

Crystal Melting and Double Quiver Algebras

from Jeffrey-Kirwan Residues

Masahito Yamazaki

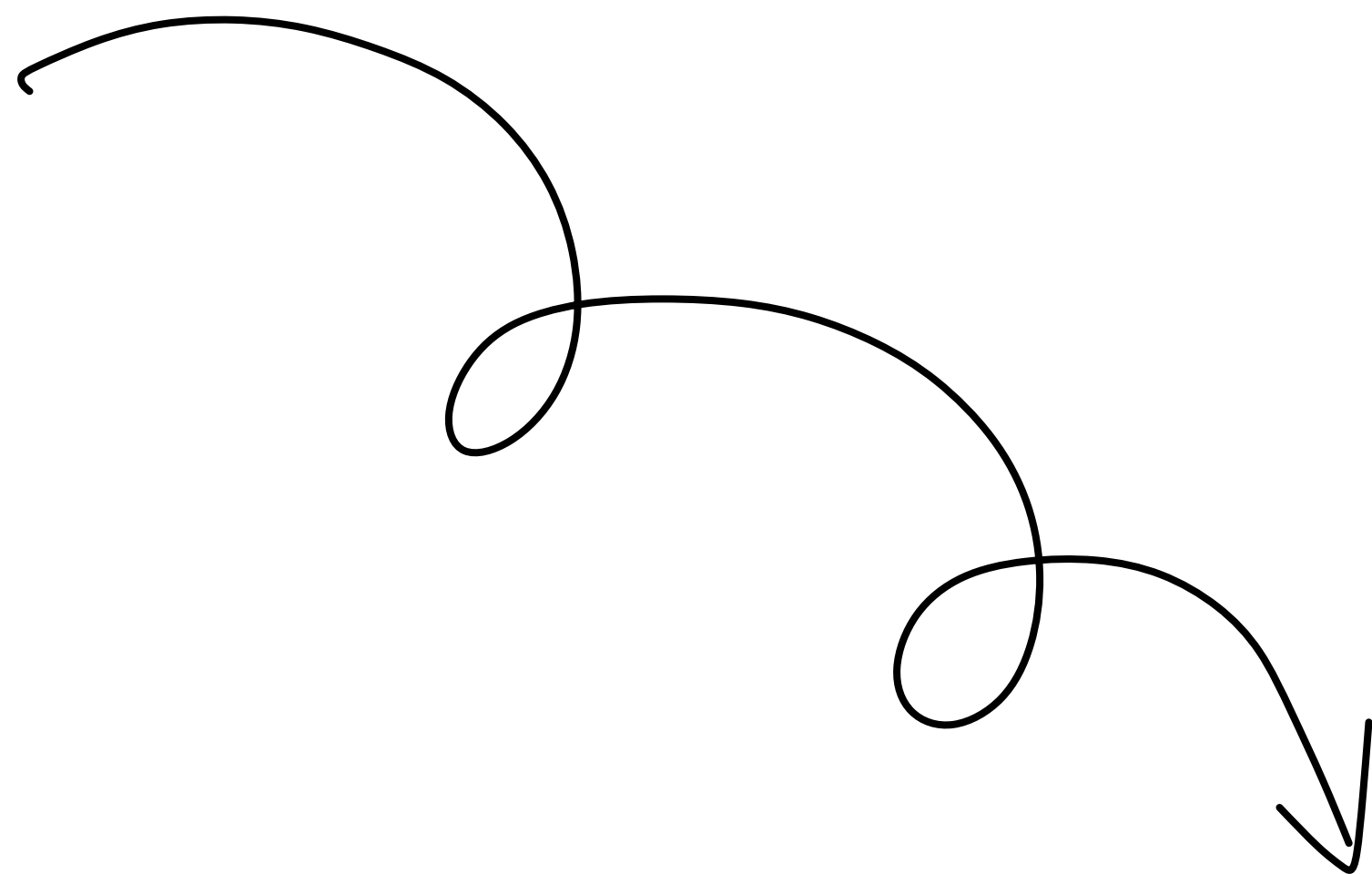


THE UNIVERSITY OF TOKYO

Jan. 7, Strings 2025 @ NYU Abu Dhabi



2008

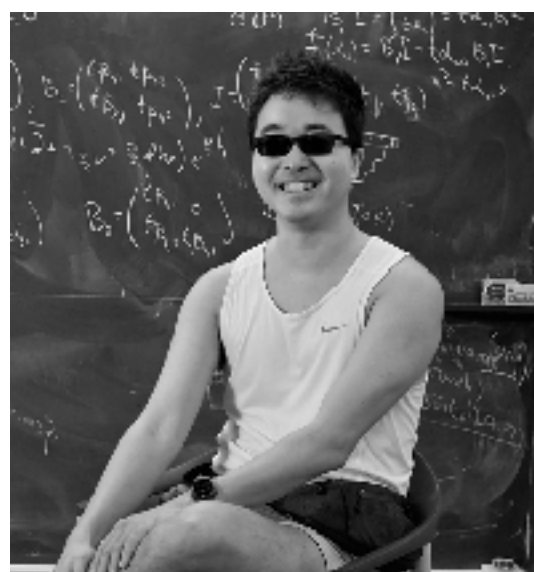


2025



الصبر مفتاح الفرج

“Patience is the key to relief.”



Counting of BPS States !!

- dualities in gauge/string theory
 - microstates of black holes
 - mathematics of enumerative invariants
- ⋮

Based on:

Jiakang Bao + MY (To Appear Tonight!!)



See also

Jiakang Bao, Rak-Kyeong Seong + MY (2401.02792)

and earlier papers with

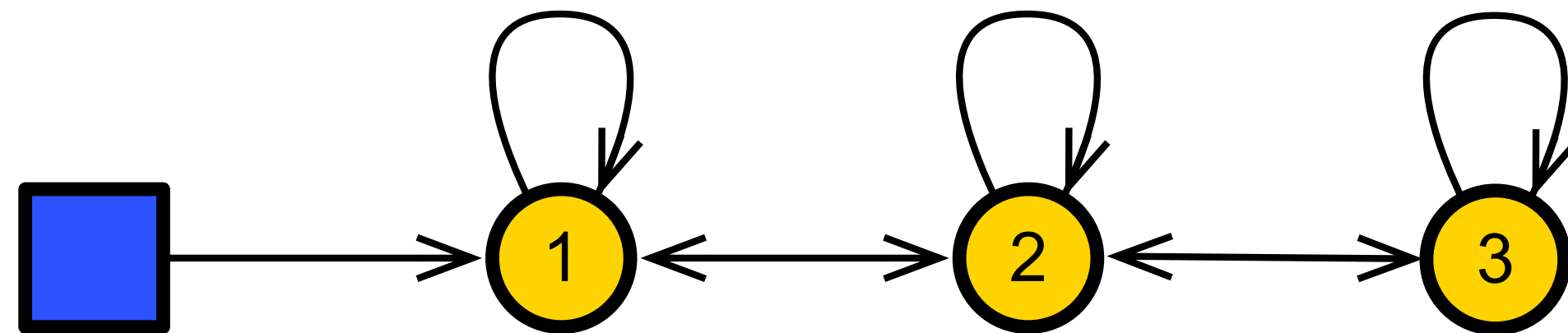
Dimitry Galakhov, Wei Li, Hiroshi Ooguri, ... (2008-)

Today: BPS / Witten indices

for $N \geq 2$ quiver gauge theories

4d $N=1$

$N=4$

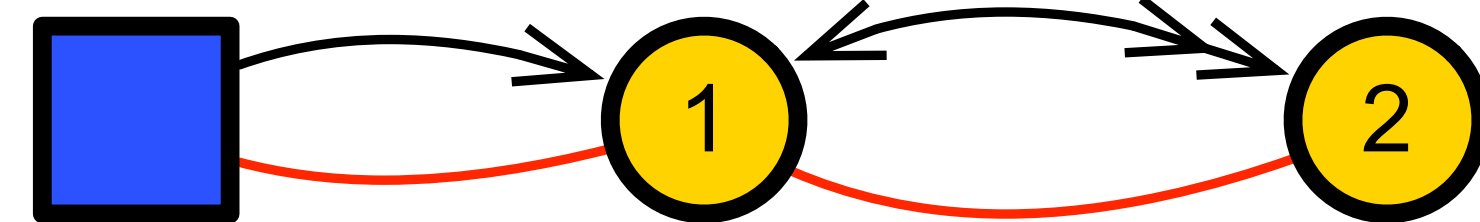


+ F-term

($\rightarrow$ chiral)

2d $N=(0,2)$

$N=2$



+ J/E-term

($\rightarrow$ chiral — Fermi)

Main Results in [Bao-MY]

① Counting

$$Z_{\text{BPS}} = Z_{\text{crystal}}$$

$$= \sum_e w(e)$$

e : crystal state
combinatorial description
 $w(e)$: weight

② Algebra

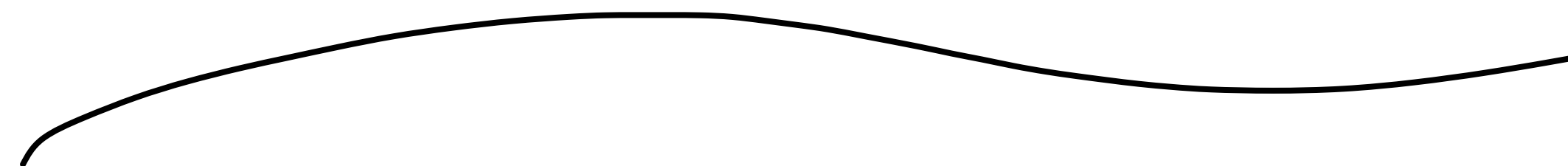
$$Z_{\text{BPS}} = \chi_Y$$

$$\tilde{Y} \supset \mathcal{R} = \{ |e\rangle \}$$

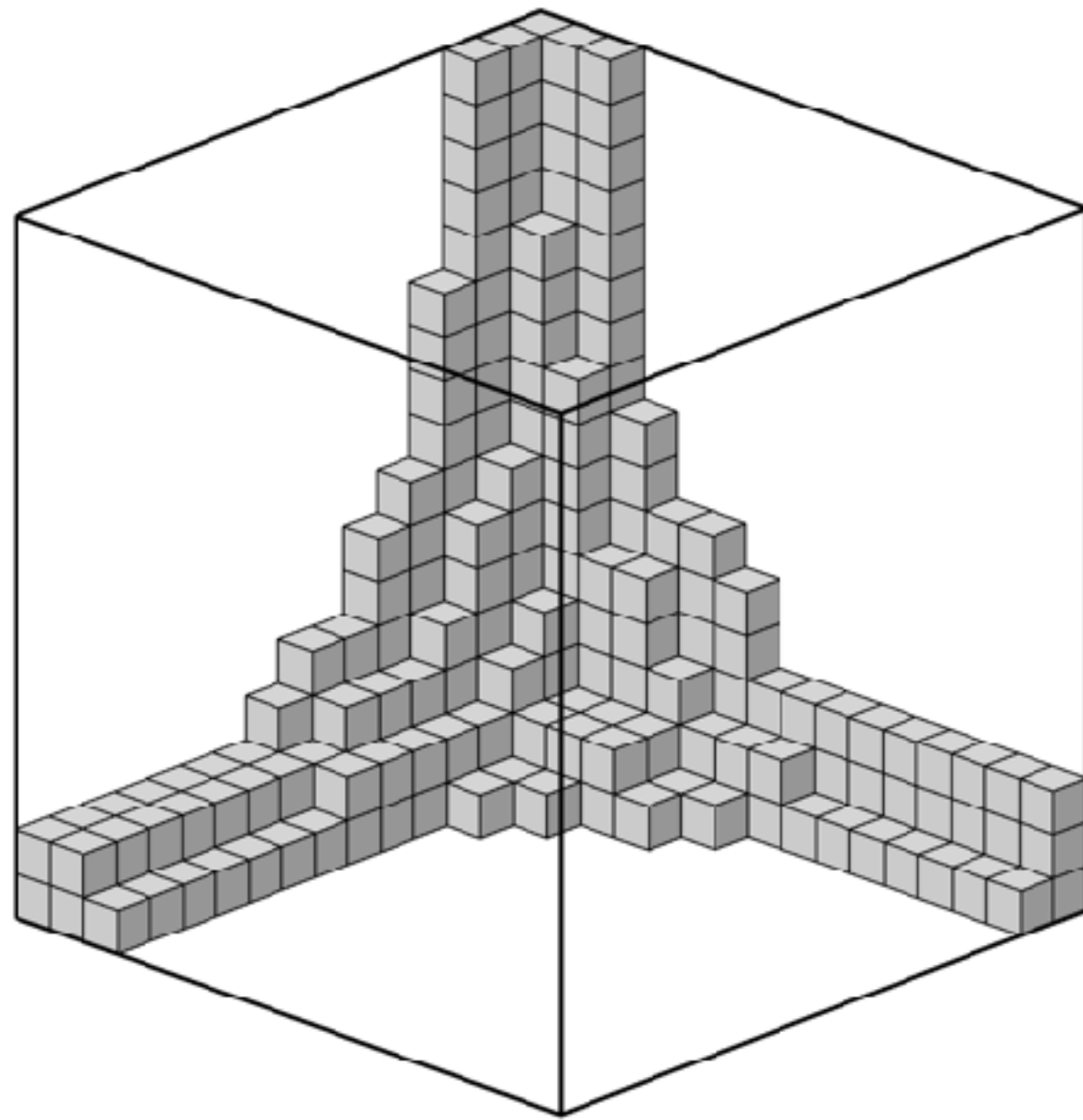
representation R

$\tilde{Y}$: new algebra
double quiver algebra

Counting



< Hint 1: Crystal Metting for toric CY_3 / CY_4 >



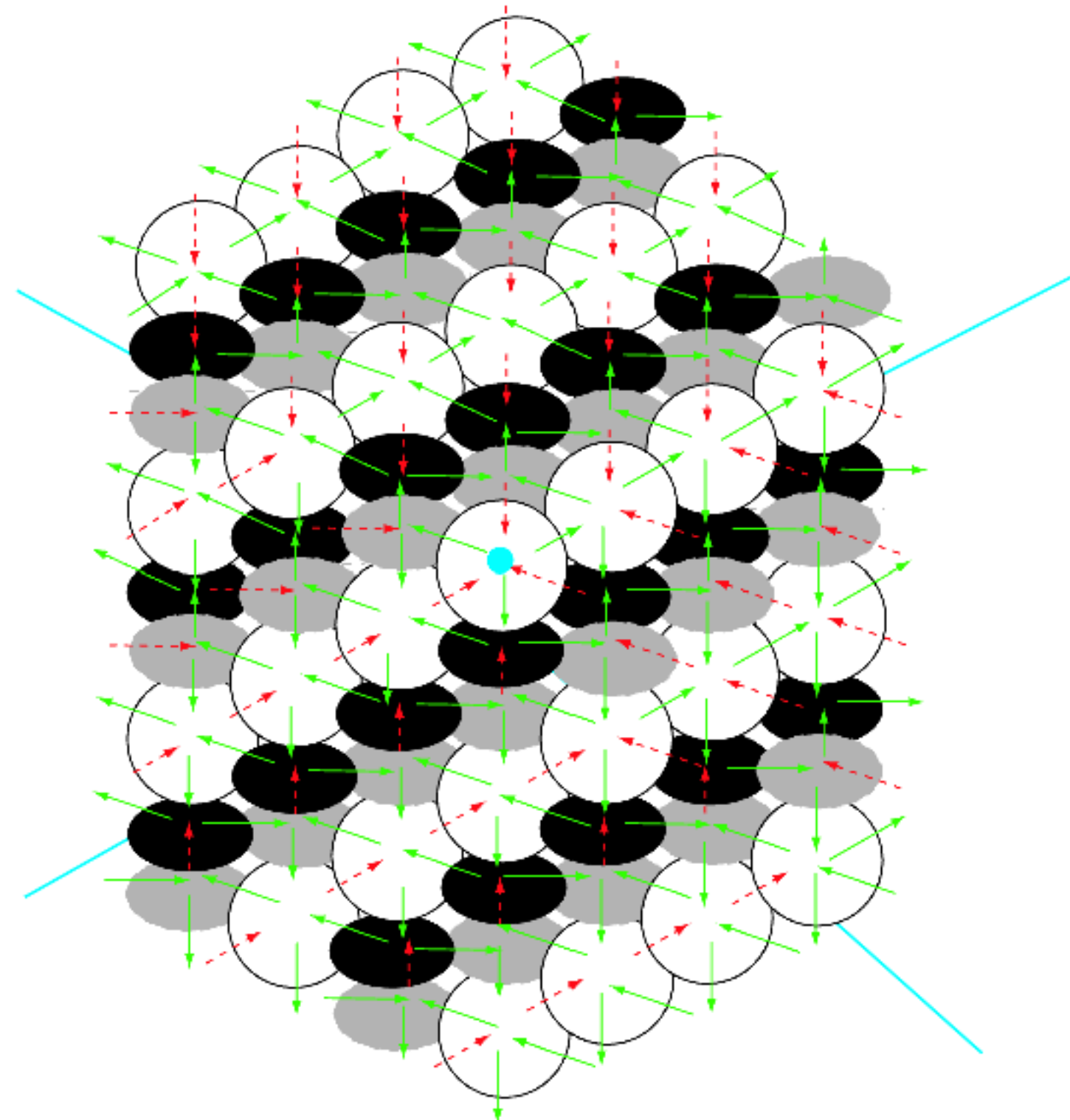
$\mathbb{C}^3$

[Okounkov - Reshetikhin - Vafa '03]

C. Vafa @ Strings 2004



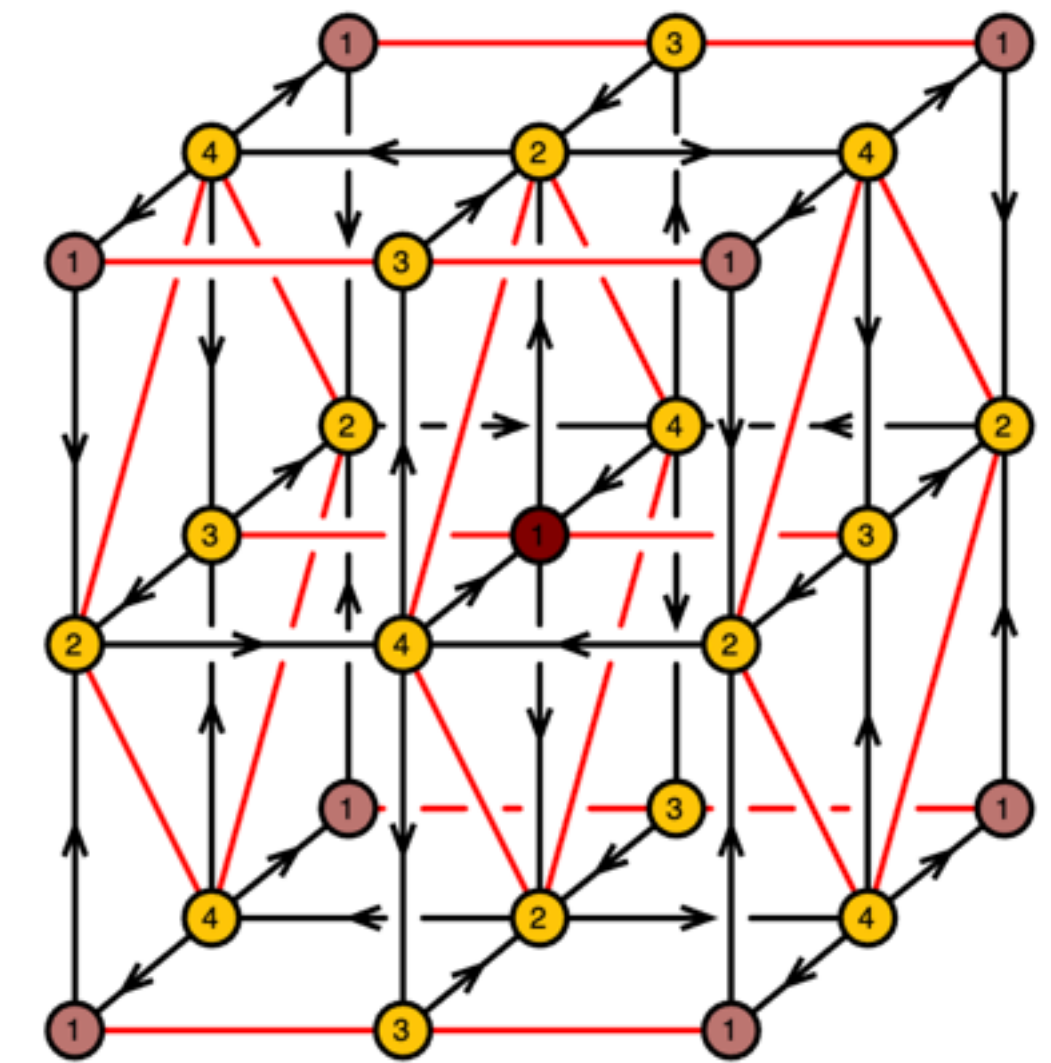
3d crystal



Toric CY_3

[Ooguri - MY '08]

H. Ooguri @ Strings 2009



toric CY_4

[Bao - Seong - MY '24]

4d crystal



< Hint 2 : JK residue formula for index >

[2d on T^2 : Benini-Eager-Hori-Tachikawa '13]
[1d on S^1 : Hori-Kim-Yi, Cordova-Shao '14]

$$Z(\epsilon) = \frac{1}{|W|} \sum_{u^* \in \mathcal{M}^*} \text{JK-Res}(\gamma) Z_{\text{1-loop}}(u, \epsilon)$$

< Hint 2 : JK residue formula for index >

[2d on T^2 : Benini-Eager-Hori-Tachikawa '13]
[1d on S^1 : Hori-Kim-Yi, Cordova-Shao '14]

$$Z(\epsilon) = \frac{1}{|W|} \sum_{u^* \in \mathcal{U}^*} \underbrace{JK\text{-Res}(\gamma)}_{u=u^*} Z_{\text{1-loop}}(u, \epsilon)$$

iterative residue [Szecseny-Vergne '03]

< Hint 2 : JK residue formula for index >

[2d on T^2 : Benini - Eager - Hori - Tachikawa '13]
[1d on S^1 : Hori - Kim - Yi, Cordova - Shao '14]

$$Z(\epsilon) = \frac{1}{|W|} \sum_{u^* \in \mathcal{M}^*} \text{JK-Res}(\underbrace{1}_{u=u^*}) Z_{\text{loop}}(u, \epsilon)$$

"separates particle/anti-particle"

(moduli dependence)

< Hint 2 : JK residue formula for index >

[2d on T^2 : Benini-Eager-Hori-Tachikawa '13]
 [1d on S^1 : Hori-Kim-Yi, Cordova-Shao '14]

$$Z(\epsilon) = \frac{1}{|W|} \sum_{u^* \in \mathcal{U}^*} \text{JK-Res}(\gamma) Z_{1\text{-loop}}(u, \epsilon)$$

flavor
chemical
potential

singular point
in the plane of

$$u = \{u_i\}_{i=1}^N \in h \subset g$$

1-loop
determinant

< Hint 2 : JK residue formula for index >

[2d on T^2 : Benini-Eager-Hori-Tachikawa '13]
[1d on S^1 : Hori-Kim-Yi, Cordova-Shao '14]

$$Z(\epsilon) = \frac{1}{|W|} \sum_{u^* \in \mathcal{U}^*} \underbrace{JK\text{-Res}(u)}_{u=u^*} Z_{\text{loop}}(u, \epsilon)$$

iterative residue [Szenes-Vergne '03]

Sum over
crystal state

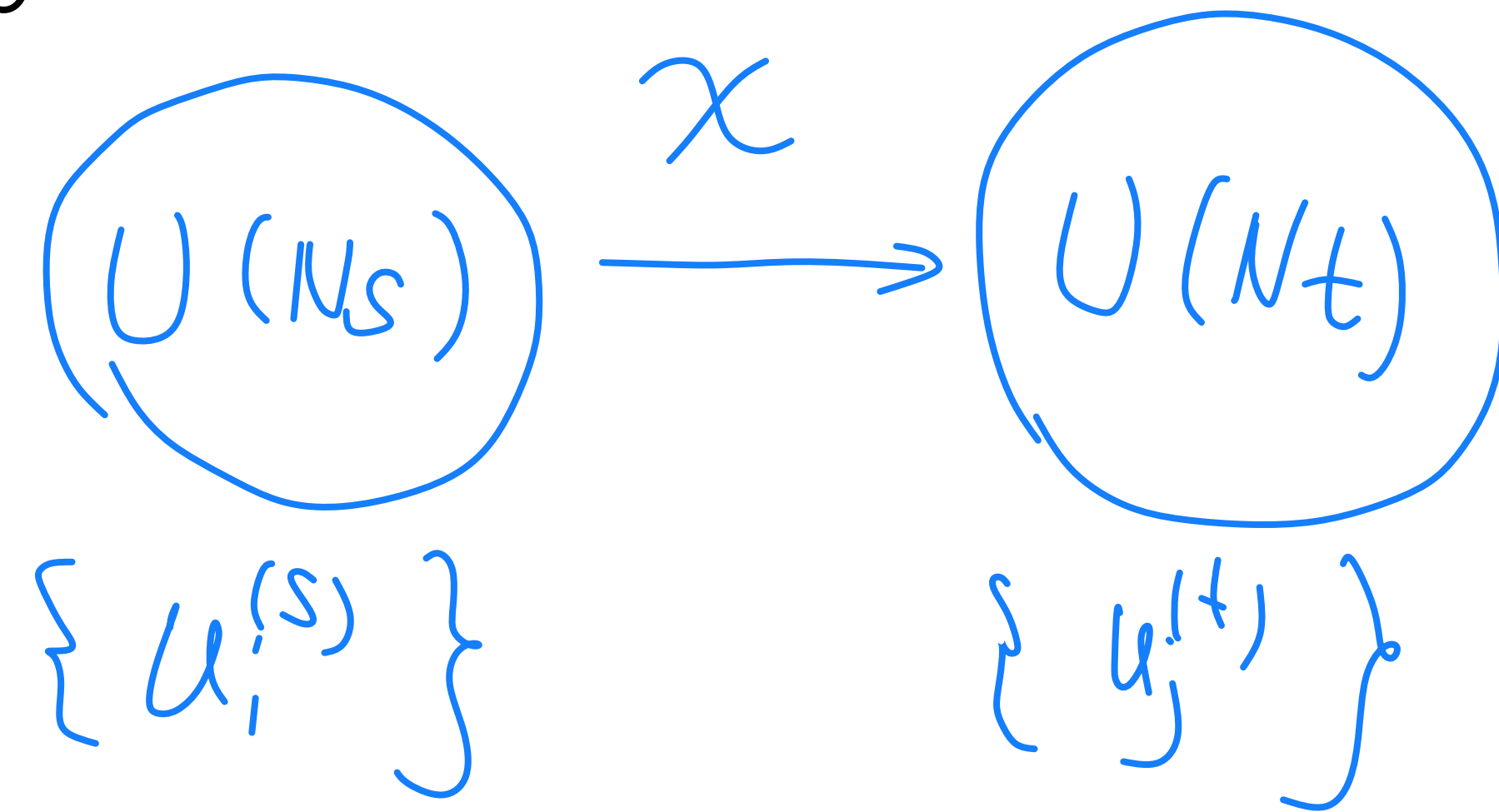
growth of crystal molecules
from atoms

Z_{1-loop} contains contributions from different multiplets

e.g. bifundamental $N=2$ chiral w/ charge ϵ_χ

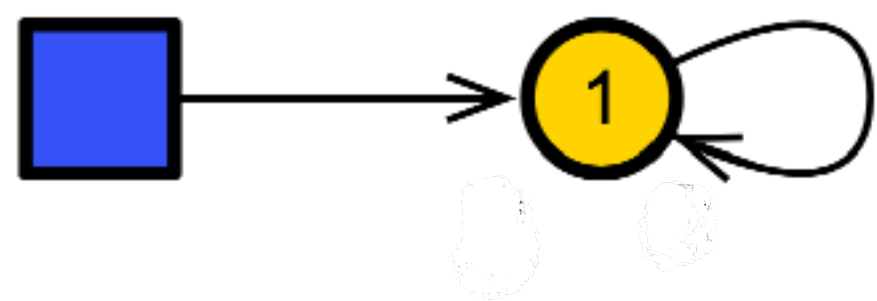
$$Z_\chi(\epsilon, u) = \prod_{i=1}^{N_s} \prod_{j=1}^{N_t} \frac{1}{\zeta(u_j^{(t)} - u_i^{(s)} - \epsilon_\chi)}$$

Pole @ $u_j^{(t)} - u_i^{(s)} - \epsilon_\chi = 0$



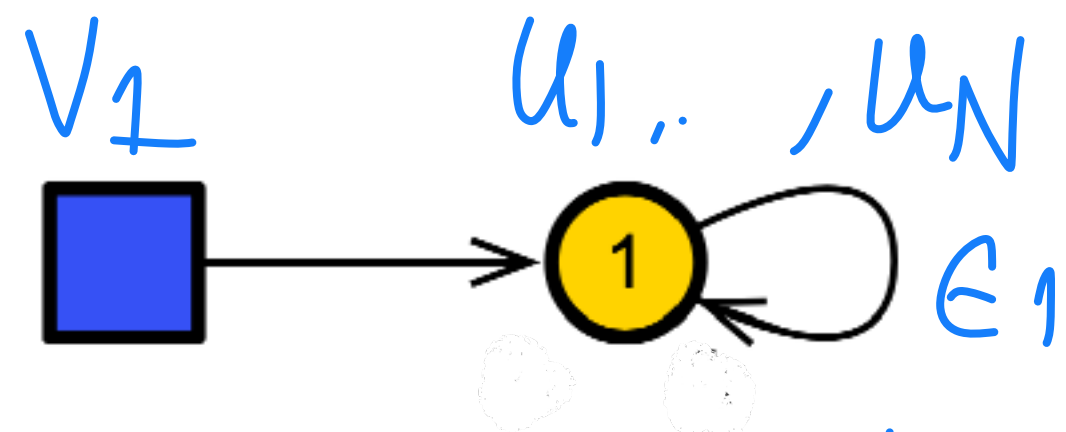
$$\left(\zeta(z) = \begin{cases} \frac{i\theta_1(\tau, z)}{\eta(\tau)}, & \leftarrow \text{ell.} & (2d \text{ on } T^2) \\ 2i \sin(\pi z), & \leftarrow \text{trig.} & (1d \text{ on } S^1) \\ z, & \leftarrow \text{rational} & (0d) \end{cases} \right)$$

Example



($N=4$ quiver, $W=\sigma$)

Example



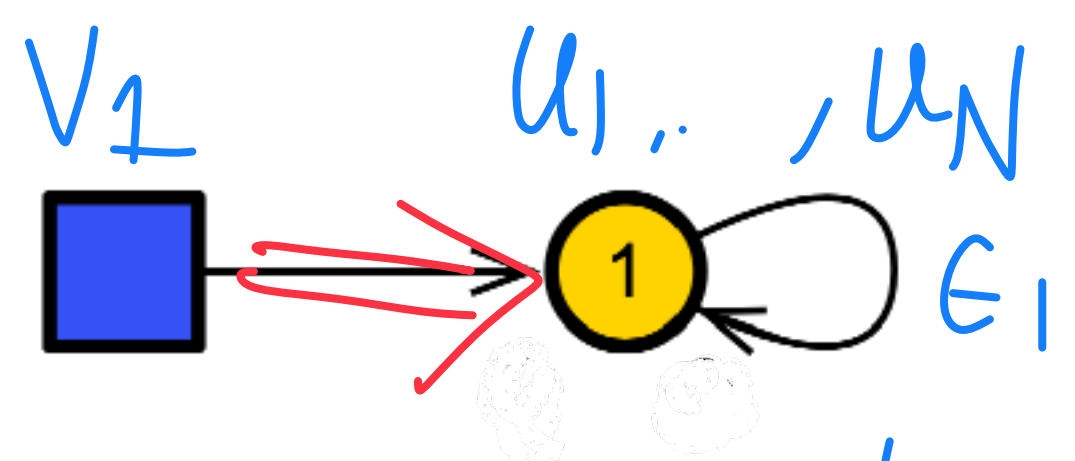
($N=4$ quiver, $W=\emptyset$)

$$\prod_{i=1}^N \frac{-\zeta(u_i + \epsilon - v_1)}{\zeta(u_i - v_1)}$$

$$\frac{-\zeta(\epsilon - \epsilon_1)}{\zeta(-\epsilon_1)} \prod_{i \neq j}^N \frac{-\zeta(u_i - u_j + \epsilon - \epsilon_1)}{\zeta(u_i - u_j - \epsilon_1)}$$

change of
chiral.

Example



($N=4$ quiver, $W=\emptyset$)

change of
chiral.

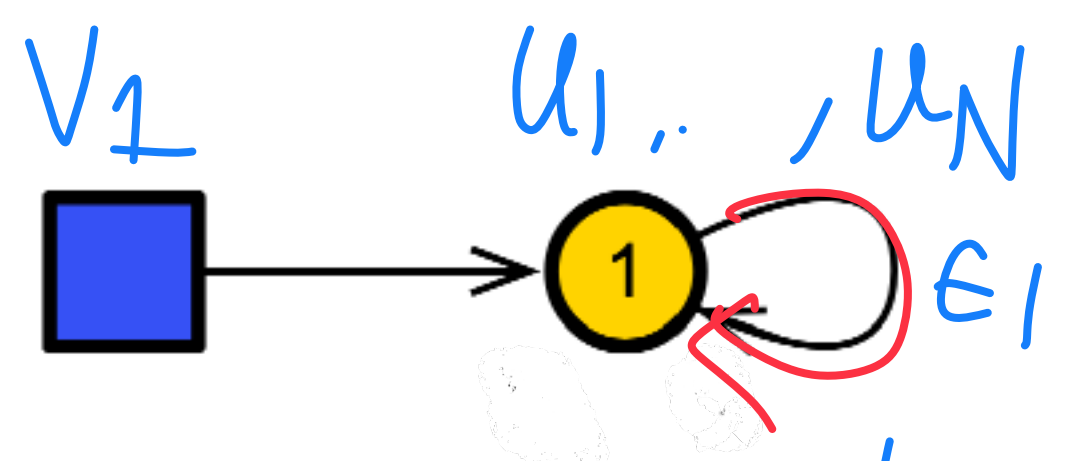
$$\prod_{i=1}^N \frac{-\zeta(u_i + \epsilon - v_1)}{\zeta(u_i - v_1)}$$

$$\frac{-\zeta(\epsilon - \epsilon_1)}{\zeta(-\epsilon_1)} \prod_{i \neq j}^N \frac{-\zeta(u_i - u_j + \epsilon - \epsilon_1)}{\zeta(u_i - u_j - \epsilon_1)}$$

$N=1$ $\{ u_1 - v_1 = 0 \}$

$u_i^* = v_1$

Example



($N=4$ quiver, $W=\emptyset$)

change of
chiral.

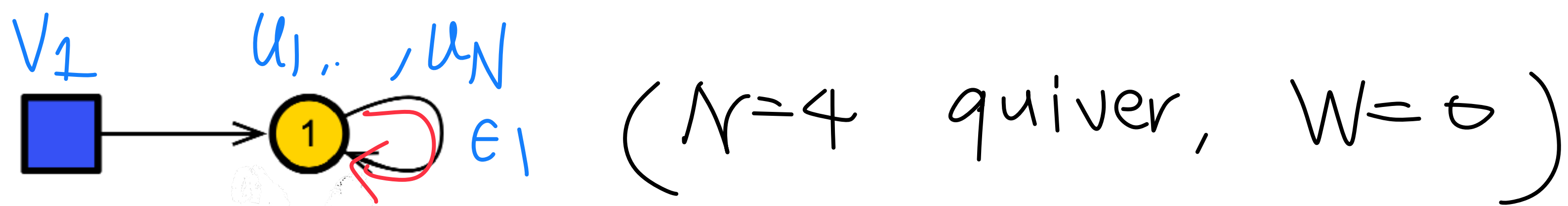
$$\prod_{i=1}^N \frac{-\zeta(u_i + \epsilon - v_1)}{\zeta(u_i - v_1)}$$

$$\frac{-\zeta(\epsilon - \epsilon_1)}{\zeta(-\epsilon_1)} \prod_{i \neq j}^N \frac{-\zeta(u_i - u_j + \epsilon - \epsilon_1)}{\zeta(u_i - u_j - \epsilon_1)}$$

$N=1$ $\{u_1 - v_1 = 0\}$
 $N=2$ $\{u_1 - v_1 = 0, u_2 - u_1 = \epsilon_1\}$

$u_1^* = v_1$
 $u_2^* = v_1 + \epsilon_1$

Example



change of
chiral.

$$\prod_{i=1}^N \frac{-\zeta(u_i + \epsilon - v_1)}{\zeta(u_i - v_1)}$$

$$\frac{-\zeta(\epsilon - \epsilon_1)}{\zeta(-\epsilon_1)} \prod_{i \neq j}^N \frac{-\zeta(u_i - u_j + \epsilon - \epsilon_1)}{\zeta(u_i - u_j - \epsilon_1)}$$

$N=1$

$\{u_1 - v_1 = 0\}$

$u_1^* = v_1$

$N=2$

$\{u_1 - v_1 = 0, u_2 - u_1 = \epsilon_1\}$

$u_2^* = v_1 + \epsilon_1$

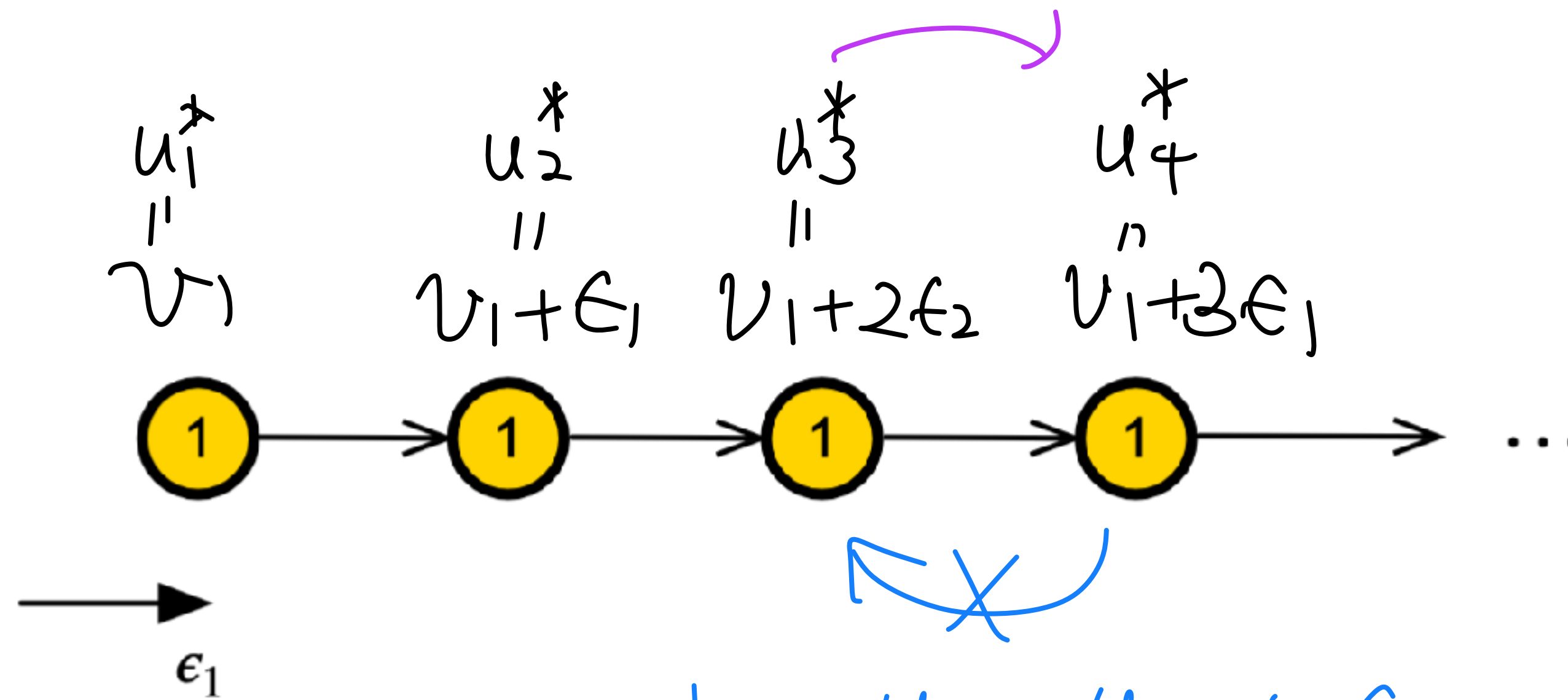
$N=3$

$\{u_1 - v_1 = 0, u_2 - u_1 = \epsilon_1, u_3 - u_2 = \epsilon_1\}$

$u_3^* = v_1 + 2\epsilon_1$

$$\text{pole } u_4 - u_3 - \epsilon_1 = 0$$

"particle"

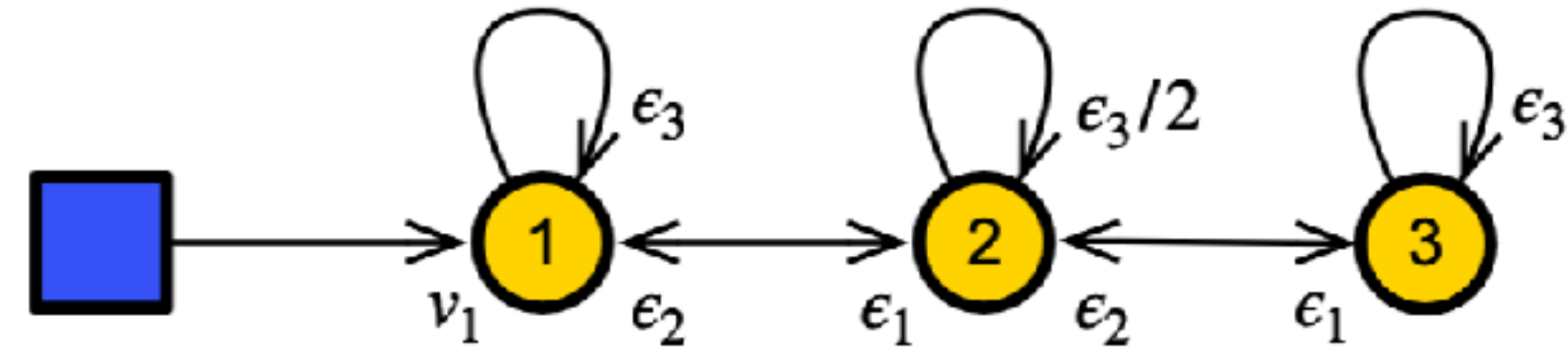
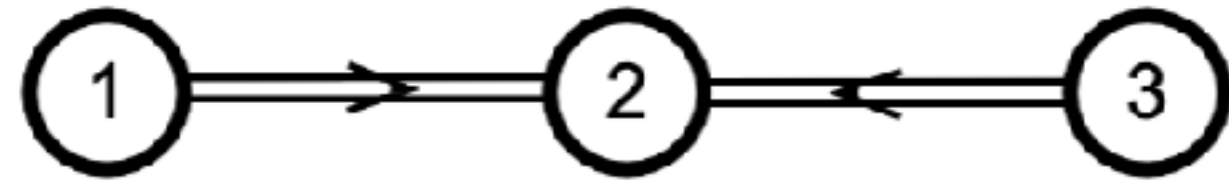


$$\text{pole } u_4 - u_3 + \epsilon_1 = 0$$

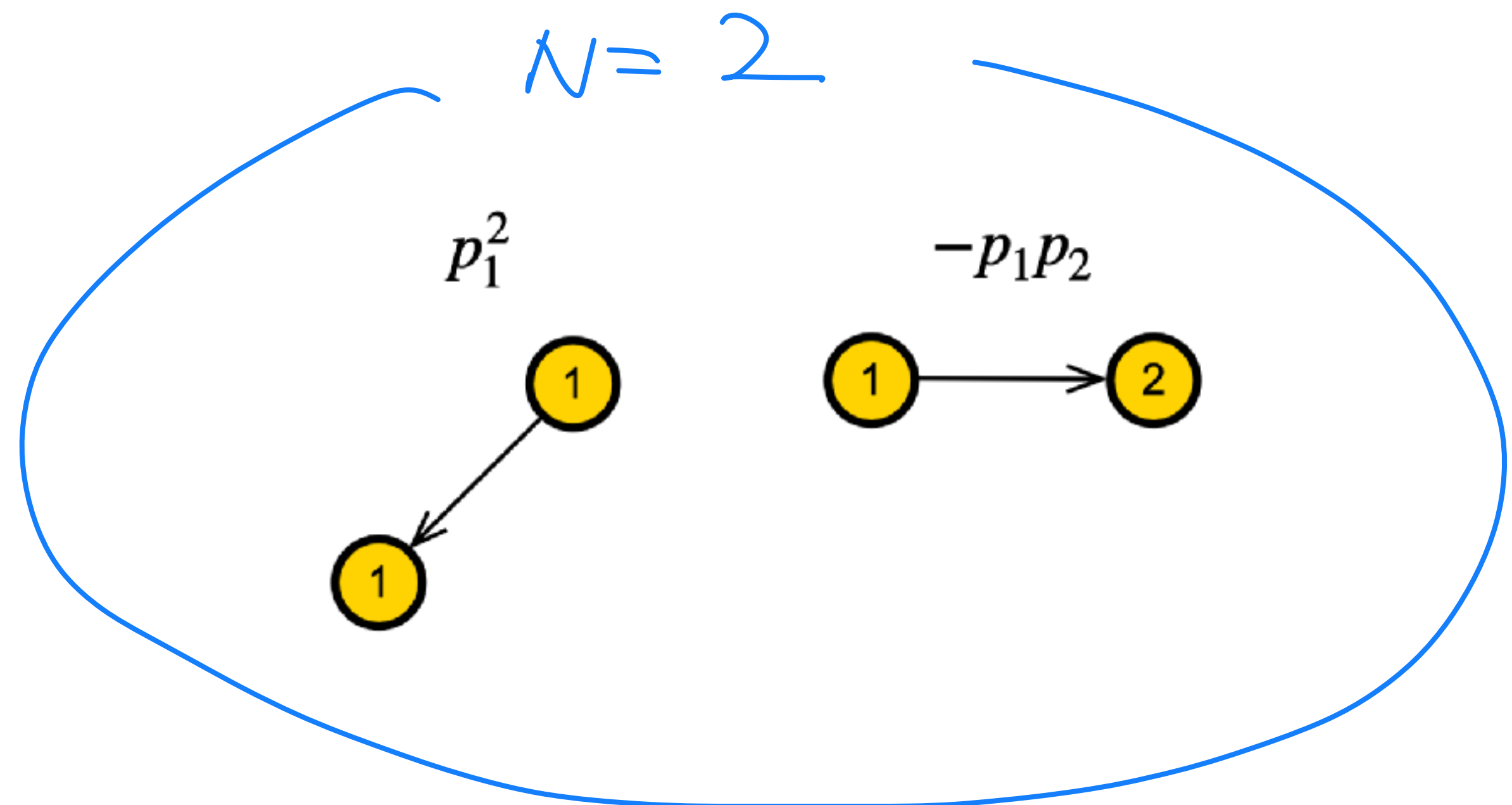
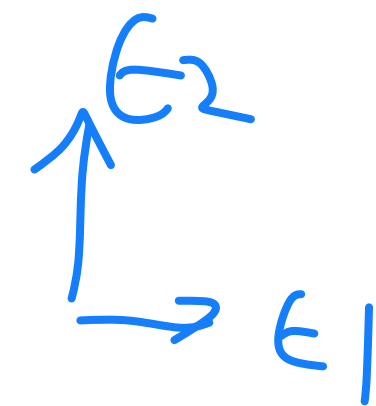
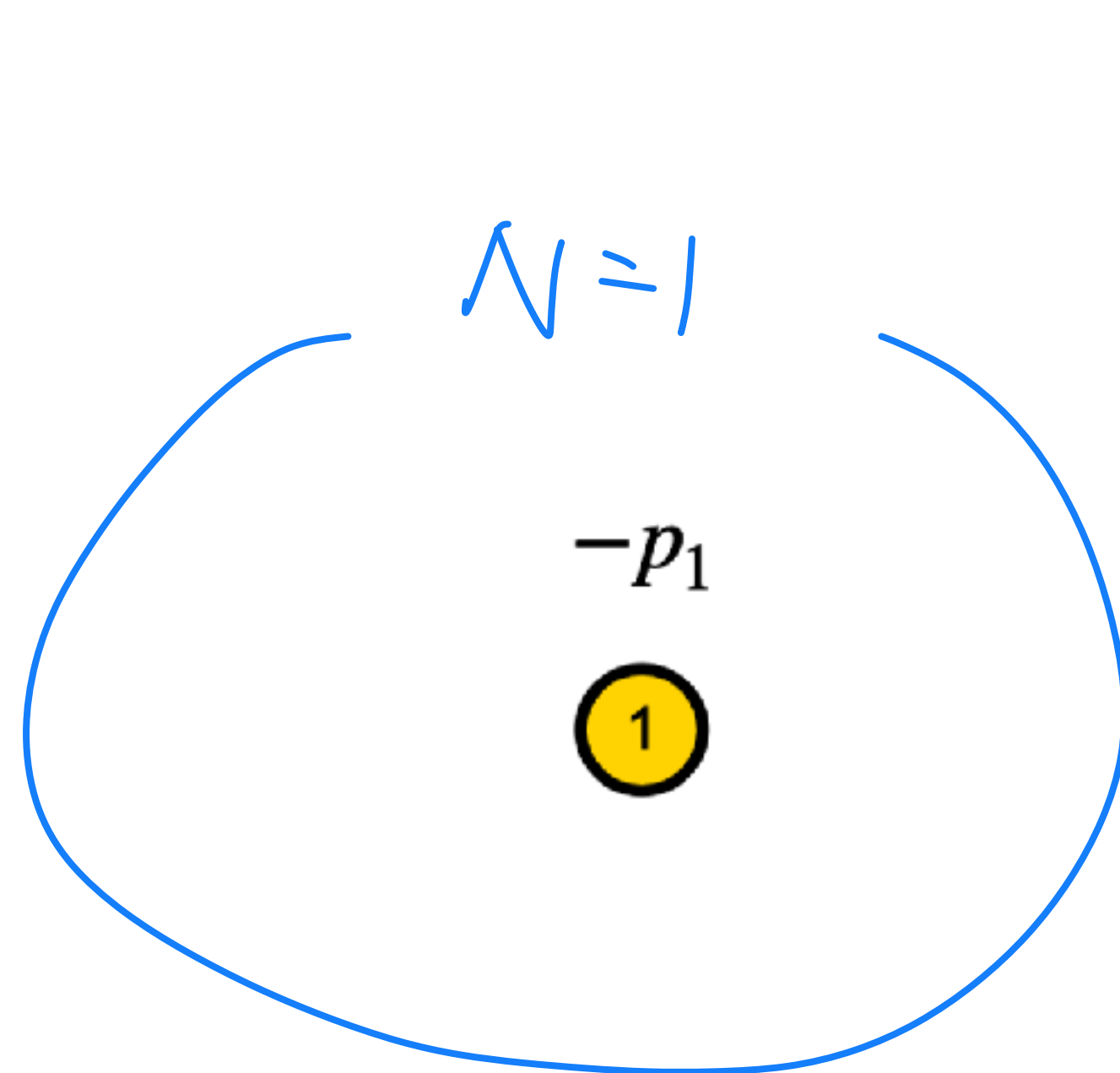
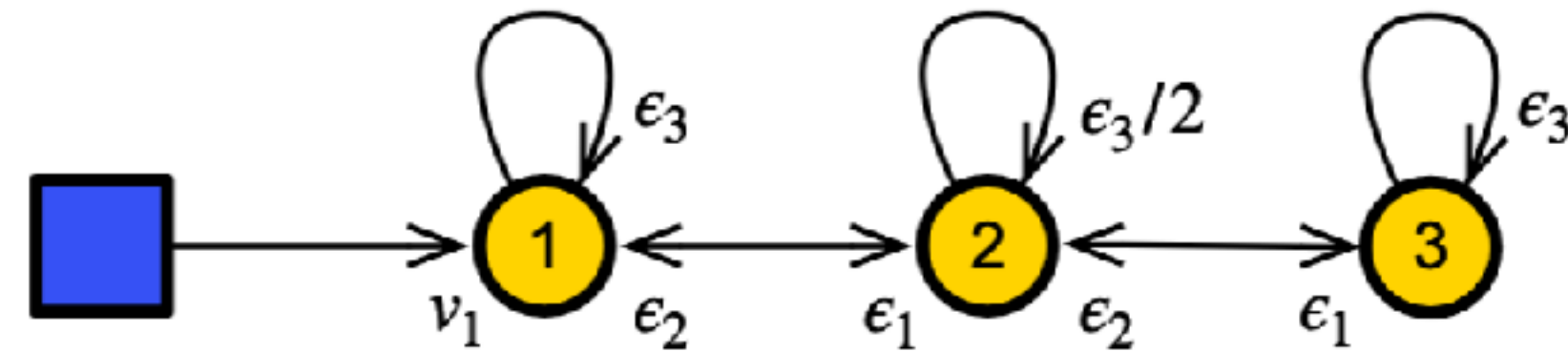
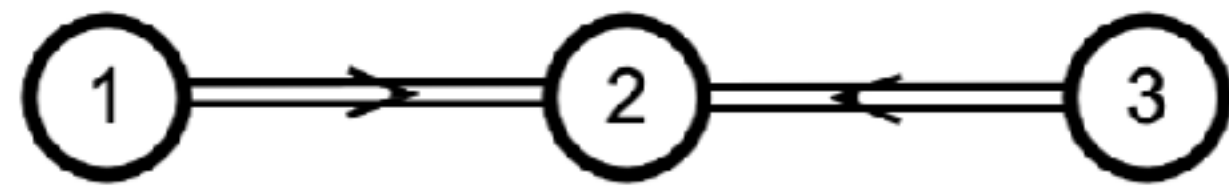
"anti-particle"

but not allowed under η

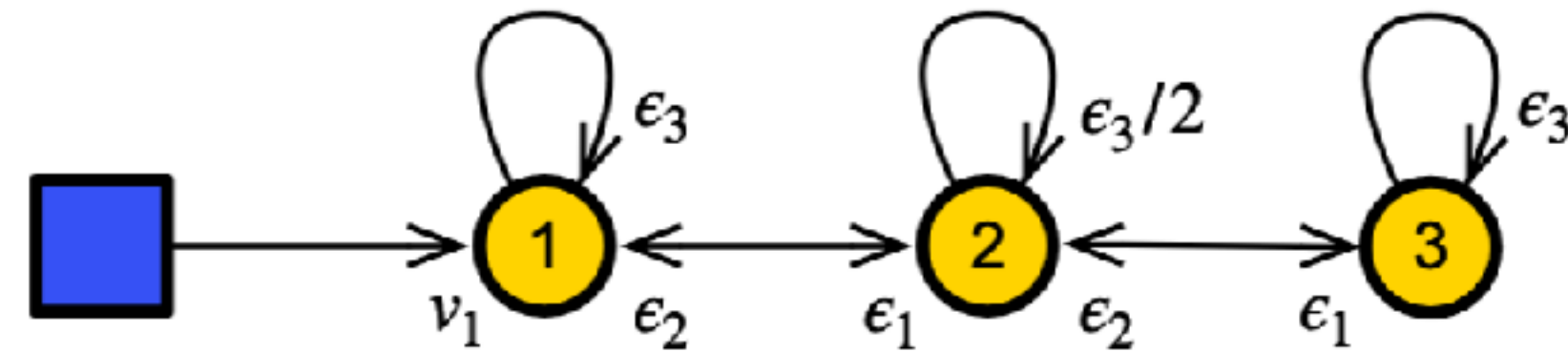
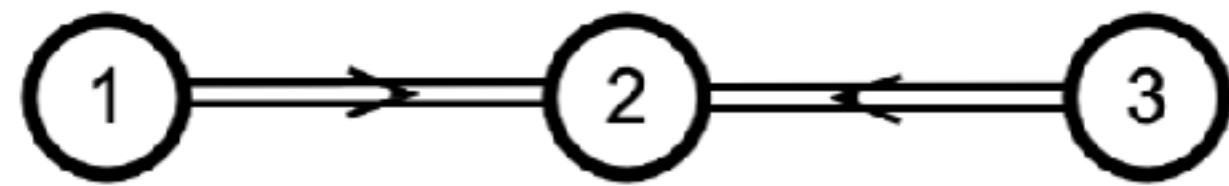
Example : $C_2^{(1)}$ Theory



Example : $C_2^{(1)}$ Theory

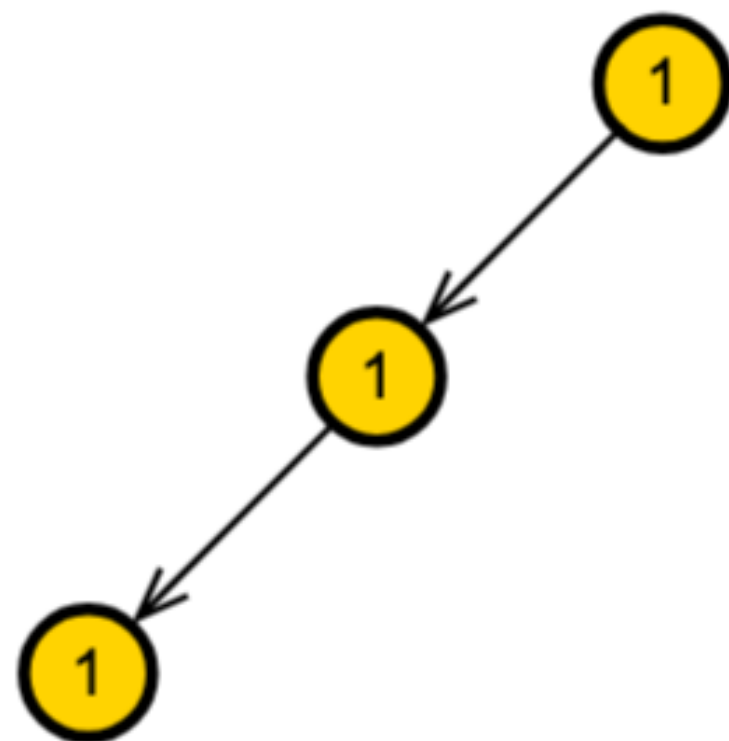


Example: $C_2^{(1)}$ Theory

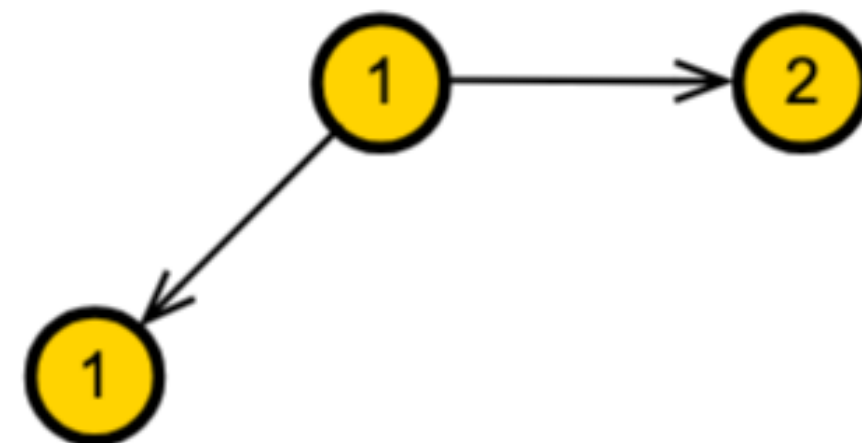


$N=3$

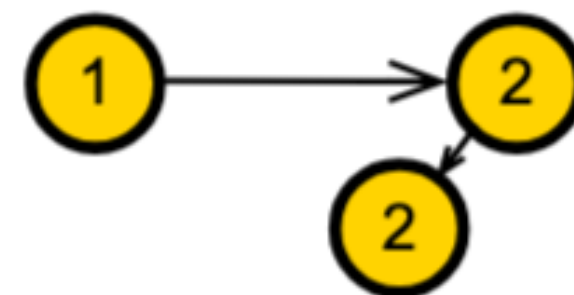
$-p_1^3$



$p_1^2 p_2$



$-p_1 p_2^2$



$-p_1 p_2 p_3$

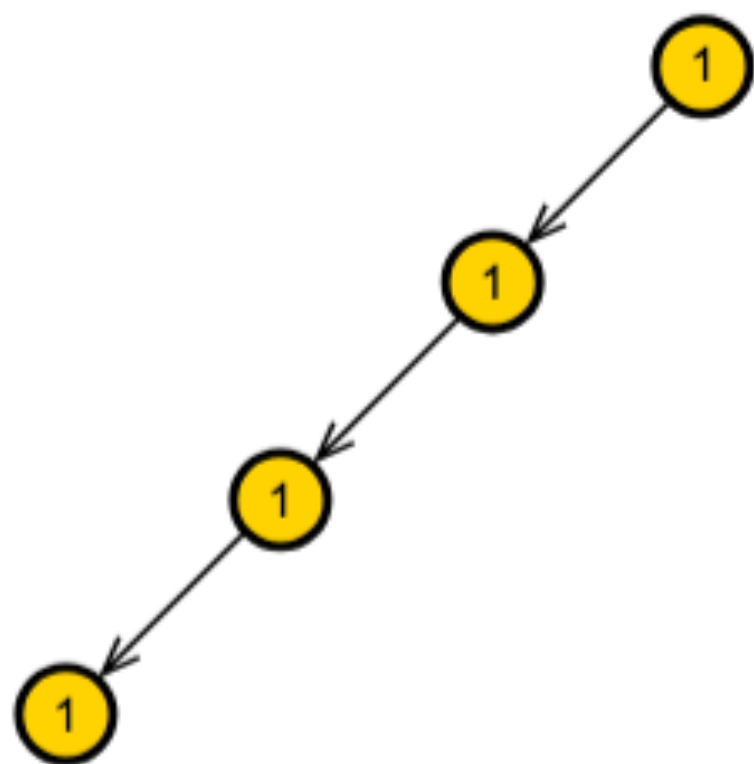


ϵ_2
 ϵ_1

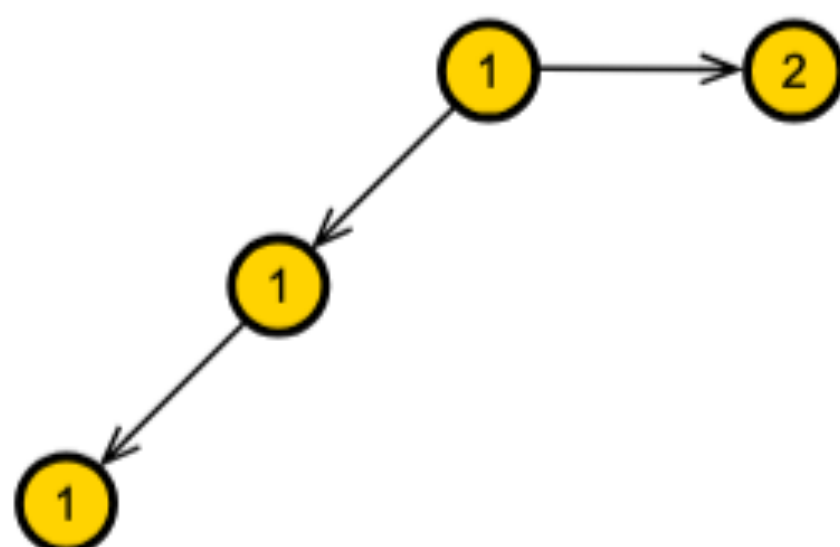
Example: $C_2^{(1)}$ Theory

$N=4$

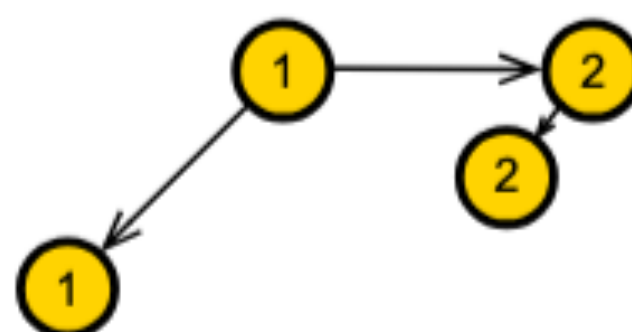
p_1^4



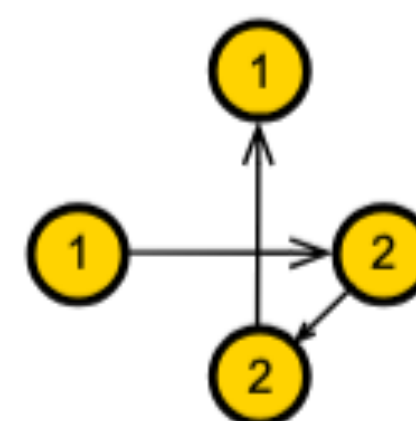
$-p_1^3 p_2$



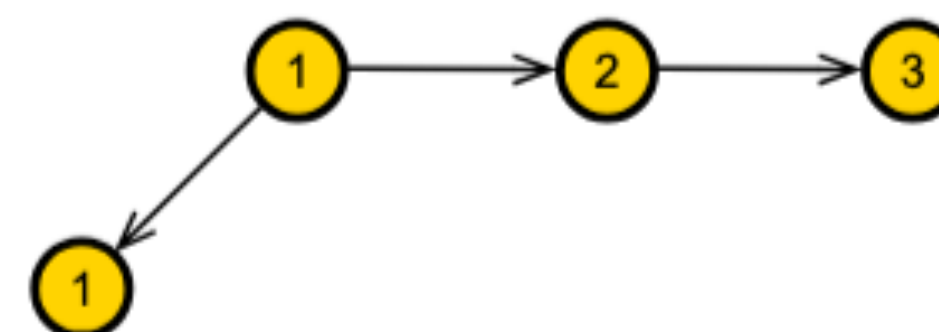
$p_1^2 p_2^2$



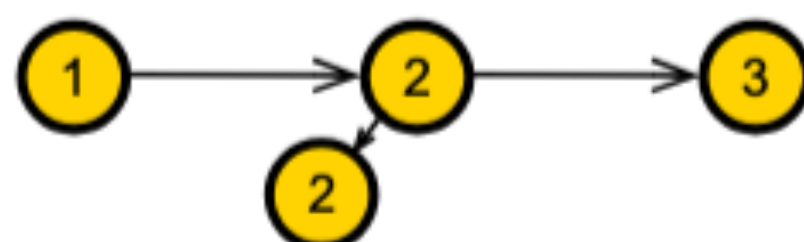
$p_1^2 p_2^2$



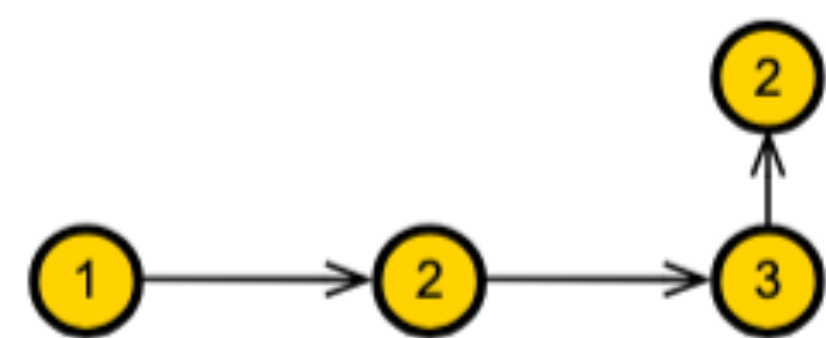
$p_1^2 p_2 p_3$



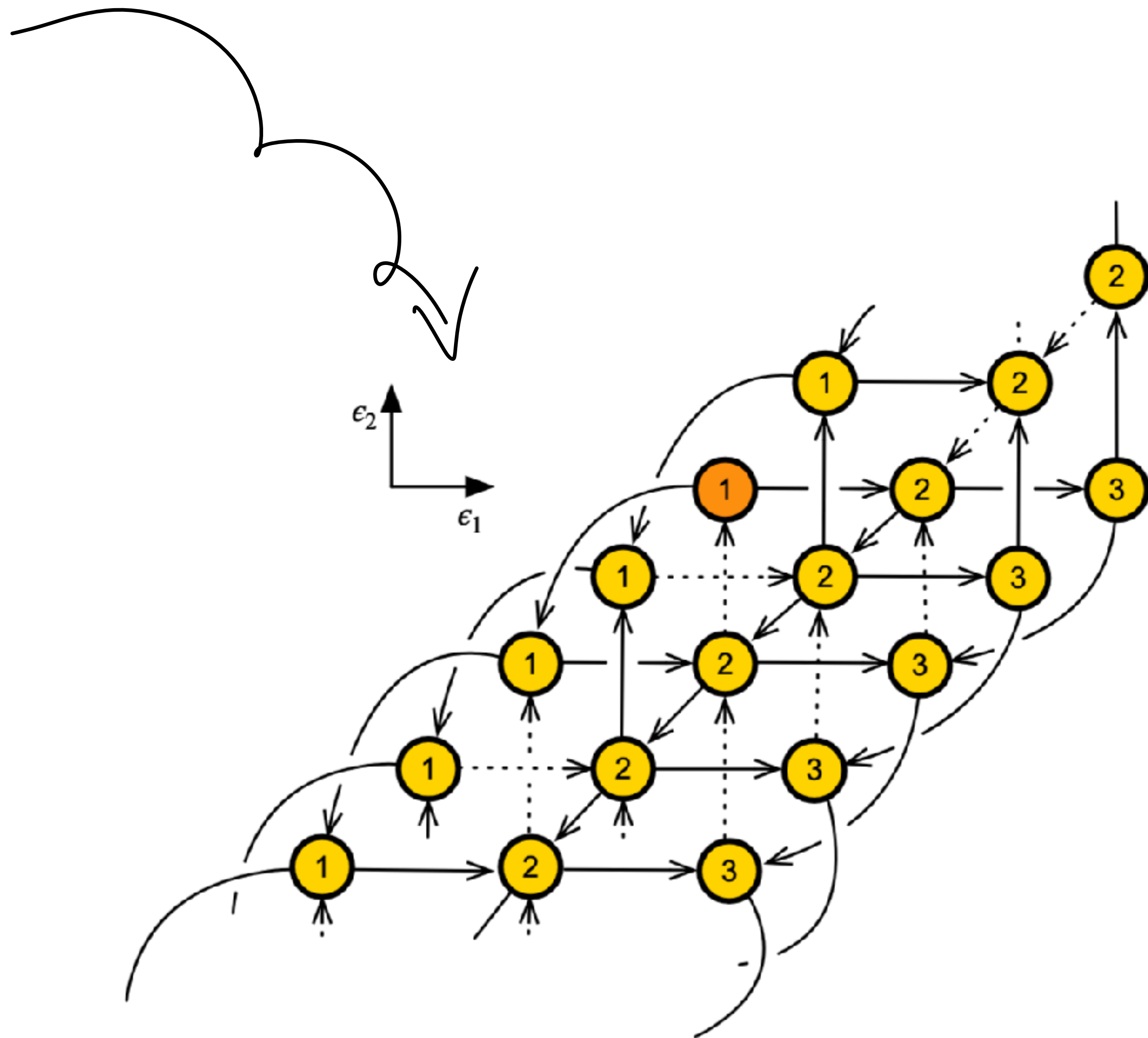
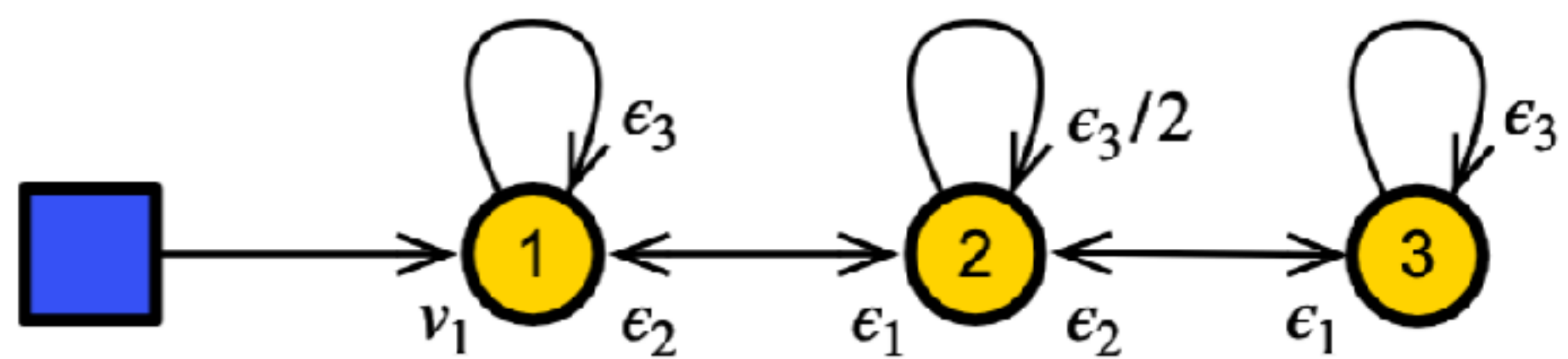
$-p_1 p_2^2 p_3$



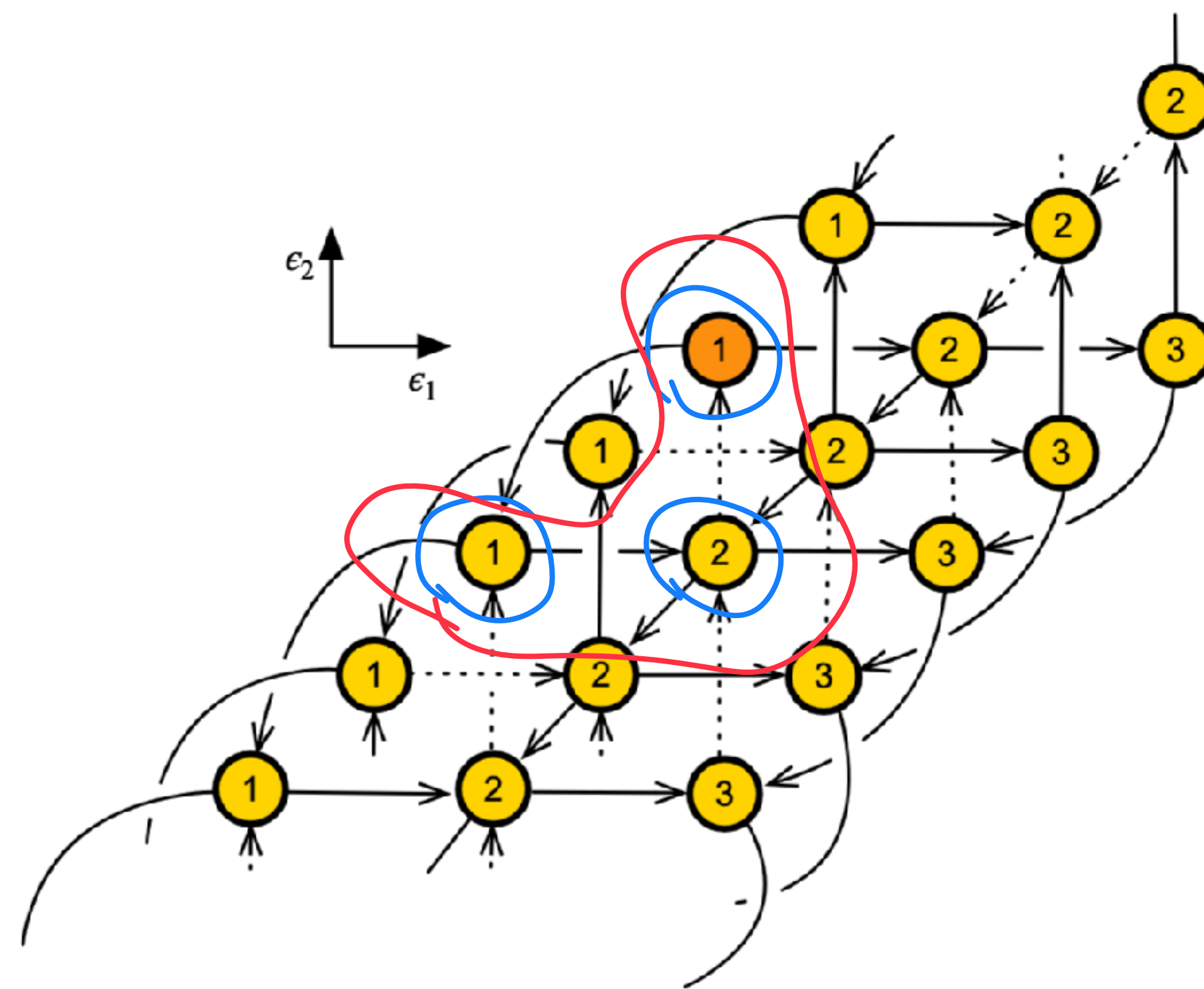
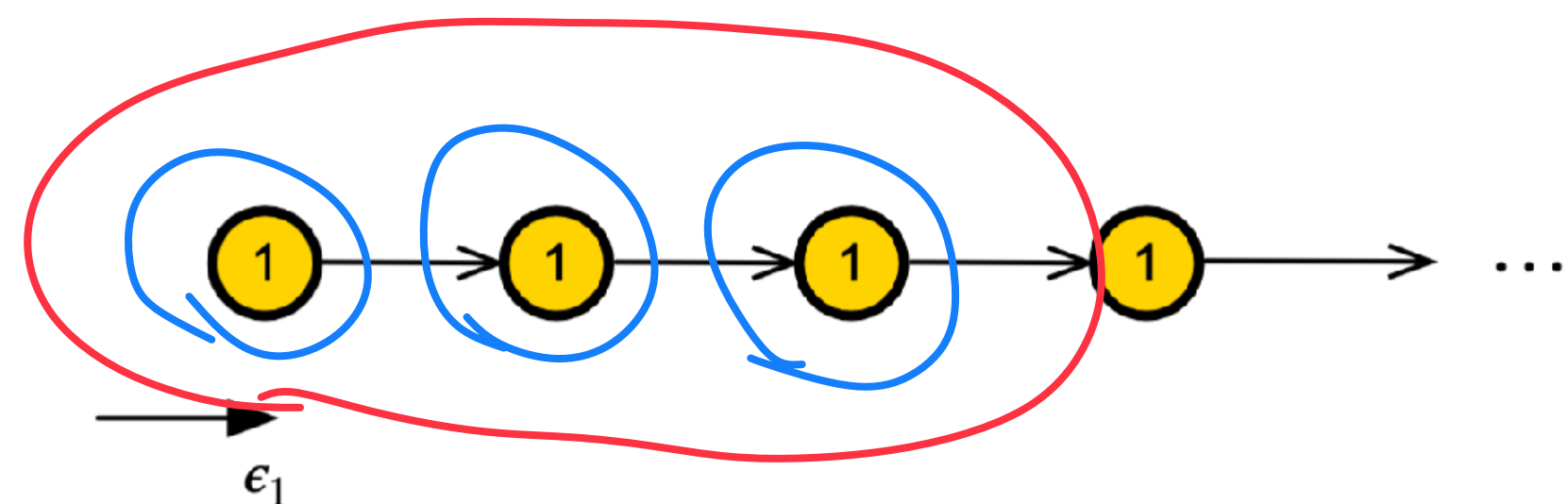
$-p_1 p_2^2 p_3$



ϵ_2
 ϵ_1



BPS state = ("molecules growing from origin")



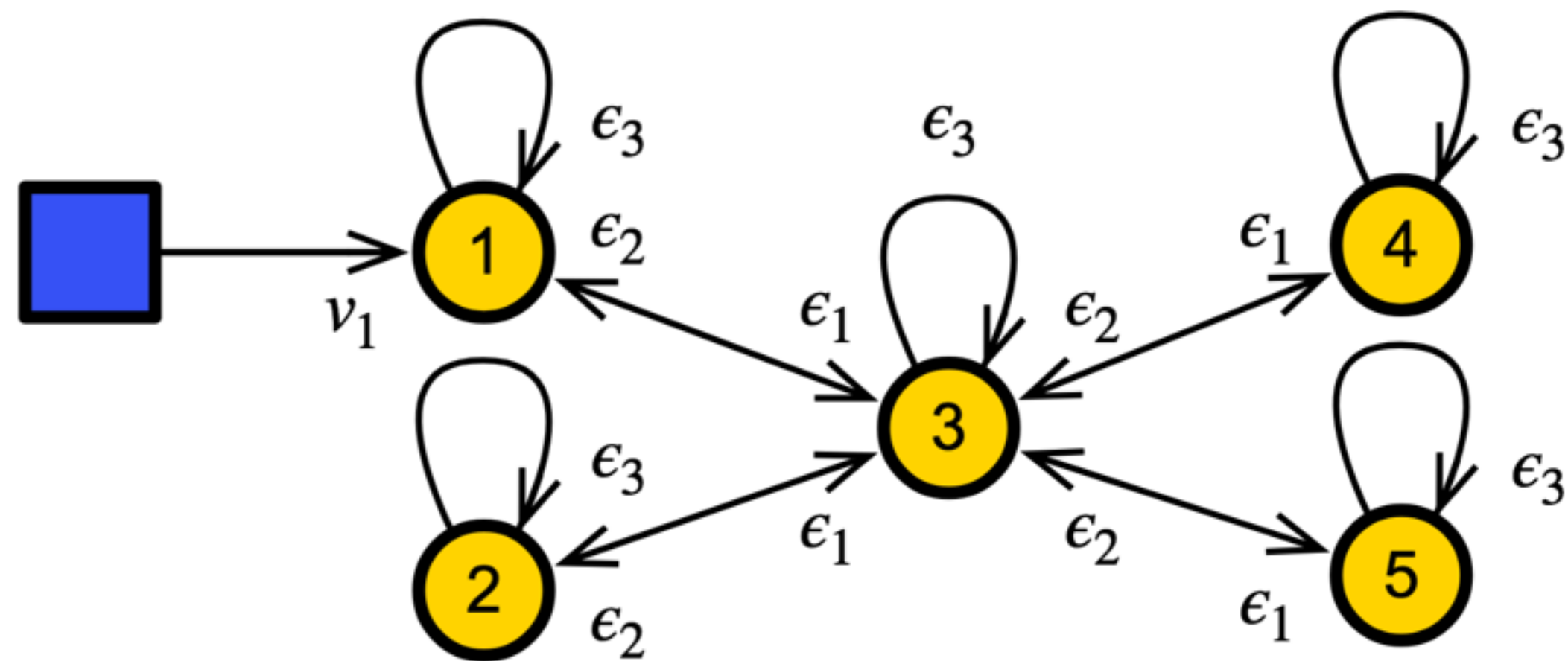
- This should NOT work for arbitrary quiver + relations
(e.g. no flavor sym, ~~SUSY~~, ...)



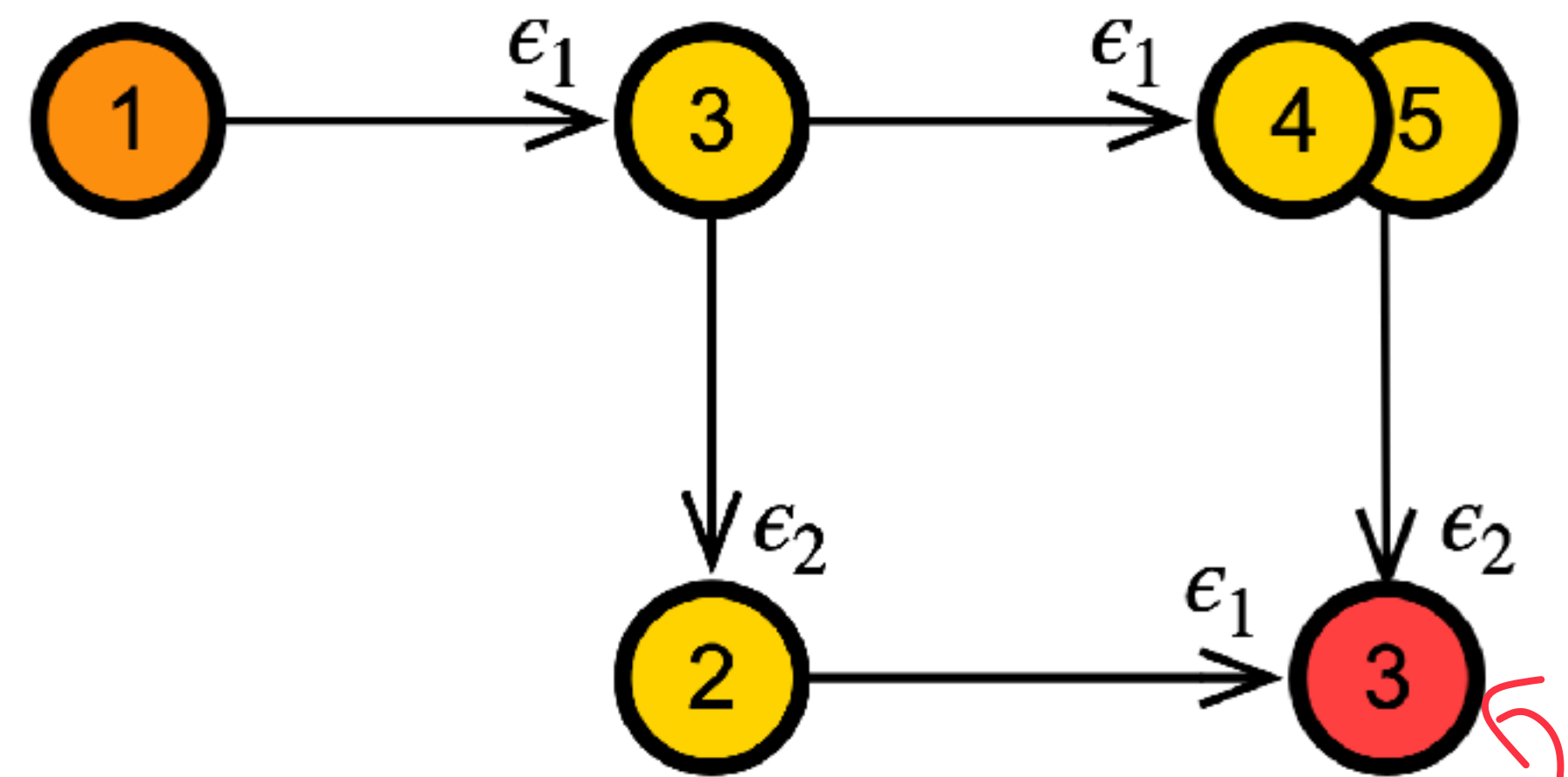
- We here assume non-degeneracy of the residue
("no higher-order pole")

This is violated when $\exists$ "overlaps in crystal"

"No - Overlap Condition"



$D_4^{(1)}$



double pole

Double Quiver Algebra

< Hint : Quiver Algebra > $\begin{bmatrix} Li-MY \quad '20 \\ Galakhov-Li-MY \quad '21 \end{bmatrix}$
Single

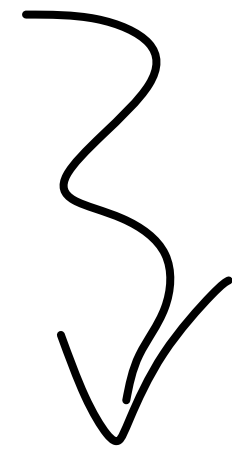
For $N=4$ quivers associated w/ toric CT_3

$\exists$ known crystal rep. of quiver algebra
single

$\left[\begin{array}{l} \text{generalizes} \\ \text{affine Yangian} \\ \text{quantum toroidal alg} \end{array} \right) \text{ for } gl_N$

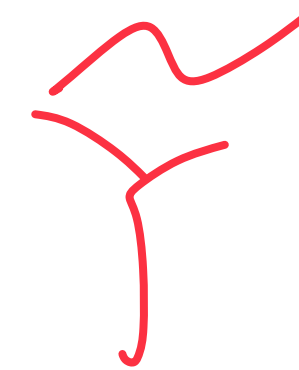
known algebra:

(single) quiver algebra



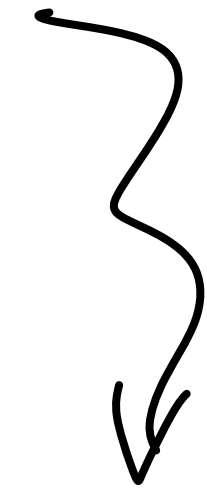
new algebra:

double quiver algebra



more direct connection to JK residue

// particle //



// particle
+
anti-particle //

Double Quiver Algebra

• generators : $\tilde{e}^a(z), \tilde{f}^a(z), \tilde{\psi}^a(z), \tilde{w}^a(z)$ (a : quiver vertex)

• "bond factor"

$$\tilde{\phi}^{a \leftarrow b}(z) := \begin{cases} \frac{\zeta(z)\zeta(-z)}{\zeta(z+\epsilon)\zeta(-z+\epsilon)} \left(\prod_{I \in \{a \rightarrow a\}} \frac{-\zeta(\epsilon - \epsilon_I)}{\zeta(-\epsilon_I)} \frac{\zeta(z + \epsilon - \epsilon_I)\zeta(z - \epsilon + \epsilon_I)}{\zeta(z - \epsilon_I)\zeta(z + \epsilon_I)} \right), & b = a, \\ \left(\prod_{I \in \{a \rightarrow b\}} \frac{-\zeta(z - \epsilon + \epsilon_I)}{\zeta(z + \epsilon_I)} \right) \left(\prod_{I \in \{b \rightarrow a\}} \frac{-\zeta(z + \epsilon - \epsilon_I)}{\zeta(z - \epsilon_I)} \right), & b \neq a. \end{cases}$$

$$\zeta(z) = \begin{cases} \frac{i\theta_1(\tau, z)}{\eta(\tau)}, \\ 2i \sin(\pi z), \\ z, \end{cases}$$

OPE-like relations

$$\tilde{\omega}^{(a)}(z)\tilde{\omega}^{(b)}(w) \simeq \tilde{\omega}^{(b)}(w)\tilde{\omega}^{(a)}(z) ,$$

$$\tilde{\psi}_+^{(a)}(z)\tilde{\psi}_+^{(b)}(w) \simeq \tilde{\psi}_+^{(b)}(w)\tilde{\psi}_+^{(a)}(z) ,$$

$$\tilde{\psi}_-^{(a)}(z)\tilde{\psi}_-^{(b)}(w) \simeq \tilde{\phi}^{a \Leftarrow b}(z-w+2c)\tilde{\phi}^{a \Leftarrow b}(z-w-2c)^{-1}\tilde{\psi}_-^{(b)}(w)\tilde{\psi}_-^{(a)}(z) ,$$

$$\tilde{\psi}_+^{(a)}(z)\tilde{\psi}_-^{(b)}(w) \simeq \tilde{\phi}^{a \Leftarrow b}(z-w+c)\tilde{\phi}^{a \Leftarrow b}(z-w-c)^{-1}\tilde{\psi}_-^{(b)}(w)\tilde{\psi}_+^{(a)}(z) ,$$

$$\tilde{\psi}_\pm^{(a)}(z)\tilde{\omega}^{(b)}(w) \simeq \tilde{\phi}^{a \Leftarrow b}(z-w+c \mp c/2)^{-1}\tilde{\phi}^{a \Leftarrow b}(z-w-c \pm c/2)\tilde{\omega}^{(b)}(w)\tilde{\psi}_\pm^{(a)}(z) ,$$

$$\tilde{\psi}_\pm^{(a)}(z)\tilde{e}^{(b)}(w) \simeq \tilde{\phi}^{a \Leftarrow b}(z-w+c \mp c/2)\tilde{e}^{(b)}(w)\tilde{\psi}_\pm^{(a)}(z) ,$$

$$\tilde{\psi}_\pm^{(a)}(z)\tilde{f}^{(b)}(w) \simeq \tilde{\phi}^{a \Leftarrow b}(z-w-c \pm c/2)^{-1}\tilde{f}^{(b)}(w)\tilde{\psi}_\pm^{(a)}(z) ,$$



bond factors

OPE-like relations

$$\delta(z-w)\tilde{\phi}^{d\leftarrow a}(u-z-c)\tilde{e}^{(d)}(u)\tilde{\omega}^{(a)}(z) + \delta(u-w)\tilde{\phi}^{a\leftarrow d}(z-w-c)\tilde{e}^{(a)}(z)\tilde{\omega}^{(d)}(w)$$

$$\simeq \delta(z-w)\tilde{\omega}^{(a)}(z)\tilde{e}^{(d)}(u) + \delta(u-w)\tilde{\omega}^{(d)}(u)\tilde{e}^{(a)}(z) ,$$

$$\delta(z-w)\tilde{\phi}^{d\leftarrow a}(u-z-c)^{-1}\tilde{f}^{(d)}(u)\tilde{\omega}^{(a)}(z) + \delta(u-w)\tilde{\phi}^{a\leftarrow d}(z-w-c)^{-1}\tilde{f}^{(a)}(z)\tilde{\omega}^{(d)}(w)$$

$$\simeq \delta(z-w)\tilde{\omega}^{(a)}(z)\tilde{f}^{(d)}(u) + \delta(u-w)\tilde{\omega}^{(d)}(u)\tilde{f}^{(a)}(z) ,$$

$$\tilde{e}^{(a)}(z)\tilde{e}^{(b)}(w) \simeq (-1)^{|a||b|}\tilde{e}^{(b)}(w)\tilde{e}^{(a)}(z) ,$$

$$\tilde{f}^{(a)}(z)\tilde{f}^{(b)}(w) \simeq (-1)^{|a||b|}\tilde{f}^{(b)}(w)\tilde{f}^{(a)}(z) ,$$

$$\tilde{\phi}^{a\leftarrow b}(z-w-c)\tilde{e}^{(a)}(z)\tilde{f}^{(b)}(w) - (-1)^{|a||b|}\tilde{f}^{(b)}(w)\tilde{e}^{(a)}(z)$$

$$\simeq \delta_{ab} \left(\delta(z-w-c)\tilde{\psi}_+^{(a)}(w+c/2) - \delta(z-w+c)\tilde{\psi}_-^{(a)}(z+c/2) - \delta(z-w)\tilde{\omega}^{(a)}(z) \right) .$$

Representation of $\tilde{\gamma}$ on crystal

$$\tilde{\psi}_{\pm}^{(a)}(z)|\mathcal{C}\rangle = \begin{cases} \tilde{\Psi}_{\mathcal{C}}^{(a)}(z)|\mathcal{C}\rangle, & \text{rational,} \\ \left[\tilde{\Psi}_{\mathcal{C}}^{(a)}(Z)\right]_{\pm}|\mathcal{C}\rangle, & \text{trigonometric/elliptic,} \end{cases}$$

$$\tilde{\omega}^{(a)}(z)|\mathcal{C}\rangle = \sum_{\text{Inad}(\Psi_{\mathcal{C}}^{(a)}(z), \eta)} \delta(z - \epsilon_a) \lim_{x \rightarrow \epsilon_a} \zeta(x - \epsilon_a) \tilde{\Psi}_{\mathcal{C}}^{(a)}(x)|\mathcal{C}\rangle,$$

diagonal

$$\tilde{e}^{(a)}(z)|\mathcal{C}\rangle = \sum_{a \in \text{Add}(\mathcal{C})} \pm \delta(z - \epsilon_a) \left(\pm \lim_{x \rightarrow \epsilon_a} \zeta(x - \epsilon_a) \tilde{\Psi}_{\mathcal{C}}^{(a)}(x) \right)^{1/2} |\mathcal{C} + a\rangle,$$

← create

$$\tilde{f}^{(a)}(z)|\mathcal{C}\rangle = \sum_{a \in \text{Rem}(\mathcal{C})} \pm \delta(z - \epsilon_a) \left(\pm \lim_{x \rightarrow \epsilon_a} \zeta(x - \epsilon_a) \tilde{\Psi}_{\mathcal{C}}^{(a)}(x) \right)^{1/2} |\mathcal{C} - a\rangle,$$

← annihilate

poles inadmissible by η

$$\tilde{\psi}_{\pm}^{(a)}(z)|\mathcal{C}\rangle = \begin{cases} \tilde{\Psi}_{\mathcal{C}}^{(a)}(z)|\mathcal{C}\rangle, & \text{rational,} \\ \left[\tilde{\Psi}_{\mathcal{C}}^{(a)}(Z)\right]_{\pm}|\mathcal{C}\rangle, & \text{trigonometric/elliptic,} \end{cases}$$

$$\tilde{\omega}^{(a)}(z)|\mathcal{C}\rangle = \sum_{\text{Inad}(\Psi_{\mathcal{C}}^{(a)}(z), \eta)} \delta(z - \epsilon_a) \lim_{x \rightarrow \epsilon_a} \zeta(x - \epsilon_a) \tilde{\Psi}_{\mathcal{C}}^{(a)}(x)|\mathcal{C}\rangle,$$

$$\tilde{e}^{(a)}(z)|\mathcal{C}\rangle = \sum_{a \in \text{Add}(\mathcal{C})} \pm \delta(z - \epsilon_a) \left(\pm \lim_{x \rightarrow \epsilon_a} \zeta(x - \epsilon_a) \tilde{\Psi}_{\mathcal{C}}^{(a)}(x) \right)^{1/2} |\mathcal{C} + a\rangle,$$

$$\tilde{f}^{(a)}(z)|\mathcal{C}\rangle = \sum_{a \in \text{Rem}(\mathcal{C})} \pm \delta(z - \epsilon_a) \left(\pm \lim_{x \rightarrow \epsilon_a} \zeta(x - \epsilon_a) \tilde{\Psi}_{\mathcal{C}}^{(a)}(x) \right)^{1/2} |\mathcal{C} - a\rangle,$$

poles allowed by η

Representation of $\tilde{\gamma}$ on crystal

defined from

JK residue

$\mathbb{Z}$ 1-loop

$$\tilde{\psi}_{\pm}^{(a)}(z)|\mathcal{C}\rangle = \begin{cases} \tilde{\Psi}_{\mathcal{C}}^{(a)}(z)|\mathcal{C}\rangle, & \text{rational,} \\ [\tilde{\Psi}_{\mathcal{C}}^{(a)}(Z)]_{\pm}|\mathcal{C}\rangle, & \text{trigonometric/elliptic,} \end{cases}$$

$$\tilde{\omega}^{(a)}(z)|\mathcal{C}\rangle = \sum_{\text{Inad}(\Psi_{\mathcal{C}}^{(a)}(z), \eta)} \delta(z - \epsilon_a) \lim_{x \rightarrow \epsilon_a} \zeta(x - \epsilon_a) \tilde{\Psi}_{\mathcal{C}}^{(a)}(x)|\mathcal{C}\rangle,$$

$$\tilde{e}^{(a)}(z)|\mathcal{C}\rangle = \sum_{\mathbf{a} \in \text{Add}(\mathcal{C})} \pm \delta(z - \epsilon_a) \left(\pm \lim_{x \rightarrow \epsilon_a} \zeta(x - \epsilon_a) \tilde{\Psi}_{\mathcal{C}}^{(a)}(x) \right)^{1/2} |\mathcal{C} + \mathbf{a}\rangle,$$

$$\tilde{f}^{(a)}(z)|\mathcal{C}\rangle = \sum_{\mathbf{a} \in \text{Rem}(\mathcal{C})} \pm \delta(z - \epsilon_a) \left(\pm \lim_{x \rightarrow \epsilon_a} \zeta(x - \epsilon_a) \tilde{\Psi}_{\mathcal{C}}^{(a)}(x) \right)^{1/2} |\mathcal{C} - \mathbf{a}\rangle,$$

Summary

JK-residue

$N=2$ quiver
+ $J/\bar{E}$ -terms

crystal
melting

$|e\rangle$

representation

refined
counting

double quiver
algebra $\tilde{Y}$

$\sim \#$
enhanced
double QA

$N=4$ quiver
+ F -terms

(single) quiver algebra
 Y

Lessons on QG?

Crystal

||

Quantum Foam

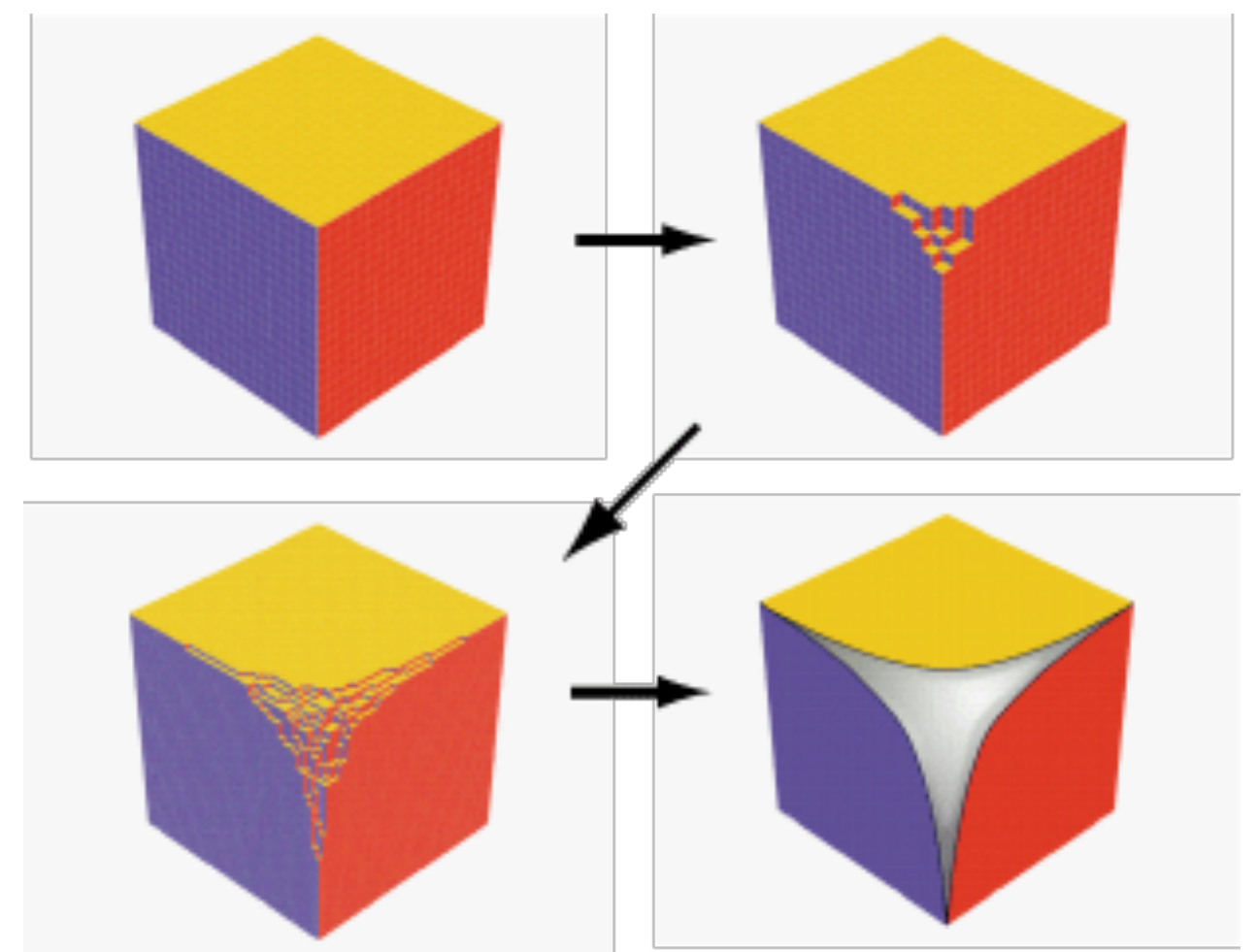
(Double) Quiver Algebra

||

Algebra for creating/annihilating
foams

[Iqbal - Nekrasov
- Okounkov - Vafa '03]

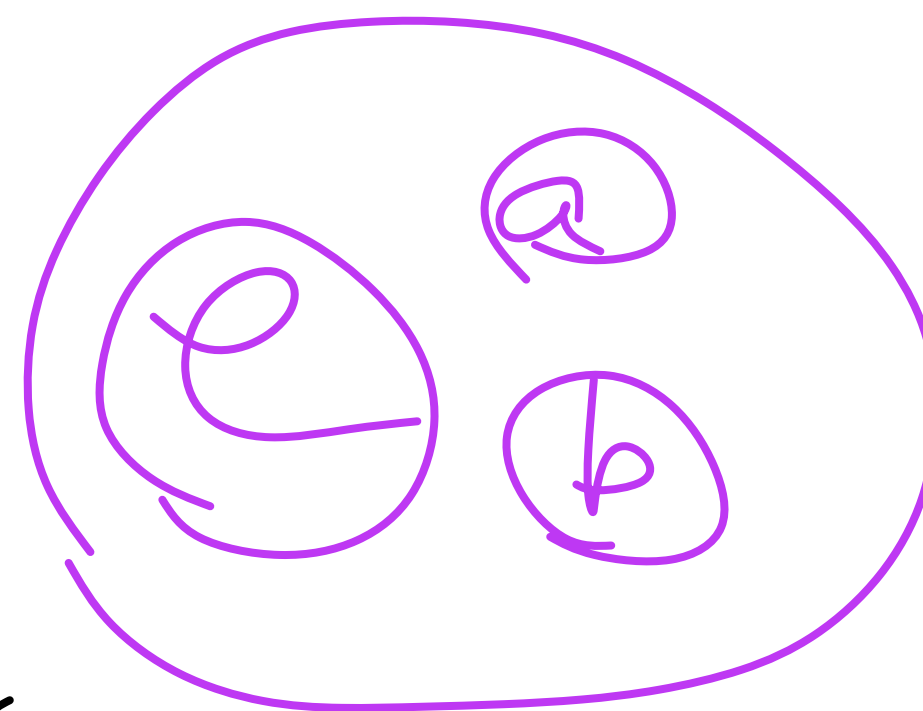
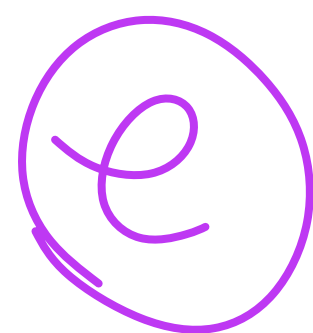
[Ooguri - MT '09]



"Confinement of Spacetime foam?"

- η : cyclic

$$e \rightarrow e+a \rightarrow e+a+b$$



- η : non-cyclic

$$e \not\rightarrow e+a \not\rightarrow e+a+b$$

