus

T-invariance: $Z(\tau) = Z(\tau + 1)$

S-invariance: $Z(\tau) = Z(1/\tau)$

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S-invariance $\longleftrightarrow$ Completeness

Modular invariance as completeness

Idea: Entropies agree for commutant algebras

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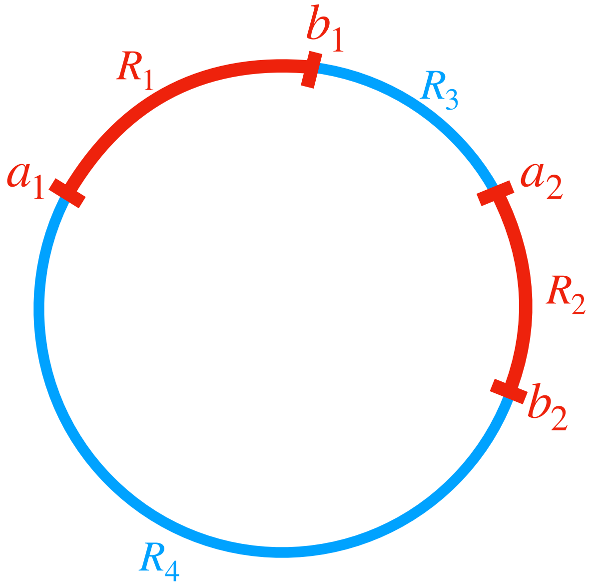

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Consider a two interval region in the circle

$$x \equiv \frac{(b_1 - a_1)(b_2 - a_2)}{(a_2 - a_1)(b_2 - b_1)} \in (0,1)$$

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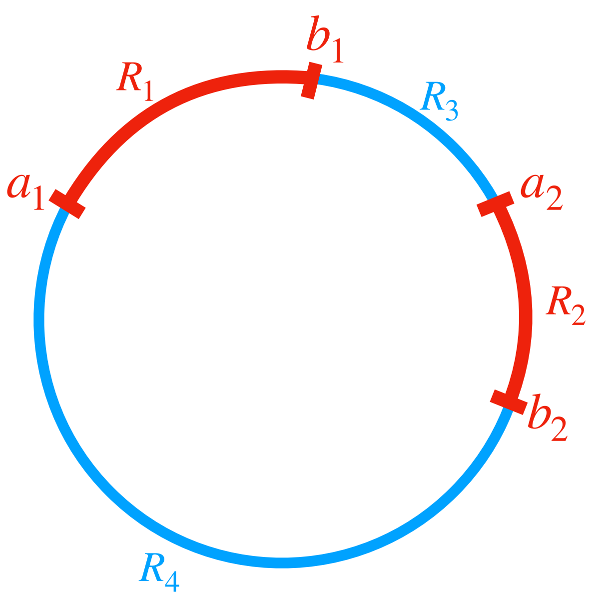

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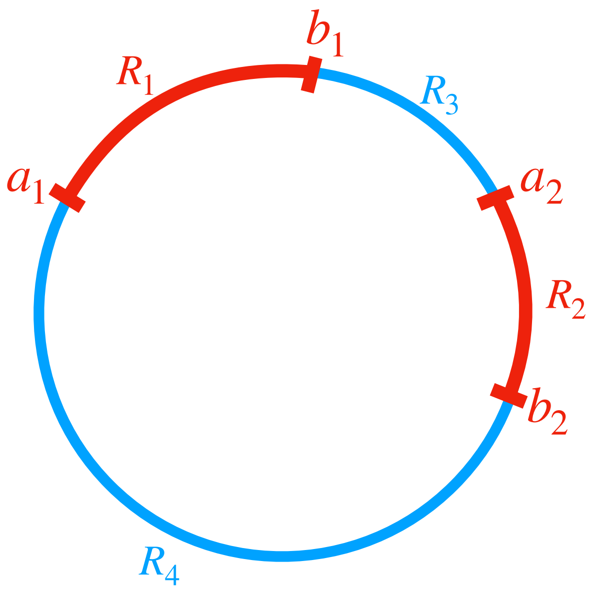

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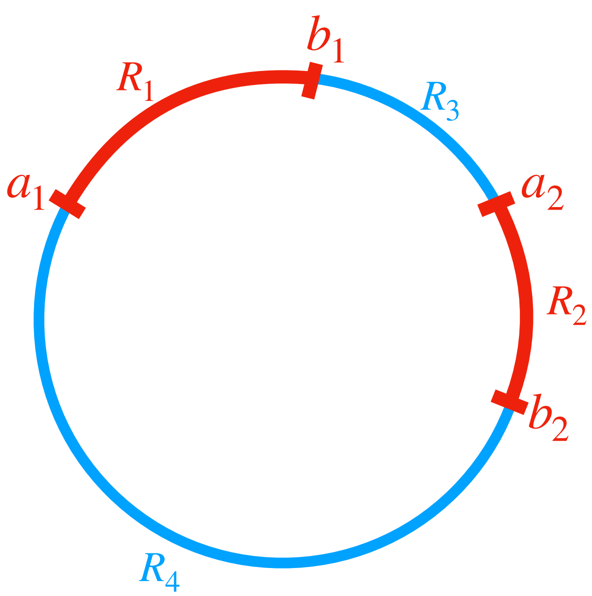

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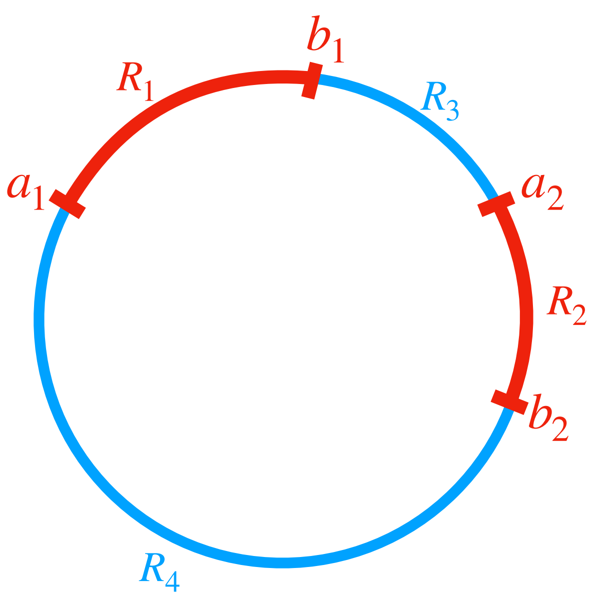

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The two interval Renyi mutual information is defined as

$$I_n \equiv S_n(R_1) + S_n(R_1) - S_n(R_1 \cup R_2) \equiv -\frac{(n+1)c}{6n} \log(1-x) + U_n(x)$$

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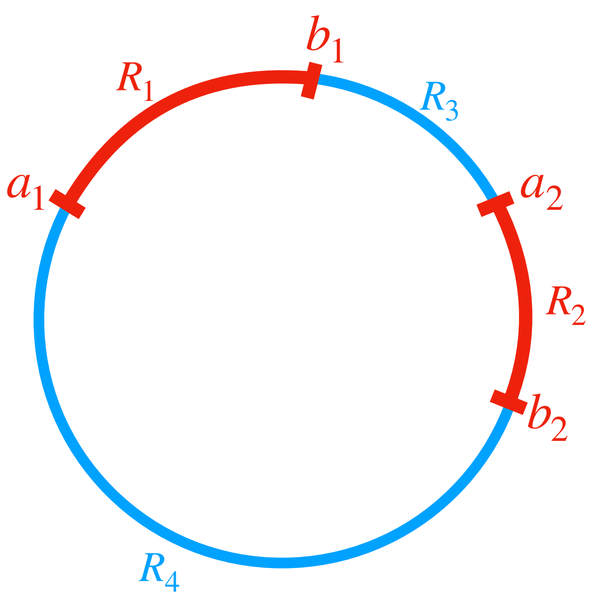

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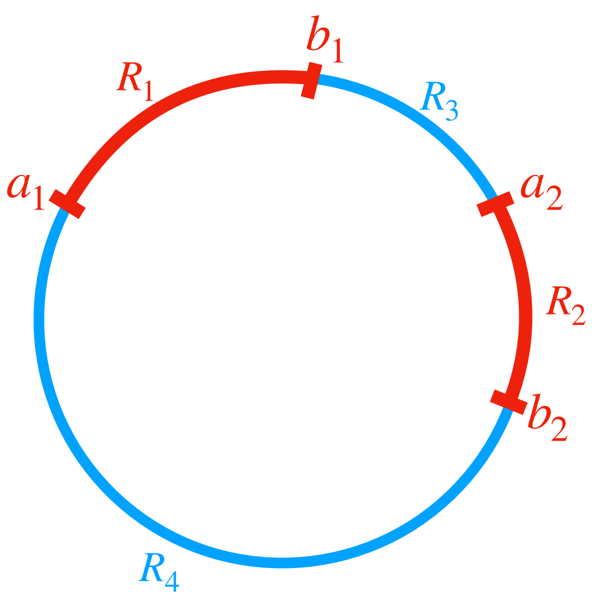

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$U_n(x) = U_n(1-x) \longleftrightarrow$ Completeness

Modular invariance as completeness

For $n = 2$ the manifold has genus one. It can be conformally mapped to a torus of radius 1 of height l

[Headrick, 2010]

$$I_2(x) = \log\, Z(il) - \frac{c}{12}\log\left(\frac{2^8(1-x)}{x^2}\right) \qquad\qquad x = \left(\frac{\theta_2(il)}{\theta_3(il)}\right)^4$$

Then we have $x \longleftrightarrow 1-x$ is equivalent to $l \longleftrightarrow 1/l$ and most important

$$U_2(x) - U_2(1-x) = \log\, Z(il) - \log\, Z(i/l)$$

$\updownarrow$ $\qquad\qquad$ $\updownarrow$

Violation of completeness $\longleftrightarrow$ Violation of S-invariance

Modular invariance as completeness

The violation of S-invariance is simple in the limit $x \to 1$. Consider a generic model

$$Z = \sum M_{rs} \chi_r(\tau) \bar{\chi}_s(\bar{\tau})$$

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This leads to the main workhorse formula

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This number has an intrinsic meaning. It determines the Jones index μ that measures the relative size between the maximal and additive algebras

$$\mathscr{A}^{max}_{R_1 \cup R_2} \supset \mathscr{A}_{R_1 \cup R_2} \quad \longrightarrow \quad \text{Relative Size} = \text{Jones index} = \mu \quad \longrightarrow \quad \mu = \left(\frac{\sum_i d_i^2}{\sum_{ij} M_{ij} d_i d_j} \right)^2$$

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Related to the category of superselection sectors

Finer classification of minimal models

The classification of modular invariant minimal models
(2d CFTs with $c < 1$) is known as the ADE classification

[Capelli, Itzykson, Zuber, 1987]

S-invariance ⟶ Completeness ⟶ Not mandatory!!

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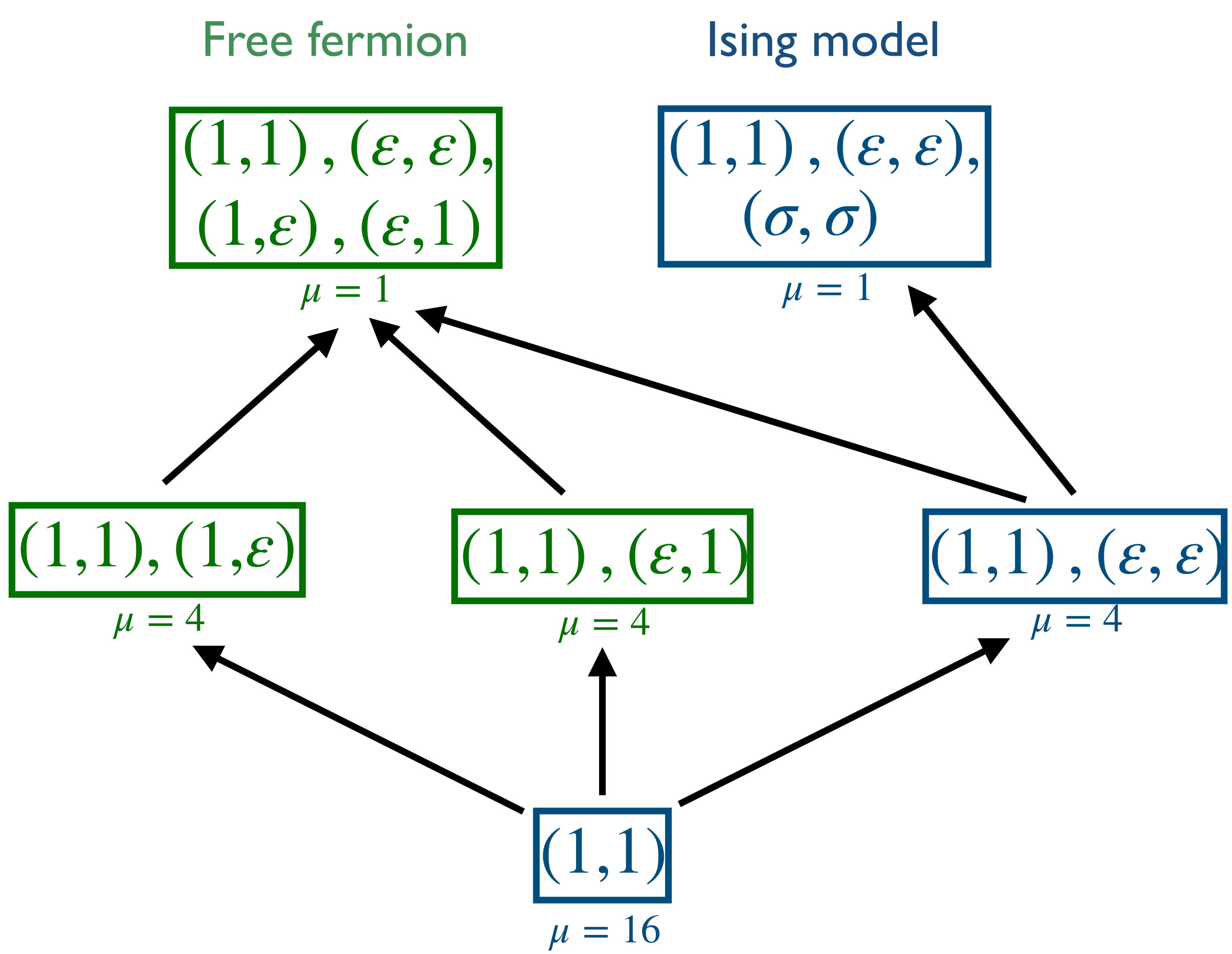
S-invariance $\longrightarrow$ Completeness $\longrightarrow$ Not mandatory!!

This suggests a finer classification. The rules of the game are:

- Inclusion of the stress tensor net
- Locality (T-invariance is necessary)
- Closure of the operator algebra

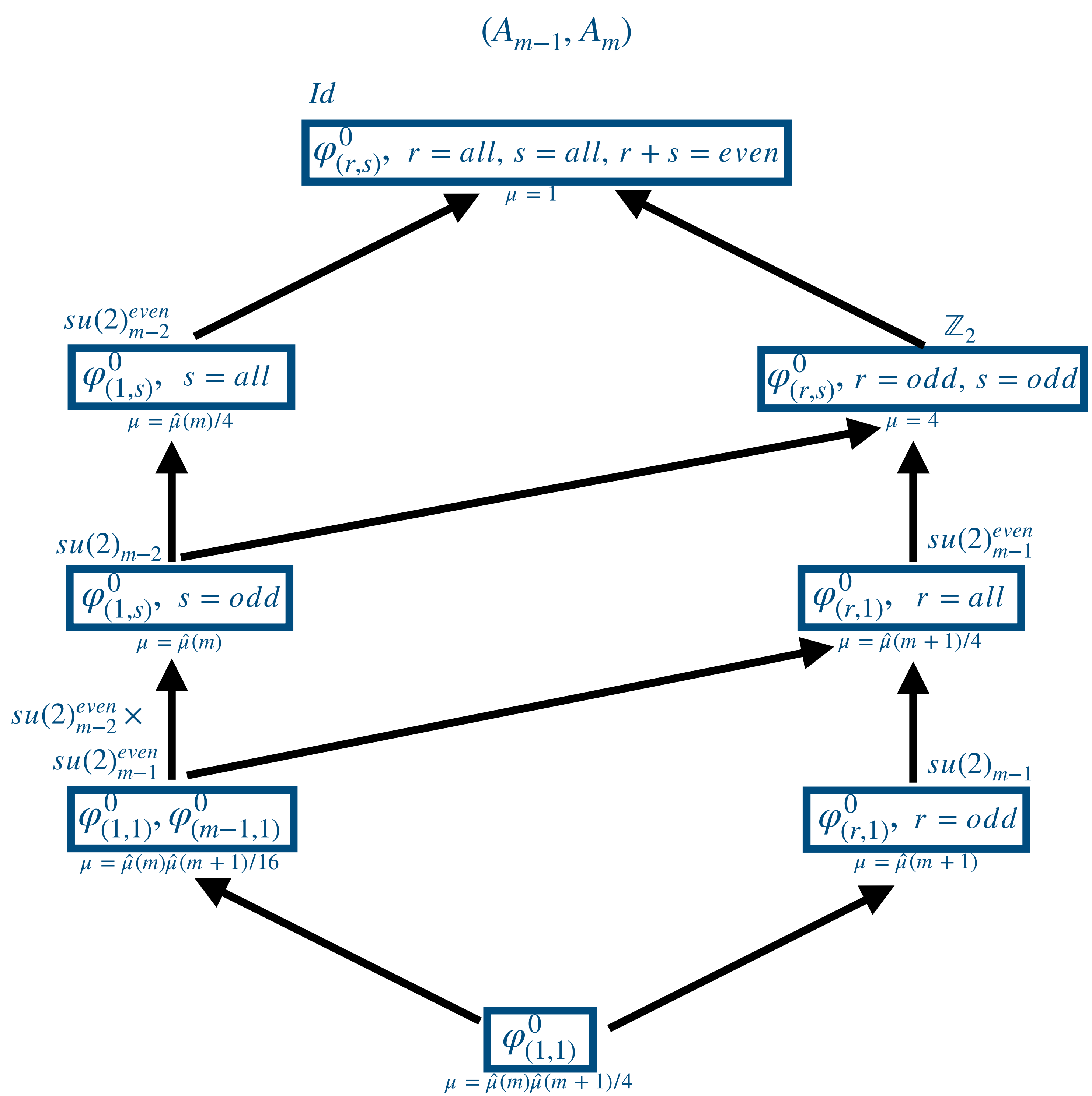
Finer classification of minimal models

The complete zoo of local models for $c = 1/2$ is then



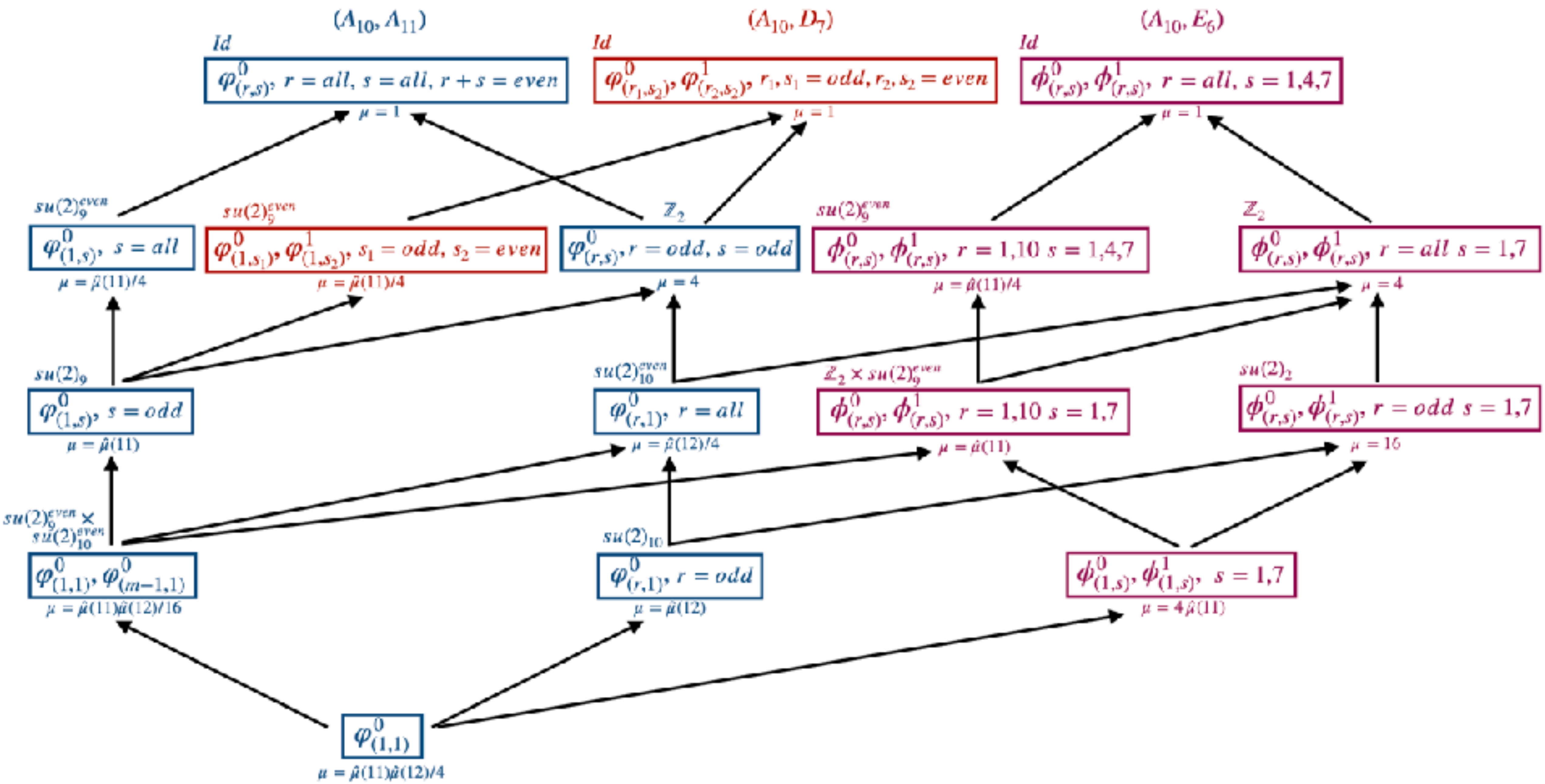
Finer classification of minimal models

The submodels of the (A,A) series for general m. Defining $\hat{\mu}(m) \equiv \frac{m^2}{4} \sin^{-4}\left(\frac{\pi}{m}\right)$, for odd m we have



Finer classification of minimal models

The case $m = 11$



Finer classification of minimal models

This classification is consistent (and indeed re-derives) with the classification of possible superselection sector categories found by

[Kawahigashi, Longo, 2005]

- We classified field theories while they classify allowed symmetries
- They used algebraic (endormorphisms) techniques while we used standard OPEs
- Both classifications are almost the same: symmetries almost determine minimal models

Selection rules for RG flows

Starting observation: If flow is triggered by φ, then we can think the theory where the flow is happening as the one with the stress tensor T plus φ. This defines a local model:

Neutral Theory: $\mathcal{N} \equiv T \vee \varphi$

The neutral theory is typically not modular invariant

The whole algebraic structure above the node acts as a witness

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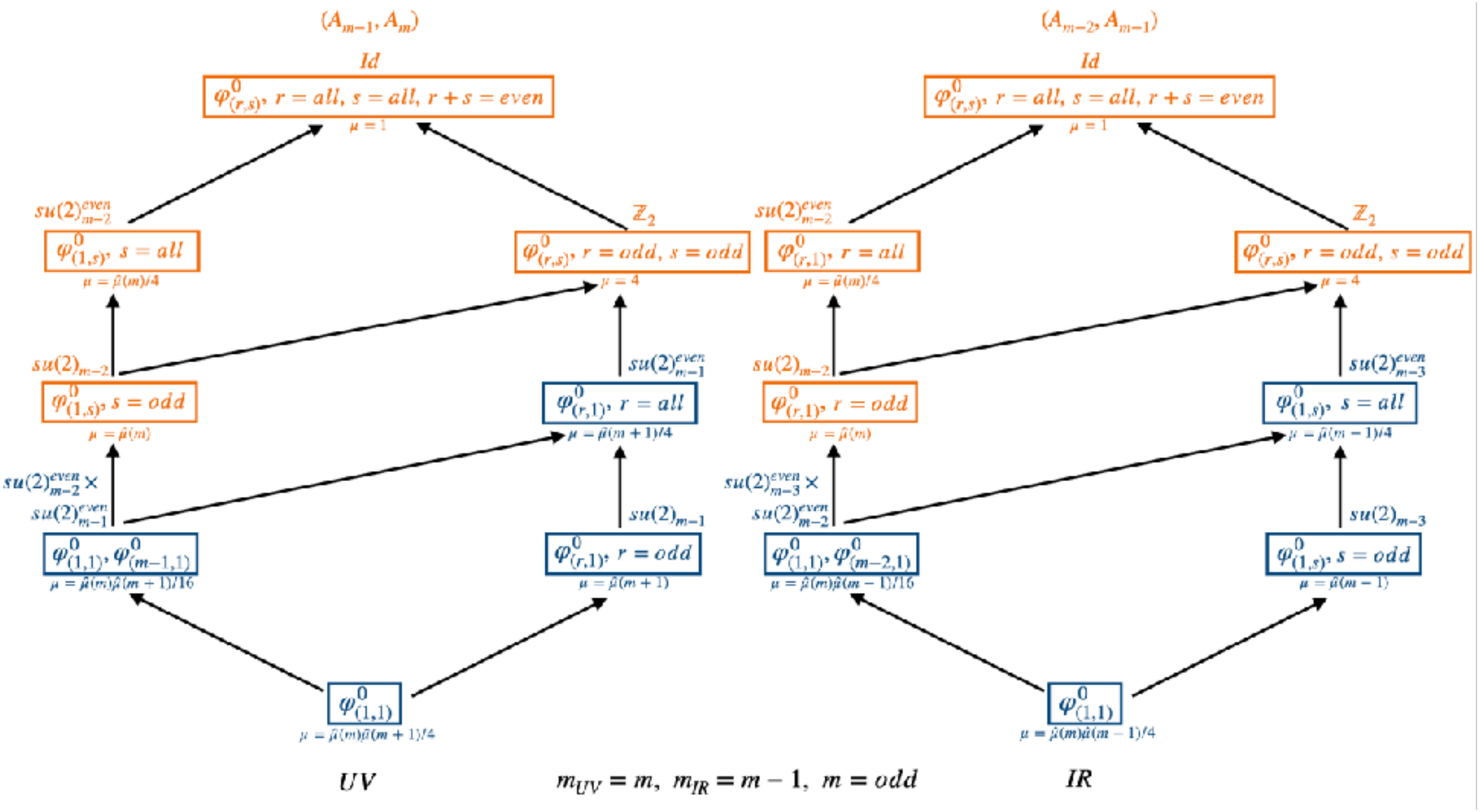
The whole algebraic structure above the node acts as a witness

More precisely, if the flow is not massive

- Global and relative Jones indices above the perturbed node are preserved, e.g. $\mu_{IR} = \mu_{UV}$
- The structure of possible completions above the perturbed node is preserved
- The category of superselection sectors of the perturbed node is preserved: $DHR_{UV} = DHR_{IR}$

Selection rules for RG flows

Example: (A,A) Zamolodchikov flow for m even



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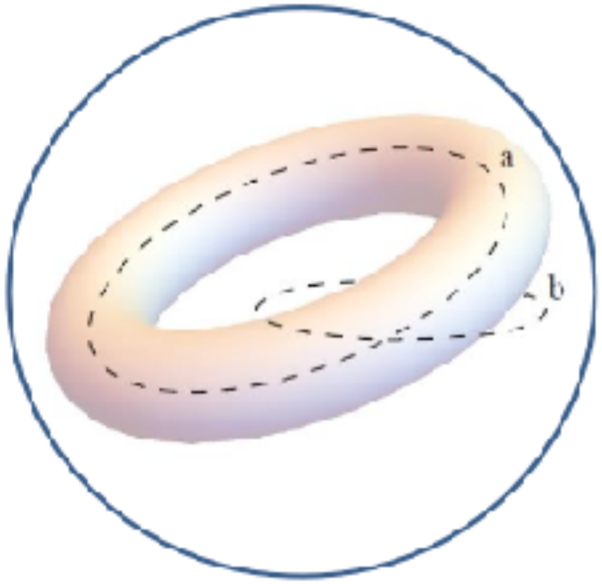
What is the imprint of non-local operators in the bootstrap data?



Symmetries and superselection sector categories from bootstrap data.

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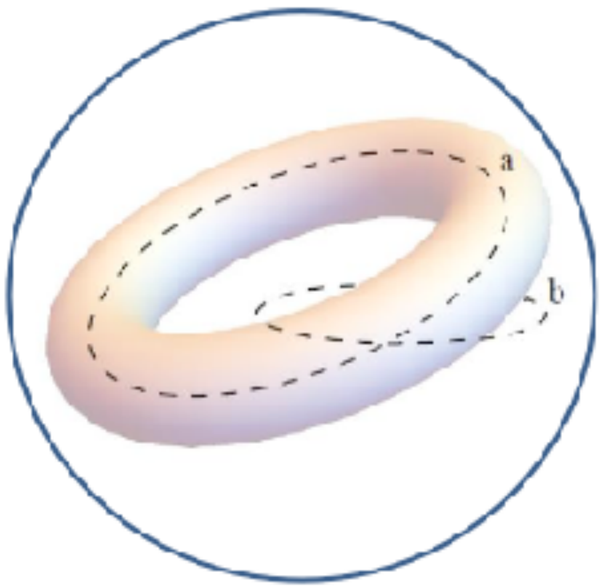

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Future: confinement and bootstrap

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