Chiral algebras from twistorial quantum field theories

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Based on work in collaboration with

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2412.17168

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2408.11092

Building off earlier work in collaboration with K. Costello
(See also Strings 22 + Amplitudes 22 talk, KC's Amplitudes 24 talk)

Plan of today's talk

1. Introduce twistorial QFTs
2. Motivation for studying such special theories
3. 2d chiral algebras govern scattering in twistorial QFTs
4. Obtaining collinear scattering to all-loop order (w/ Fernandez)
5. Scattering in the presence of line defects (w/ Garner)
6. Conclusions & Future work

Part 1: Twistorial QFTs

Today our interest is in a special class of non-SUSY'c, integrable QFTs in four dimensions, called twistorial QFTs.

Vanishing scattering amplitudes but nonzero *form factors*

They are (closely related to) self-dual gauge theories

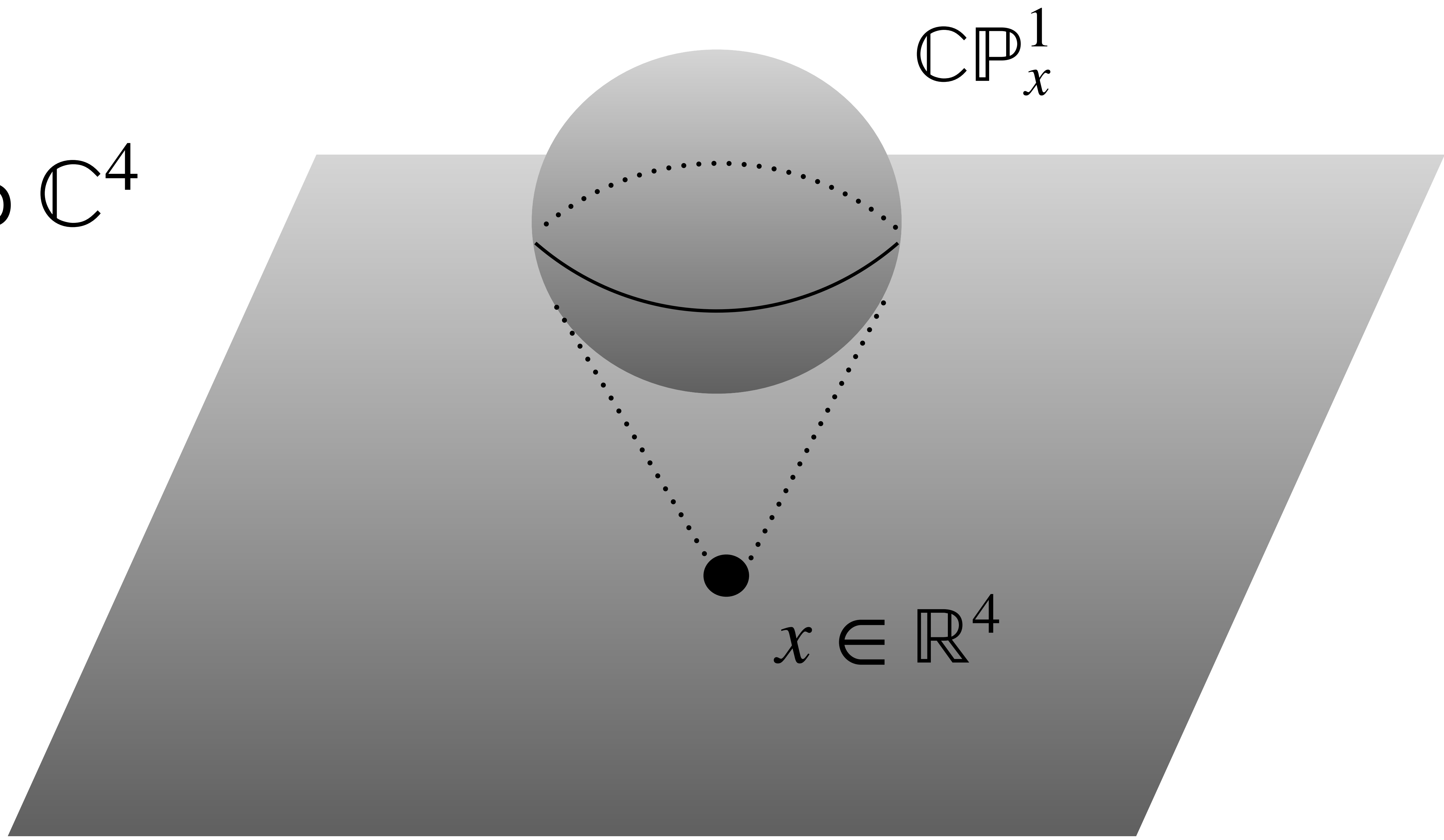
Classical integrability can be encoded by the existence of a ***local*** holomorphic lift to the twistor space of the underlying spacetime manifold [Ward, textbook: Mason-Woodhouse]

$$\int_{\mathbb{PT}} \mathrm{Tr}(\mathscr{B}\mathscr{F}^{(0,2)}(\mathscr{A}))$$

$$\downarrow$$

$$\int_{\mathbb{R}^4} \mathrm{Tr}(BF(A))$$

Twistor space assoc. to $\mathbb{C}^4$



$\mathbb{PT} \simeq \mathbb{R}^4 \times \mathbb{CP}^1$

Quantum mechanically, a 6d gauge anomaly furnishes an obstruction to 4d integrability

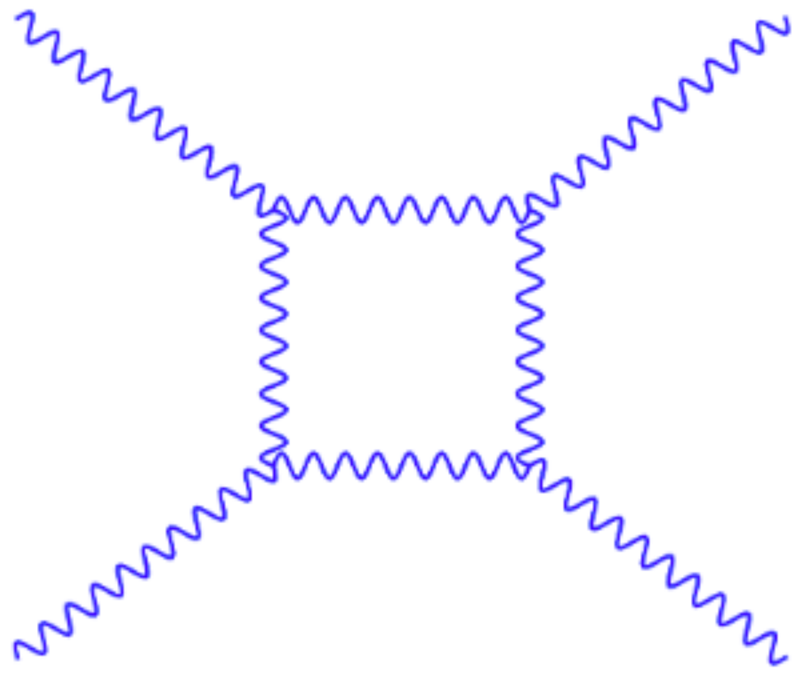
[Costello-Li, Costello]

Anomaly may be cancelled by:

$$\mathrm{Tr}_{\mathrm{adj}}(X^4) - \mathrm{Tr}_R(X^4) = \lambda^2_{\mathfrak{g},R}\mathrm{Tr}_{\mathrm{fun}}(X^2)^2$$

$X \in \mathfrak{g}$

[Okubo]



1. Supersymmetrization
2. Green-Schwarz mechanism (for certain $\mathfrak{g}$)
3. Addition of certain real fermionic matter (for certain $\mathfrak{g}, R$)
4. Suitable combination of points 2, 3

One example of an anomaly-free 6d theory is coupled SO(8) hol'c CS + Kodaira-Spencer theory.

Admits a string theory interpretation type I B-model topological string, useful for twisted holography

[Costello-NMP-Sharma x2, Sharma's Strings 23 talk]

Today's twistorial QFTs: self-dual YM theories coupled to various matter sectors

- SU(2) SDYM w/ $N_f = 8$, SU(3) SDYM w/ $N_f = 9$
- SU(N) SDYM w/ $8F \oplus 8\bar{F} \oplus \wedge^2 F \oplus \wedge^2 \bar{F}$
- SU(2), SU(3), SO(8), $E_{6,7,8}$ SDYM w/ fourth-order "axion" ρ
- SU(N) SDYM w/ $N_f = N$ and ρ
- SO(N) SDYM w/ $N_f = N - 8$ and ρ
- etc.

see also: [Bittleston-Costello-Zeng] for topological string interpretations

Part 2: Why study twistorial QFTs?

Still challenging to bootstrap general amplitudes/form factors in 4d non-SUSY QFTs
Search for theoretical/mathematical structures to assist

Twistorial QFTs have *rational* form factors with poles only when $\langle ij \rangle \to 0$

Singularities fixed by chiral algebra OPE, higher-loop form factors → lower loop/lower pt

Easy to use induction in chiral algebra picture to general n-pt answers

In gauge theory, some observables in SDYM compute YM observables with special external helicity configurations (coupled to special anomaly-cancelling matter content)

[Dixon-Morales x2]: Used twistorial anomaly condition to find relations among 1-loop n-gluon QCD subamplitudes

Self-duality and special helicity configurations

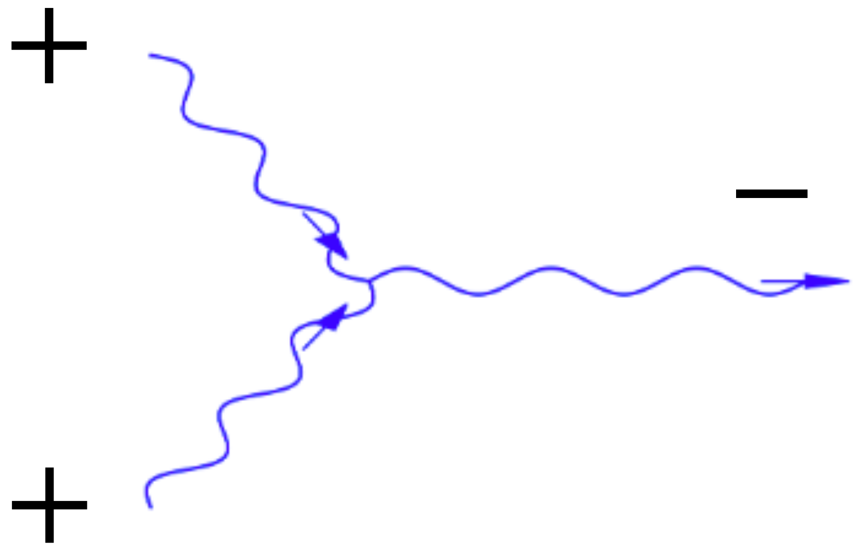
$$S[A,B] = \int_{\mathbb{R}^4} \text{tr}\left(B \wedge F(A)\right) \longrightarrow \text{add } \frac{1}{2} g_{YM}^2 \int \text{tr}(B \wedge B) \text{ then integrate out } B \longrightarrow$$

ordinary YM
(+ θ-angle)

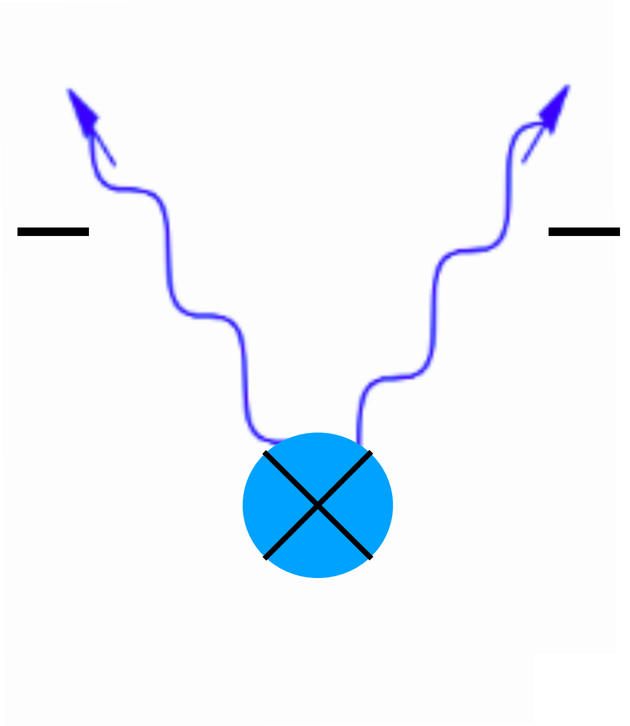
$B \in \Omega^2_-(\mathbb{R}^4, \mathfrak{g})$ [Chalmers-Siegel]

(+-) propagator

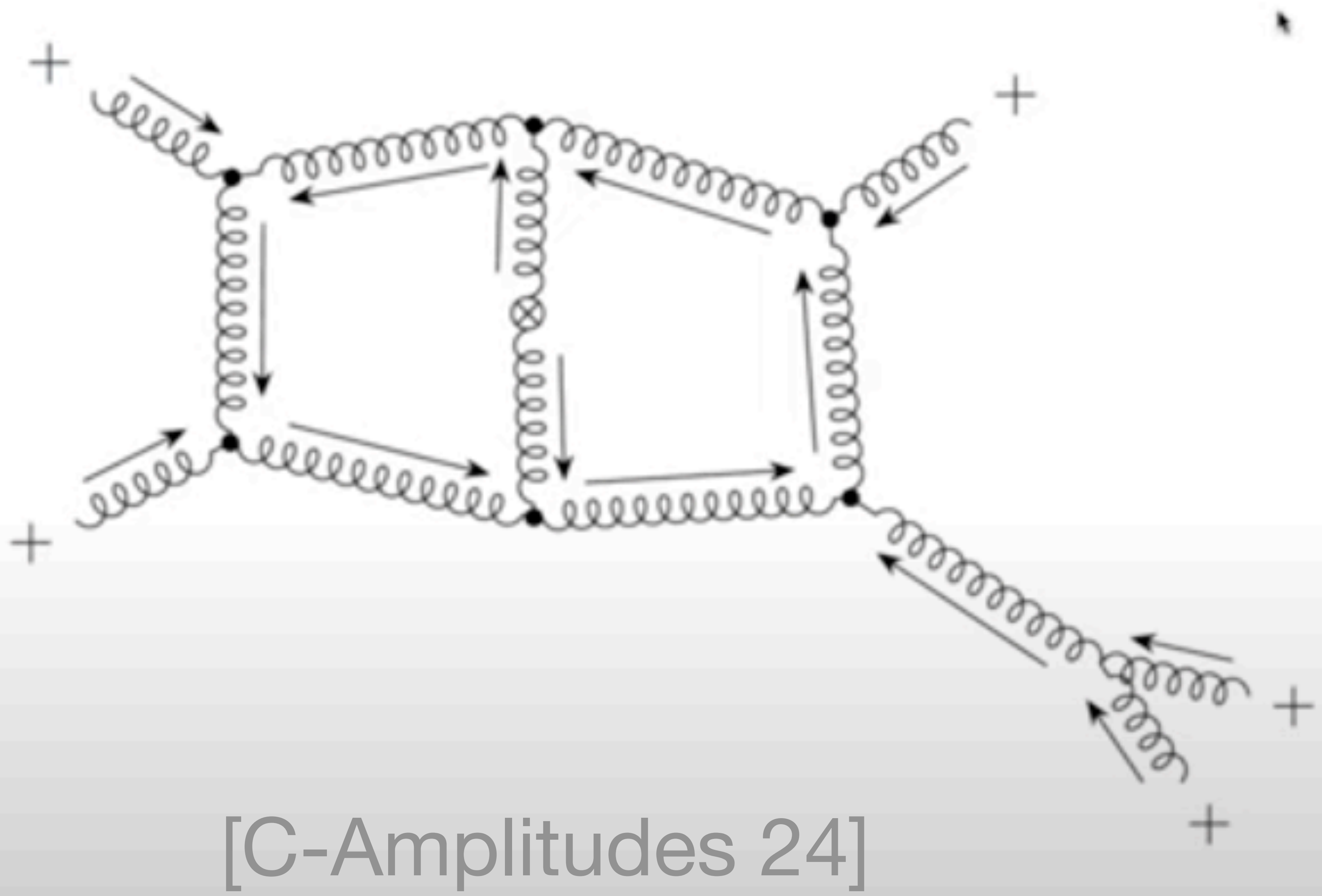
(-++) vertex



(- -) vertex in form factor



$tr(B^2)$ form factor in SDYM
= certain diagrams in YM



[C-Amplitudes 24]

[C-NMP x2]: Tree-level, 2 -, n + gluons (Parke-Taylor)

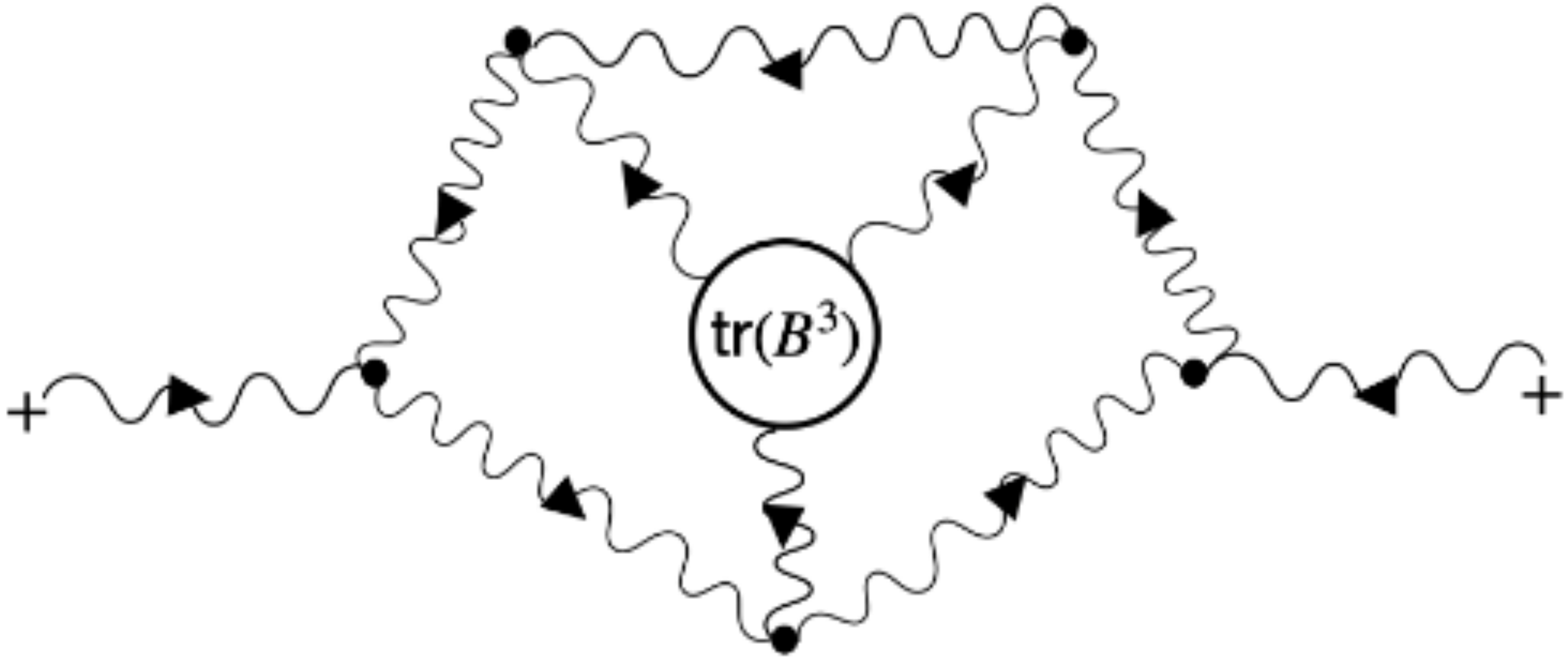
1-loop, 1 -, n + gluons (B-D-K)

[Costello]: 2-loop, n + gluons

} chiral algebra

[Dixon-Morales]: Verify 2-loop 4-pt amplitude } gen. unitarity, mass regulator

Higher loop form factors for YM theory



More generally: $tr(B^n)$ form factors in SDYM compute $tr(F_-^n)$ YM form factors

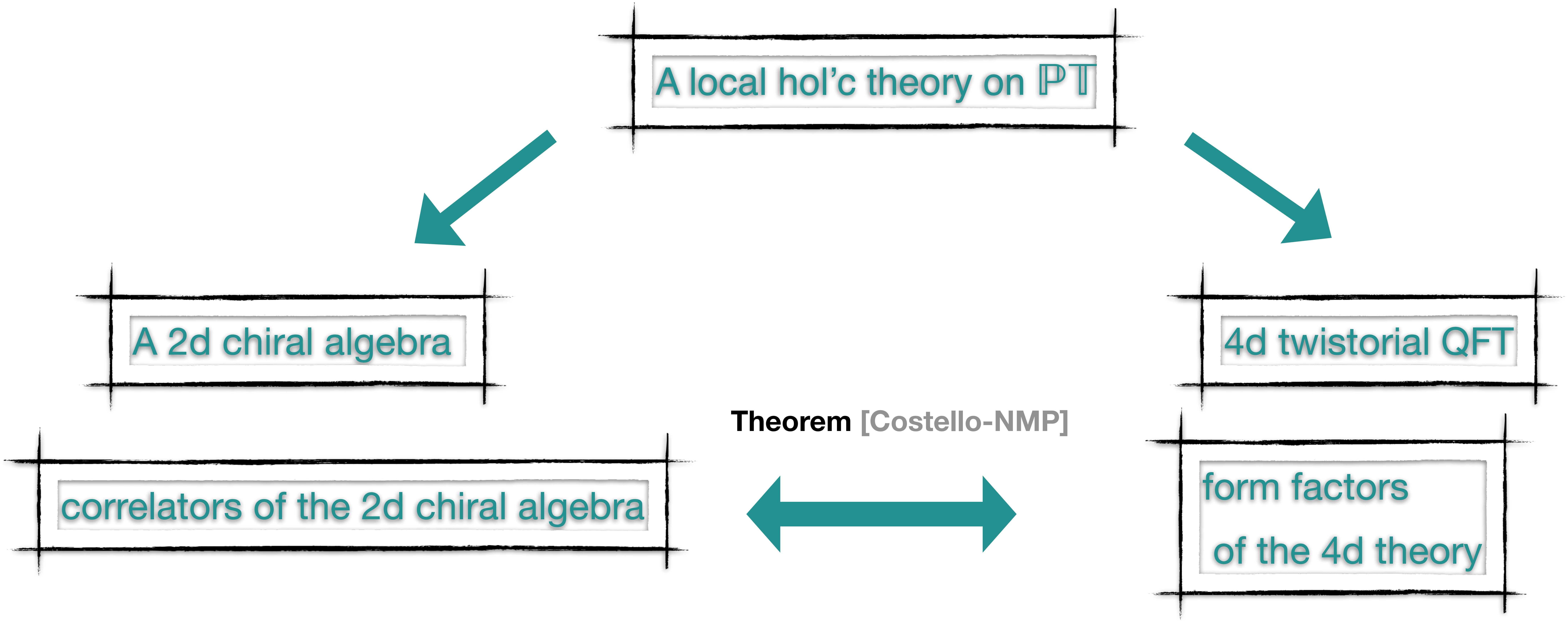
l loops, (n-l) negative helicity external states

(i.e. up to n-loops)

Pert. solvability of a small subsector of massless QCD coupled to special matter

Part 3: Chiral algebras underlie twistorial QFTs

For all twistorial QFTs, the following theorem applies



"Chiral algebra bootstrap": For the theories to which this applies, trade loop-level amplitude computations for algebraic manipulations

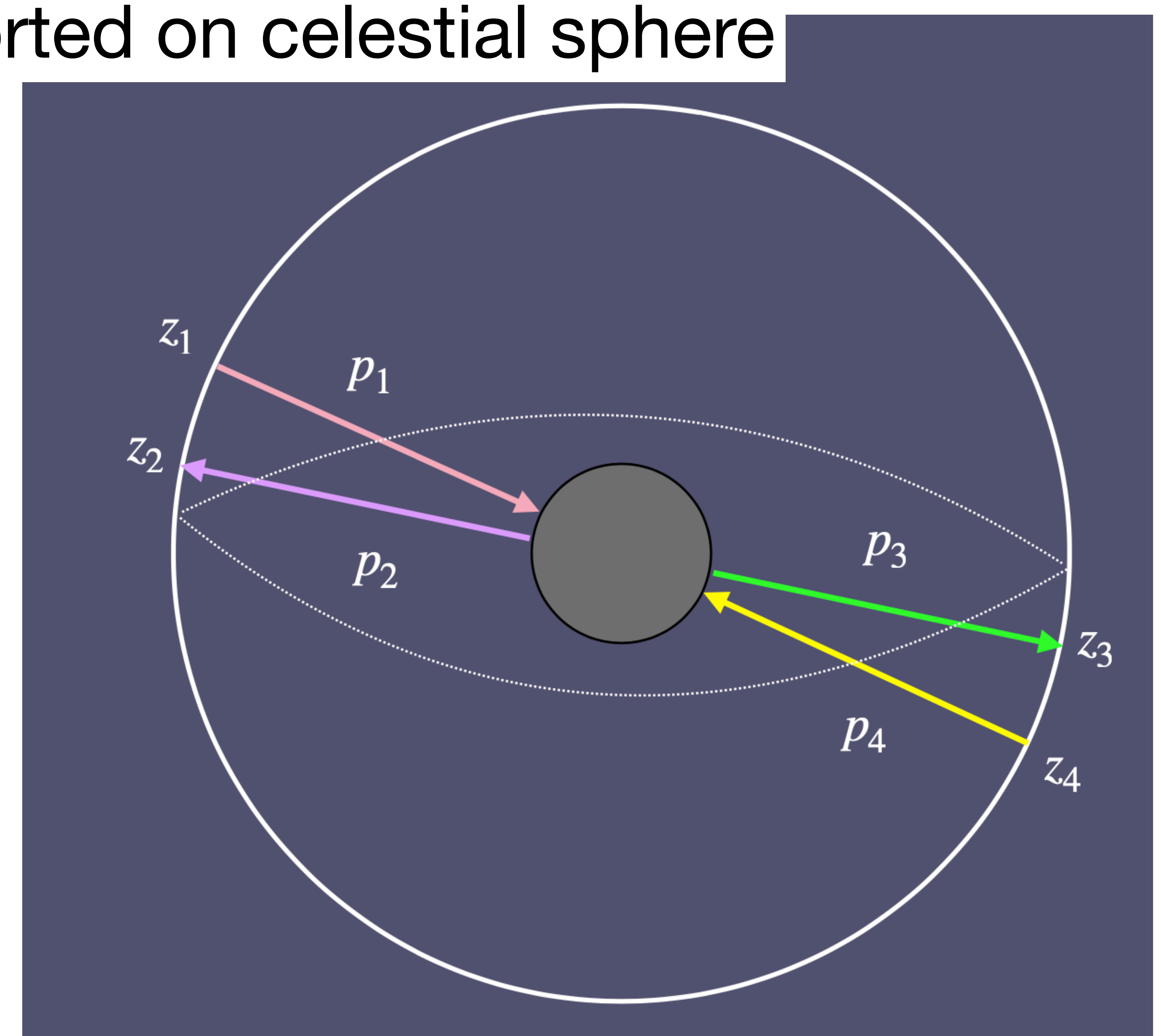
- Form factors are rational, with poles only in $\langle ij \rangle$
- polar part is itself a form factor at lower loop-order or insertion #, determined by OPE

2d chiral algebra	4d theory
conf. primary operators	massless on-shell states
OPEs	holomorphic collinear limits
conformal blocks (cf. CS/WZW)	local operators
correlation functions	form factors

Useful to think of chiral algebra as supported on celestial sphere

cf. [Guevara-Himwich-Pate-Strominger]
[Strominger]

(See other talks in this session!)



The chiral algebra

Consider coupling the holc 6d theory to some general holc 2d theory along a $\mathbb{CP}^1$

Assuming the 6d/2d theories are themselves BRST invariant, what are the conditions for BRST invariance of the coupling?

e.g.

$$\mathrm{Exp}\left(\sum_{k_1,k_2\geq 0} \int_{\mathbb{CP}^1} \frac{1}{k_1!k_2!} \partial_{v_1}^{k_1} \partial_{v_2}^{k_2} \mathscr{A}_{\bar{z}}^a J_a[k_1,k_2](z) \right)$$

$k_1, k_2 \in \mathbb{Z}_{\geq 0}$

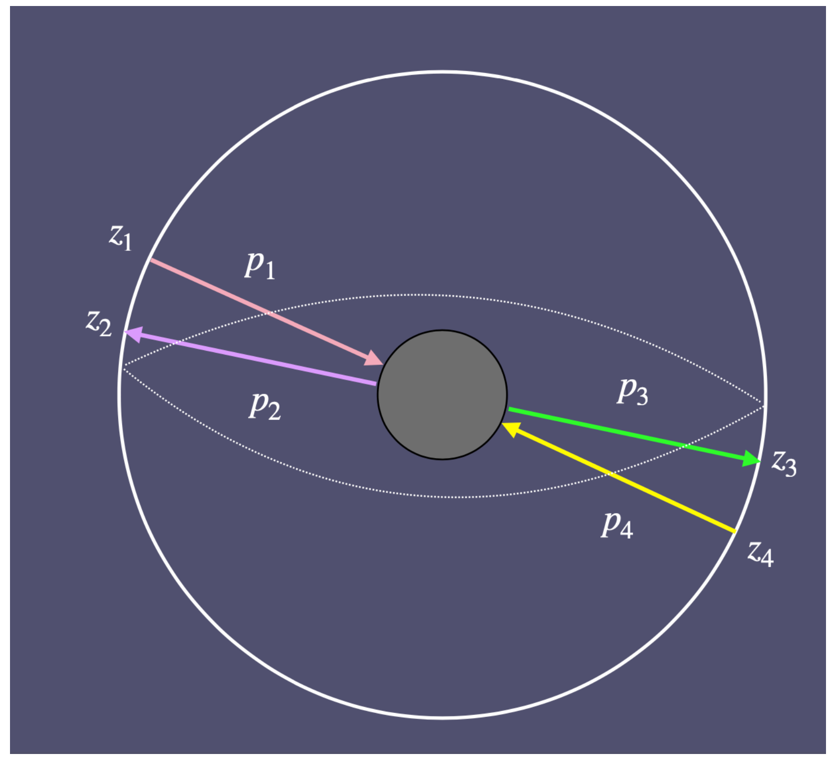
v_1, v_2 transverse to $\mathbb{CP}^1$

$\mathscr{B} \leftrightarrow \tilde{J}$

Order by order, can derive constraints on the OPE among the 2d currents.

Recall the twistor space identification of $\mathbb{CP}^1$ with the celestial sphere:

4d interpretation of collinear limits



(Best done in the BV-BRST formalism)

[Costello-NMP]:

If the 6d theory has a gauge/BRST anomaly, this chiral algebra fails to associate at one-loop

On the other hand, given a non-anomalous 6d theory, we *used* associativity to find some one-loop deformations of the OPE

(Verified in [Fernandez])

$$J_a[1,0](z_1)J_b[0,1](z_2) \sim \frac{1}{z_{12}^2} f_{ab}^c \tilde{J}_c[0,0](z_1) - \frac{1}{z_{12}} f_{ab}^c \partial_z \tilde{J}_c[0,0](z_1)$$

(++-) one-loop splitting

$$-CK^{fe}\frac{1}{z_{12}}(f_{ae}^c f_{bf}^d + f_{ae}^d f_{bf}^c) : J_c[0,0]\tilde{J}_d[0,0] : (z_1)$$

Conjecture: One may use knowledge of the 6d/2d coupling on twistor space + associativity to determine the chiral algebra to arbitrary order

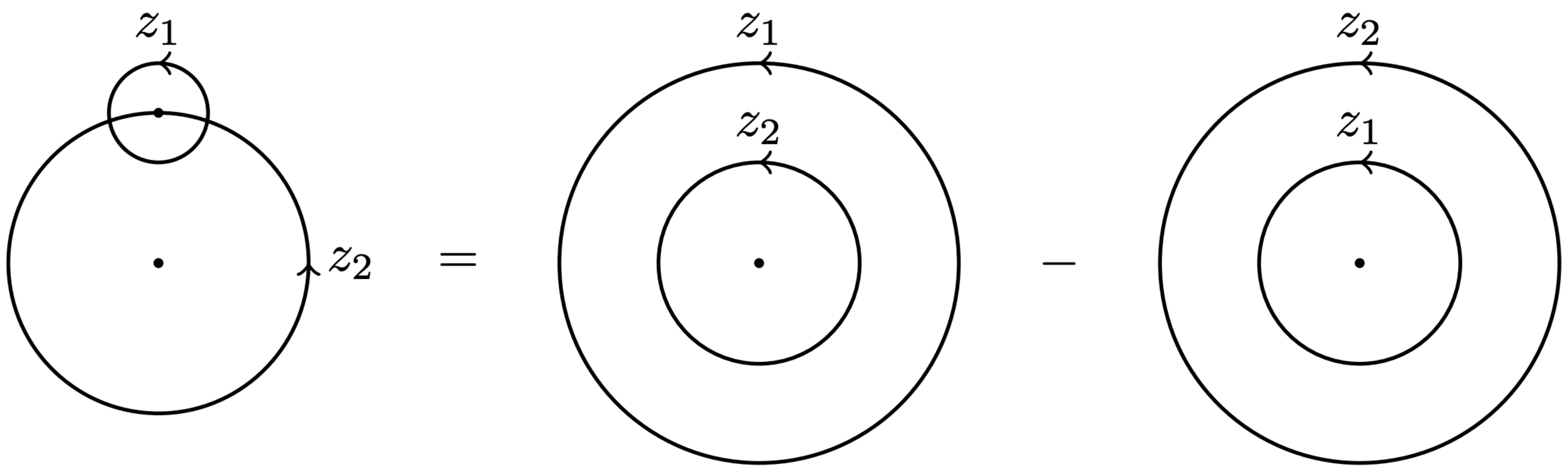
Part 4: Results at arbitrary loop order

We now know the chiral algebra (collinear splitting fns) to arbitrary loop order using constraints from symmetry and associativity [Fernandez-NMP]

To fix the OPE corrections

- Step 1: Determine the general form of the OPE corrections (bulk/defect couplings, symmetries)
- Step 2: Determine conditions on the undetermined numerical coefficients from associativity
- Step 3: Solve those equations to fix the coefficients

$$\oint_{|z_2|=2} dz_2 z_2^l \left(\oint_{|z_{12}|=1} \phi_1(z_1)\phi_2(z_2)\right)\phi_3(0) = \oint_{|z_1|=2} dz_1 \phi_1(z_1) \left(\oint_{|z_2|=1} dz_2 z_2^l \phi_2(z_2)\phi_3(0)\right) - (-1)^{F_1F_2}\oint_{|z_2|=2} dz_2 z_2^l \phi_2(z_2)\left(\oint_{|z_1|=1} dz_1\phi_1(z_1)\phi_3(0)\right)$$



$$\phi_i(z)\phi_j(w) \sim \sum_{n\geq 1} \frac{\{\phi_i\phi_j\}_n(w)}{(z-w)^n}$$

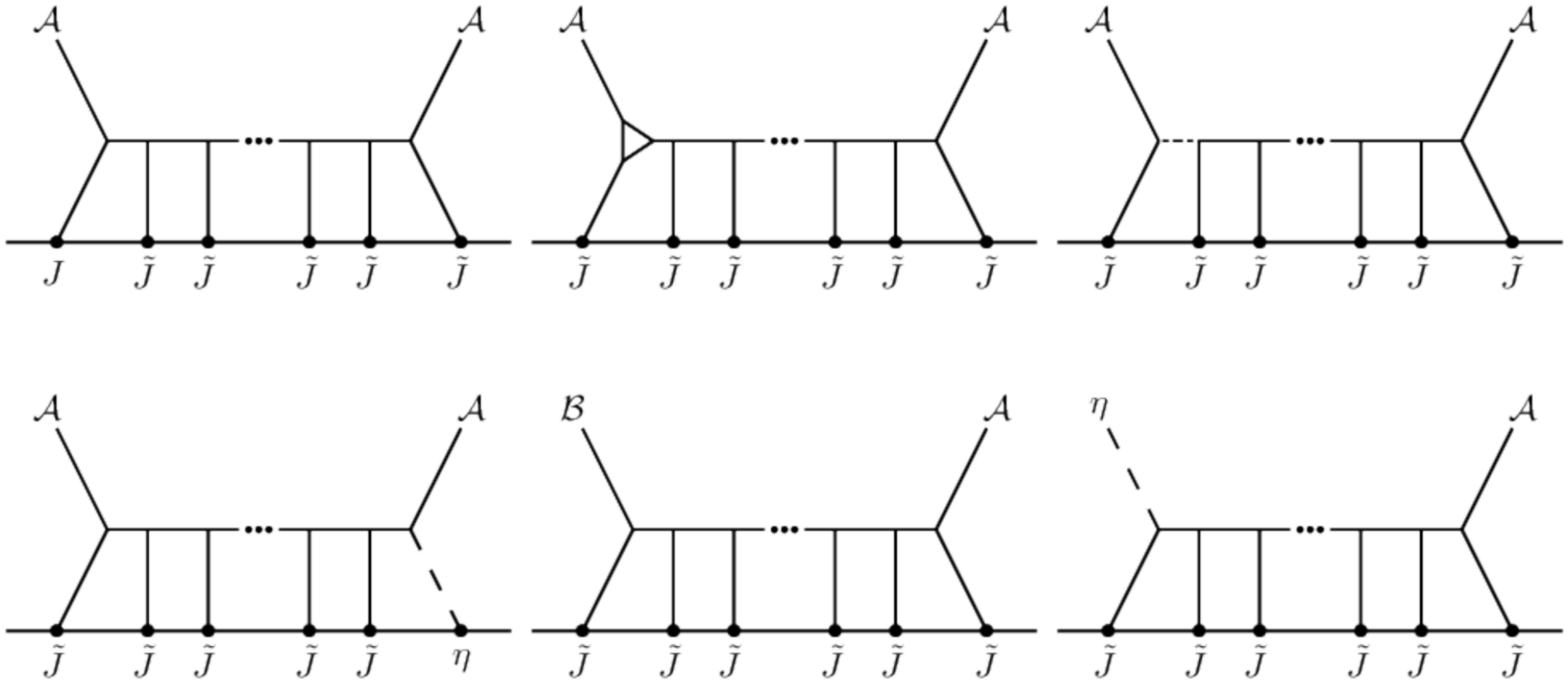
$$\{\{\phi_1\phi_2\}_1\phi_3\}_{l+1} = \{\phi_1\{\phi_2\phi_3\}_{l+1}\}_1 - (-1)^{F_1F_2}\{\phi_2\{\phi_1\phi_3\}_1\}_{l+1}$$

To determine which diagrams contribute to a given OPE, we use:

- Interactions in the twistorial Lagrangian are only cubic in form
- Only two type of vertices: $\mathscr{B}\mathscr{A}^2$ and $\eta\mathscr{A}^2$
- Invariance under rescaling of $\hbar$

e.g. SDYM + axion-like theory

$$\hbar \to \alpha\hbar \qquad \mathscr{B} \to \alpha\mathscr{B} \qquad \eta \to \sqrt{\alpha}\eta$$



Here are some of the terms expressed using unknown coefficients.

$$\tilde{J}_a[t](z)J_b[r](0) \sim \frac{1}{z}\sum_{m\geq 1}\sum_{k_j^i\geq 0}^{\sum_{j=1}^{m+1}k_j=t+r-m} \hbar^m \underset{(t,r)}{\overset{(m)}{f}}[k_1,...,k_{m+1}]_{ab}^{i_1\cdots i_{m+1}} :\prod_{j=1}^{m+1}\tilde{J}_{i_j}[k_j]:$$

$$E[t](z)J_b[r](0) \sim \frac{1}{z}\sum_{m\geq 1}\sum_{k_j^i\geq 0}^{\sum_{j=1}^{m+1}k_j=t+r-m-1} \hat{\lambda}_{\mathfrak{g}}\hbar^{m+\frac{1}{2}} \underset{(t,r)}{\overset{(m)}{j}}[k_1,...,k_{m+1}]_{b}^{i_1\cdots i_{m+1}} :\prod_{j=1}^{m+1}\tilde{J}_{i_j}[k_j]:$$

$$F[t](z)J_b[r](0) \sim \sum_{m\geq 1}\sum_{k_j^i\geq 0}^{\sum_{j=1}^{m+1}k_j=t+r-m} \hat{\lambda}_{\mathfrak{g}}\hbar^{m+\frac{1}{2}}\left(\frac{1}{z^2}\underset{(t,r)}{\overset{(m)}{k}}[k_1,...,k_{m+1}]_{b}^{i_1\cdots i_{m+1}}\right.$$

$$\left. + \frac{1}{z}\underset{(t,r)}{\overset{(m)}{l}}[k_1,...,k_{m+1}]_{b}^{i_1\cdots i_{m+1}}\hat{\partial}_1\right) :\prod_{j=1}^{m+1}\tilde{J}_{i_j}[k_j]:$$

- Use associativity to fix OPE coefficients in terms of the $\overset{(m)}{f}$ coefficients.

- We obtained a recursion relation for $\overset{(m)}{f}$ at arbitrary m.

- We used the recursion relation to find a closed-form expression for $\overset{(1)}{f}$, and a recursive expression for $\overset{(m)}{f}$ with $m > 1$.

$$K_{ab}^{i_1\cdots i_{m+1}} = -f_{aj_1}^{i_1} K^{j_1 j_2} f_{j_2 j_3}^{i_2} \cdots f_{j_{2m-2} j_{2m-1}}^{i_m} K^{j_{2m-1} j_{2m}} f_{j_{2m} b}^{i_{m+1}}$$

$$\alpha(t,k) = t^2(k^1+1) - t^1(k^2+1) \qquad \beta(t) = t^1 + t^2$$

$$\underset{(t,r)}{\overset{(m)}{j}}[k_1, ..., k_{m+1}]_b^{i_1\cdots i_{m+1}} = -\left(\frac{\alpha(t,k_1)}{\beta(t)}\right) \underset{(t-1,r)}{\overset{(m)}{f}}(k_1, ..., k_{m+1}) K^{i_1 j} K_{jb}^{i_2\cdots i_{m+1}}$$

$$\underset{(t,r)}{\overset{(m)}{k}}[k_1, ..., k_{m+1}]_b^{i_1\cdots i_{m+1}} = -\left(\frac{\beta(k_1+1)}{\beta(t+1)}\right) \underset{(t,r)}{\overset{(m)}{f}}(k_1, ..., k_{m+1}) K^{i_1 j} K_{jb}^{i_2\cdots i_{m+1}}$$

$$\underset{(t,r)}{\overset{(m)}{l}}[k_1, ..., k_{m+1}]_b^{i_1\cdots i_{m+1}} = -\underset{(t,r)}{\overset{(m)}{f}}(k_1, ..., k_{m+1}) K^{i_1 j} K_{jb}^{i_2\cdots i_{m+1}}.$$

Some terms with single poles were also determined more formally by homotopy transfer methods [Zeng]

Recursive Expression for $\overset{(m>1)}{f}$

$$\underset{(r^1,r^2)(t^1,t^2)}{\overset{(m)}{f}}[k_1; ...; k_{m+1}] = -\sum_{j=1}^{t^1} \underset{(r^1,r^2)(t^1-j,t^2)}{\overset{(m-1)}{f}}[k_1; ...; k_{m-1}; l] \underset{(l^1,l^2)(1,0)}{\overset{(1)}{f}}[k_m; (k^1_{m+1} + 1 - j, k^2_{m+1})]$$

$$+\sum_{j=1}^{t^1} \underset{(r^1,r^2)(l^1,l^2)}{\overset{(m-1)}{f}}[k_1; ...; k_m] \underset{(t^1-j,t^2)(1,0)}{\overset{(1)}{f}}[l; (k^1_{m+1} + 1 - j, k^2_{m+1})]$$

$$-\sum_{j=1}^{t^2} \underset{(r^1,r^2)(0,t^2-j)}{\overset{(m-1)}{f}}[k_1; ...; k_{m-1}; l] \underset{(l^1,l^2)(0,1)}{\overset{(1)}{f}}[k_m; (k^1_{m+1} - t^1, k^2_{m+1} + 1 - j)].$$

Also a nice closed form solution [Zeng] in terms of Clebsch-Gordon coefficients, Wigner 6j symbols

Part 5: Scattering in the presence of defects

Extend to “self-dual line defects”

Consider $\mathbb{R}^4 \backslash l$, l the Euclidean wordline of a heavy charged particle, say x^4-axis

Sources a field, e.g. a charged scalar field gives a 4d field config $\sim \dfrac{1}{r}$

$$r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$$

double-valued when cplxifying to $\mathbb{C}^4$

[Bailey]: A line in spacetime lifts to a quadric Q in $\mathbb{PT}$

The right twistor space is $\mathbb{PT}_Q = \mathbb{PT} \cup_{Q \neq 0} \mathbb{PT}$

- We extended the formulation of 6d hol’c gauge theories to these twistor spaces
- Adding bundles on $\mathbb{PT}_Q$ that are lifts of charged sources (e.g. self-dual dyon)
- Discuss Penrose transform (relative cohomology)
- Define the chiral algebra carefully on $\mathbb{CP}^1_{Q,x}$
- (BV formalism essential for imposing kinematical constraints due to gluing)

Takeaways:

Singular part of ghost # 0 OPEs in the presence of these self-dual backgrounds is undeformed* from flat bckgd.

Conformally soft generators resum to analogue of plane wave states in this background

Form factors with a $tr(B^2)$ insertion in the presence of this dyon background using algebra OPEs

2-point function: more intricate than in flat background, but we derive it in two ways.

Recover n-pt tree-level MHV scattering of gluons in a self-dual dyon background

[Adamo-Bogna-Mason-Sharma]

$$\langle \mathrm{Tr}(B^2)(x) | J^{(m_1)}_{a_1}[\tilde{\lambda}_1](z_1) \dots \widetilde{J}^{(m_r)}_{a_r}[\tilde{\lambda}_r](z_r) \dots \widetilde{J}^{(m_s)}_{a_s}[\tilde{\lambda}_s](z_s) \dots J^{(m_n)}_{a_n}[\tilde{\lambda}_n](z_n) \rangle_{CO}$$
$$= \left(\prod_{i=1}^{n} q_+(x, z_i)^{m_i + e_i} q_-(x, z_i)^{m_i} e^{ik_i \cdot x} \right) \frac{(z_r - z_s)^4}{(z_1 - z_2) \dots (z_n - z_1)} \mathrm{Tr}(T_{a_1} \dots T_{a_n})$$

Part 6: What’s Next?

Summary

- Twistorial QFTs are a class of integrable, not-necessarily-SUSY 4d theories characterized by a local uplift to twistor space.
- Their basic observables are form factors. Form factors with a single local operator insertion are controlled, and computable, by a 2d chiral algebra
- We have shown that the chiral algebra OPEs can be fixed by symmetries and associativity. This effectively solves a small subsector of massless QCD with certain matter content.
- We have also extended the ``chiral algebra bootstrap" to scattering in the presence of defects sourcing self-dual field configurations (tree-level MHV scattering in self-dual dyon).

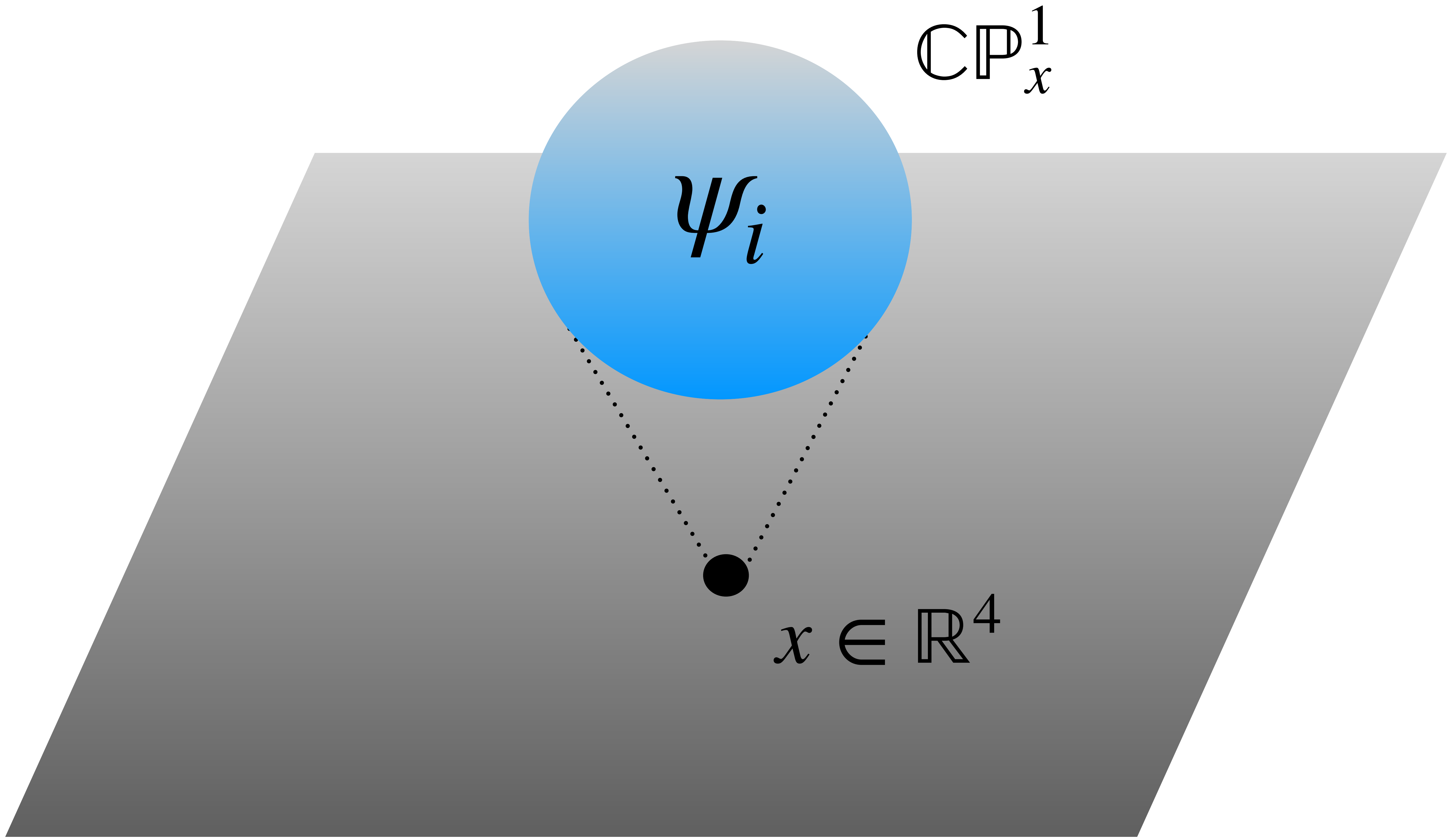
Future directions

- Form factors with multiple local operator insertions?
- Twistorial theories at large-N: single-trace formulas independent of matter?
- Useful organizing principle for scattering around more general field configs/ defects? Ordinary Wilson lines (discussions w/ Garner & Mason)
- CSW rules from the chiral algebra bootstrap for higher-loop form factors? (discussions w/ Costello & Morales)
- Other applications to (rational terms of) massless QCD amplitudes?

Thank you!

A heuristic picture:

$J[k,l](\psi), \tilde{J}[k,l](\psi), \ldots$

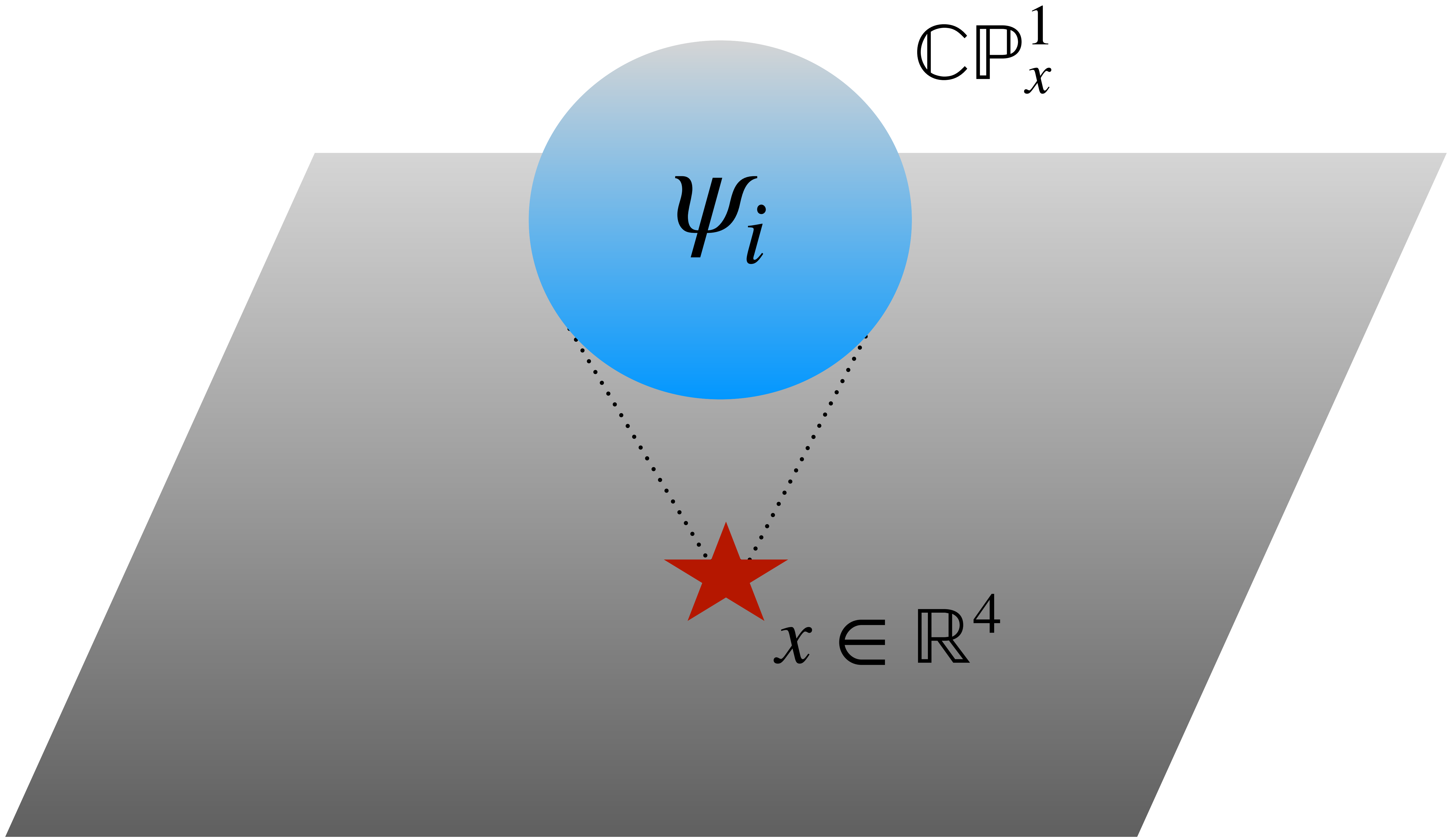


$$\langle J[k,l] \ldots \tilde{J}[r,s] \rangle = \int \mathscr{D}\psi e^{\int \psi \bar{\partial} \psi} J[k,l] \ldots \tilde{J}[r,s]$$

unique conformal block

A heuristic picture:

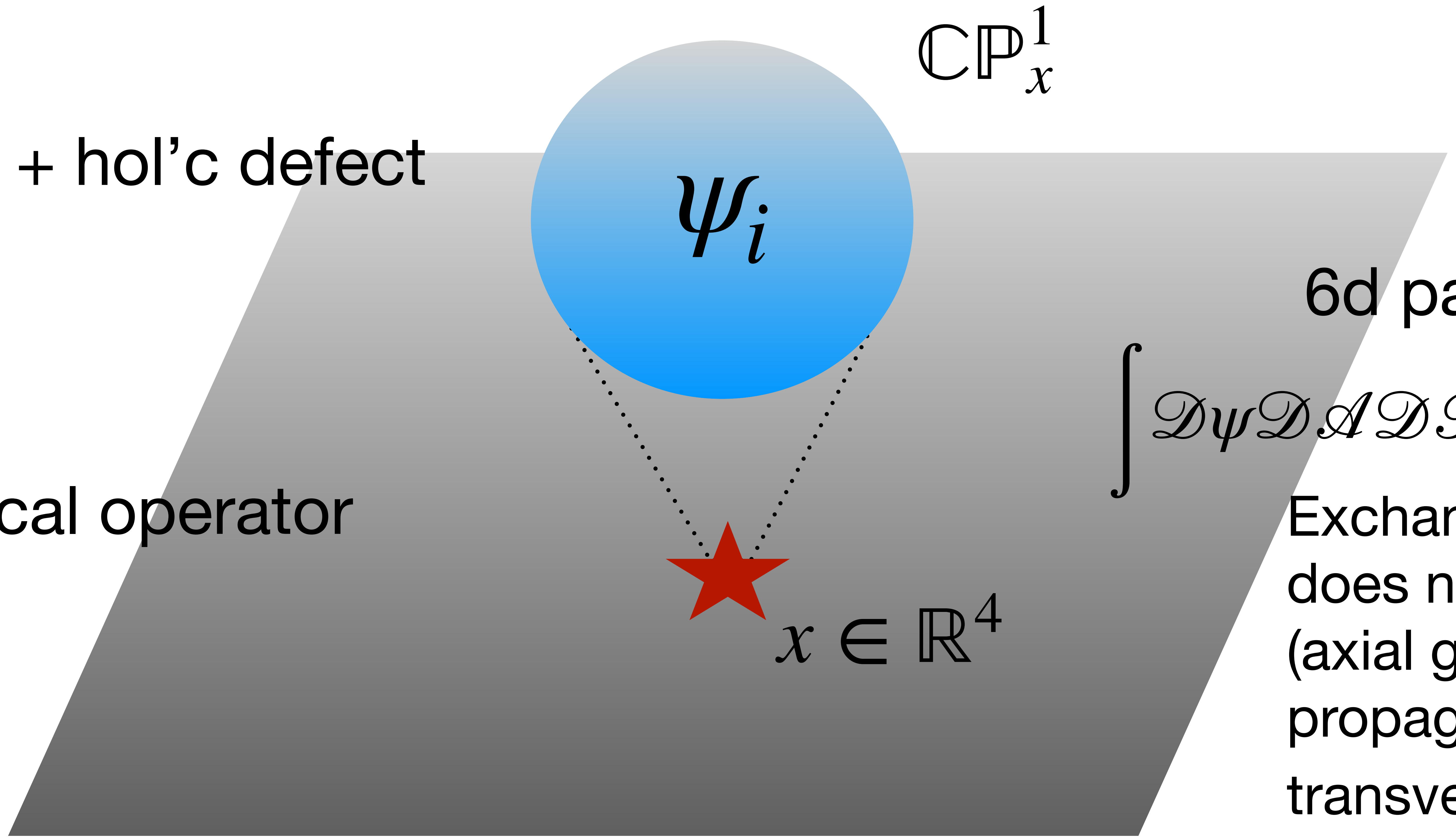
Integrate out fermions to obtain a local operator in 4d



A heuristic picture:

6d hol'c thy + hol'c defect

4d thy + local operator



6d path integral

$$\int \mathscr{D}\psi \mathscr{D}\mathscr{A}\mathscr{D}\mathscr{B} e^{S_{tot}} J[k,l]\ldots\tilde{J}[r,s]$$

Exchange of 6d fields does not contribute (axial gauge: propagate only transverse to $\mathbb{CP}^1_z$)

||

$$\langle J[k,l]\ldots\tilde{J}[r,s]\rangle = \int \mathscr{D}\psi e^{\int \psi\bar{\partial}\psi} J[k,l]\ldots\tilde{J}[r,s]$$

Here are some of the terms expressed using unknown coefficients.
Pole order: fixed by matching combined dilatation symmetry

$\frac{1}{z}$ and ∂_z have combined dilatation = 1

$J, \tilde{J}, E, F$ have combined dilatation = 1, 0, 0, 1 respectively

$$\tilde{J}_a[t](z)J_b[r](0) \sim \frac{1}{z}\sum_{m\geq 1}\sum_{k_j^i\geq 0}^{\sum_{j=1}^{m+1}k_j=t+r-m} \hbar^m \overset{(m)}{\underset{(t,r)}{f}}[k_1,...,k_{m+1}]_{ab}^{i_1\cdots i_{m+1}} :\prod_{j=1}^{m+1}\tilde{J}_{i_j}[k_j]:$$

$$E[t](z)J_b[r](0) \sim \frac{1}{z}\sum_{m\geq 1}\sum_{k_j^i\geq 0}^{\sum_{j=1}^{m+1}k_j=t+r-m-1} \hat{\lambda}_{\mathfrak{g}}\hbar^{m+\frac{1}{2}} \overset{(m)}{\underset{(t,r)}{j}}[k_1,...,k_{m+1}]_{b}^{i_1\cdots i_{m+1}} :\prod_{j=1}^{m+1}\tilde{J}_{i_j}[k_j]:$$

$$F[t](z)J_b[r](0) \sim \sum_{m\geq 1}\sum_{k_j^i\geq 0}^{\sum_{j=1}^{m+1}k_j=t+r-m} \hat{\lambda}_{\mathfrak{g}}\hbar^{m+\frac{1}{2}}\Bigg(\frac{1}{z^2}\overset{(m)}{\underset{(t,r)}{k}}[k_1,...,k_{m+1}]_{b}^{i_1\cdots i_{m+1}}$$
$$+\frac{1}{z}\overset{(m)}{\underset{(t,r)}{l}}[k_1,...,k_{m+1}]_{b}^{i_1\cdots i_{m+1}}\hat{\partial}_1\Bigg) :\prod_{j=1}^{m+1}\tilde{J}_{i_j}[k_j]:$$

- Use associativity to fix OPE coefficients in terms of the $\overset{(m)}{f}$ coefficients.

$$\{\tilde{J}_a[t]\{J_b[r]J_c[s]\}_1\}_1 = \{J_b[r]\{\tilde{J}_a[t]J_c[s]\}_1\}_1 - \{J_c[s]\{\tilde{J}_a[t]J_b[r]\}_1\}_1$$

$$\{\{J_a[t]J_b[r]\}_1 J_c[s]\}_1 = \{J_a[t]\{J_b[r]J_c[s]\}_1\}_1 - \{J_b[r]\{J_a[t]J_c[s]\}_1\}_1$$

$$\{F[t]\{J_b[r]J_c[s]\}_1\}_2 = \{J_b[r]\{F[t]J_c[s]\}_2\}_1 - \{J_c[s]\{J_b[r]F[t]\}_1\}_2$$

- We obtained a recursion relation for $\overset{(m)}{f}$ at arbitrary m.

- We used the recursion relation to find a closed-form expression for $\overset{(1)}{f}$, and a recursive expression for $\overset{(m)}{f}$ with $m > 1$.

$$K_{ab}^{i_1 \cdots i_{m+1}} = -f_{a j_1}^{i_1} K^{j_1 j_2} f_{j_2 j_3}^{i_2} \cdots f_{j_{2m-2} j_{2m-1}}^{i_m} K^{j_{2m-1} j_{2m}} f_{j_{2m} b}^{i_{m+1}}$$

$$\alpha(t,k) = t^2(k^1+1) - t^1(k^2+1) \qquad \beta(t) = t^1 + t^2$$

$$\underset{(t,r)}{\overset{(m)}{j}}[k_1, \ldots, k_{m+1}]_b^{i_1 \cdots i_{m+1}} = -\left(\frac{\alpha(t,k_1)}{\beta(t)}\right) \underset{(t-1,r)}{\overset{(m)}{f}}(k_1, \ldots, k_{m+1}) K^{i_1 j} K_{jb}^{i_2 \cdots i_{m+1}}$$

$$\underset{(t,r)}{\overset{(m)}{k}}[k_1, \ldots, k_{m+1}]_b^{i_1 \cdots i_{m+1}} = -\left(\frac{\beta(k_1+1)}{\beta(t+1)}\right) \underset{(t,r)}{\overset{(m)}{f}}(k_1, \ldots, k_{m+1}) K^{i_1 j} K_{jb}^{i_2 \cdots i_{m+1}}$$

$$\underset{(t,r)}{\overset{(m)}{l}}[k_1, \ldots, k_{m+1}]_b^{i_1 \cdots i_{m+1}} = - \underset{(t,r)}{\overset{(m)}{f}}(k_1, \ldots, k_{m+1}) K^{i_1 j} K_{jb}^{i_2 \cdots i_{m+1}}.$$

Some terms with single poles were also determined more formally by homotopy transfer methods [Zeng]

$f^{(1)}$ two ways

$$\sum_{ac} = \left(\sum_{a=0}^{\min(m,l_1)} \sum_{c=1}^{\min(n,l_2+1)} \binom{l_1}{a}\binom{l_2}{c-1} - \sum_{a=1}^{\min(m,l_1+1)} \sum_{c=0}^{\min(n,l_2)} \binom{l_1}{a-1}\binom{l_2}{c} \right) \binom{m}{a}\binom{n}{c}$$

holomorphic integral [Fernandez]

$$\mathcal{M}_1 = \sum_{ac} \frac{(m+n-a-c)!(a+c-1)!a!c!(r+m-a)!(s+n-c)!(1+k_1+k_2)!(1+l_1+l_2-a-c)!}{(m+n)!(1+m+r+n+s-a-c)!k1!k2!l1!l2!}$$

$$m(p,q;x,y;u) = \sum_{j=1}^{\mathrm{Min}[u,x+1]} \frac{(p-j)!(1+x+y-j)!}{(1+p+q-j)!(1+x-j)!}.$$

associativity [Fernandez-NMP]

$$\underset{(r,t)}{\overset{(1)}{f}}(k,l) = \left(\frac{1}{16\pi^2}\right)\frac{(r^2+t^2)!(1+k^1+k^2)!}{k^1!k^2!l^2!} m(r^1+t^1, r^2+t^2, l^1, l^2; t^1)$$
$$- \left(\frac{1}{16\pi^2}\right)\frac{t^2!(1+k^1+k^2-r^1-r^2)!}{(k^1-r^1)!(k^2-r^2)!l^2!} m(t^1, t^2, l^1, l^2; t^1)\theta(k^1-r^1)\theta(k^2-r^2)$$
$$- \left(\frac{1}{16\pi^2}\right)\frac{r^1!(1+k^1+k^2)!}{k^1!k^2!(l^1-t^1)!} m(r^2+t^2, r^1, l^2, l^1-t^1; t^2)\theta(l^1-t^1).$$

It turns out, these two heinous expressions are indeed equal!