Exact tidal anomalous dimensions

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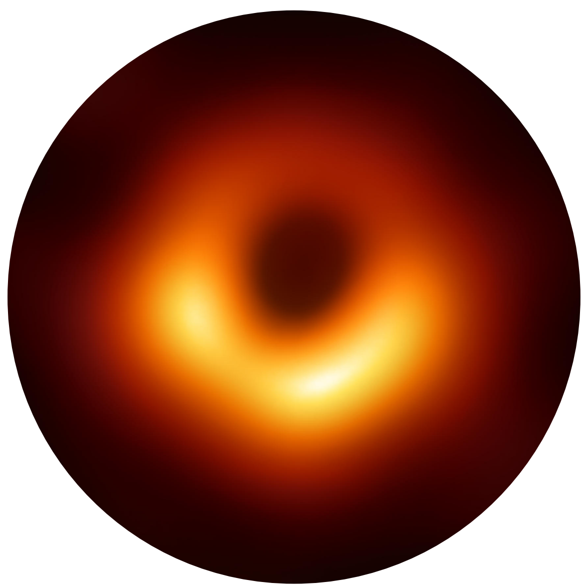
w/ Ivanov, Li, Zhou

Strings 2025 @ NYU, Abu Dhabi

Motivation and Setup

How do we tell apart…



vs.

if we can only probe $\omega \ll 1/R$?

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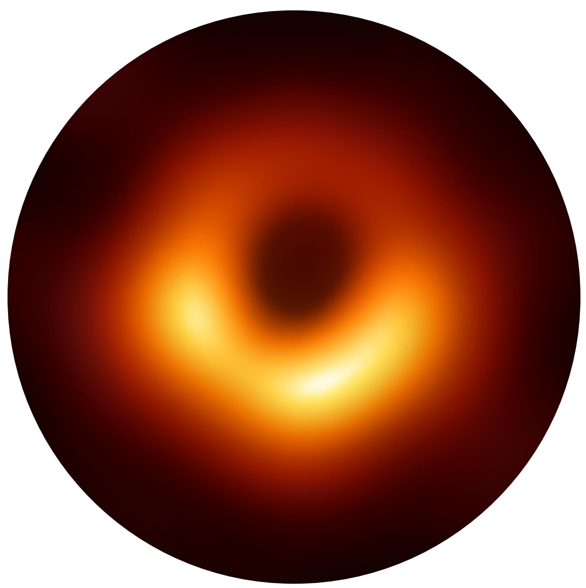

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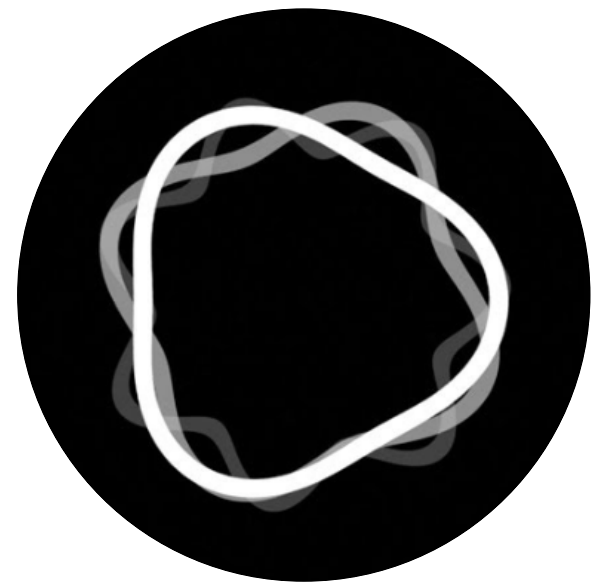

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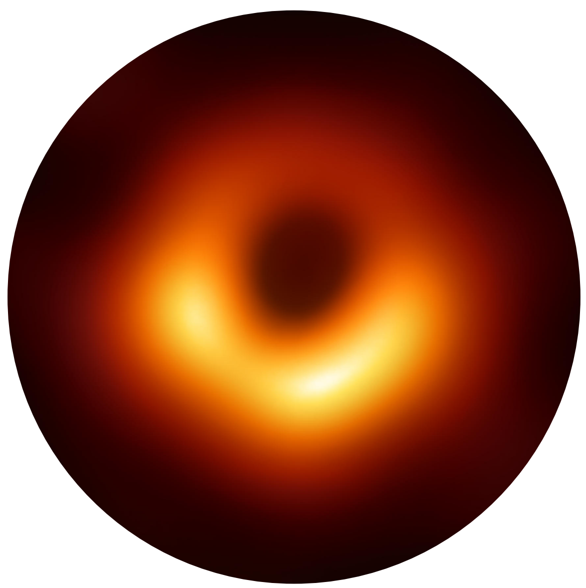

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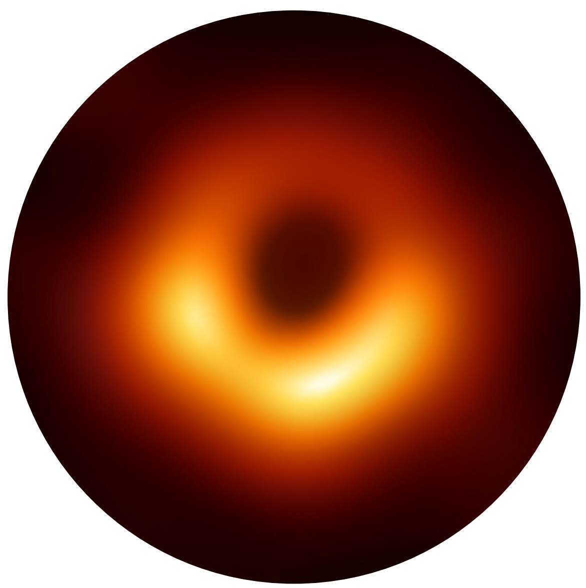

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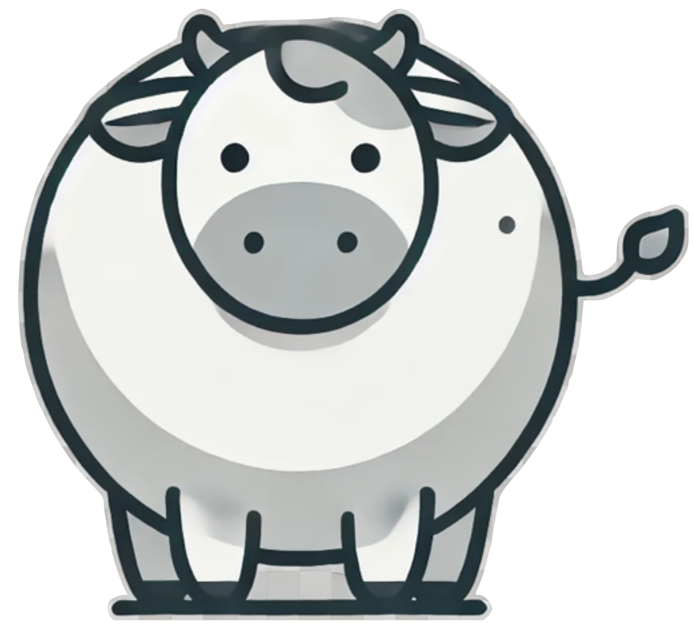
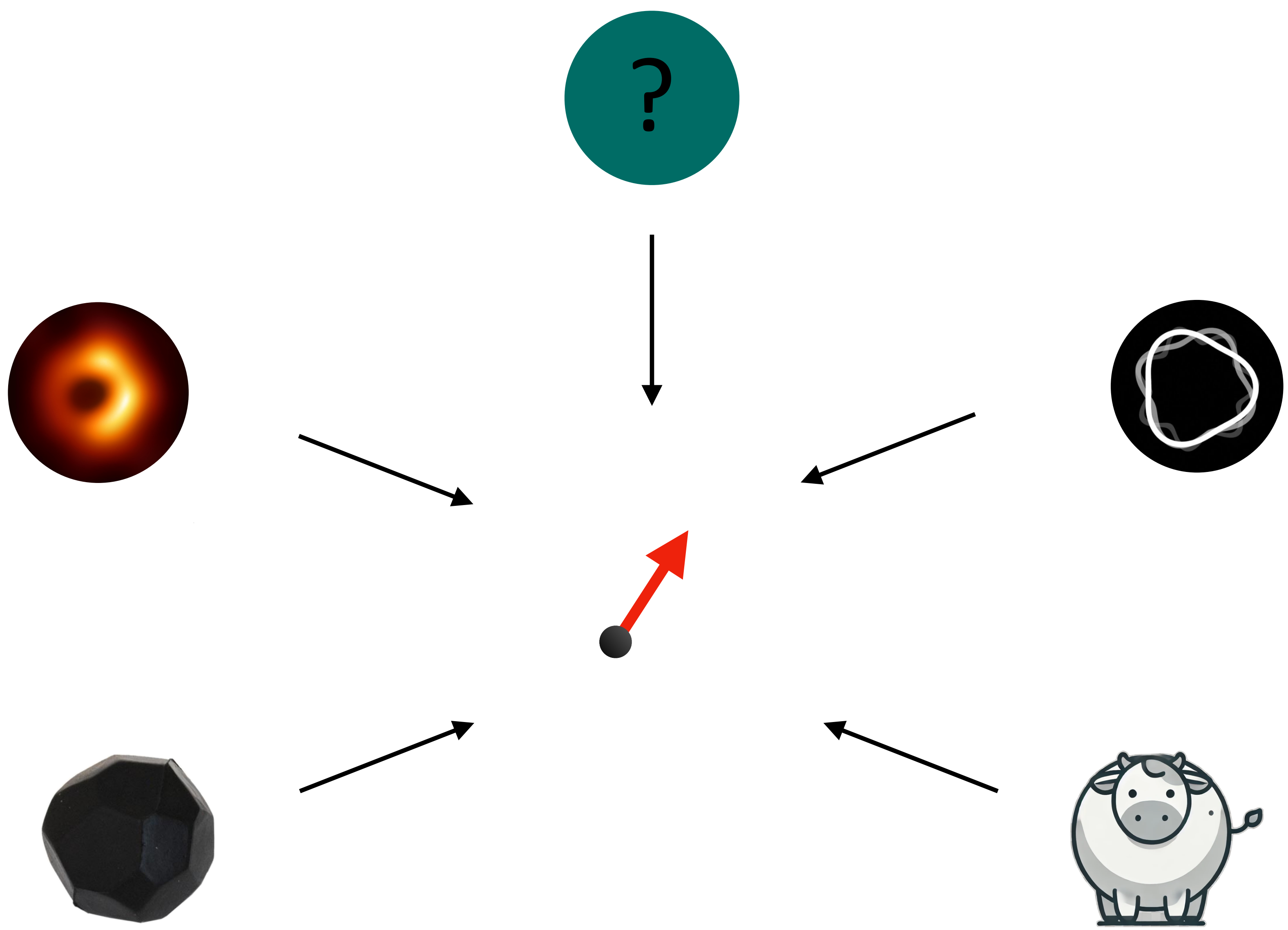


if we can only probe $\omega \ll 1/R$?

The simplest EFT

$$S = m \int d\tau \sqrt{g_{\mu\nu}(x)\dot{x}^{\mu} x^{\nu}} \quad + \quad \text{spin} \quad + \quad \mathcal{O}(R\omega)^{\#}$$



The simplest EFT

Point particle must be augmented by dynamical multipoles

[Goldberger, Rothstein; Porto]

$$S = \int d\tau \left[m(\tau) + \omega_i L^i + Q_{ij}(\tau) E^{ij} + Q_{ijk}(\tau) \nabla^{(i} E^{jk)} \right.$$

spin

octupole

mass/energy

quadrupole

$$\left. + c_2 E_{ij} L^i L^j + \mathcal{O}(L^3) + \text{magnetic} + \cdots \right]$$

spin-ind multipoles

$$E_{ij} = C_{i0j0}$$
$$B_{ij} = {}^*C_{i0j0}$$

Multipoles encode microscopic/UV degrees of freedom

dim. analysis: $Q_\ell \sim m R^\ell$ $\quad c_\ell \sim m^{1-\ell}$

Tidal effects in the EFT

Finite-size effects encoded by correlations

$$\lambda_\ell(\omega) = \int d\tau\, e^{i\omega\tau}\, \theta(\tau)\, \langle [Q_\ell(\tau), Q_\ell(0)] \rangle \quad = \underbrace{\lambda_\ell}_{\text{Static tide}} + \underbrace{i\lambda_{\ell\omega}\, \omega + \lambda_{\ell\omega^2}\, \omega^2 + \cdots}_{\text{Dynamical tides}}$$

This includes effects such as:

$$\lambda_{\ell\omega^n} \sim mR^{2\ell+n}$$

- Absorption by horizon/hydro.
- Tidal deformations

n even: conservative

n odd: dissipative

Tidal effects in the EFT

Non-minimal couplings by integrating out multipoles

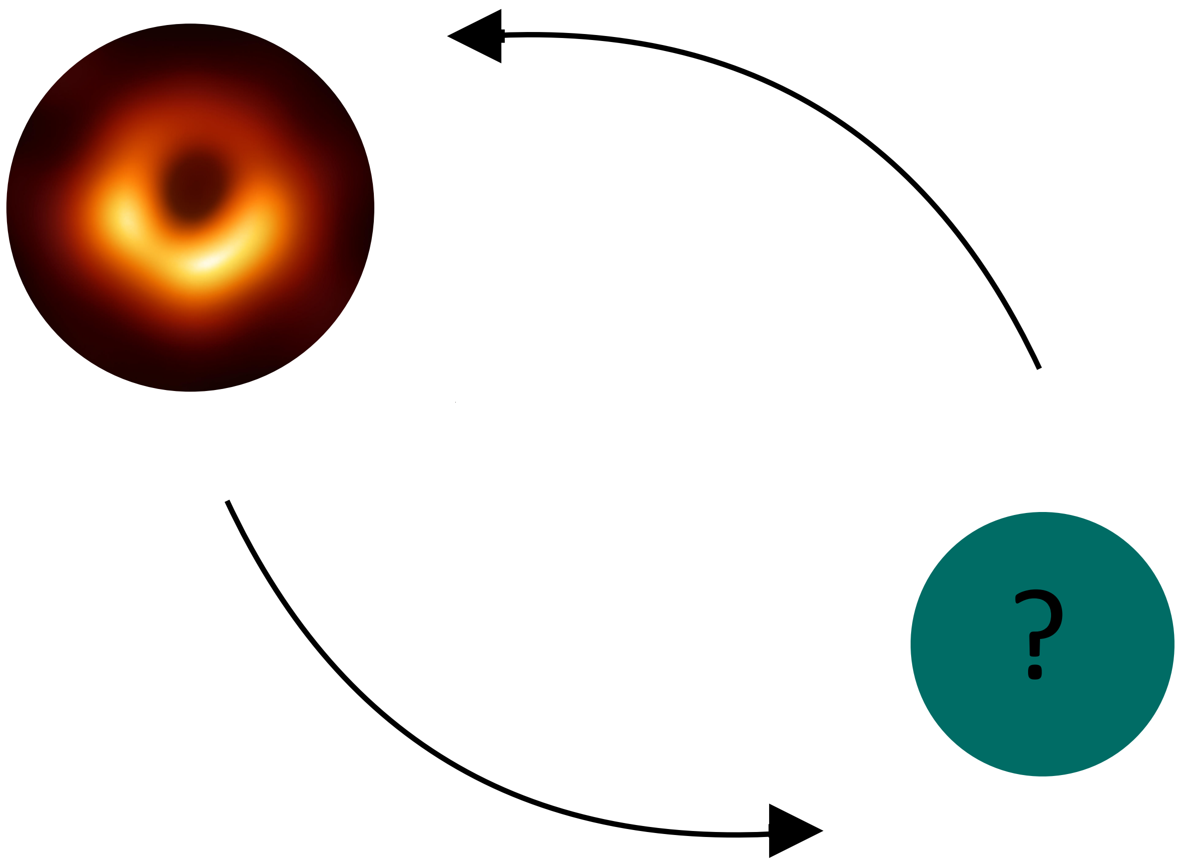
$$\Delta S^{\text{con.}} = \lambda_2 \int d\tau E_{\mu\nu}^2 + \lambda_{2\omega^2} \int d\tau (\dot{E}_{\mu\nu})^2 + \text{magnetic} + \cdots$$

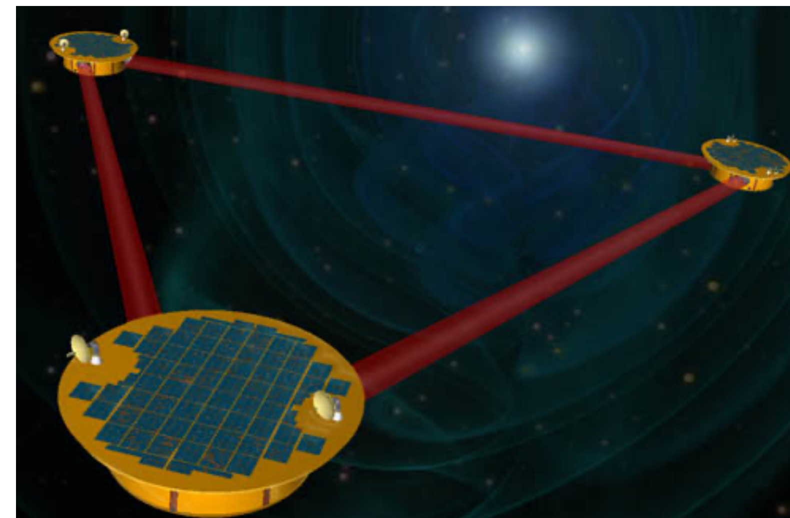
and dissipation (in-in effective action)

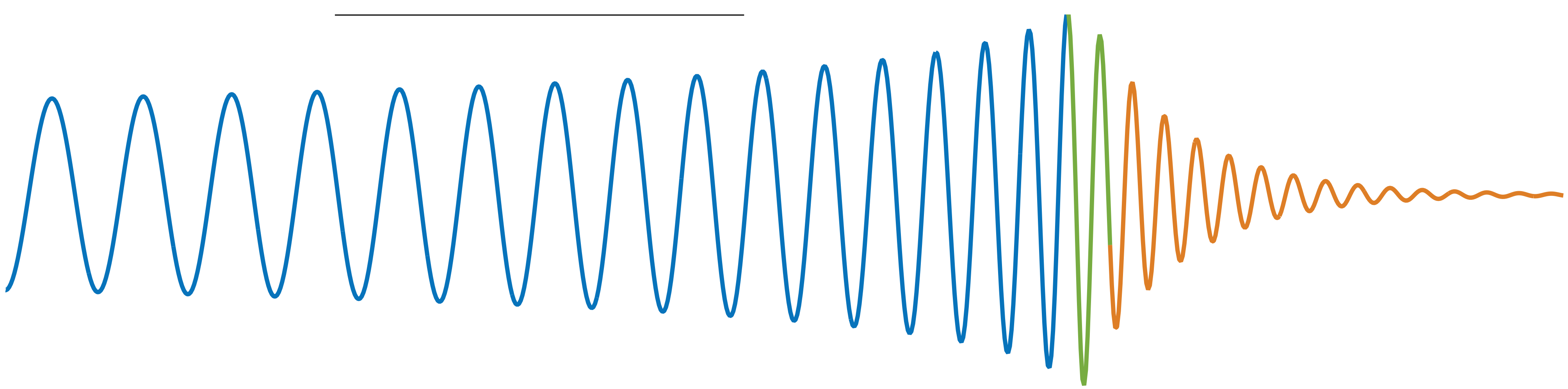
$$\Delta S^{\text{dis.}} = \lambda_{2\omega} \int d\tau \, E_{-\mu\nu} \dot{E}_{+}^{\mu\nu} + \text{magnetic} + \cdots$$

Why might one care?

Experiments!

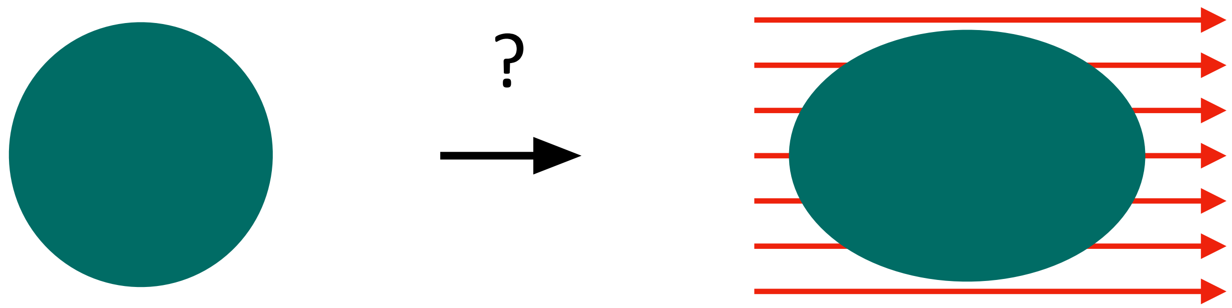






Tidal Love numbers

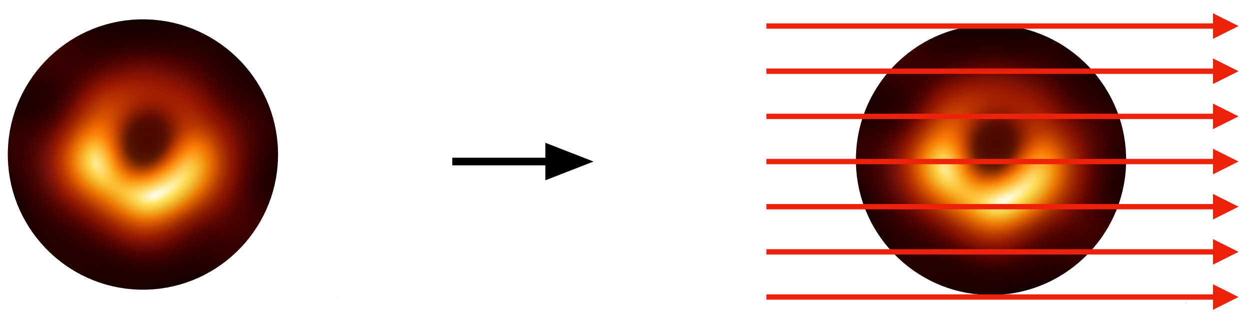
Linear response to applied external gravitational field.



"gravitational polarizability"

$$Q_{ij}^{\text{ind}} = \lambda_2 E_{ij}^{\text{ext.}}$$

Static responses are zero for BH in D=4 GR! [Damour, Nagar; Binnington, Poisson; …]



Excellent possible window into new physics! [Cardoso, Franzin, …]

Note: dynamical responses are not! [Ivanov, Li, **JPM**, Zhou]

Why are these interesting?

The modify the post-Newtonian potential & fluxes

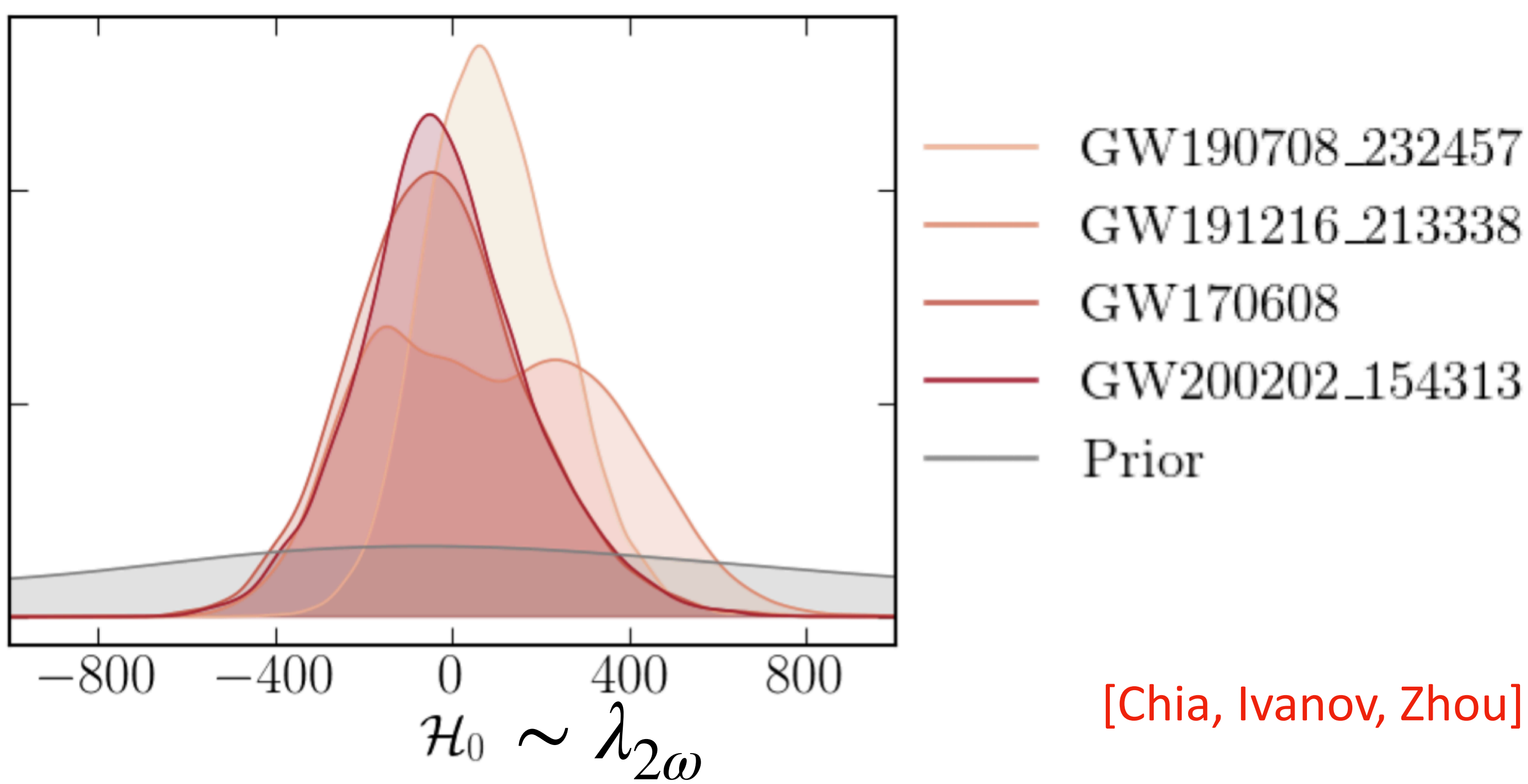
e.g.,

$$V(r) \sim \lambda_2 \frac{R^4}{r^4} \frac{R_s^2}{r^2}$$

"5PN"

[Damour; Cheung, Solon; Bern, **JPM**, Roiban, Sawyer, Shen; Porto, many others]

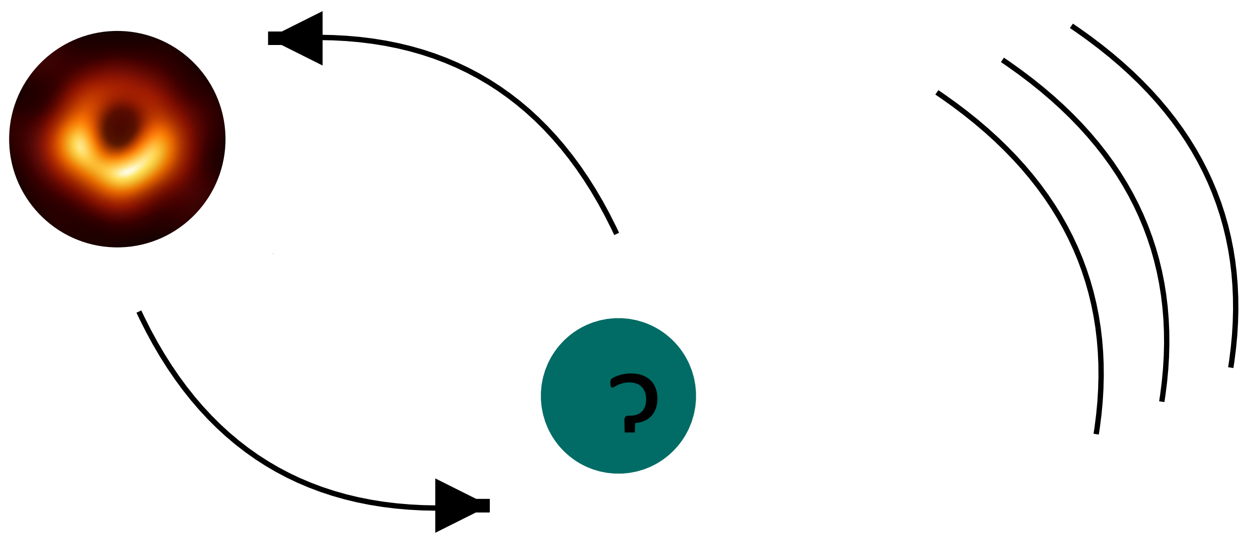
So we should be able to measure with gravitational wave detectors. Equation of state of NS? [Flanagan, Hinderer]

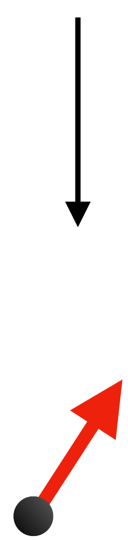


[Chia, Ivanov, Zhou]

Radiation from a binary

Binary itself is pointlike for $\omega \ll 1/r$





(E, L, Q_ℓ)

c.f. quadrupole formula

$$\frac{dE}{d\omega} = \frac{G}{5}\dddot{Q}_2^2 + \cdots$$

Quantum Black Holes

Black hole horizons can decohere quantum superpositions

[Danielson, Satishchandran, Wald]

Point particle EFT was recently used to show that this effect is generic in the vicinity of any compact object and due to dissipative tides

[Biggs, Maldacena]

$$\langle Q_\ell(\tau) Q_\ell(\tau') \rangle \sim \lambda_2 \delta(\tau - \tau') + \lambda_{2,\omega} \partial_\tau \delta(\tau - \tau') + \cdots$$

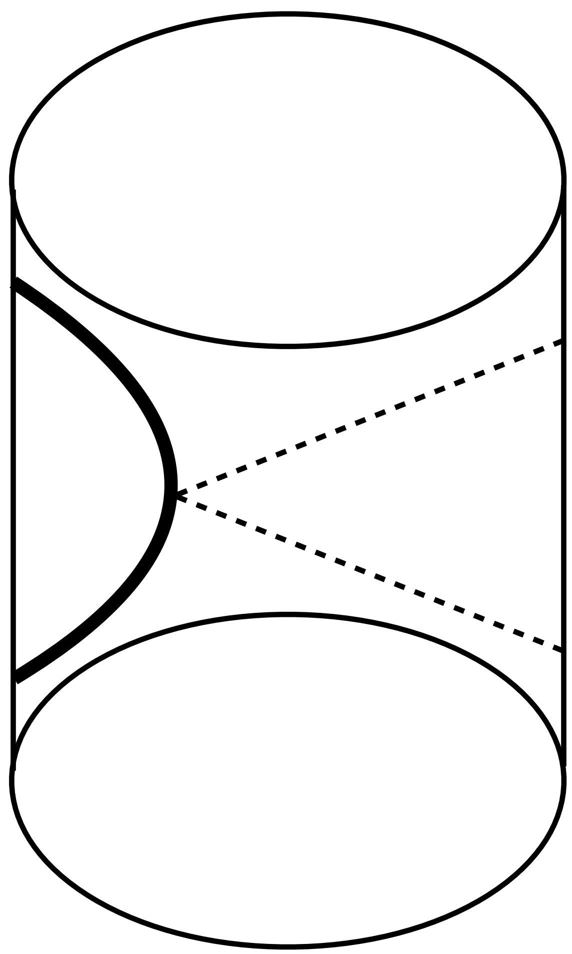
One can also use the EFT to study the long-distance effects of Hawking radiation in various settings.

[Goldberger, Rothstein]

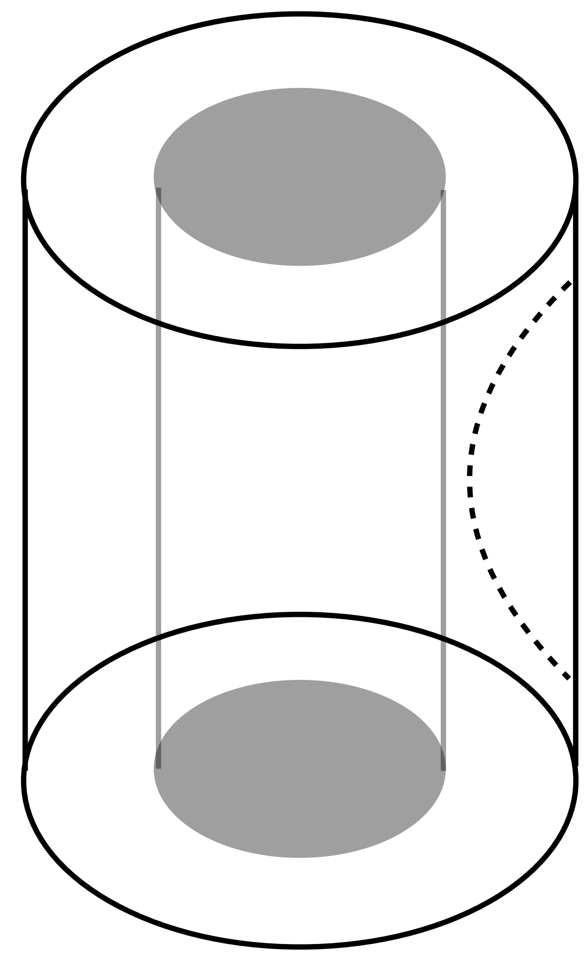
E.g., they give contributions to cross sections comparable to loops of gravitons!

Heavy states vs. Black holes

$\langle \mathcal{O}_H \mathcal{O}_L \mathcal{O}_L \mathcal{O}_H \rangle$ vs. $\langle \mathcal{O}_L \mathcal{O}_L \rangle_T$



vs.



To which order in $1/N$ do these coincide? How do we tell BH from other heavy operators with same dim. and spin?

EFT predictions

Universality

$$S = \int d\tau \left[E(\tau) + \omega_{ij} L^{ij} + \mathcal{O}(L^2) + \mathcal{O}(R^5) \right]$$

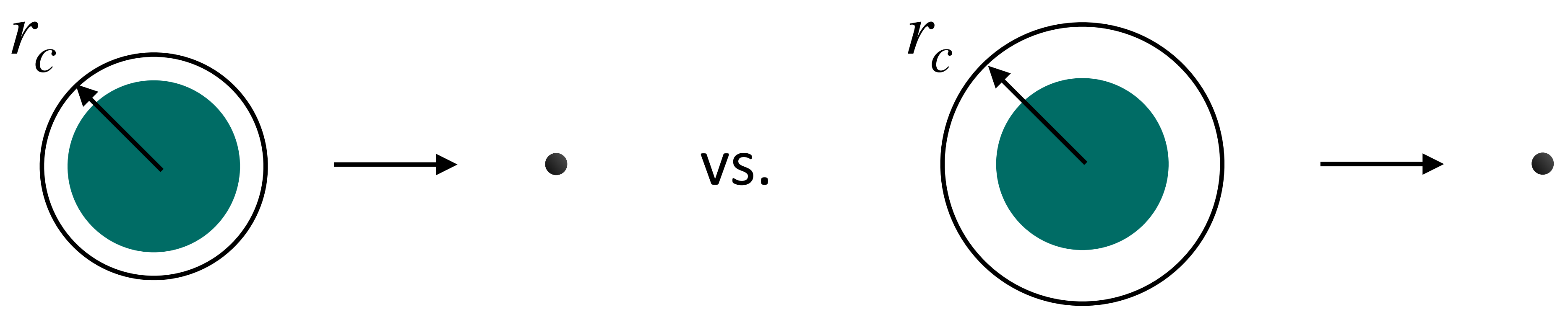
Universal!

Non-universal

Leading tide is $\lambda_2 \sim R^5$, so EFT predicts that we cannot tell apart point particle from BH until at least $O(R^5, RL^2)$

Tidal coefficients run!

A compact object cannot be cleanly separated from its gravitational field



Hence, tides depend on “how much of the spacetime is integrated out together with the microscopic d.o.f of the object”

$$\lambda_\ell(\omega, r_c) = \lambda_\ell + \lambda_{\ell\omega}(r_c)\,\omega + \lambda_{\ell\omega^2}(r_c)\,\omega^2 + \cdots$$

In the EFT this manifests itself in UV divergences, even classically!

UV divergences

Classical GR coupled to a point particle is non-renormalizable! We should expect classical UV divergences

$$\mathbb{O}(r_c \sim 1/\mu, \omega) = \frac{1}{\epsilon} + \lambda_\ell(\omega) + 2\gamma_\ell(\omega)\,\lambda_\ell(\omega)\,\log(r_c\omega) + \beta_\ell(\omega)\,\log(r_c\omega) + \cdots$$

These will be absorbed by tidal coefficients

$$\lambda_\ell(r_c, \omega) = \bar{\lambda}_\ell(\omega) + 2\gamma_\ell(\omega)\,\bar{\lambda}_\ell(\omega)\,\log(R/r_c) + \beta_\ell(\omega)\,\log(R/r_c) + \cdots$$

"Bare coupling" "Anomalous dimension" "Beta function"

$$\mu\frac{dQ_\ell(\omega)}{d\mu} = \gamma_\ell(\omega)Q_\ell(\omega)$$

$$\mu\frac{d}{d\mu}[(\mu^2)^{\gamma_\ell}\lambda_\ell(\omega)] = \beta_\ell(\omega)$$

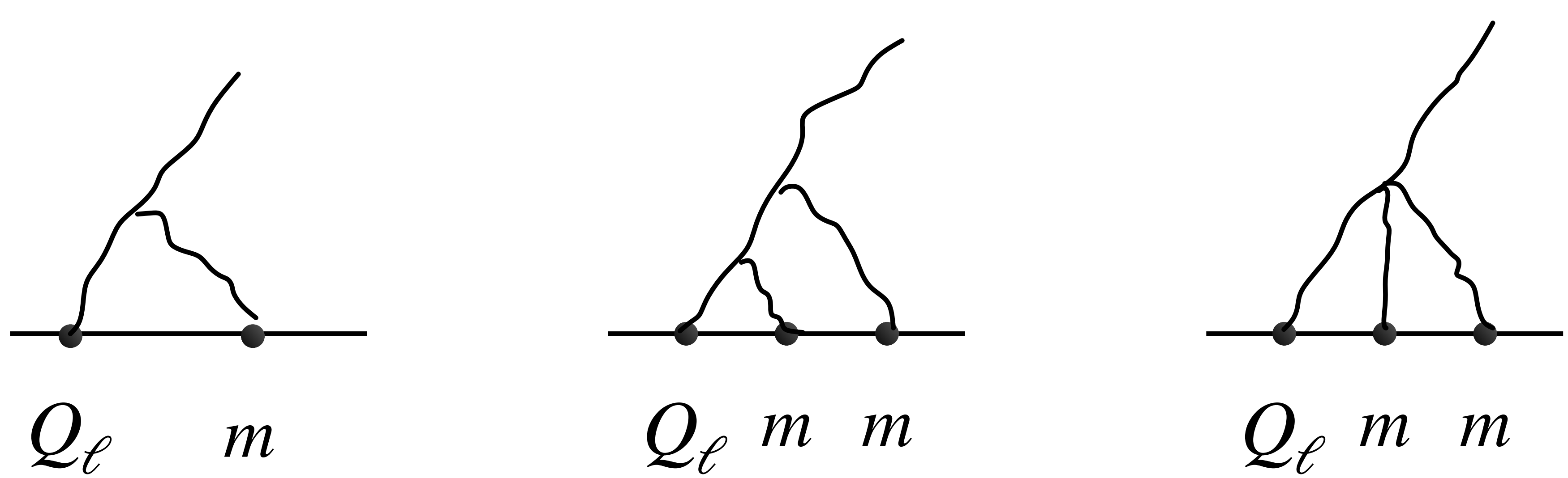
Multipole form factor

To compute the anomalous dimension one could study the multipole form factor

$$F_\ell(\omega) = \langle h | Q_\ell | BH \rangle \qquad Q_\ell \to Q_\ell^{\mathrm{ren}} = (R\omega)^{\gamma(\omega)} Q_\ell$$

It has the general structure

$$F_\ell(\omega) \sim Gm|\omega| + G^2 m^2 \omega^2 \log(\omega/\mu) + G^3 m^3 |\omega|^3 + \cdots$$



Universality in anomalous dimensions

EFT does not know Wilson coefficients, however it predicts RG running!

$$\lambda_\ell(r_c,\omega) = \bar{\lambda}_\ell(\omega) + 2\gamma_\ell(\omega)\,\bar{\lambda}_\ell(\omega)\,\log(R/r_c) + \beta_\ell(\omega)\,\log(R/r_c) + \cdots$$

Not universal

Universal(ish)!

Anomalous dimensions and beta functions are almost the same for any object (BH, NS, Coal, Cow, etc)

$$S = \int d\tau\,\left[\,E(\tau) + \omega_{ij}L^{ij} + \mathcal{O}(L^2) + \mathcal{O}(R^5)\,\right]$$

This is a classical RG, can we understand it to all orders?

$$\mu \frac{dQ_\ell(\omega)}{d\mu} = \gamma_\ell(\omega) Q_\ell(\omega)$$

Exact tidal anomalous dimensions

Results:

[Ivanov, Li, **JPM**, Zhou]

$a = |L|/m$

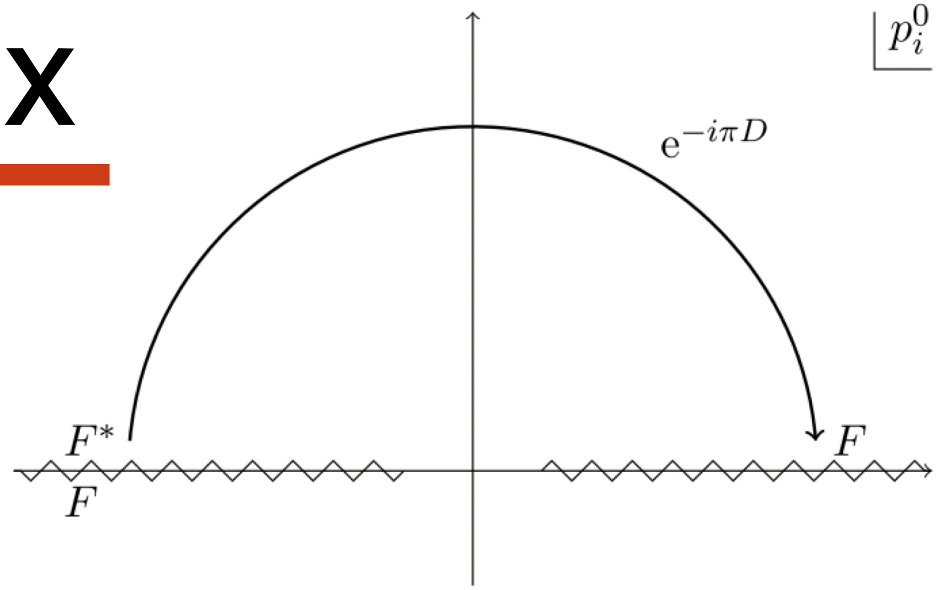
1. $\gamma_\ell(\omega, a) = -\frac{2}{\pi}\Big(\delta_\ell(\omega, a) + \delta_\ell(-\omega, -a)\Big)$

"Far zone" scattering phase shift

2. $\gamma_\ell^{\mathrm{BH}}(\omega) = -\left(a(\omega) + \frac{1}{2} + \ell\right)$

Quantum SW A-period for $SU(2)$ $\mathcal{N} = 2$ SYM with $N_f = 3$ hypers

Review: RG from S-matrix



Key relation: [Caron-Huot, Wilhelm]

$$e^{-i\pi\gamma}F^* = SF^*S^\dagger$$

Analyticity + dimensional analysis:

$$F(p_i) = F^*(e^{i\pi}p_i) = e^{-i\pi D}F^*(p_i) = e^{-i\pi\gamma}F^*$$

Unitarity:

$$SS^\dagger = 1 \rightarrow SF^* - FS^\dagger = 0 \rightarrow F = SF^\star$$

Allows, on-shell computation of RG running!

γ from δ

Instead consider symmetric Green's function $\quad O(\tau) = Q_\ell E^\ell$

$$G_S(\omega) = \tfrac{1}{2}\langle \mathrm{BH}\,|\,\{O(\omega), O(-\omega)\}\,|\,\mathrm{BH}\rangle \sim \mathrm{Im} G_R$$

Analyticity + dim. Analysis:

$$e^{i\pi D} G_S(\omega) = G_S(e^{i\pi}\omega) = G_S(\omega)^* = \tfrac{1}{2}\langle \mathrm{BH}\,|\,\{O^\dagger(\omega), O^\dagger(-\omega)\}\,|\,\mathrm{BH}\rangle$$

Unitarity: $\quad O^\dagger = S^\dagger O S^\dagger \qquad G_S(\omega)^* = e^{-2i(\delta(\omega)+\delta(-\omega))} G_S(\omega)$

$$\boxed{\gamma_\ell(\omega) = -\frac{2}{\pi}(\delta_\ell(\omega) + \delta_\ell(-\omega))\,\Big|^{FZ}}$$

Scattering phase-shift in BHPT

Scattering amplitude:

$$f_s(\theta) = \frac{2\pi}{i\omega} \sum_{\ell=s}^{\infty} {}_{-s}S_\ell^s(1, a\omega)\, {}_{-s}S_\ell^s(\cos\theta, a\omega)(\eta_{\ell s} e^{2i\delta_{\ell s}} - 1)$$

Phase shifts receive contributions from "near zone" ($r \sim R_s$) and "far zone" $r \gg R_s$

$$\delta_{\ell s} = \delta_{\ell s}|^{NZ} + \delta_{\ell s}|^{FZ}$$

Contains information about tides

Computable in EFT, modulo counterterms

Wave scattering off BH

Regge-Wheeler/Teukolsky equation $\Box_{BH}\, h_{\mu\nu} = 0$

$$\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{d}{dr}\,{}_sU_{\ell m}(r)\right) + V_{eff}(r)\,{}_sU_{\ell m}(r) = 0$$

incoming b.c. $\quad {}_{-s}U_{\ell m}(r) \to B^{(\mathrm{inc})}_{-s\ell m} r^{-1} e^{-i\omega r} + B^{(\mathrm{refl})}_{-s\ell m} r^{-1+2s} e^{i\omega r}$

"Connection coefficients for
the confluent Heun equation"

$$\eta_{\ell s} e^{2i\delta_{\ell s}} \sim \frac{B^{(\mathrm{refl})}_{-s\ell m}}{B^{(\mathrm{inc})}_{-s\ell m}}$$

BHPT and SW theory

Old method: matched asymptotic expansions [Mano, Suzuki, Takasugi]

Recent observation: [Aminov, Grassi, Hatsuda; …]

Radial Teukolksy Equation **=** Quantum Curve for $SU(2)$ $\mathcal{N}=2$ SYM with $N_f = 3$ hypers

$$m_1 = i\frac{m\chi - 2Gm\omega}{\sqrt{\chi^2-1}} \qquad m_2 = -s - i2Gm\omega \qquad m_3 = s - i2Gm\omega$$

$$L = -2iGm\omega\sqrt{\chi^2-1}$$

Allows a complete solution of the connection problem via Nekrasov-Shatashvili functions (∼semiclassical Liouville conformal blocks)!

Scattering Solution

[Bautista, Bonelli, Iossa, Tanzini, Zhou]

Coulomb phase

TS constants

Renormalized ℓ

$$\delta_\ell\big|^{FZ} = \epsilon\log(2|\epsilon|) - \frac{1-\kappa}{2}\epsilon + \frac{1}{2}\mathrm{Im}\,\partial_{m_3}F + \frac{1}{2}\mathrm{Arg}[A_s^P] + \frac{1}{2}\mathrm{Arg}\frac{\Gamma(1+\nu-m_3)}{\Gamma(1+\nu+m_3)} + \boxed{\frac{\pi}{2}(\ell-\nu)}$$

NS function

NS function, F, can be computed combinatorially, and also A-period

$$a^2 - \frac{1}{4} = \nu(\nu+1) = L\partial_L F(\nu) - u$$

“Matone relation”

$f(m, \chi, s)$

Near zone similar (but rational), so won’t show.

All orders BH anomalous dimensions

$$\gamma_\ell(\omega) = \frac{2}{\pi}(\delta_\ell(\omega) + \delta_\ell(-\omega))\,|^{FZ}_{BHPT}$$

Plugging in we find that this is precisely the SW quantum period or renormalized angular momentum!

$$\boxed{\gamma_\ell(\omega) = (\nu(\omega) - \ell\,) = -\left(a(\omega) + \frac{1}{2} + \ell\right)}$$

Which can be computed from combinatorially to all orders! Even numerically non-perturbatively.

$$\gamma = \frac{1}{2\ell+1}\left(-2 - \frac{s^2}{\ell(\ell+1)} + \frac{[(\ell+1)^2 - s^2]^2}{(2\ell+1)(2\ell+2)(2\ell+3)} - \frac{(\ell^2 - s^2)^2}{(2\ell-1)2\ell(2\ell+1)}\right)(R_s\omega)^2 + \mathcal{O}((R_s\omega)^3)$$

An application

Waveform resumation

We can use it to resum logarithms in the multipolar binary waveform itself! This is the observable measure in experiment.

$$h_{\ell m} \sim (r\omega)^{\nu(\omega)} \, h_{\ell m}^{\text{finite}}$$

For quasi-circular binaries this allows us to propose a formula for "tail-resummed" multipolar waveform

$$h_{\ell m} = (-ir\omega)^{\nu(\omega)} e^{i2iGE\omega \log(2r\omega) + GE\omega\pi} \frac{\Gamma(\nu + 1 - 2iGE\omega)}{\Gamma(\nu + 1)} h_{\ell m}^{\text{finite}}$$

In the probe limit $m_1 \ll m_2$, this agrees with a recent calculation, which also uses relation to SW theory

[Fucito, Morales, Russo]

Comparison with state-of-the-art

We can compare with state-of-the-art post-Newtonian waveforms used for LIGO/Virgo/Kagra (4PN)

[Blanchet et al. 2023]

$$
\begin{aligned}
H_{22} = 1 &+ \left(-\frac{107}{42} + \frac{55}{42}\nu\right)x + \boxed{2\pi x^{3/2}} + \left(-\frac{2173}{1512} - \frac{1069}{216}\nu + \frac{2047}{1512}\nu^2\right)x^2 + \left[\boxed{-\frac{107\pi}{21}} + \left(\boxed{\frac{34\pi}{21}} - 24\,\mathrm{i}\right)\nu\right]x^{5/2} \\
&+ \left[\frac{27027409}{646800} \boxed{-\frac{856}{105}\gamma_\mathrm{E}} + \boxed{\frac{428\,\mathrm{i}\,\pi}{105}} + \boxed{\frac{2\pi^2}{3}} + \left(-\frac{278185}{33264} + \frac{41\pi^2}{96}\right)\nu - \frac{20261}{2772}\nu^2 + \frac{114635}{99792}\nu^3 \boxed{-\frac{428}{105}\ln(16x)}\right]x^3 \\
&+ \left[\boxed{-\frac{2173\pi}{756}} + \left(\boxed{-\frac{2495\pi}{378}} + \frac{14333\,\mathrm{i}}{162}\right)\nu + \left(\boxed{\frac{40\pi}{27}} - \frac{4066\,\mathrm{i}}{945}\right)\nu^2\right]x^{7/2} \\
&+ \left[-\frac{846557506853}{12713500800} + \boxed{\frac{45796}{2205}\gamma_\mathrm{E}} - \boxed{\frac{22898}{2205}\mathrm{i}\pi} - \frac{107}{63}\pi^2 + \boxed{\frac{22898}{2205}\ln(16x)}\right. \\
&\quad + \left(-\frac{336005827477}{4237833600} + \frac{15284}{441}\gamma_\mathrm{E} - \frac{219314}{2205}\mathrm{i}\pi - \frac{9755}{32256}\pi^2 + \frac{7642}{441}\ln(16x)\right)\nu \\
&\quad \left. + \left(\frac{256450291}{7413120} - \frac{1025}{1008}\pi^2\right)\nu^2 - \frac{81579187}{15567552}\nu^3 + \frac{26251249}{31135104}\nu^4\right]x^4 + \mathcal{O}\left(x^{9/2}\right).
\end{aligned}
\tag{6.17}
$$

"tail-of-memory"

Same for other values of (ℓ, m)

Conclusions

Amplitudes/Analiticity + EFT/RG + SUSY Gauge Theory