How the Hilbert space of two-sided black holes factorises

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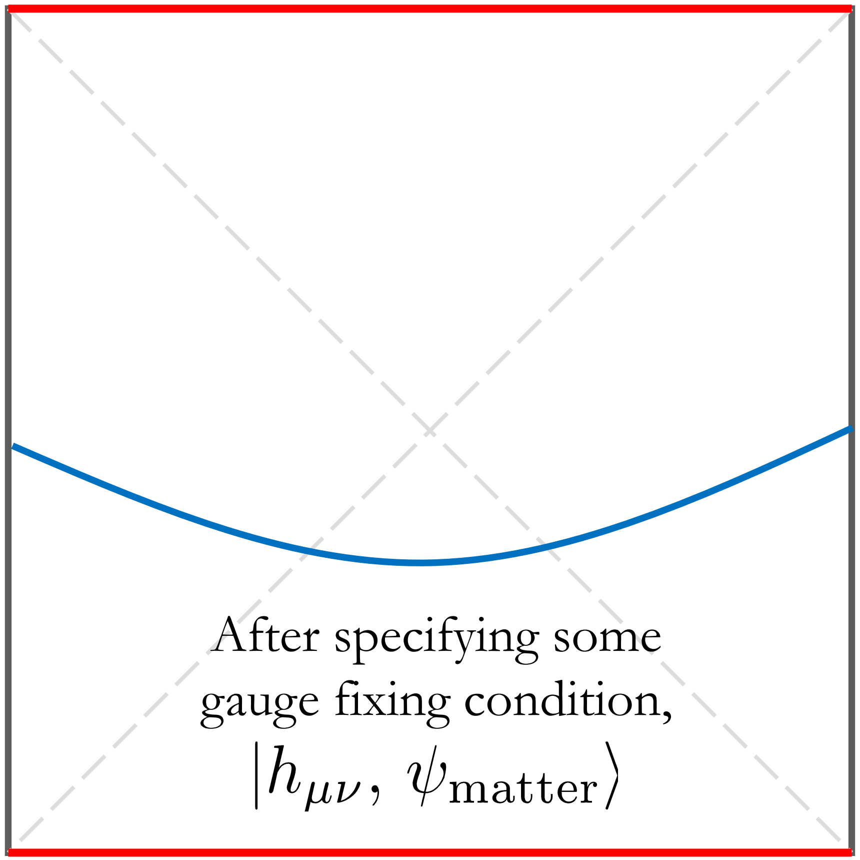
Work with Jan Boruch, Guanda Lin, and Cynthia Yan.
Teaser for work with Ahmed Abdalla, Stefano Antonini, and Adam Levine.

The Hilbert space of two-sided black holes: The factorisation puzzle



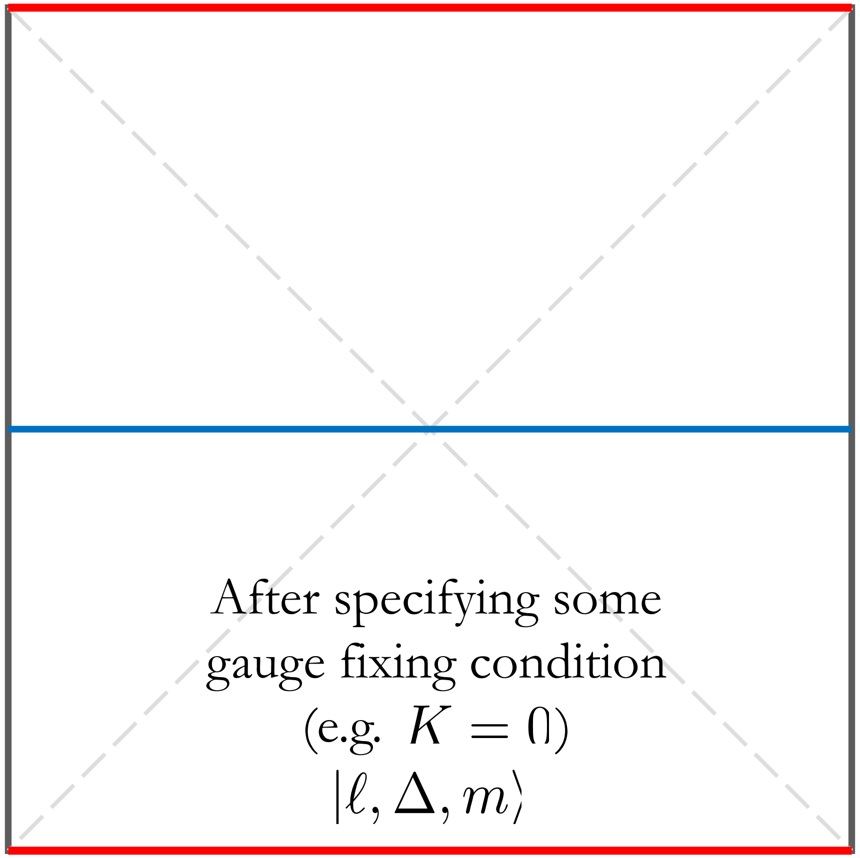
Consider the Hilbert space of two-sided black holes.

In AdS/CFT, according to ER=EPR
$\mathcal{H}_{\text{bulk}} = \mathcal{H}_L \otimes \mathcal{H}_R$

$\neq$

Bulk Hilbert space, in a G_N^{-1} expansion

An example that makes things concrete:
2D gravity coupled to matter



For example, in the simplest model of gravity (JT gravity) coupled to matter,

$$\mathcal{H}_{\text{bulk}}^{\text{pert}} = L^2(\mathbb{R}) \otimes \mathcal{H}_{\text{matter}}$$

$\mathcal{H}_{\text{matter}}$ can be decomposed into unitary irreps of $\widetilde{SL}(2,\mathbb{R})$

Different basis choices,

$$|\ell, \Delta, m\rangle \;\Leftrightarrow\; |E_L, E_R, \Delta\rangle \;\Leftrightarrow\; |\beta_L, \beta_R, \Delta\rangle$$

[Lin, Maldacena, Zhao `19, Penington Witten `23, Kolchmeyer `23]

The fact that the Hilbert space of two-sided black holes seems different than the factorised boundary Hilbert space is known as the factorisation puzzle.
It was often thought that resolving this puzzle requires a detailed knowledge about the UV.

[Harlow `16, Guica, Jafferis `17, Harlow, Jafferis `18]

The purpose of this talk is to propose a resolution to this puzzle using the low-energy theory.

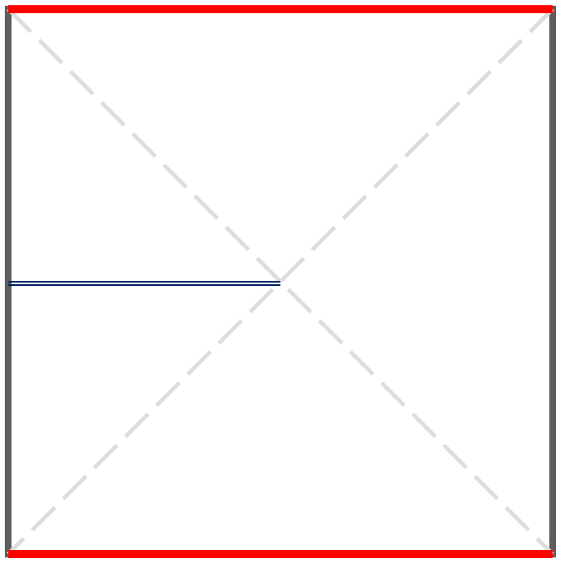
We will show that non-perturbative corrections to the gravitational path integral lead to

$$\mathcal{H}_{\text{bulk}} = \mathcal{H}_L \otimes \mathcal{H}_R$$

… different than the factorization puzzle.

$$Z(\beta_L)Z(\beta_R) \neq Z(\beta_L, \beta_R)$$

Another way to describe the lack of factorisation: Algebraic classification of gravitational observables



What is the algebra of observables in the left sub-region?

Factorisation

Type III

$G_N = 0$

Familiar notions cannot be defined:

$\mathrm{Tr},\ \rho,\ |\psi\rangle,\ S$

Type II

$G_N \to 0$

Familiar notions become better defined but still no pure states.

[for examples, see Witten `21]

Type I

G_N finite

According to AdS/CFT pure states are well defined

$\Leftrightarrow$

$\mathcal{H}_{\text{bulk}} = \mathcal{H}_L \otimes \mathcal{H}_R$

Strategy

Using the gravitational path integral, prove that

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{bulk}}}(k_L k_R) = \mathrm{Tr}_{\mathcal{H}_L}(k_L)\,\mathrm{Tr}_{\mathcal{H}_R}(k_R)\,, \quad \forall\, k_{L,R} \in \mathcal{A}_{L,\,R}$$

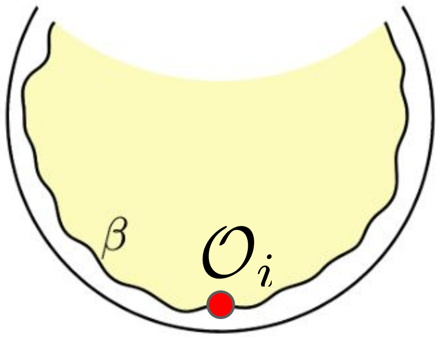
In this talk, we will only focus on

$$k_L = e^{-\beta_L H_L}\,, \quad k_R = e^{-\beta_R H_R}\,.$$

What is $\mathcal{H}_{\text{bulk}}$?

To have a hope of seeing factorisation, we need to break the gauge constraint $H_L = H_R$.

$$\mathcal{H}_{\text{bulk}}(K) \equiv \text{Span}\ \{|q_i\rangle =$$



$$\text{for } E_L,\ E_R \in \underbrace{\mathcal{E}}_{\text{Arbitrary energy interval}},\ i = 1,\ \ldots\ , K\}$$

For simplicity,
$\Delta_1 \approx \cdots \approx \Delta_K = \Delta$

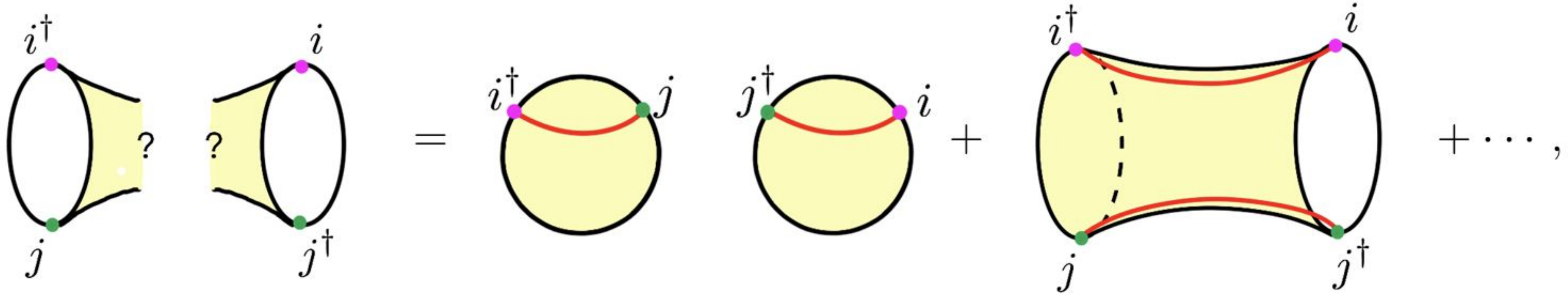
These are "Bag Of Gold" states.

Strategy: calculate the statistics of the trace

$$= \underbrace{(M^{-1})}_{\text{Generalized inverse}}{}_{ij} \langle q_i| \, e^{-\beta_L H_L} e^{-\beta_R H_R} \, |q_j\rangle \, , \qquad M_{ij} \equiv \langle q_i | q_j \rangle \; .$$

While we might not know these inner-products exactly, we now believe that we can compute their statistics:

$$\overline{|\langle q_i|q_j\rangle|^2} =$$



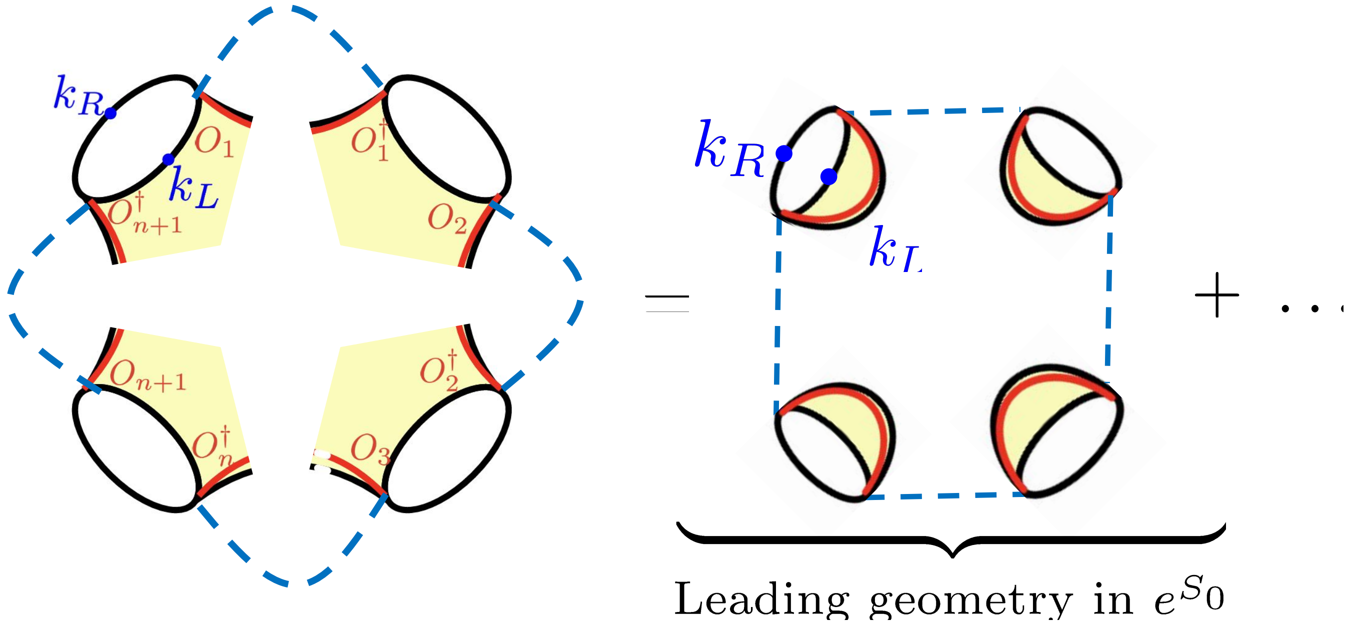
Not necessarily an ensemble, maybe a coarse-graining of a regular UV complete theory.

Strategy: calculate the statistics of the trace

Using this assumption, we can compute,

$$\lim_{n\to -1} \overline{\langle q_i | q_{i_1} \rangle \ldots \langle q_{i_{n-1}} | q_j \rangle \langle q_i | e^{-\beta_L H_L} e^{-\beta_R H_R} | q_j \rangle}$$

$$= \lim_{n\to -1}$$



No factorisation from the leading geometries in e^{S_0}

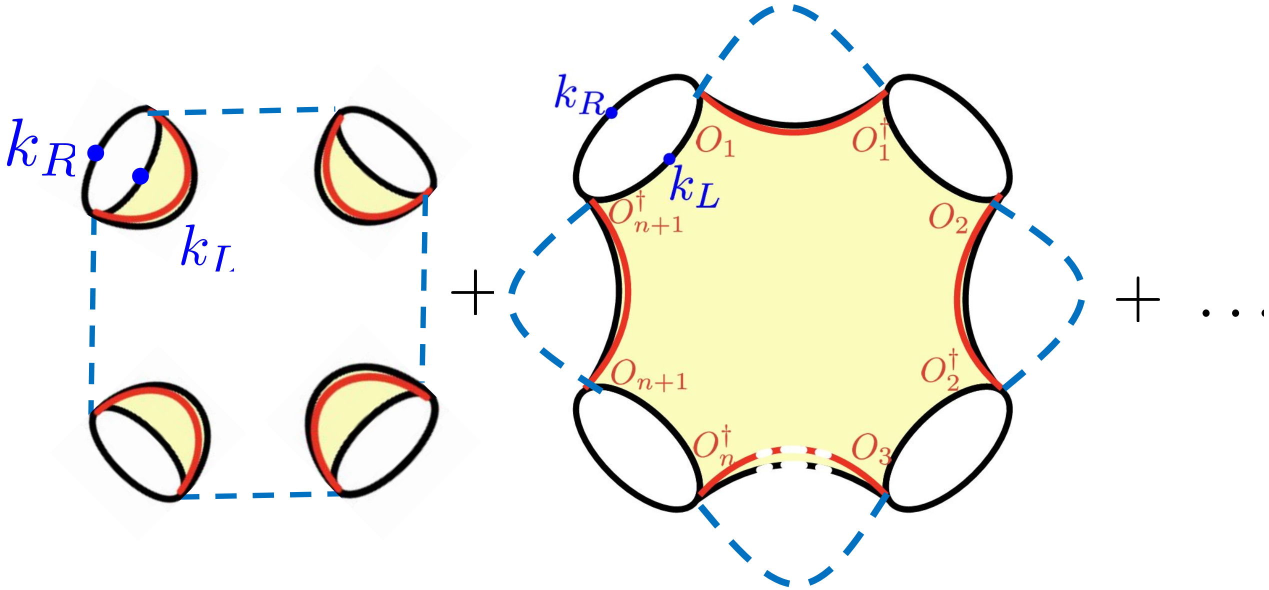
If we only included the leading geometries in e^{S_0}, then we would never see factorization:

$$= \sum_{i=1}^{K} \frac{\langle q_i | \, e^{-\beta_L H_L} e^{-\beta_R H_R} \, | q_i \rangle_{\text{pert}}}{\langle q_i | q_i \rangle_{\text{pert}}} \propto K \, .$$

This verifies that perturbatively we will never see that the Hilbert space factorises.
This is because when we cannot ignore subleading contributions in e^{-S_0}.

Which geometries contribute?

$$\overline{\mathrm{Tr}_{\mathcal{H}_{\mathrm{bulk}}(K)}(k_L\, k_R)} = \lim_{n\to -1}$$



$$Ke^{4S_0} \quad \sim \quad K^4 e^{-2S_0}$$

Therefore, to probe factorisation we need to resum all wormhole geometries in the limit

[similar computations analysis seen in in
Pennington, Shenker, Stanford, Yang `20, **Iliesiu**, Hsin, Yang `21,
Balasubramanian, Lawrence, Magan, Sasieta `22, Boruch, **Iliesiu**, Yan `23]

Strategy: from statistics to exact results

How do we prove factorization through statistics?

Step 1. To leading order in $e^{\# G_N^{-1}}$,

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{bulk}}(K)}(e^{-\beta_L H_L}\, e^{-\beta_R H_R}) = \begin{cases} \mathrm{Tr}_{\mathcal{H}_L}(e^{-\beta_L H_L})\, \mathrm{Tr}_{\mathcal{H}_R}(e^{-\beta_R H_R})\,, \quad \forall\, \beta_L,\, \beta_R\,, & \text{when } K \geq d^2, \\ \text{Complicated non-factorising trace,} & \text{when } K < d^2\,, \end{cases}$$

We can also go beyond leading order.

Strategy: from statistics to exact results

Define,

$$(\beta_L, \beta_R) = 0 \qquad \Leftrightarrow \qquad \mathcal{H}_{\text{bulk}} = \mathcal{H}_L \otimes \mathcal{H}_R$$

(If eigenenergies are non-degenerate)

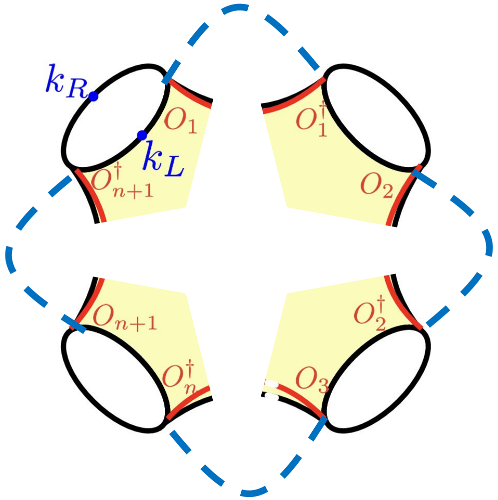
Step 2. To all orders in $e^{\# G_N^{-1}}$, $\quad \overline{(\beta_L, \beta_R)} = 0\,.$

Step 3. To all orders in $e^{\# G_N^{-1}}$, $\quad \overline{(\ (\beta_L, \beta_R))^2} = 0\,.$

Today: We will mostly focus on Step 1.

Step 1: factorisation to leading order

To resum all geometries,



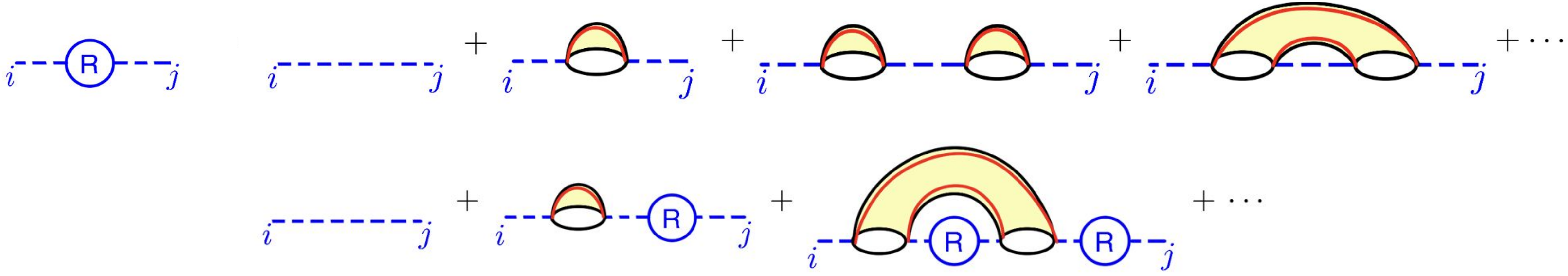
we define the resolvent for the matrix $M_{ij} \equiv \langle q_i | q_j \rangle$

$$\mathbf{R}_{ij}(\lambda) = \left(\frac{1}{\lambda - M} \right)_{ij}$$

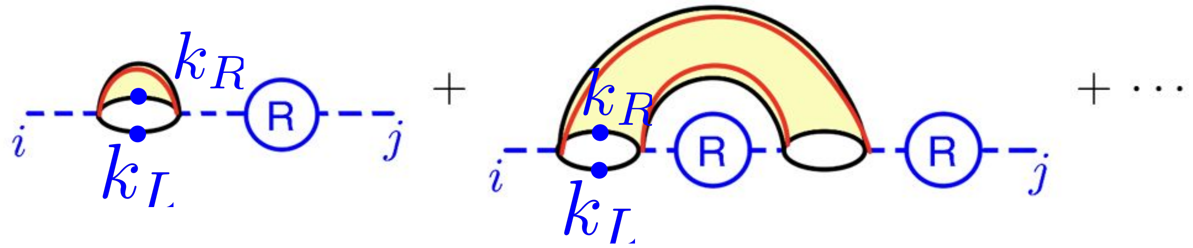
$$\overline{(M^n)_{ij} \langle q_i | e^{-\beta_L H_L} e^{-\beta_R H_R} | q_j \rangle} = \frac{1}{2\pi i} \oint_{C_0} \mathrm{d}\lambda \, \lambda^n \, \overline{\mathbf{R}_{ij}(\lambda) \langle q_i | e^{-\beta_L H_L} e^{-\beta_R H_R} | q_j \rangle}$$

A Schwinger-Dyson equation

The resolvent can be expanded in a geometric series,



$$\overline{\mathbf{R}_{ij}\,\langle q_i|\,k_L k_R\,|q_j\rangle} =$$



A Schwinger-Dyson equation

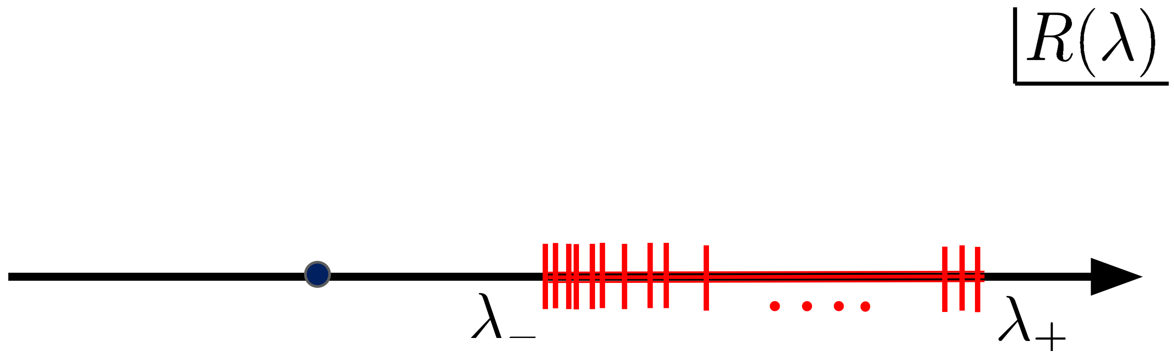
For the resolvent,

$$R(\lambda)$$

For the trace,

The analytic structure of the Schwinger-Dyson equation

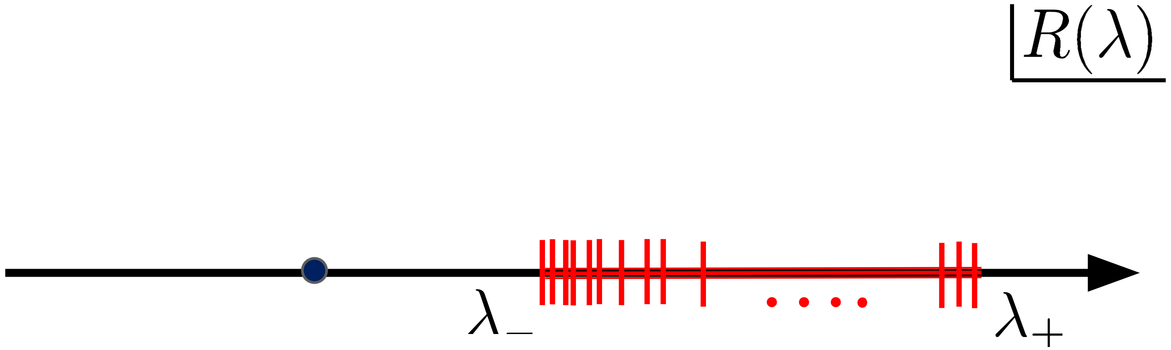
While we cannot solve the Schwinger-Dyson exactly, we can constrain the analytic structure of the resolvent.



$$R(\lambda) \sim \frac{K-d^2}{\lambda}\Theta(K-d^2) + R_0 \quad \text{where} \quad d^2 = e^{2S_0}\int_{E_L,E_R\in\mathcal{E}} \rho(E_L)\rho(E_R)\mathrm{d}E_L\mathrm{d}E_R$$

…and that no other non-analyticities appear on the negative real axis.
We have a semi-positive definite inner-product even non-perturbatively.

The Hilbert space is finite-dimensional

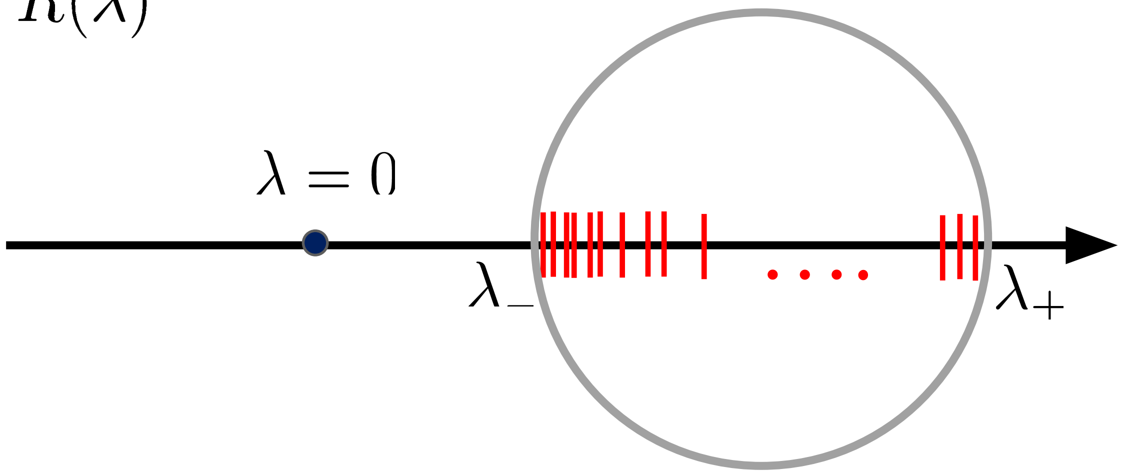


$$R(\lambda) \sim \frac{K - d^2}{\lambda}\Theta(K - d^2) + R_0$$

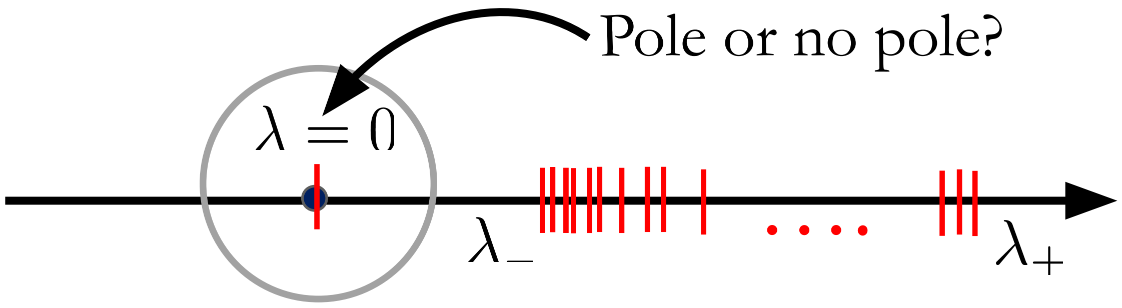
($\exists$ null states)

The physical mechanism for factorisation

$R(\lambda)$



$\Rightarrow$



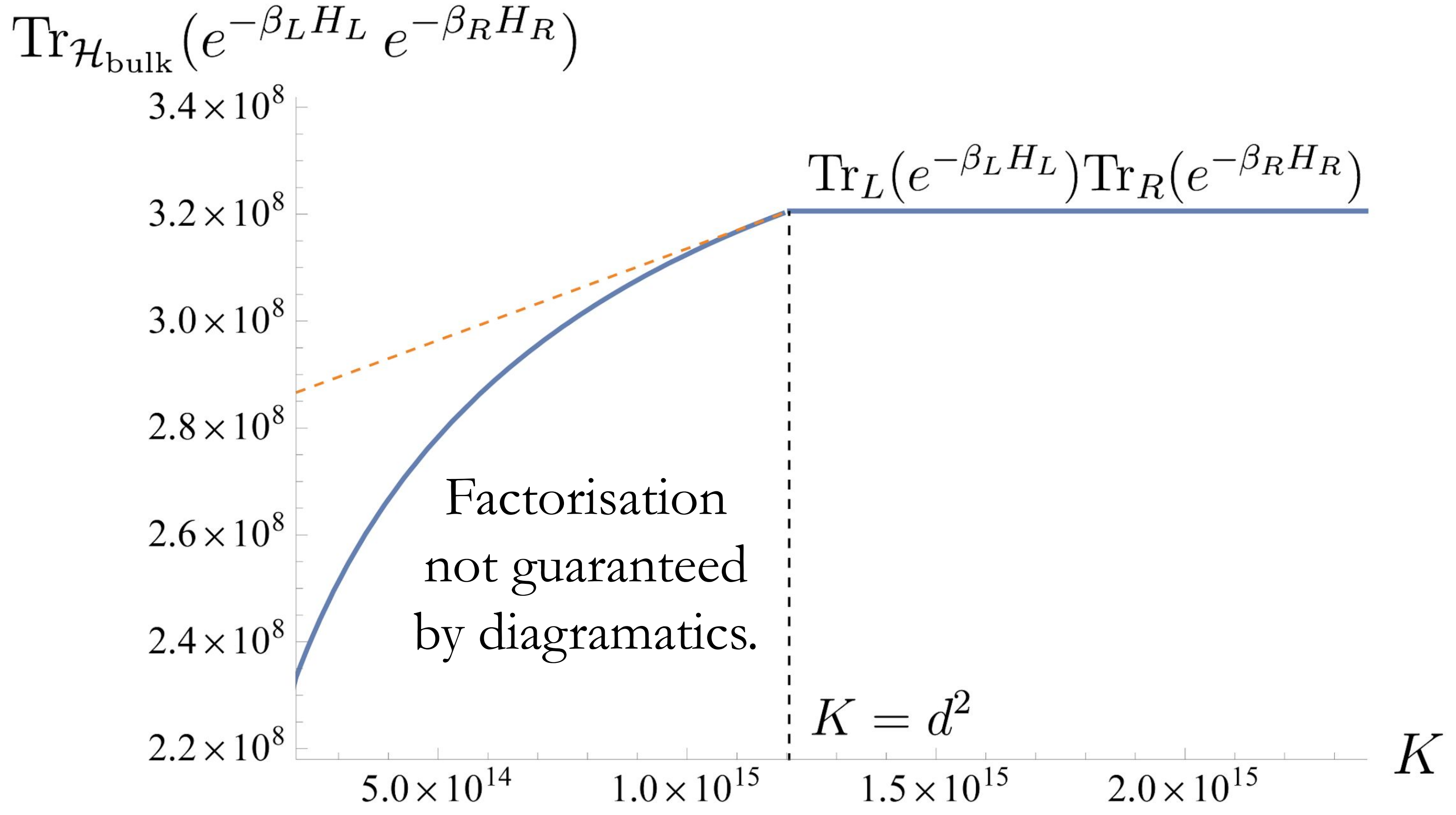
$=$

$\Rightarrow$ No null states $\Rightarrow$ Ugly non-factorising answer!

$\Rightarrow$ $\exists$ null states $\Rightarrow$ Factorising answer!

The physical mechanism for factorisation

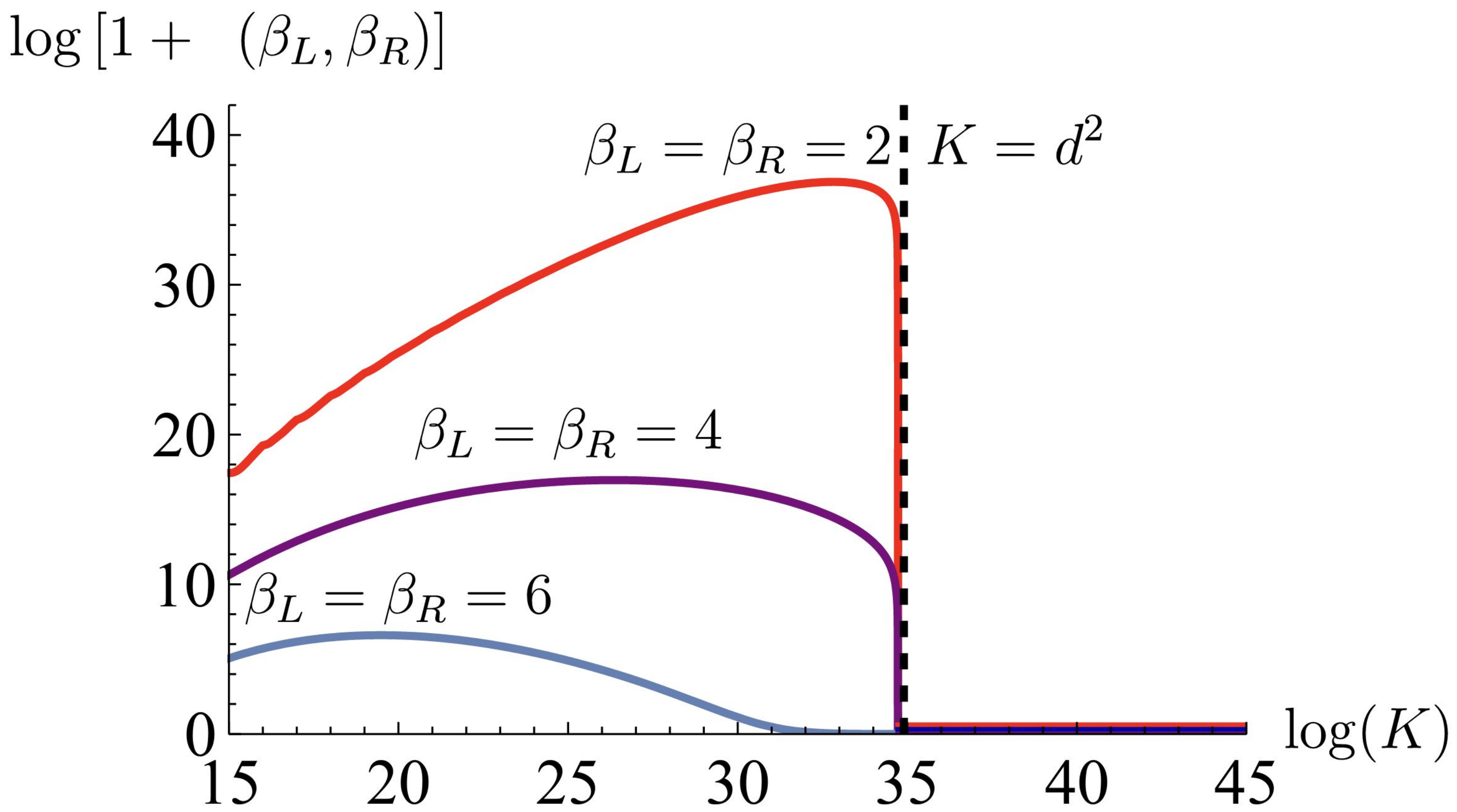
The "Page curve" of factorisation



Just like the Page curve: a consistency condition for $\mathcal{H}_{\mathrm{bulk}} = \mathcal{H}_L \otimes \mathcal{H}_R$

Tracking the differential

We can analyze whether factorisation is exact by studying,



Proving steps 2 and 3

For large enough , we can also show that

$$\overline{\mathrm{Tr}_{\mathcal{H}_{\mathrm{bulk}}(K)}(e^{-\beta_L^{(1)}H_L}e^{-\beta_R^{(1)}H_R})\dots\mathrm{Tr}_{\mathcal{H}_{\mathrm{bulk}}(K)}(e^{-\beta_L^{(n)}H_L}e^{-\beta_R^{(n)}H_R})} =$$

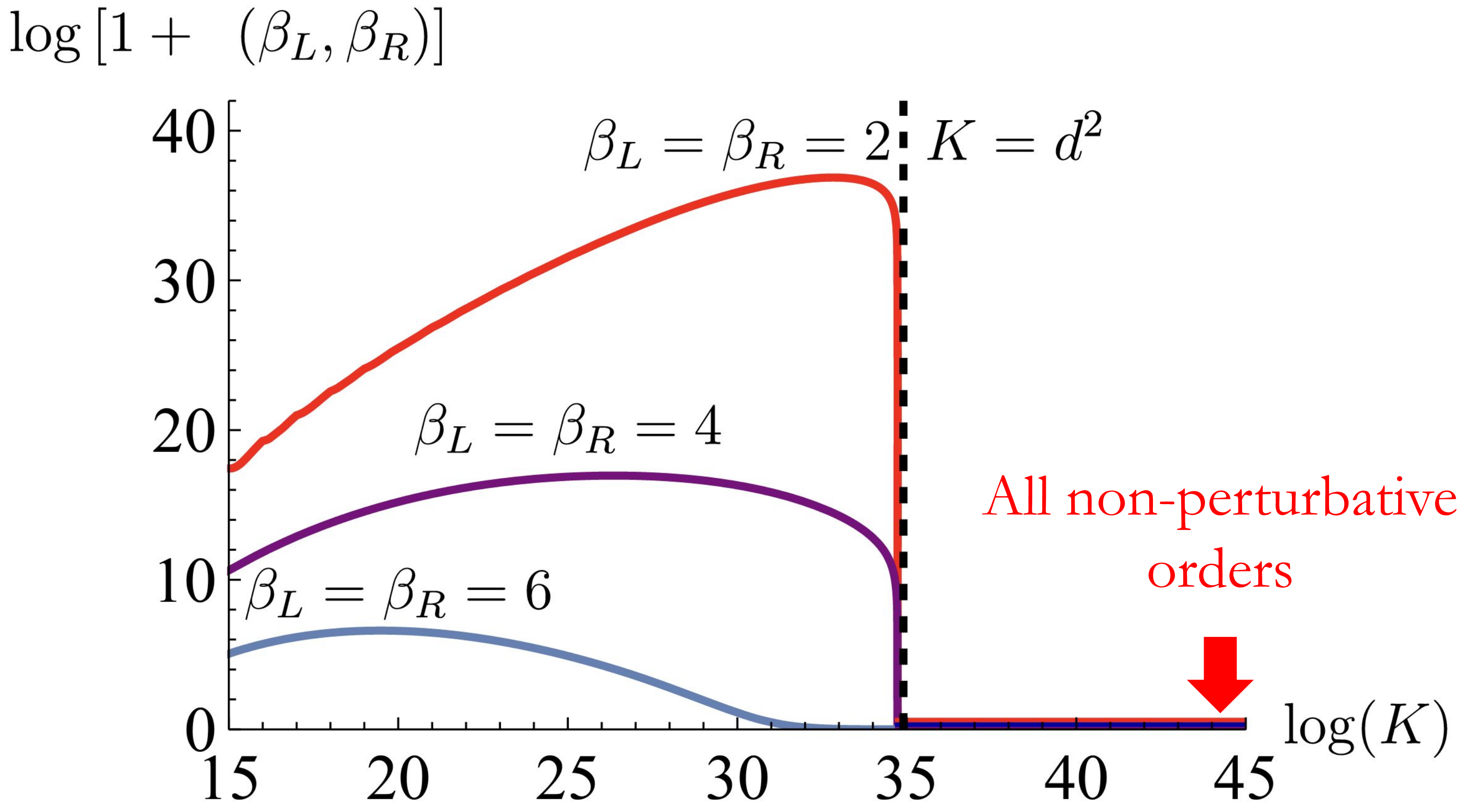
$$= \overline{\mathrm{Tr}_L(e^{-\beta_L^{(1)}H_L})\,\mathrm{Tr}_R(e^{-\beta_R^{(1)}H_R})\dots\mathrm{Tr}_L(e^{-\beta_L^{(n)}H_L})\,\mathrm{Tr}_R(e^{-\beta_R^{(n)}H_R})}$$

to all orders in

Conclusion

Despite only having access to statistics, we can prove that factorisation occurs exactly before coarse-graining.

$$\overline{\quad(\beta_L, \beta_R)} = 0, \quad \sigma_{\ (\beta_L,\beta_R)} = 0\,.$$



Final comments

1) The one-sided trace is now well defined,

$$\mathrm{Tr}_{\mathcal{H}_{bulk}}(K_L K_R) = (\mathrm{Tr}_{\text{one-sided}} K)^2, \qquad K_L = K_R = K \qquad \Rightarrow$$

Requirement for Type I algebra

2) Step 1 only involves saddle-points of the Euclidean path integral $\Rightarrow$ Generalizable to higher dimensions

[Balasubramanian et al. `24, Li `24]

Final comments

3) We assumed there are no degeneracies of the eigenenergies. What if there is a boundary global symmetry (and a bulk gauge symmetry)?

$$\mathrm{Tr}_{\mathcal{H}_{\text{bulk}}}\left(e^{-\beta_L H_L-\alpha_L^{(i)}Q_L^{(i)}}e^{-\beta_R H_R-\alpha_R^{(i)}Q_R^{(i)}}\right)=\mathrm{Tr}_{\mathcal{H}_L}\left(e^{-\beta_L H_L-\alpha_L^{(i)}Q_L^{(i)}}\right)\mathrm{Tr}_{\mathcal{H}_R}\left(e^{-\beta_R H_R-\alpha_R^{(i)}Q_R^{(i)}}\right)$$

To define $\mathcal{H}_{\text{bulk}}$ we need to have particles in all irreps of the gauge group $\Rightarrow$ Completeness conjecture

4) There are exceptions to having degeneracies that can be split by bosonic charges: e.g. BPS states.

$$\mathrm{Tr}_{\mathcal{H}_{\text{bulk}}}(k_L k_R)=\mathrm{Tr}_{\mathcal{H}_L}(k_L)\,\mathrm{Tr}_{\mathcal{H}_R}(k_R)\,,\qquad k_{L,R}=e^{-\alpha_{L,R}\overbrace{\pi_{\text{BPS}}\mathcal{O}_{L,R}\pi_{\text{BPS}}}^{\text{Simple operator}}}.$$

we can prove that this occurs exactly when $K>d_{\text{BPS}}^2$. $\Rightarrow$ Factorisation for two-sided BPS BHs

Final comments

5) Null states galore! $|E_L,\, E_R, \Delta\rangle,\ |\ell, \Delta, m\rangle \quad \Rightarrow \quad |E_L^{(i)}, E_R^{(j)}\rangle$

[see **Iliesiu**, Levine, Lin, Maxfield, Mezei `24]

$$\Downarrow$$

No linear operator measures connectedness and the length stops having a geometric meaning.

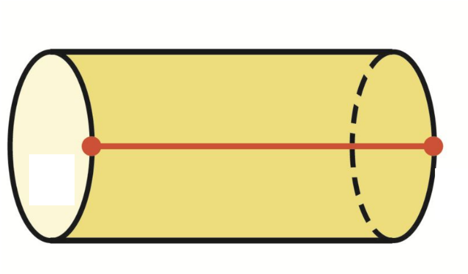
What does this imply for an infalling observer?

Do the observables they measure correspond to linear operators on

Final comments

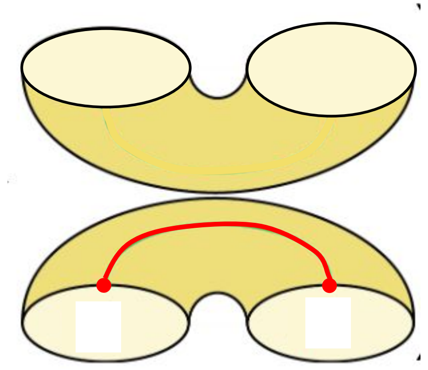
Issue: No good non-perturbative notion of an observer with respect to which we can dress the observables that they measure.

In two-sided black holes,



[see **Iliesiu**, Levine, Lin, Maxfield, Mezei `24 for examples]

Even more problematic, in a closed universe,



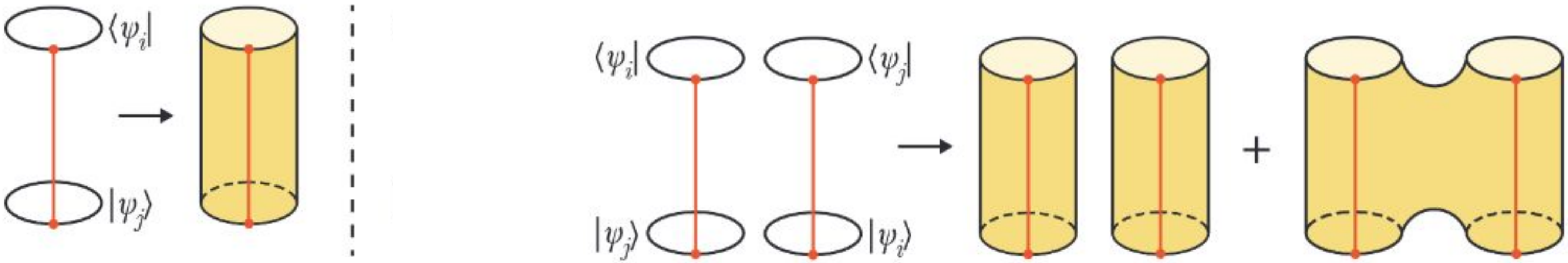
[Marolf, Maxfield `20, Usatyuk, Wang, Zhao `24]

Final comments

[Abdalla, Antonini, **Iliesiu**, Levine `25]
[see also Harlow, Usatyuk, Zhao `25]

The inner-products in the Hilbert space that describes the physics seen by an observer,

Closed universe example



The same resolvent computation from before now yields,

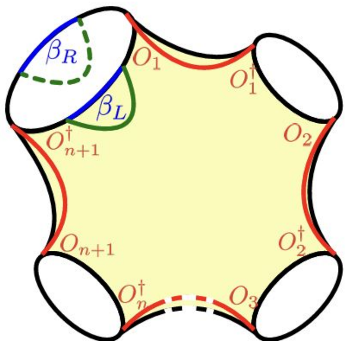
Concrete framework to quantify the non-perturbative quantum gravity effects seen by a gravitating observer.

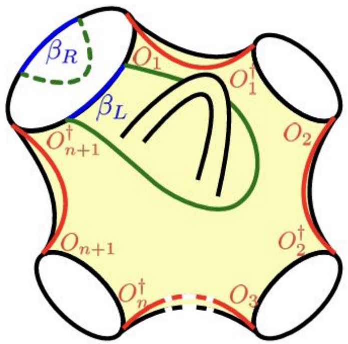
Thank you!

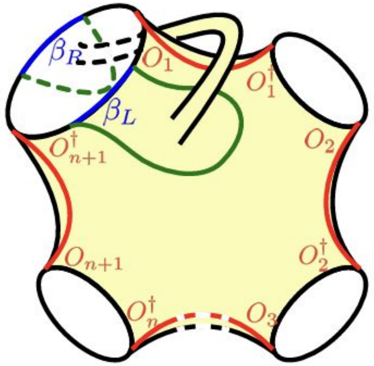
Additional details

Going to even higher orders in

We can include higher genus corrections,

$$\overline{\mathbf{R}_{ij} \langle q_i | \, k_L k_R \, | q_j \rangle} =$$ 

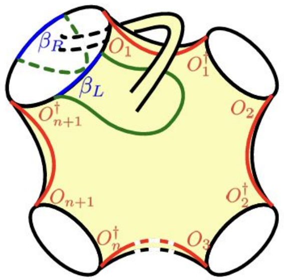
$+$ 

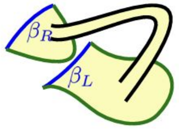
$+$ 

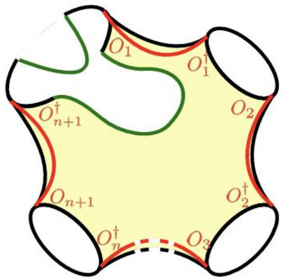
$+ \cdots$

but suppress non-planar index contractions by taking $K \to \infty$ first.

We can perform a decomposition into geometries we've encountered before:



$= \int d\ell_L d\ell_R e^{\ell_L} e^{\ell_R}$ 

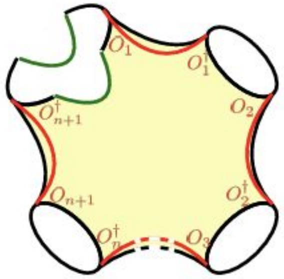
$\times$ 

,

The same pinwheel geometry was encountered at finite K:

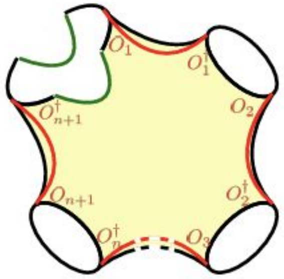
$$Tr(k_L k_R) = \langle \beta_L | \hat{O} | \beta_R \rangle = \int d\ell_L d\ell_R e^{\ell_L} e^{\ell_R}$$

$\times$



Projector onto infinite temperature state

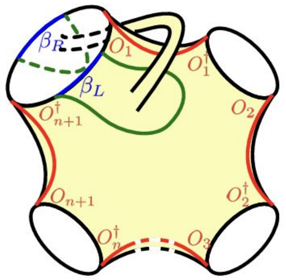
Using the results,



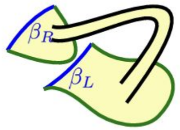
$$= e^{2S_0} \varphi_{\beta=0}(\ell_L) \varphi_{\beta=0}(\ell_R),$$

when $K > d^2$.

Step 1: Going to even higher orders



$$= \int d\ell_L d\ell_R e^{\ell_L} e^{\ell_R}$$



$$\times$$

$$\Downarrow$$

$$\overline{\mathrm{Tr}_{\mathcal{H}_{bulk}(K>d^2)}\left(e^{-\beta_L H_L} e^{-\beta_R H_R}\right)} = \underbrace{\overline{\mathrm{Tr}_{\mathcal{H}_L}\left(e^{-\beta_L H_L}\right) \mathrm{Tr}_{\mathcal{H}_R}\left(e^{-\beta_R H_R}\right)}}_{\int_{\mathcal{E}} dE_L dE_R \overline{\rho(E_L)\,\rho(E_R)}\, e^{-\beta_L E_L - \beta_R E_R}} + O\left(\frac{1}{K}\right).$$

Consistent with factorisation before coarse-graining. However, we want to probe factorisation more directly.

Step 2 & 3: Going to even higher orders

We can similarly show that,

$$\overline{\mathrm{Tr}_{\mathcal{H}_{\mathrm{bulk}}(K)}(e^{-\beta_L^{(1)}H_L}e^{-\beta_R^{(1)}H_R})\dots\mathrm{Tr}_{\mathcal{H}_{\mathrm{bulk}}(K)}(e^{-\beta_L^{(n)}H_L}e^{-\beta_R^{(n)}H_R})} =$$

$$= \overline{\mathrm{Tr}_L(e^{-\beta_L^{(1)}H_L})\,\mathrm{Tr}_R(e^{-\beta_R^{(1)}H_R})\dots\mathrm{Tr}_L(e^{-\beta_L^{(n)}H_L})\,\mathrm{Tr}_R(e^{-\beta_R^{(n)}H_R})}$$

Considering two 2-sided traces:

$$\overline{d(\beta_L,\beta_R)} \equiv \mathcal{D}_{\beta_L,\beta_R,\beta'_L,\beta'_R}\overline{Z_{\mathrm{bulk}}(\beta_L,\beta_R)Z_{\mathrm{bulk}}(\beta'_L,\beta'_R)}|_{\beta'_{L/R}=\beta_{L/R}} \qquad \mathcal{D}_{\beta_L,\beta_R,\beta'_L,\beta'_R} \equiv \partial_{\beta_L}\partial_{\beta_R} - \partial_{\beta_L}\partial_{\beta'_R}$$

$$\Downarrow$$

$$\overline{d(\beta_L,\beta_R)} = 0$$

Considering four 2-sided traces:

$$\overline{d(\beta_L,\beta_R)^2} = 0$$

Details about the dimension of

The same resolvent computation from before now yields,