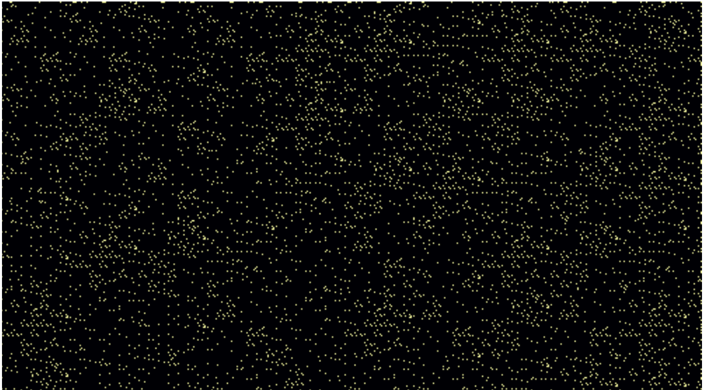
The Landau Paradigm: generalized symmetries in condensed matter

John McGreevy (UCSD)

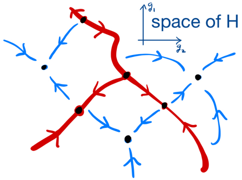
based partly on 2204.03045



Unity of purpose between hep-th and cond-mat

A big goal of condensed matter physics is to understand possible **phases of matter**.

A **phase of matter** is the basin of attraction of a fixed point of the renormalization group (RG).
(A *stable* phase has a fixed point with no relevant operators.)

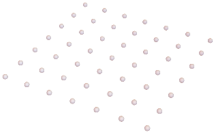


This definition a priori has no relation to symmetry.

Landau Paradigm.

(basis of most condensed matter understanding!)

Phases of matter are classified by how they represent their symmetries.



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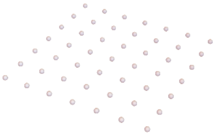

Consequence(?): gapless excitations or degeneracy (in a phase) are Goldstone modes for spontaneously broken symmetries.

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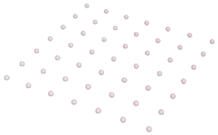

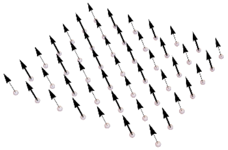

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This is a conjecture!

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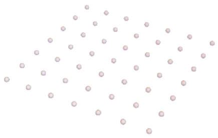

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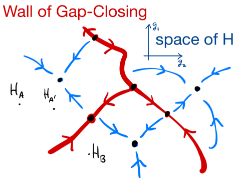
There are some apparent exceptions!

- topological insulators and integer quantum Hall states.
- topological order. [Wegner, Wen]
- other deconfined states of gauge theory (*e.g.* Coulomb phase of E&M).
- fracton phases.
- superconductors.
- (Landau) Fermi liquid.

Brief non-symmetry accounting of gapped phases.

A **gapped** phase is an equivalence class of gapped Hamiltonians, $H_A \simeq H_{A'}$ if their groundstates are related by adiabatic evolution and/or inclusion of product states.

Crucial Q: How to label phases?



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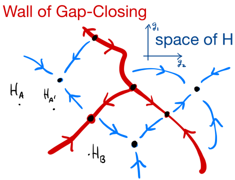
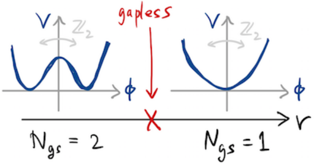


A quantity is *topological* if it doesn't change under continuous deformations.

They can break a discrete symmetry. Landau.

(Possible response: even SSB phases are distinguished by topology.)



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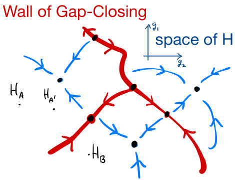

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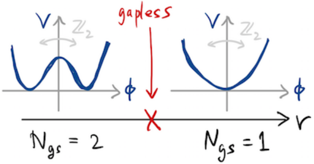

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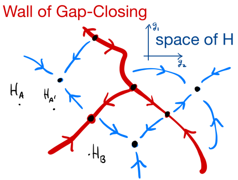

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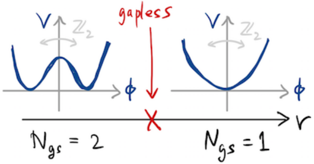

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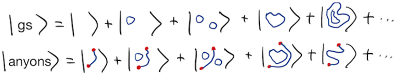
Topological phases can be divided into two classes: those with *topological order* and those without.

Brief non-symmetry accounting of gapped phases.



Topological order [Wen]: localized excitations that can't be created by any local operator (anyons).

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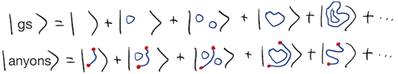

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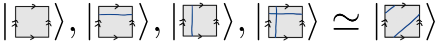
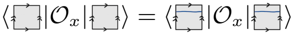
Other important symptoms:

- Topology-dependent groundstate degeneracy

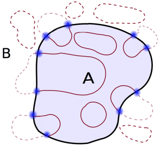


These groundstates are *locally indistinguishable*:



$\forall$ local ops $\mathcal{O}_x$.

- Long-range entanglement

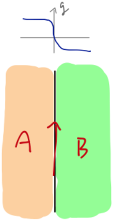


[Fig: Tarun Grover]

Brief non-symmetry accounting of gapped phases.

Even without TO, there can still be phases distinct from the trivial phase. One way in which they can be distinguished is by what happens if we put them on a space with boundary.

very rough idea:



e.g.: integer quantum Hall (IQH) states, topological insulators, symmetry-protected topological states (SPTs) such as Haldane phase of spin-1 chain, polyacetylene

Generalized Landau paradigm.

The idea is that by suitably refining and generalizing our notions of symmetry, we can incorporate all of these “beyond-Landau” examples into a *Generalized Landau Paradigm*.

[Wen Gaiotto Seiberg Kapustin Willett Hofman Iqbal JM Cordova Schafer-Nameki ...]

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Two important steps:

- Generalized symmetries
- ('t Hooft) Anomalies

New Ingredient 1:

('t Hooft) Anomalies

't Hooft anomalies.

$$H = \frac{p^2}{2m} \to H_A = \frac{(p+A)^2}{2m}$$

Given a system with a symmetry, there is a procedure for coupling to background fields.

This process involves some arbitrary choices.

$$A \to A + d\alpha$$

An anomaly is when the result
(*e.g.* the partition function $Z \equiv \mathrm{tr} e^{-\beta H}$)
depends on these choices.

$$Z \to e^{\mathbf{i} \int \alpha \mathcal{A}} Z$$

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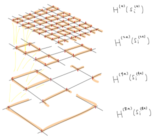

$Z \to e^{\mathbf{i} \int \alpha \mathcal{A}} Z \implies$ The anomaly is an RG invariant!

Any anomaly in the UV description must be realized somehow in the correct IR description.

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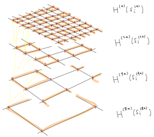

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Useful perspective: an anomaly is an obstruction to gauging the symmetry.

Anomaly inflow and SPTs.

SPT (Symmetry-Protected Topological phase) $\equiv$ nontrivial phase of matter (with some symmetry G) without SSB or topological order.

Can be characterized by its edge states (interface with vacuum).
The idea is that the edge theory has to represent an anomaly for G; this anomaly that labels the bulk phase.

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e.g.: topological insulators, integer quantum Hall (IQH), polyacetylene, Haldane phase of spin-1 chain.

An effective field theory for IQH, regarded as an SPT for charge conservation symmetry:
Solve $d \star j = 0$ by $j = da$.
$S_{\text{IQH}}[a, A] = \frac{1}{4\pi} \int_M (ada + A \star j)$
Under $A \to A + d\lambda$, $\delta S_{\text{IQH}} = \frac{1}{4\pi} \int_{\partial M} f\lambda$.
This is the contribution to the chiral anomaly from a single right-moving edge mode.

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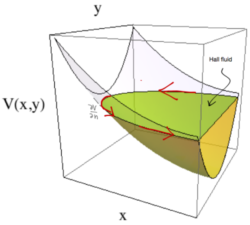

The variation of the bulk action cancels the anomaly of the edge theory.

[Callan-Harvey 85]

Anomaly as a label on SPTs.

[Chen-Gu-Wen, Vishwanath-Senthil, Kitaev, Kapustin, ... review: 1405.4015]

The edge theory cannot be trivial: it has to be either

- gapless
- symmetry-broken
- or topologically ordered.

(Such a statement is called an LSMOH theorem
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The Point (for our purposes): We are still using the realization of symmetries to label these phases!

See Maissam's talk for a better symmetry understanding of SPTs.

New Ingredient 2:

Generalized Symmetries

What's a symmetry of a quantum many-body system?

Noether's theorem relates symmetries to **topological defect operators** $U_g(\Sigma)$.

Conservation $\implies$ topological.

Group law $\implies$ Fusion rule: $U_g(\Sigma)U_{g'}(\Sigma) = U_{gg'}(\Sigma)$ (up to phases).

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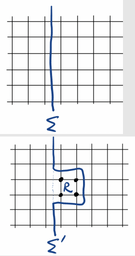
Example of topological defect operator:

Ising model, Euclidean, any D: $S[\sigma] = \sum_{\langle xy \rangle} J_{xy}\sigma_x\sigma_y$

$U_{-1}(\Sigma)$ is an instruction to flip the sign of J for any bond crossing Σ.

If $\Sigma' - \Sigma = \partial R$, $U_{-1}(\Sigma)$ and $U_{-1}(\Sigma')$ are related by redefining $\sigma_x \to -\sigma_x$ for $x \in R$.

$\implies$ σ_x is charged.



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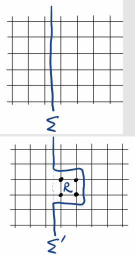


Useful reverse perspective:

Topological defect operators are a sufficient condition for symmetry.

- Continuous and discrete symmetries on equal footing.
- Noether symmetries and topological symmetries on equal footing.
- Allows generalizations!

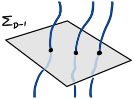
Higher-form symmetries

[Gaiotto-Kapustin-Seiberg-Willett, Sharpe, Hofman-Iqbal, Lake...]

($D \equiv d + 1 =$ number of spacetime dimensions.)

0-form symmetry:

$\partial^\mu J_\mu = 0$ (*i.e.* $d \star J = 0$)
$\implies Q = \int_{\Sigma_{D-1}} \star J$ is independent of time-slice Σ, *i.e.* is topological.



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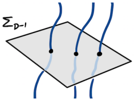

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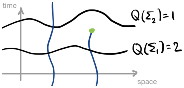
Charged particle worldlines can't end (except on charged operators).



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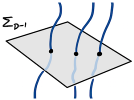

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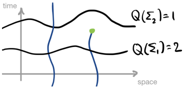

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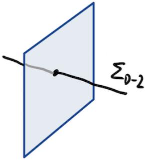
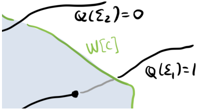
1-form symmetry:

$J_{\mu\nu} = -J_{\nu\mu}$ with $\partial^\mu J_{\mu\nu} = 0$ (*i.e.* $d \star J = 0$)
$\implies Q_\Sigma = \int_{\Sigma_{D-2}} \star J$ depends only on the topological class of Σ_{D-2}.



In particular $U_\Sigma(\alpha) \equiv e^{\mathbf{i}\alpha Q_\Sigma}$ commutes with H.

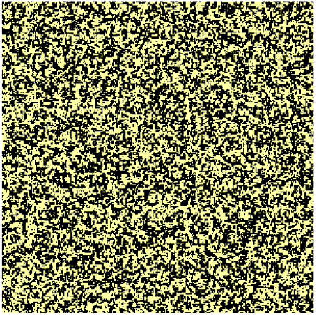
Charged string worldsheets can't end (except on charged operators).



Charged objects are loop operators, create strings:

$W[C] \to e^{\mathbf{i}\oint_C \Gamma} W[C], \quad d\Gamma = 0.$

A consequence of generalized 0-form symmetry



Coarsening: Start with a $T = \infty$ configuration of the Ising model.

Evolve by metropolis rule (Glauber dynamics) at temperature $T < T_c$.

Domains grow and try to minimize their surface area.

In a convex domain: 2 steady state configurations.

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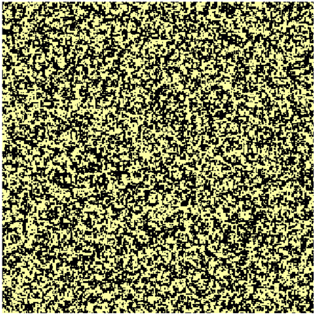
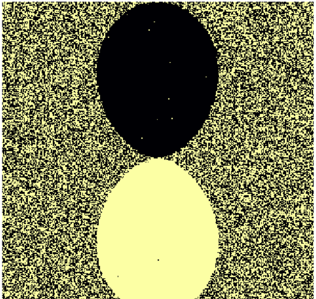


A literal bottleneck to coarsening: Only do the updates in a nearly-disconnected domain.

In this domain: 4 approximate steady state configurations.



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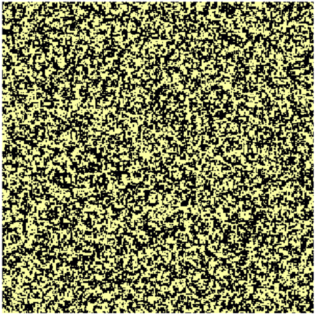
Domains grow and try to minimize their surface area.

In a convex domain: 2 steady state configurations.

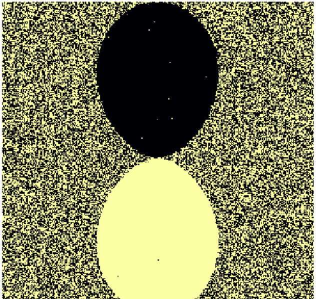


A literal bottleneck to coarsening: Only do the updates in a nearly-disconnected domain.

In this domain: 4 approximate steady state configurations.

The extra long-lived configs are explained by:

$d\alpha = 0 \implies \alpha$ is constant on each component of space.



Physics examples of exact one-form symmetries:

- Maxwell theory with only electric charges:
  $J^{\mu\nu}_{(m)} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} = (d\tilde{A})^{\mu\nu}$ is conserved: $\nabla_\mu J^{\mu\nu}_{(m)} = 0$ (no monopoles).
  The symmetry operator is $U^{(m)}_\alpha(\Sigma) = e^{\frac{\mathbf{i}\alpha}{2\pi}\int_\Sigma F}$. (Charged operator is the 't Hooft line, $W^E = e^{\mathbf{i}\oint_C \tilde{A}}$, $\tilde{A} \to \tilde{A} + \Gamma, d\Gamma = 0$.)
  Without electric charge: $J_{(e)} = F$ is also conserved.
  Symmetry op: $U^{(e)}_\alpha(\Sigma_2) = e^{\mathbf{i}\frac{2\alpha}{g^2}\int_{\Sigma_2}\star F}$.
  (The charged operator is the Wegner-Wilson loop $e^{\mathbf{i}\oint_C A}$,
  $A \to A + \Gamma, d\Gamma = 0$.)

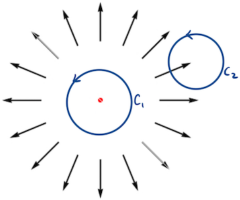
- Pure $\mathsf{SU}(N)$ gauge theory
  or $\mathbb{Z}_N$ gauge theory
  or $\mathsf{U}(1)$ gauge theory with charge-N matter
  has a $\mathbb{Z}_N$ 1-form symmetry ('center symmetry').
  (Charged line operator is the Wegner-Wilson line in the minimal irrep,
  $W[C] = \mathrm{tr} P e^{\mathbf{i}\oint_C A}$.)

Physics examples of higher-form symmetries:

Many condensed matter systems have **emergent** higher-form symmetries.

- Superfluids (and other ordered phases): When we spontaneously break a 0-form $\mathsf{U}(1)$ symmetry in $d = 2$, there is an emergent 1-form $\mathsf{U}(1)$ symmetry whose charge counts the winding number of the Goldstone phase φ around an arbitrary closed loop C, $Q[C] = \oint_C \star J = \oint_C d\varphi/(2\pi)$.
(In d spatial dimensions, this is a $(d-1)$-form $\mathsf{U}(1)$ symmetry.)
The charged operator creates a vortex (in $d = 2$, or a vortex line or sheet in $d > 2$).

Not an exact symmetry: broken by vortices.



- Spin liquids
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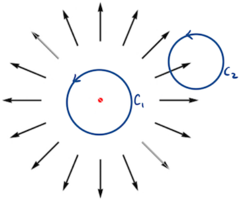

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This idea has been very fruitful: any question we can ask for ordinary symmetries, we can also ask for generalized symmetries...

Higher-form symmetries can be broken spontaneously.

[Kovner-Rosenstein, Nussinov-Ortiz, Gaiotto-Kapustin-Seiberg-Willett, Hofman-Iqbal, Lake]

0-form symmetry:

Unbroken phase: correlations of charged operators are short-ranged, decay when the charged object (S^0 = two points) grows.

$\langle \mathcal{O}(x)^\dagger \mathcal{O}(0) \rangle \sim e^{-m|x|}$

($|x| = \text{Area}(S^0(x))$.)

Broken phase for 0-form sym:

$\langle \mathcal{O}(x)^\dagger \mathcal{O}(0) \rangle = \langle \mathcal{O}^\dagger \rangle \langle \mathcal{O} \rangle + \ldots$

independent of size of S^0.

Particle condensation.

$|\text{gs}\rangle \sim |0 \text{ particles}\rangle + |1 \text{ particle}\rangle + |2 \text{ particles}\rangle + \cdots$

1-form symmetry:

Unbroken phase: correlations of charged operators are short-ranged, decay when the charged object grows.

$\langle W(C) \rangle \sim e^{-T_{p+1}\text{Area}(C)+\ldots}$

For E&M, area law for $\langle W^E(C) \rangle$ is the superconducting phase.

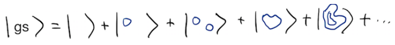
Broken phase for 1-form sym:

$\langle W(C) \rangle = e^{-T_p \text{Perimeter}(C)+\ldots}$

(set to 1 by counterterms local to C: large loop has a vev)

(or Coulomb law)

String condensation.



Spontaneous symmetry breaking.

SSB $\Leftrightarrow$ LRO.

$|\psi\rangle$ is not stationary under the symmetry (SSB)

if and only if

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Proof: $\boxed{\Leftarrow}$ Suppose the state is stationary under the symmetry

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Then for any charged operator $O = e^{\mathbf{i}\gamma}S^\dagger OS$, $\gamma \notin 2\pi\mathbb{Z}$,

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which says $\langle\psi|O|\psi\rangle = 0$.
$\boxed{\Rightarrow}$ [T Grover] Reduced density matrix of a region X:
$\rho_X = \mathrm{tr}_{\bar{X}}|\psi\rangle\langle\psi| = \sum_I \langle O_I\rangle O_I$, $\{O_I\}$ ON basis of ops on X: $\mathrm{tr} O_I O_J = \delta_{IJ}$.
If no charged operator has an expectation value, then the sum only contains neutral operators.

But then $S\rho_X S^\dagger = \rho_X$, the state is invariant.

Landau was even more right than we thought.

[Kovner-Rosenstein, Gaiotto et al, Hofman-Iqbal, Lake]

- The gaplessness of the photon can be understood as required by spontaneously broken $\mathsf{U}(1)$ 1-form symmetry.

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If we couple to a bg field $\Delta L = j_\mu \mathcal{A}^\mu$,

$$\mathcal{L}_{\text{eff}} = \frac{1}{4\pi g}\Bigg(\underbrace{d\varphi}_{\text{Goldstone}} + \mathcal{A}\Bigg)^2 .$$

The goldstone transforms nonlinearly $\varphi \to \varphi + \lambda, \mathcal{A} \to \mathcal{A} - d\lambda$. This is a global symmetry if $d\lambda = 0$.

(By $(\text{form})^2$ I mean $(\text{form}) \wedge \star(\text{form})$.)

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The goldstone transforms nonlinearly $a \to a + \lambda, \mathcal{B} \to \mathcal{B} - d\lambda$. This is a global symmetry if $d\lambda = 0$.

Maxwell term for a. g^{-2} = stiffness.

$$\langle 0 | j_{\mu\nu}(x) | \zeta, p\rangle = (\zeta_\mu p_\nu - \zeta_\nu p_\mu) f e^{\mathbf{i}px}$$

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[Nussinov-Ortiz 07, Gaiotto et al 14, Wen 18]

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$\equiv$ degenerate groundstates that are *locally indistinguishable*.

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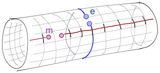
$$U^m(M)V^n(C) = e^{2\pi \mathbf{i} \frac{mn}{p} \#(C,M)} V^n(C)U^m(M). \quad (\#(C,M) \equiv \text{intersection } \#)$$

$U^m(M)$ = symmetry operator, $V^n(C)$ =charged object, also topological.

This is the algebra of electric and magnetic flux surfaces in $\mathbb{Z}_p$ gauge theory.



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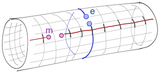
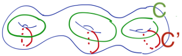


- eg 2 (Laughlin FQHE): in $D = 2+1$, $\mathbb{Z}_k^{(1)}$ 1-form symmetry with an 't Hooft anomaly

$$U^m(C)U^n(C') = e^{\frac{2\pi \mathbf{i} mn \#(C,C')}{k}} U^n(C')U^m(C).$$



(The flux carries charge.) Gives k groundstates on T^2.

Counterexample

[Chamon-Castelnuovo 07, Huxford-Nguyen-Kim 23]

$$H_\beta = + \sum_{\text{vertices } i} Q_i - \sum_{\text{plaquettes } p} B_p$$

$B_p \equiv \prod_{\ell \in \partial p} Z_\ell$ is the usual plaquette term,

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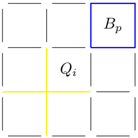

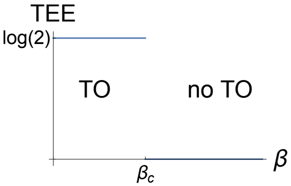

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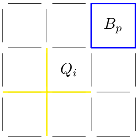
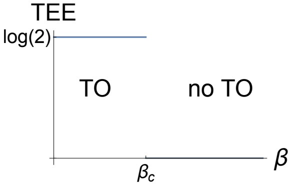
- But for $\beta > \beta_c$, $|\text{gs}(\beta)\rangle$ for various windings are distinguishable by local operators!





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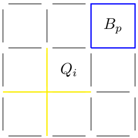

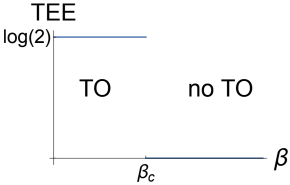

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A (mixed) state with the same properties arises upon subjecting the toric code to strong enough decoherence.

[Dennis-Landahl-Kitaev-Preskill 01, Bao et al, Chen-Grover]

"pre-modular" TO [Wang-Wu-Wang, Ellison-Cheng, Sohal-Prem]

Anomalies of higher-form symmetries.

Example: [Gaiotto et al, Hsin-Lam-Seiberg, ...] Abelian anyons in $D = 2 + 1$.

Gauging a symmetry involves summing over background fields

$\ni$ arbitrary insertions of symmetry operators.

For a 1-form symmetry in $D = 2 + 1$, this means summing over anyon worldlines = *anyon condensation.* [Bais-Slingerland Kong Burnell]

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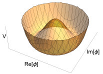
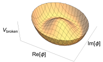
Mutual statistics between two anyon types is a mixed anomaly.

The generalized symmetry that emerges in a groundstate with TO is *always* anomalous, by braiding nondegeneracy (a theorem of Entanglement Bootstrap): every anyon braids nontrivially with another.

Robustness of higher-form symmetries.

Consequences of emergent (aka accidental) symmetries are approximate: Explicitly breaking a 0-form symmetry gives a mass to the Goldstone boson.





Q: The existence of magnetic monopoles with $m = M_{\text{monopole}}$ explicitly breaks the 1-form symmetry of electrodynamics:

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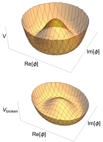

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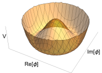

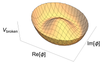

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Cheap explanation #2: By dimensional reduction.

$m_\gamma \overset{[\text{Polyakov}]}{\sim} e^{-M_{\text{monopole}} R} \overset{R \to \infty}{\to} 0.$

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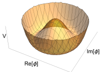

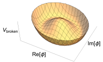

Q: The existence of magnetic monopoles with $m = M_{\text{monopole}}$ explicitly breaks the 1-form symmetry of electrodynamics:

$$\partial^\mu J^E_{\mu\nu} = j^{\text{monopole}}_\nu \ .$$

If the photon is a Goldstone for this symmetry, does this mean the photon gets a mass? No!

Cheap explanation #1: By dimensional analysis (take $m_e \to \infty$). $m_\gamma \to 0$ when $M_{\text{monopole}} \to \infty$.

Cheap explanation #2: By dimensional reduction. $m_\gamma \overset{[\text{Polyakov}]}{\sim} e^{-M_{\text{monopole}} R} \overset{R\to\infty}{\to} 0$.

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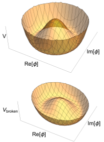
Robustness of higher-form symmetries.



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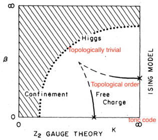
General conclusion: Higher form symmetries are rarely exact, but when the emerge they are more robust than 0-form symmetries.

Robustness of higher-form symmetries.

- Discrete analog: In the toric code, the discrete 1-form symmetries are exact, but in the rest of the deconfined (broken) phase, they are *emergent* (and still spontaneously broken). A rigorous proof of this [Hastings-Wen 04] constructs the string operators by quasi-adiabatic continuation.



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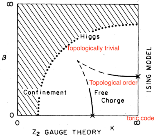
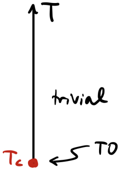


- Known forms of topological order in $D \leq 3+1$ have the property that at any $T > 0$ they are smoothly connected to $T = \infty$ (a trivial product state).

For the goal of using TO to store quantum information: *active* error correction is required at any $T > 0$.



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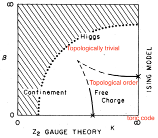

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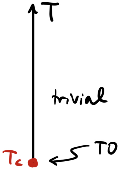
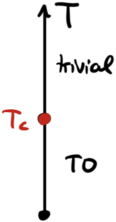


- $\exists$ stable TO at $T > 0$: 2-form toric code in 4+1d.

[Landahl-Dennis-Kitaev-Preskill, 01]

U(1) version: 2-form gauge field is massless even at finite temperature $0 < T < T_c$.

Why: a theory with a 2-form symmetry on a circle still has a 1-form symmetry.



Some applications of this perspective

Higher-form symmetries are usually emergent.

Nevertheless, we can consider the subset of $\{H\}$ where they are exact.

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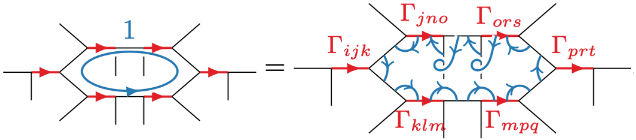
Nevertheless, we can consider the subset of $\{H\}$ where they are exact.

- SPTs protected by generalized symmetries. [Thorgren-Keyserslink 15, Zhu-Lan-Wen, Xu-You, Ye-Gu, Wan-Wang]

- Lattice models with exact anomalous 1-form symmetry.

[Inamura-Ohmori, Eck-Fendley 24]



Provide a systematic generalization of Kitaev's honeycomb model, understandable spin liquid groundstates (some without gappable boundaries).

0-form symmetry : mean field theory

$\vdots \quad \vdots$

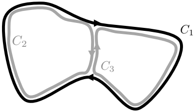
p-form symmetry : ?

Mean String (Brane) Field Theory. [N. Iqbal-JM, Hidaka-Kawana]

All terms consistent with basic principles in (area) derivative expansion:

$$S_{\mathrm{LGW}}[\psi] = \int [dC] \left(V\left(|\psi[C]|^2\right) + \frac{1}{2L[C]} \oint ds \frac{\delta\psi^\star[C]}{\delta C_{\mu\nu}(s)} \frac{\delta\psi[C]}{\delta C^{\mu\nu}(s)} + \cdots \right) + S_r[\psi],$$

$V(x) \equiv rx + ux^2 + \cdots,$ $\frac{\delta}{\delta C^{\mu\nu}}$: area derivative [Migdal, Polyakov]



Topology-changing recombination terms:

$$S_r[\psi] = \int [dC_{1,2,3}] \delta[C_1 - (C_2 + C_3)] \left(\lambda\psi[C_1]\psi^\star[C_2]\psi^\star[C_3] + h.c. \right)$$

$+\cdots$ also respect p-form symmetry.

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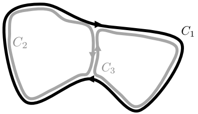

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$+ \cdots$ also respect p-form symmetry.

- Important disclaimer: Not UV complete, no gravity.
- It gives an interesting new perspective on phase transitions of gauge theories. The presence of the cubic recombination term gives a reason that they are often first order.
- It motivates an interesting analogy between 4d U(1) gauge theory and the Kosterlitz-Thouless transition in 2d. [Cardy 1980!]
- Homotopy classification of “topological defects”. [Pace-Liu 2311.09293]
- What is a gauge theory? [Polyakov, Migdal, Makeenko, Banks, Yoneya]

Another application

[Sal Pace, 2308.05730]

In a phase that spontaneously breaks an ordinary (0-form) G symmetry, there's an emergent generalized symmetry associated with conservation of homotopy defects.

e.g. superfluid in D dimensions (= SSB of $\mathsf{U}(1)^{(0)}$ symmetry) has an emergent $\mathsf{U}(1)^{(D-2)}$ symmetry, counting vortices.

In the ordinary SSB phase for G, this new symmetry is unbroken.

But now we can ask about proximate phases where this emergent symmetry is spontaneously broken!

Such phases have been described in the past as 'restoring the symmetry but suppressing defects'.

e.g. In an ordinary superfluid, $\mathsf{U}(1)^{(D-2)}$ cannot be spontaneously broken because of the HWMC theorem.
But suppose we study SSB of $\mathsf{G} = \mathsf{U}(1) \times \mathsf{U}(1)$ symmetry.
$\rightarrow$ emergent $\mathsf{U}(1)^{(D-2)} \times \mathsf{U}(1)^{(D-2)}$ symmetry with currents $\star d\phi_1$ and $\star d\phi_2$.
There is also a new $\mathsf{U}(1)^{(D-3)}$ symmetry whose current is $\star(d\phi_1 \wedge d\phi_2)$ [Brauner, 2012.00051].

This can be spontaneously broken, leading to a Coulomb phase.

Further Generalizations of the Notion of Symmetry

Generalizations of symmetry

[Table from Shu-Heng Shao]

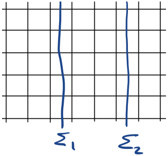
Key insight: we don't really need a transformation of the dofs, we just need a locality-preserving operator O with $HO = OH$.

Properties of symmetry operator	Ordinary symmetry	Higher-form symmetry	Subsystem symmetry	Non-invertible symmetry
Codimension in spacetime	1	> 1	> 1	≥ 1
How topological is it?	fully	fully	not completely	fully
Fusion rule	group $g_1 \cdot g_2 = g_3$	group $g_1 \cdot g_2 = g_3$	group $g_1 \cdot g_2 = g_3$	category $\mathcal{D} \cdot \mathcal{D}^\dagger \neq 1$

Subsystem symmetries and fracton phases, briefly.

Symmetry $\not\Rightarrow$ fully-topological defect operators.

So far: symmetry operators were fully topological. But there can exist operators $U(\Sigma)$ with $[U(\Sigma), H] = 0$, but which are not topological.



For example:

$\Sigma_1 \simeq \Sigma_2$ but $\langle U(\Sigma_1)...\rangle \neq \langle U(\Sigma_2)...\rangle$.

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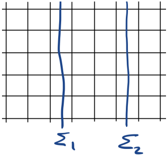

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Subsystem (or 'faithful') symmetry: symmetry operators act independently on *rigid* subspaces.

• Gapped fracton phases: spontaneously break a discrete subsystem higher-form symmetry. [...Qi-Hermele-Radzihovsky, Rayhaun-Williamson]

Charged objects are stuck where the symmetry acts.

Counterexamples to lore that gapped phase is described by TQFT.

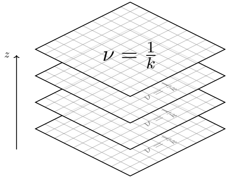
Part of quest for low-dimensional finite-T quantum memory. [Haah 11]

Problematize notions of 'phase of matter'. [Chen, Hermele, ...]

• Multipole symms: (e.g. $\dot{J}^0 + \partial_i \partial_j J^{ij} = 0$) $\xrightarrow{\text{SSB}}$ gapless fracton phases.

[Pretko Seiberg Shao Gorantla Gromov Bulmash Barkeshli ...]

Fracton examples.



Trivial fracton example: stack 2+1d top. states.
e.g. stack abelian quantum Hall states (xy planes) at $z = Ia$, $I = 1..L$.

$$S[a_I] = \sum_I \int_{x,y} \frac{k}{4\pi} a_I \wedge da_I .$$

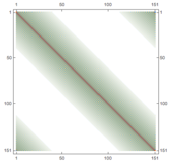
Anyons cannot escape their layer ('planeons').

w/ PBC, GSD $\sim k^L$: $\log GSD \propto L$.

More interesting: couple the layers [Qiu et al, Ma-Chen et al 2010.08917]

$$S[a_I] = \sum_{IJ} \frac{K_{IJ}}{4\pi} \int_{x,y} a_I \wedge da_J .$$

$K_{IJ} \in \mathbb{Z}$, quasi-diagonal: can arise as an effective description of coupled layers of quantum Hall states, and sometimes is gapped. $\log GSD(L)$ is more interesting, but still has a linear envelope.



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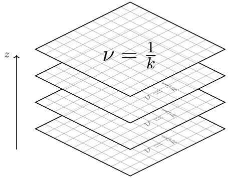
Fraction examples.



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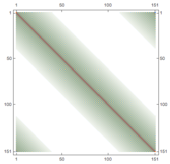

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$\exists$ more isotropic gapped fracton phases. [Haah-Fu-Vijay, ...]

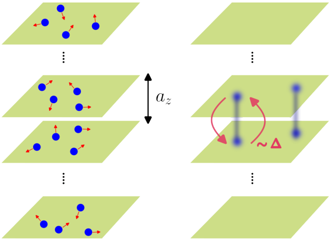
NFL from subsystem symmetries.

[Panigrahi-Kumar, 2411.08091]

Take 2d layers of Fermi liquid (a), with separately conserved $\mathsf{U}(1)$, couple with $\Delta H = \sum_i \int d^2x \rho_i(x)\rho_{i+1}(x)$.

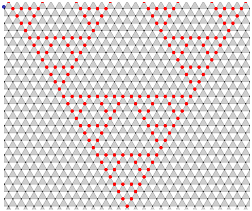
$\implies \langle c_i^\dagger c_{i+1} \rangle = \Delta \neq 0.$

Goldstone modes for SSB of subsystem symmetry destroy the quasiparticles (b), resulting in a non-Fermi liquid.



(a) (b)

Fractal symmetry and Cantor symmetry.



- The subsystem Σ could be a fractal:
  e.g. $H = \sum_{\Delta(ijk)} Z_i Z_j Z_k + g \sum_i X_i$

[Newman Moore Yoshida Williamson Zhou Zhang Pollmann You Devakul Burnell Sondhi, Sfairopoulos-Causer-Mair-Garrahan]

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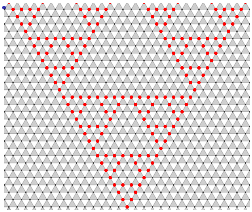

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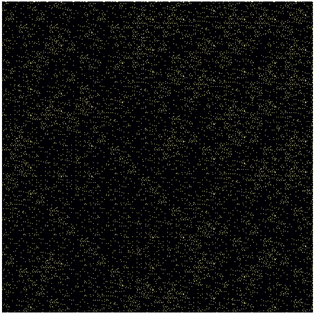
[Newman Moore Yoshida Williamson Zhou Zhang Pollmann You Devakul Burnell Sondhi, Sfairopoulos-Causer-Mair-Garrahan]

In fact, we can have symmetries whose support has no obvious pattern!

e.g. there is a translation-invariant Hamiltonian with 14 states on each link of a square lattice with a symmetry operator whose support is this pattern: $\longrightarrow$

[Ting-Chun David Lin 2411.03115]



This is an ingredient in our current best attempt at a (explicit, non-holographic) translation-invariant stable quantum memory in $d \leq 3$ space dimensions.

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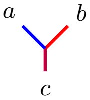
Non-invertible symmetries.

[Moore-Seiberg 89 Fuchs Runkel Schwiegert Frohlich...Chang-Lin-Shao-Wang-Yin 18, reviews: Shao 2308.00747, Schafer-Nameki 2305.18296]

(also known as: categorical symms
or algebraic higher symms
or fusion category symms)

Suppose we have topological operators (associated to each closed $(D-p-1)$-manifold Σ) satisfying a fusion algebra

$$T_a T_b = \sum_c N_{ab}^c T_c \qquad N_{ab}^c \neq 0 \implies$$



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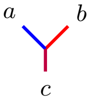

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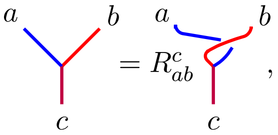
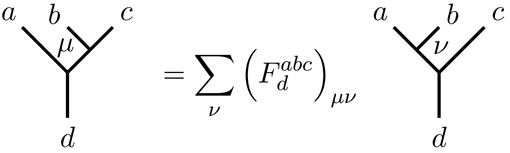
Application: Non-abelian topological order as SSB.

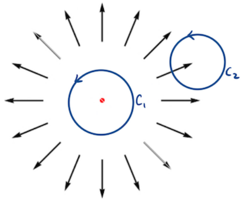




An application to nonlinear σ models

[Hsin 22, Pace-Zhu-Beaudry-Wen 24]

Nonlinear σ-model (in general spatial dimension d) with target space $\mathcal{M}$
(low energy theory of an ordered phase spontaneously breaking $G \to H, \mathcal{M} = G/H$)
can be regularized so that singular configurations are forbidden:
has a non-invertible symmetry.
("d-Rep($\mathbb{G}(d)$) symmetry": The classifying space $B\mathbb{G}(d)$ of $\mathbb{G}(d)$ is the dth Postnikov stage of $\mathcal{M}$).



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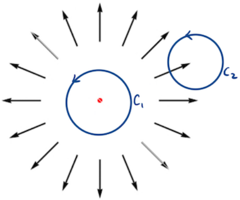


Using this symmetry: the disorded phase of the 3+1d (singularity-free) NLSM on S^2 is axion electrodynamics.

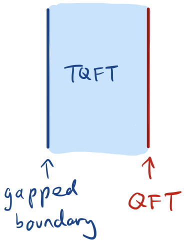
Disagrees with both (S^N (gapped) and $\mathbb{CP}^{N-1}$ (electrodynamics)) large-N extrapolations in the QFT literature.

A tool for thinking about topological operators

[in cond-mat: Wen, Kong 17;

in hep-th: Gaiotto-Kulp 20;

in math: Freed-Moore-Teleman 22]



A symmetry action on a D-dimensional QFT Q can be encoded in a $D+1$-dimensional TQFT

- that admits a gapped boundary
- and admits a boundary with Q on it.

Then the defect operators of the TQFT act on Q.

e.g. to get a discrete G action on Q, take G gauge theory.

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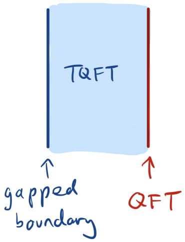
[in cond-mat: Wen, Kong 17;

in hep-th: Gaiotto-Kulp 20;

in math: Freed-Moore-Teleman 22]



A symmetry action on a D-dimensional QFT Q can be encoded in a $D+1$-dimensional TQFT

- that admits a gapped boundary
- and admits a boundary with Q on it.

Then the defect operators of the TQFT act on Q.

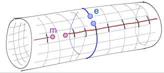
e.g. to get a discrete G action on Q, take G gauge theory.

possible gapped boundaries of this TQFT	$\leftrightarrow$	possible gapped phases in D dimensions with the associated symmetry.

More on this from others ...

Apparent exceptions to Landau revisited

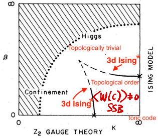
- topological insulators and integer quantum Hall states. ✓
- topological order. [Wegner, Wen] ✓



- other deconfined states of gauge theory (*e.g.* Coulomb phase of E&M, gapless spin liquids). ✓
- fracton phases. ✓
- superconductors.
- (Landau) Fermi liquid and other phases with Fermi surfaces.

Superconductors and Higgs phases.

[Thorngren, Verresen, Borla, Rakovszky, Vishwanath, 2211.01376, 2303.08136]



If we completely Higgs a gauge theory, what's left?

It breaks no symmetries (area law for loop operators), no Goldstones, mass gap. [Fradkin-Shenker 79]

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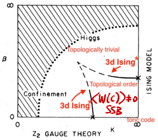


If we completely Higgs a gauge theory, what's left?

It breaks no symmetries (area law for loop operators), no Goldstones, mass gap. [Fradkin-Shenker 79]

If the system preserves the magnetic $\mathsf{U}(1)^{(d-2)}$ symmetry (no monopoles, $J_{\text{mag}} = da$) and the condensate is charged under some other symmetry $\mathsf{U}(1)_{\text{mat}}$, the Higgs phase is an SPT:

$$S[a, \phi, \mathcal{B}_{\text{mag}}, \mathcal{A}_{\text{mat}}] = \int \left(\mathcal{B}_{\text{mag}} \wedge \frac{da}{2\pi} + \lambda \wedge (d\varphi - ma - q\mathcal{A}_{\text{mat}}) \right)$$

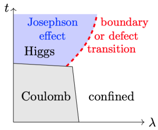
$a \simeq a + d\alpha, \varphi \simeq \varphi + m\alpha, \lambda \simeq \lambda$

$$\overset{\int Da D\varphi D\lambda}{\rightsquigarrow} S_{\text{eff}}[\mathcal{B}_{\text{mag}}, \mathcal{A}_{\text{mat}}] = \frac{q}{m} \int \mathcal{B}_{\text{mag}} \wedge \frac{d\mathcal{A}_{\text{mat}}}{2\pi}.$$

Nontrivial SPT response.

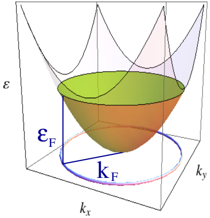
$m > 1$ (*e.g.* electronic superconductor: $m = 2$) is a Symmetry-Enriched Topological Phase.

This simple action reproduces much of the phenomenology.



Symmetries of Fermi surfaces.

[Else-Thorngren-Senthil 20]



Lattice translation symmetry + charge conservation

$\Longrightarrow$ Can define *filling fraction* ν.

If ν is continuously tunable within the phase

('compressible') $\Longrightarrow$ Very large emergent symmetry,

larger than any compact Lie group!

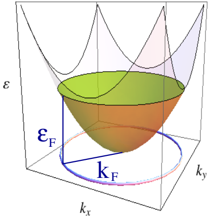
One way to saturate this LSMOH theorem in $D = 2 + 1$:

'ersatz Fermi liquid' has $\mathsf{LU}(1)$ symmetry: fermion number independently conserved at each point on the FS.

Includes many known states with a Fermi surface, with uniform implications (via anomaly matching) for phenomenology (quantum oscillations, ...)

Symmetries of Fermi surfaces.

[Else-Thorngren-Senthil 20]



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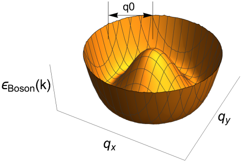
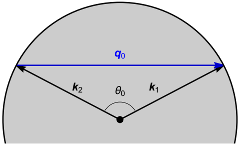
Surprising consequence for QFT:

At a direct transition between a liquid metal and solid: FS coupled to "Bose surface", couples each point to infinitely many others!



[Grover-JM to appear]



$\mathsf{LU}(1)$ must act by $(k_1 = k_2 + q_0)$

$\psi_k \to e^{\mathbf{i}\theta_k}\psi_k, \rho_{q_0} \to e^{\mathbf{i}(\theta_{k_1} - \theta_{k_2})}\rho_{q_0}$.

Final thought.

Q: Does the enlarged Landau paradigm

(including all generalizations of symmetries, and their anomalies)

incorporate all phases of matter

(and transitions between them)

as consequences of symmetry?

Some apparent exceptions:

- topological insulators have edge anomalies.
- topological order = SSB of anomalous higher-form symmetries.
- fracton phases = SSB of *subsystem* higher-form symmetries.
- Coulomb phases = SSB of continuous 1-form symmetries.
- superconductors are (sometimes) SPTs.
- (Landau) Fermi liquid emerges $\mathsf{LU}(1)$ symmetry.
- amorphous solids = SSB of replica symmetry!
- CFTs with no (symmetric) relevant operators (*e.g.* Dirac spin liquid or Stiefel liquid [Zou-He-Wang 2101.07805]).

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Landau was even more right than we thought.
This seems to be a fruitful principle.

The end.

Thanks for listening.

Thanks to Xie Chen, Tarun Grover, Diego Hofman, Nabil Iqbal, Yi-Zhuang You for helpful discussions.

What's a symmetry of a quantum many-body system?

A collection of operators $\{U_g\}$

1. $[H, U_g] = 0$.
2. $O \to U_g O U_g^\dagger$ preserves local operators. (rules out $U = |E\rangle\langle E|$.)
3. $U_g \neq 1$ on the Hilbert space. (not a gauge redundancy.)
4. (not required:) U_g is supported on a whole constant-time slice.
5. (not required:) $\{U_g\}$ represent a group $U_{g_1} U_{g_2} = U_{g_1 g_2}$ (maybe projectively).
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One more demand often fruitfully applied in the hep-th literature (*e.g.* modular invariance) is the existence of defect operators that create twisted sectors.

Q: is locality-preserving enough to guarantee this?

If $U = \prod_i u_i$ is a quantum circuit, then yes: $U(R) = \prod_{i \in R} u_i$.

But some locality-preserving operators (like translations) are nontrivial 'QCA's, which cannot be truncated.

Symmetry does not require topological operators

even with Lorentz symmetry:

Symmetry does not require topological operators

[Sasaki-Yamanaka 88, Eguchi-Yang 89, Bazhanov-Lukhanov-Zamolodchikov, hep-th/9412229]

even with Lorentz symmetry:

$$I_{2k-1} = \int_0^{2\pi} đu T_{2k}(u)$$

Consider any $1+1d$ CFT.
The Virasoro generators L_n are not symmetries, since

(local!)

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0}$$

$$I_1 = L_0 - \frac{c}{24} = H, \quad [I_k, I_l] = 0.$$

they don't commute with H.

(symmetries!)

$T_2(u) = T(u), \quad T_4(u) =: T^2(u) : \quad T_6(u) =: T^3(u) : + \frac{c+2}{12} : (T'(u))^2 :, \quad \cdots$

[Zhipei Zhang, unpublished]

The I_{2k-1} are symmetries that seem not to be topological.