Huge (half-BPS) Operators (in AdS_5/CFT_4)

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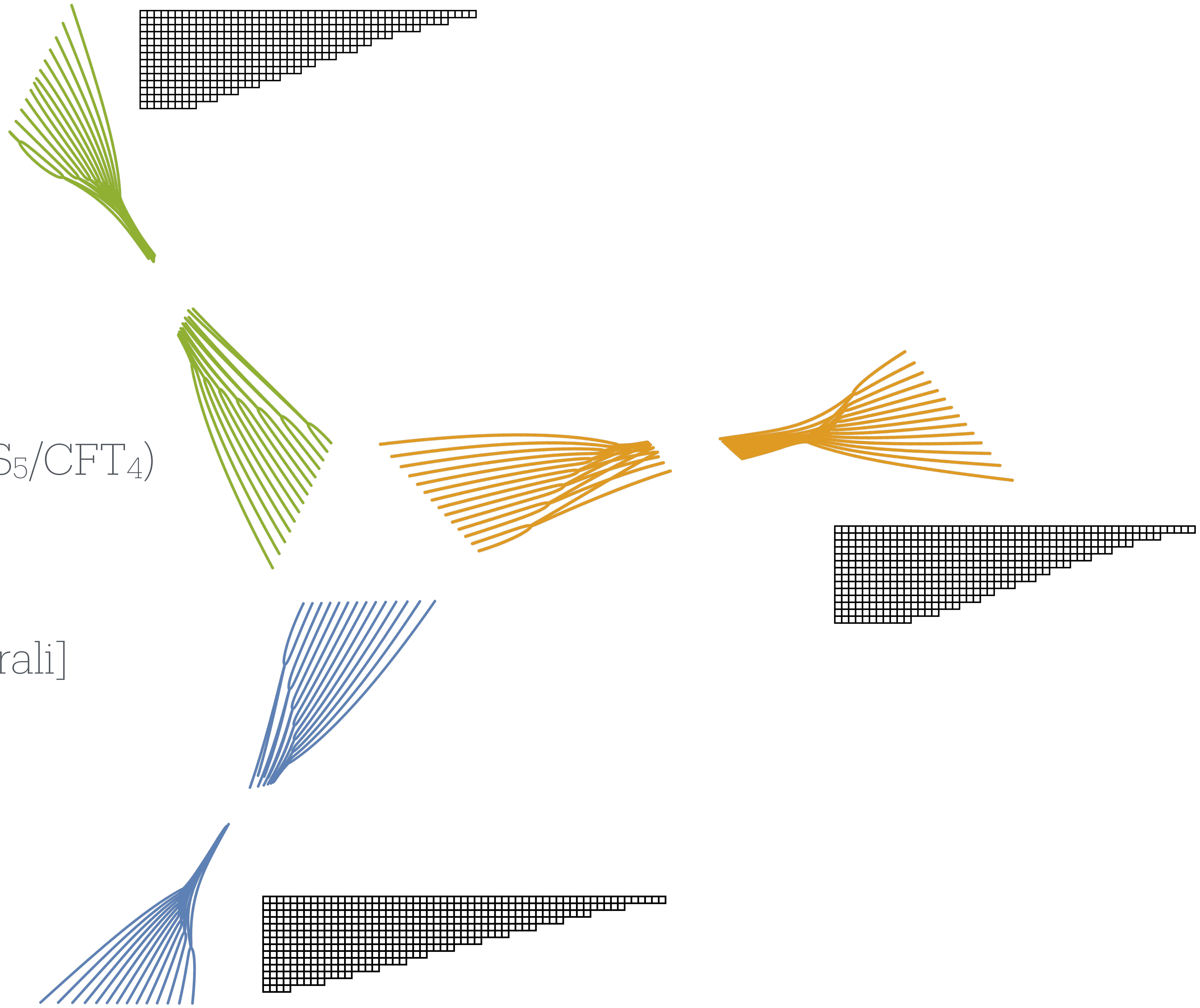
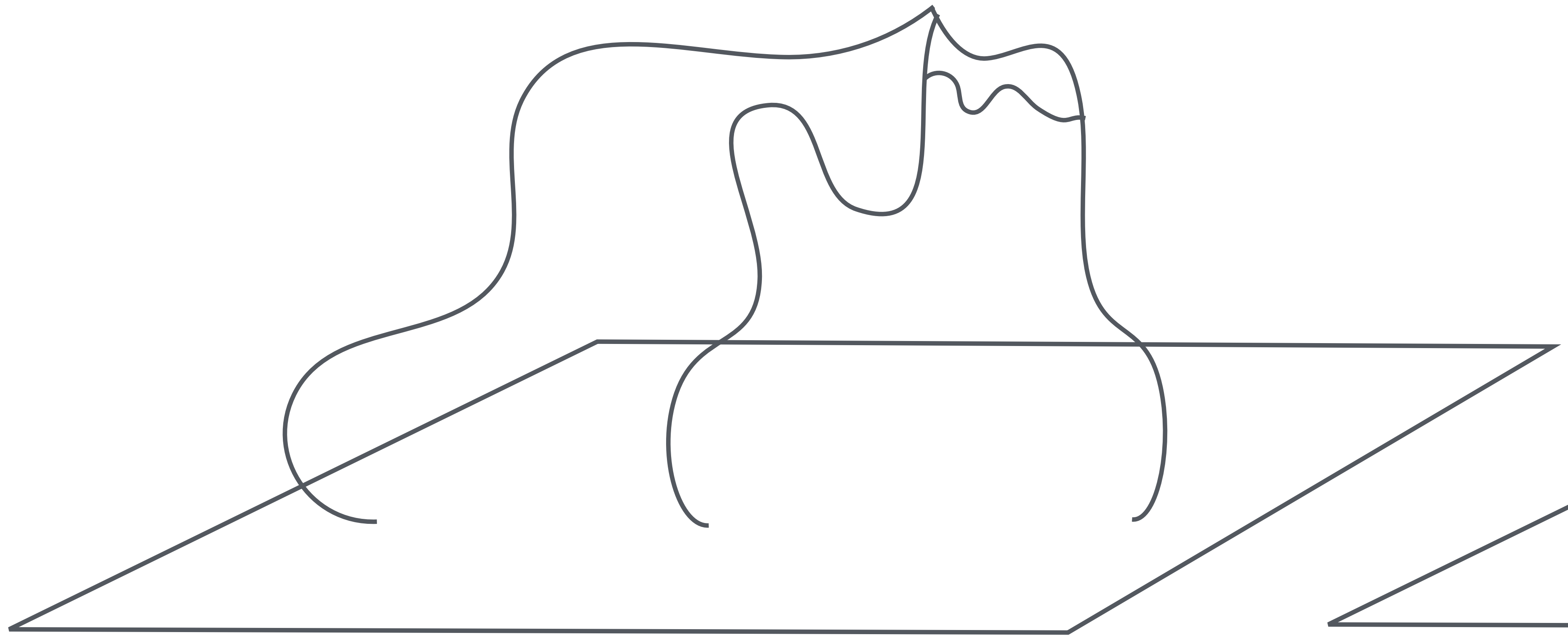
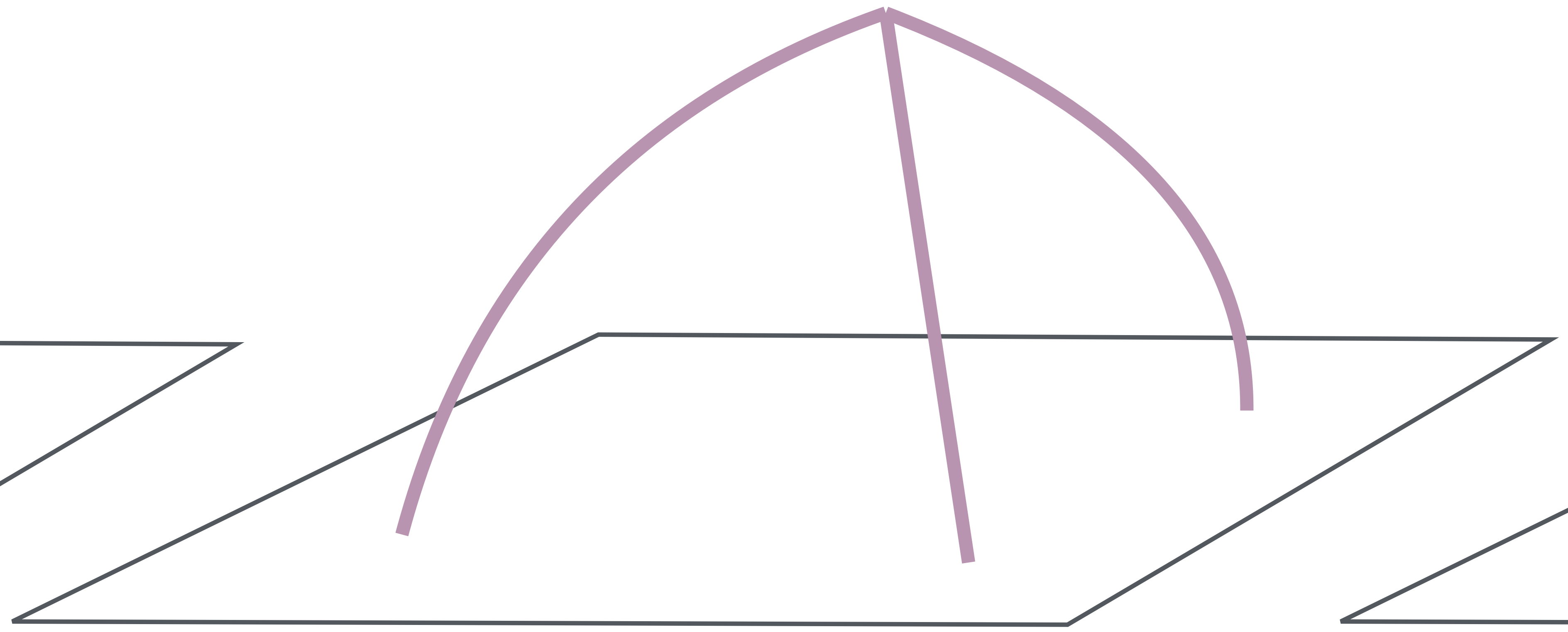


Light, heavy, huge



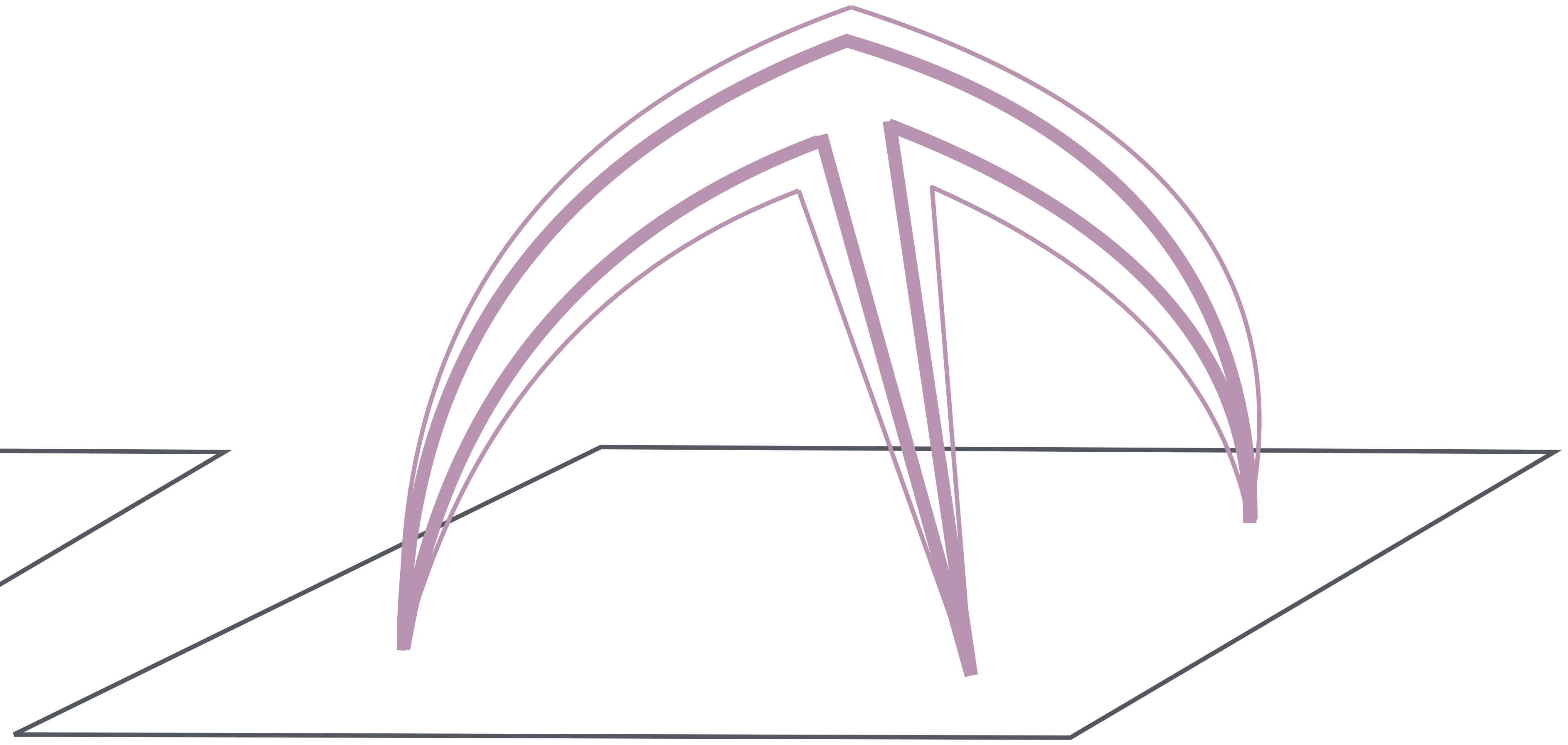
$\Delta = O(1)$

[Witten 1998; Freedman, Mathur, Matusis, Rastelli 1998; Lee, Minwalla, Rangamani, Seiberg 1998; Penedones 2010; Rastelli, Zhou 2016; Gonçalves, Pereira, Zhou, 2019;...]



$1 \ll \Delta \ll N^2$

[Janik, Surowka, Wereszczynski 2010; Roiban, Tseytlin 2010; Janik, Wereszczynski, 2011; Klose, McLoughlin 2011; Minahan 2012; Kazama, Komatsu 2012; Caetano, Toledo 2012; Aprile, PV 2000, ...]



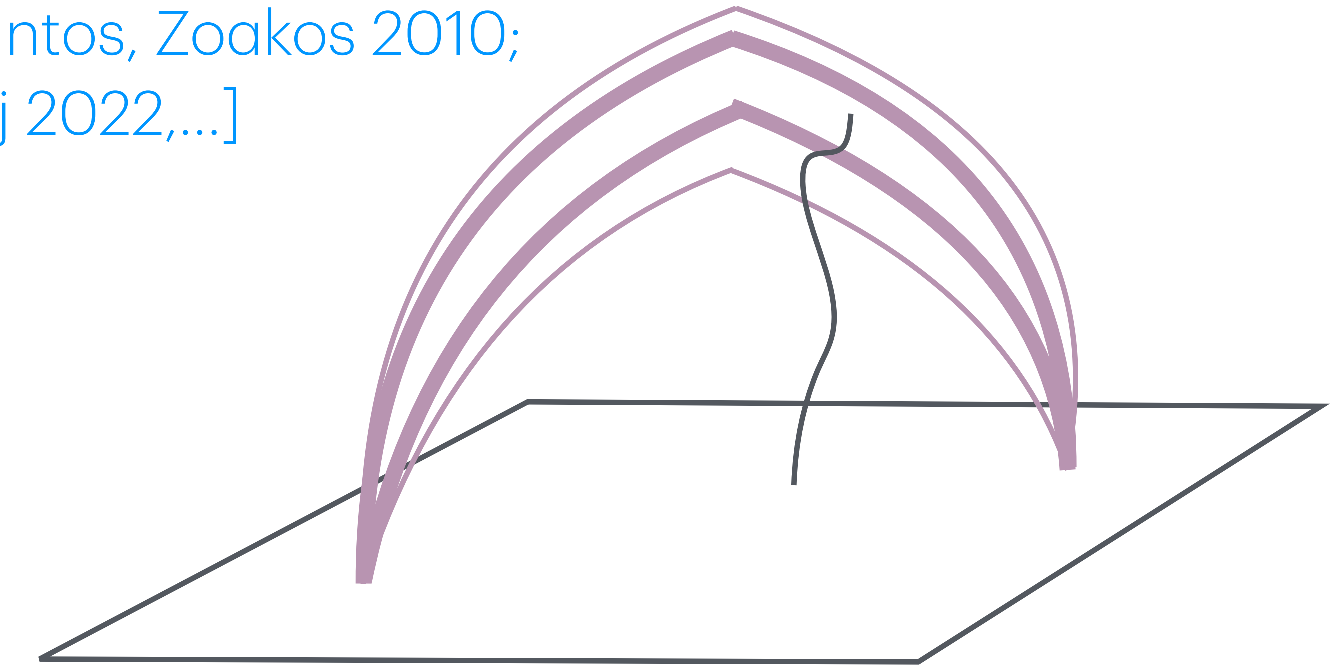
$\Delta = O(N^2)$

[Lin, Lunin, Maldacena 2004; Yamaguchi, 2011; D'Hoker, Estes, Gutperle 2007; Gomis, Matsuura, Okuda, Trancanelli 2008; Fitzpatrick, Kaplan, Walters, 2015; Chang-Lin 2016; Chandra, Collier, Hartman, Maloney 2020; Abajian, Aprile, Myers, PV, 2024; ...]

Heavy-Heavy-Light Correlators are great probes and great precision tests of AdS/CFT. [Skenderis, Taylor 2007, Zarembo 2010; Costa, Monteiro, Santos, Zoakos 2010; Paul, Perlmutter, Raj 2022,...]

$$\langle O_{\text{Heavy}} O_{\text{Heavy}} O_{\text{Light}} \rangle = \langle O_{\text{Light}} \rangle_{\text{AdS with a heavy object (string, brane,...) inside}}$$

$$\langle O_{\text{Huge}} O_{\text{Huge}} O_{\text{Light}} \rangle = \langle O_{\text{Light}} \rangle_{\text{Back-reacted geometry created by the huge operator}}$$



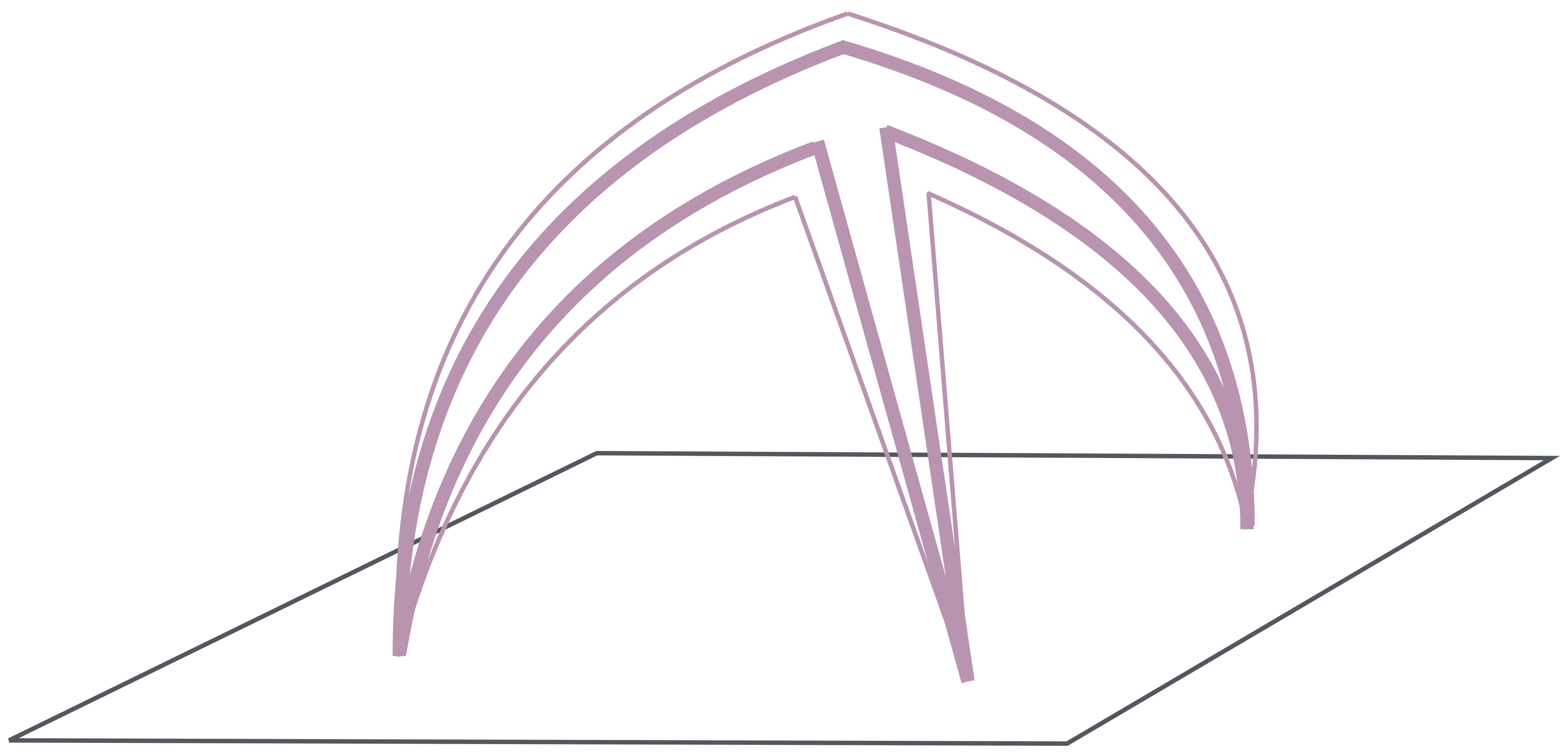
These correlators are O(1) quantities. They are *not* exponentially suppressed (or enhanced).

Three heavy objects

Heavy-Heavy-Heavy or Huge-Huge-Huge behave exponentially. They are semi-classical tunnelling processes. For example

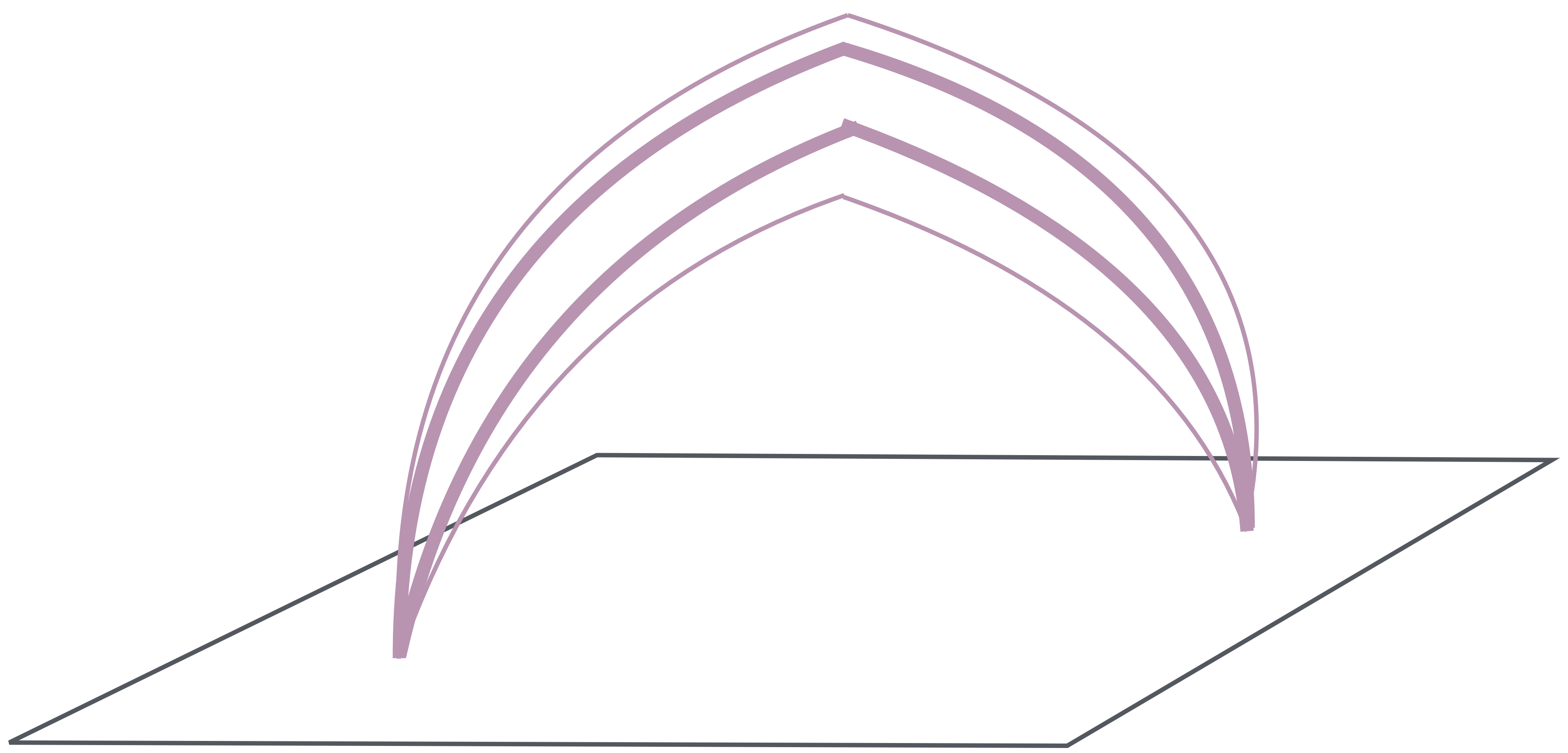
$$\langle O_{\text{Huge}} O_{\text{Huge}} O_{\text{Huge}} \rangle \simeq e^{N^2 (\texttt{semi-classical tunnelling geometry action}) + (\texttt{quantum corrections})}$$

Hard to find:



$\neq$

Easy:



Three-legged space-time banana geometry. Close to each puncture the geometry is similar to the two-point function but globally it is quite complicated.

Space-time banana. We can start with a geometry in global AdS and perform a few conformal transformations and Wick rotations to bring infinite past and future to two finite points in the boundary of Poincare AdS [Abajian, Aprile, Myers, PV]. This strategy was proposed for big classical strings in [Janik, Surowka, Wereszczynski 2010]

1/2 BPS operators

$$O \equiv \chi_R(\underbrace{y\cdot\phi(x)}_{\text{matrix } \Phi \text{ with eigenvalues } z_i}) \equiv \det_{i,j} z_i^{\overbrace{\lambda_j+j-1}^{h_j}} / \det_{i,j} z_i^{j-1}$$

Δ = number of scalars = number of boxes of the Young-tableaux

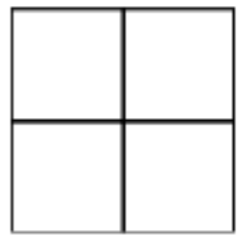
Two- and three- point functions of these operators are coupling independent due to SUSY. [Baggio, de Boer, Papadodimas, 2012]

Two point functions are orthonormal. [Corley, Kevicki, Ramgoolam 2011]

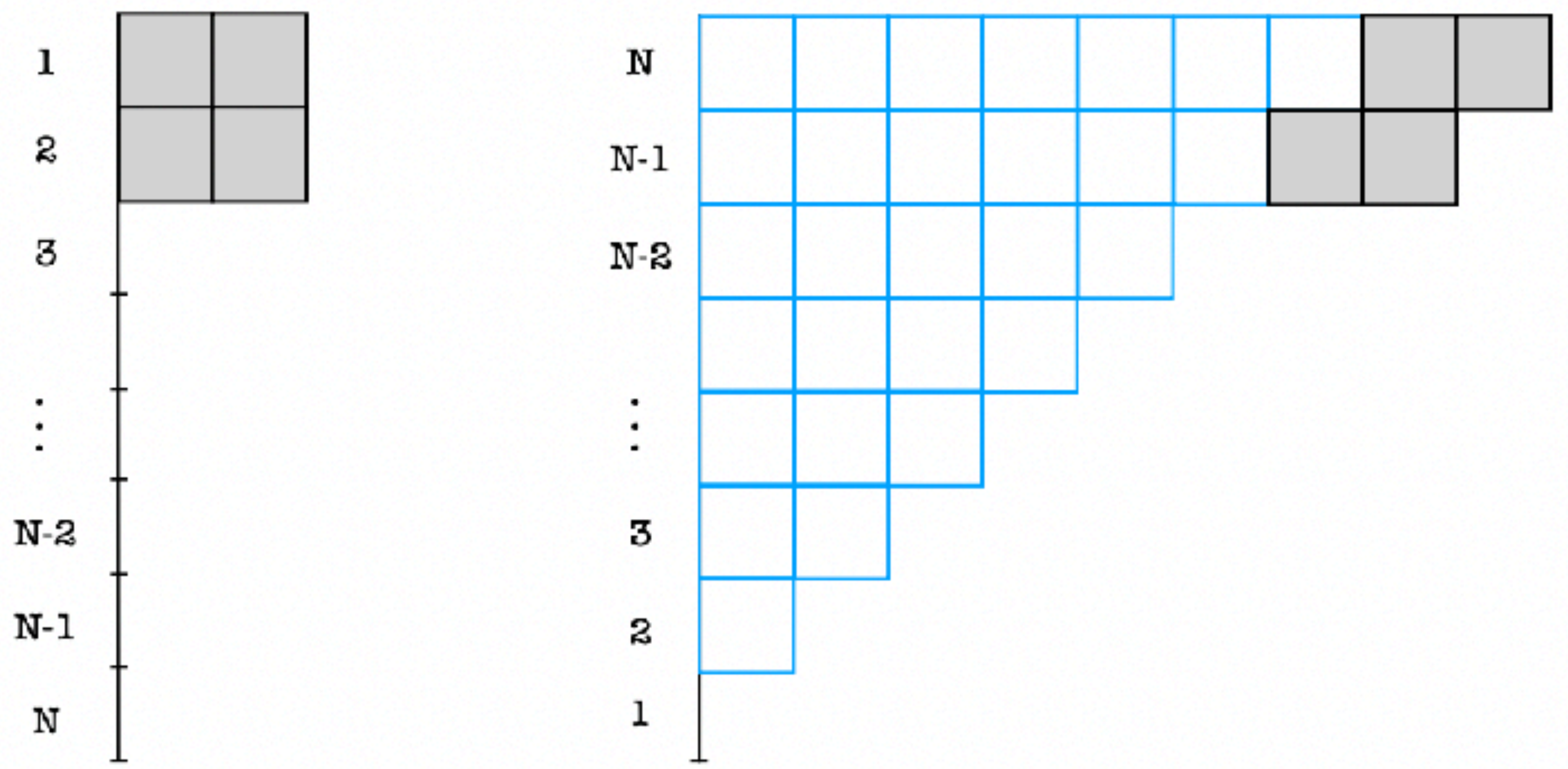
For Young-Tableaux with O(1) number of boxes — i.e. for light operators — the match between the gauge theory combinatorics and the SUGRA overlaps for 3-pt functions was one of the first powerful tests of AdS/CFT [Lee, Minwalla, Rangamani, Seiberg 1998].

There has been a lot of work on correlation functions of light 1/2 BPS operators. There is some work on heavy operators but very little for huge operators; that is what this talk is about.

Example:

$\mathcal{R} =$ , $\quad \chi_R(\Phi) = \frac{1}{8}\mathrm{tr}(\Phi)^4 + \frac{1}{4}\mathrm{tr}(\Phi^4) - \frac{1}{4}\mathrm{tr}(\Phi^2)\mathrm{tr}(\Phi)^2 - \frac{1}{8}\mathrm{tr}(\Phi^2)^2$

$\lambda = (2,2,0,0,\ldots,0)$ $\qquad h = (0,1,2,\ldots,N-4,N-3,N,N+1)$



1/2 *Huge* BPS operators

(h = O(N), # of boxes = O(N²))

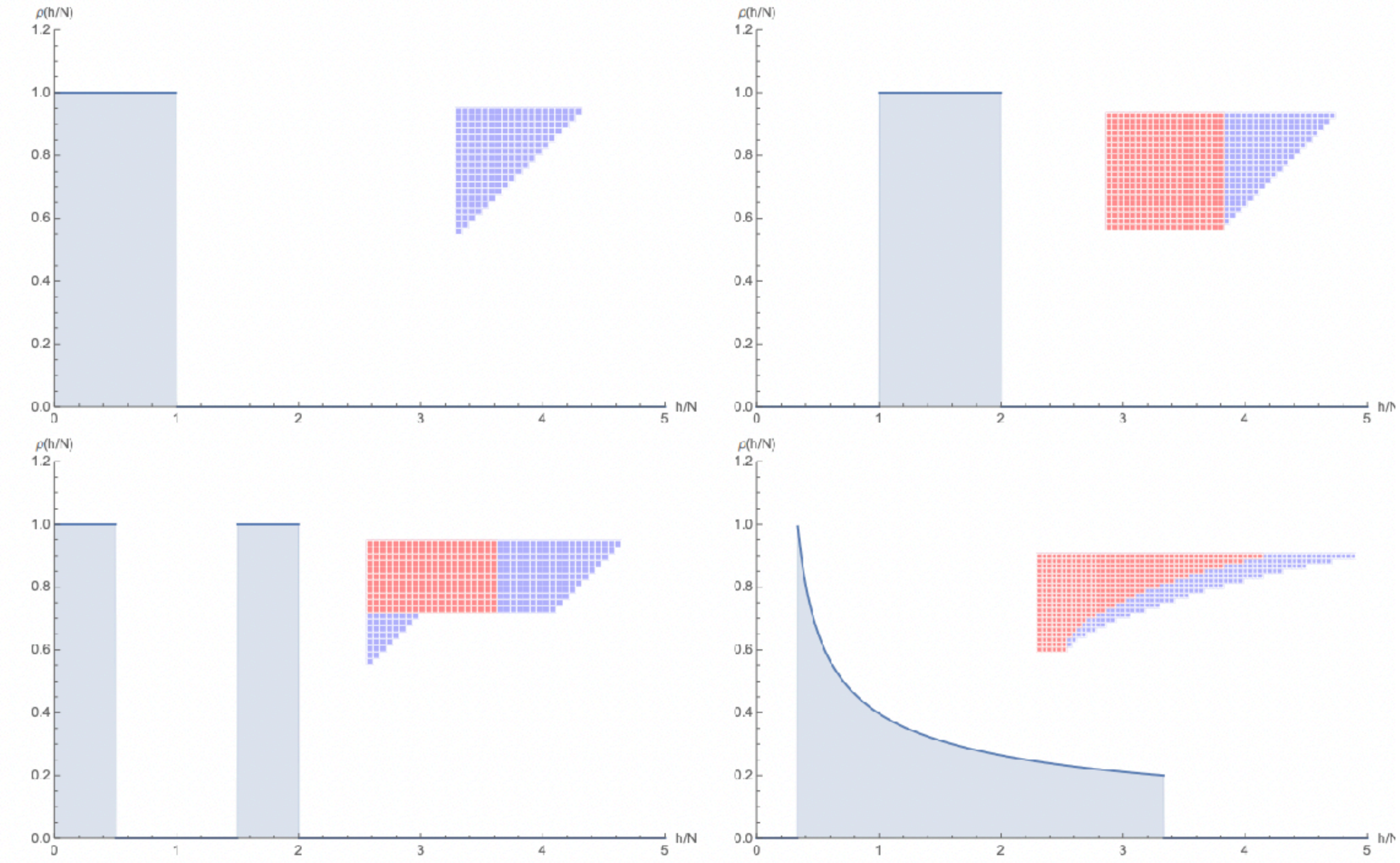


Figure 4: We consider here the large N limit and very large representations with corresponding Young Tableaux with $\mathcal{O}(N^2)$ boxes. In the smooth classical limit we are considering here these can be nicely described by the density $\rho(h)$ of heights h_j of the corresponding shifted Young Tableaux.

Today we ask

$$C_{123}[\rho_1, \rho_2, \rho_3] = ?$$

This was the problem we addressed with Harish Murali and Vladimir Kazakov about half a year ago.

Three Matrix Model Starting Point

We consider three operators with three representations R. For now they can be light, heavy or huge. The three point function is then given by $Z(\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3)\times\sqrt{Z(\phi,\phi,\phi)/\prod_{n=1}^{3} Z(\mathcal{R}_n,\mathcal{R}_n,\phi)}$ where the 3 matrix model partition function

$$Z(\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3)=\int\prod_{i=1}^{3} dM_i\,\chi_{R_i}(M_i)\,e^{-N\,\mathrm{tr}\left(\frac{1}{2}\sum_i M_i^2-\sum_{i<j} M_iM_j\right)}$$

Kinetic term such that different matrix have unit propagator while same matrices have zero propagator.

Everything would factorize if it were not for the last term. We thus introduce a simple forth matrix X to disentangle it and reduce Z to a bunch of XM factorized interactions. We then integrate over the angles between X and the M's. At the end we get an integral over 4 sets of eigenvalues (4N variables instead of the original $3N^2$).

Four Matrix *Eigenvalue* Quantum Expression

$$Z(\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3) = \int d\mu(x) \prod_{n=1}^{3} \oint d\mu(m_n) I(x, m_n) I(h_n, -\log m_n) \frac{\Delta(-\log m_n)}{\Delta(m_n)} \Delta(h_n)$$

where (with G being the Barnes G-function),

$$I(a,b) \equiv \frac{\det_{i,j} e^{N a_i b_i}}{\Delta(a)\Delta(b)} \times \frac{G(N+1)}{N^{N(N-1)/2}}$$

can also be cast as an angular integral $I(a,b) = \int dU e^{N \mathrm{tr}(AU^\dagger BU)}$ as shown by Harish-Chandra-Itzykson-Zuber.

$$d\mu(x) = \prod_{i=1}^{N} dx_i \, \Delta(x)^2 \exp\left(-\frac{N}{2} \sum_{i=1}^{N} x_i^2\right)$$

$$\Delta(x) \equiv \prod_{i>j} (x_i - x_j) = \det_{i,j} x_i^{j-1}$$

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Six I's: Three from three characters and three from the angles between X and the M's

where (with G being the Barnes G-function),

I is a simple determinant:

$$I(a,b) \equiv \frac{\det_{i,j} e^{N a_i b_j}}{\Delta(a)\Delta(b)} \times \frac{G(N+1)}{N^{N(N-1)/2}}$$

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The measure is the standard matrix model one with the usual Vandermond factors

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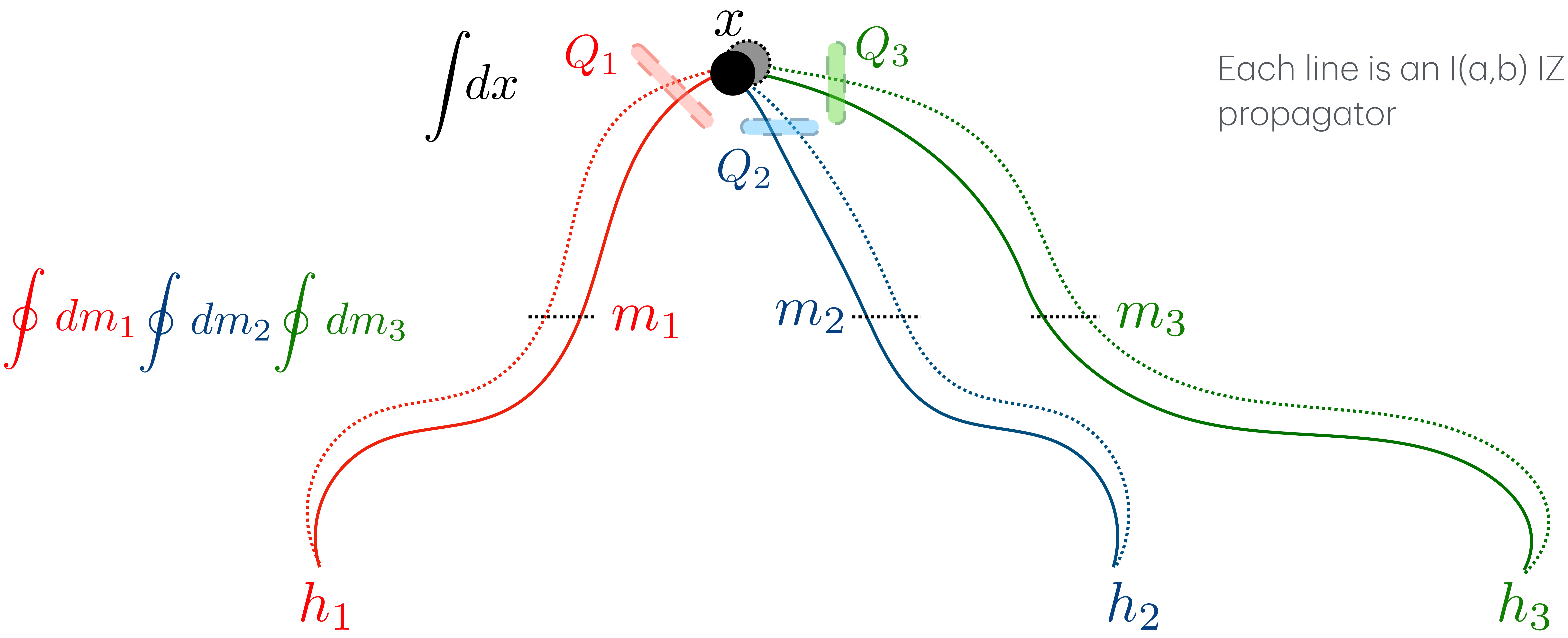
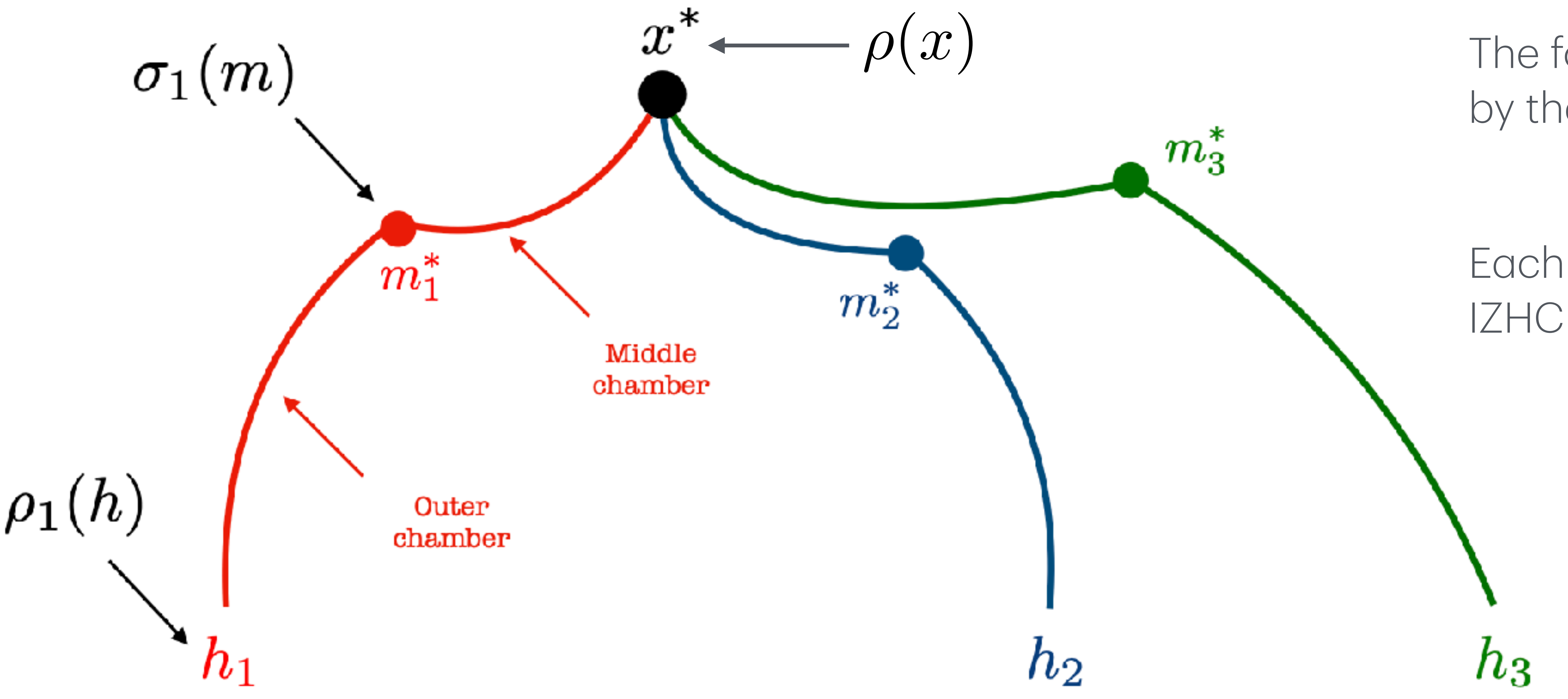
Quantum three point function in a figure



Each line is an I(a,b) IZHC propagator

Graphical representation of the four-fold integral representation . The three sets of outer variables h_n propagate into bulk variables m_n which then propagate into a common point x. The points m_n and x are integrated over; in other words, these propagations are quantum.

Huge operators



The four integrals collapse: They are given by the saddle point optimal configurations.

Each line is the **classical limit** of the I(a,b) IZHC propagator

For very large operators and large N all integrals become dominated by a leading trajectory. In other words, the quantum propagations of the previous figure become classical. This classical propagation is dominated by a fluid. Each leg of the diagram can be through as a fluid propagating in a *chamber*. At the four dots – which we dub as *inner* junctions – these fluids are glued together with some gluing conditions given by the saddle point equations for the four saddle point equations for the four variables x, m_1, m_2, m_3. In total we have six chambers and four junctions.

Fluids

$$I(a,b) \equiv \frac{\det_{i,j} e^{N a_i b_i}}{\Delta(a)\Delta(b)} \times \frac{G(N+1)}{N^{N(N-1)/2}}$$

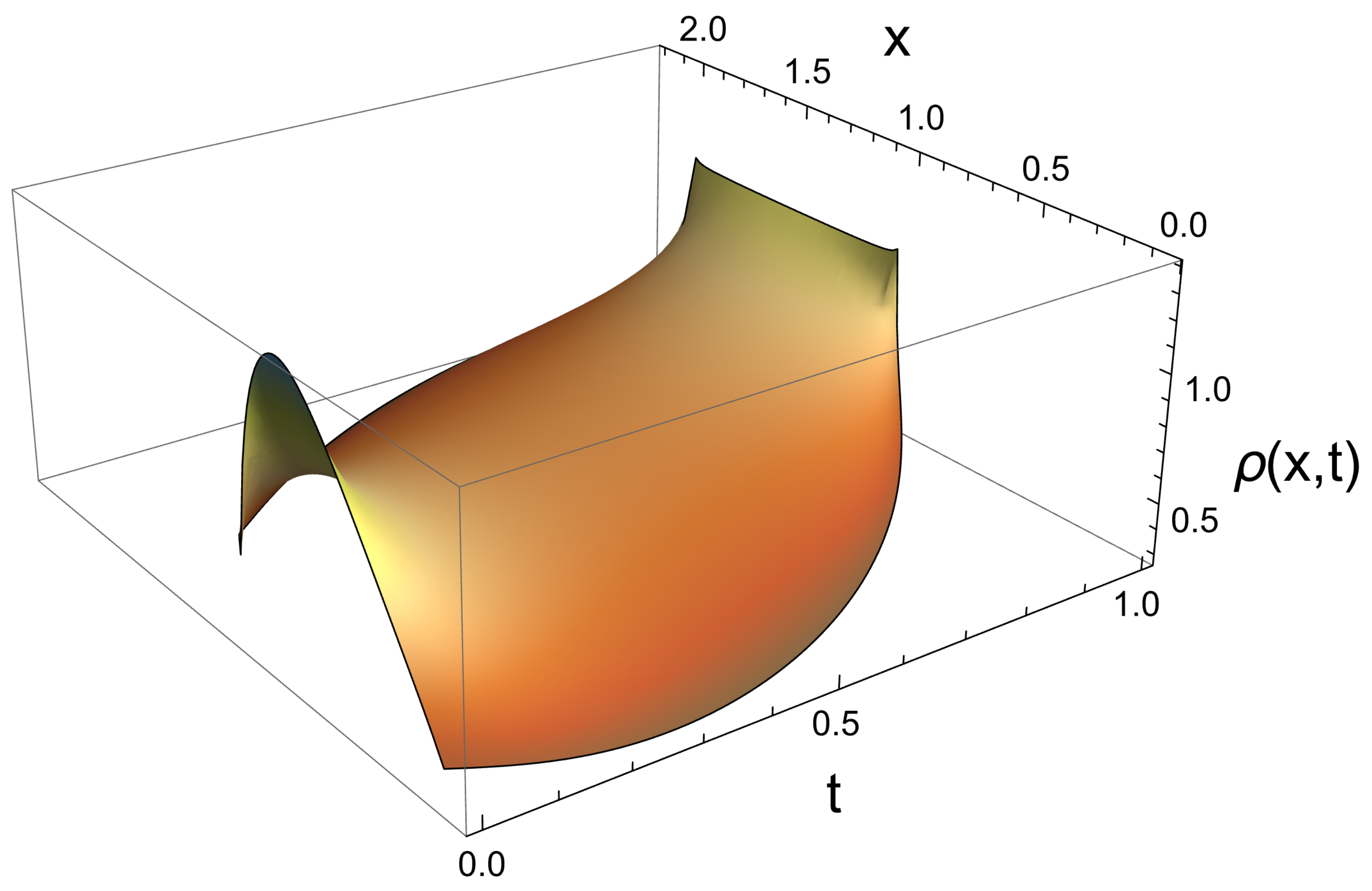
The HCIZ integral admits a beautiful classical limit in terms on an *integrable (!)* 1+1dim fluid:

$$\frac{1}{N^2}\log I(a,b) \simeq S_{\mathtt{fluid}}[\rho_a,\rho_b] + S_{\mathtt{bdy}[\rho_a,\rho_b]}$$

[Matysin 1993]

where S_{bdy} is a simple explicit expression while S_{fluid} is the action of an inviscid fluid with negative pressure $P \sim -\rho^2$,

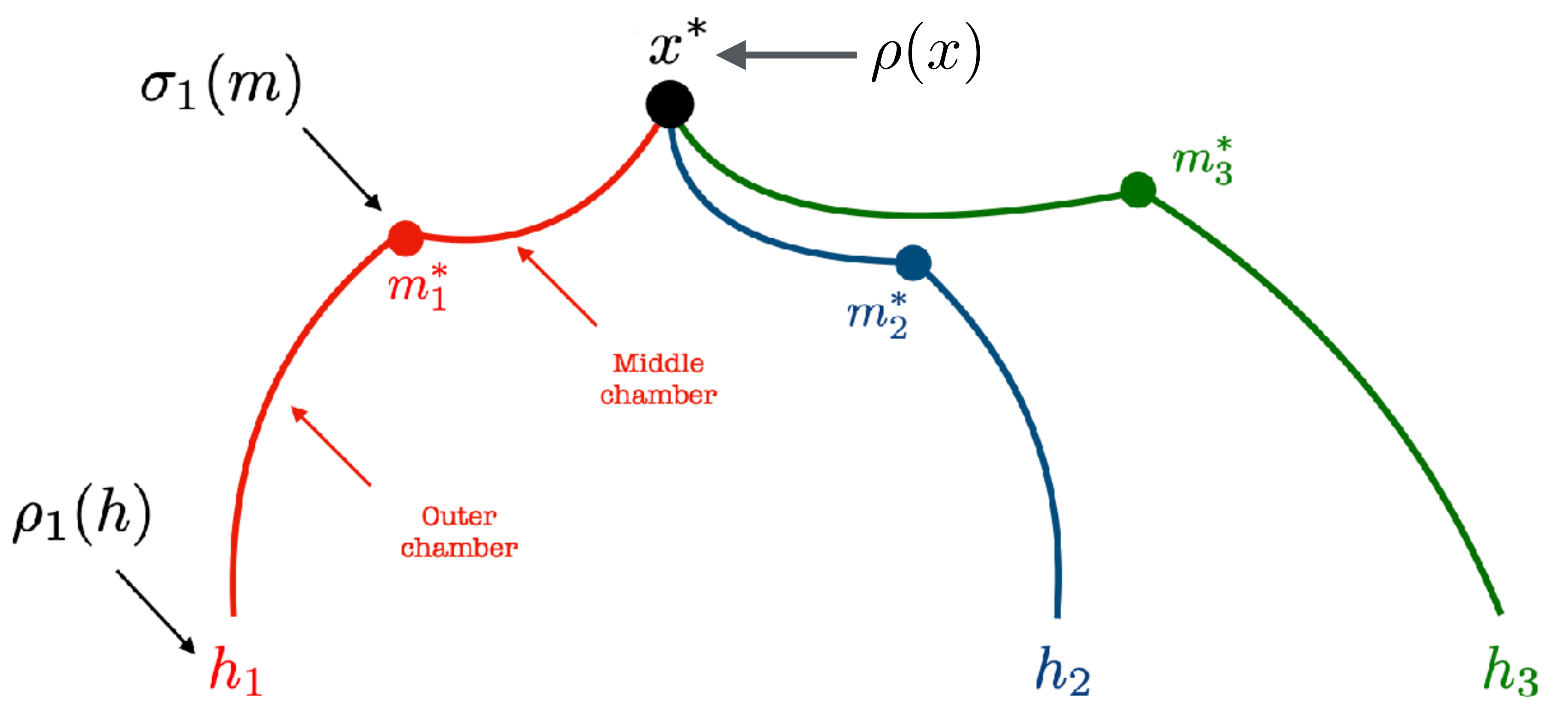
$$S_{\mathtt{fluid}}[\rho_a,\rho_b] = -\frac{1}{2}\int_0^1 dt \int dx\; \rho(x,t)\left(v(x,t)^2 + \frac{\pi^2}{3}\rho(x,t)^2\right)$$



where

$$\rho(x,t=0) = \rho_a(x) \qquad \text{into} \qquad \rho(x,t=1) = \rho_b(x)$$

Final result



$$\frac{1}{N^2}\log Z[\eta_1,\eta_2,\eta_3] \simeq \sum_{n=1}^{3} S_{\texttt{fluid}}[\rho,\sigma_n] + \sum_{n=1}^{3} S_{\texttt{fluid}}[\tilde{\sigma}_n,\eta_n] +$$

$$+\int dx\, x^2\left(\rho(x) + \frac{1}{2}\sum_{n=1}^{3}\tilde{\sigma}_n(x)\right) - \frac{1}{2}\int dx \int dy \log|x-y|\,\rho(x)\rho(y)$$

where we used $\sigma_n(x)$ and $\tilde{\sigma}_n(x) = e^{-x}\sigma_n(e^{-x})$ to indicate the densities of m_n and $-\log m_n$ respectively, at the junction n, see figure . Lastly, we have four gluing conditions which follow from the saddle point equations,

$$\begin{aligned} ⨍ dz\,\frac{\rho(z)}{x-z} &= 2x + \sum_{n=1}^{3} v_n(x,0) \\ w_n(x,0) &= -x - e^{-x}\, v_n(e^{-x},1)\,, \qquad n = 1,2,3 \end{aligned}$$

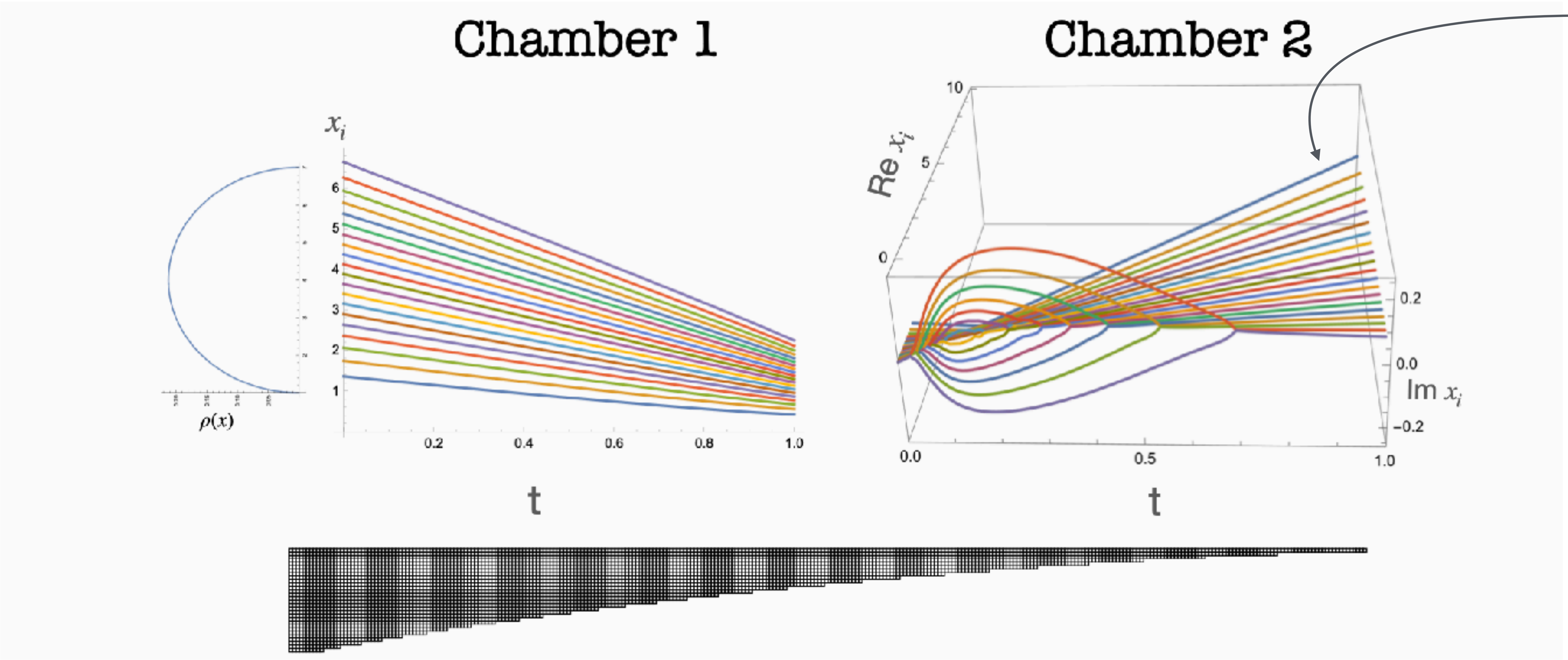
An example exploiting integrability of the fluids

Consider symmetric flows with three identical operators.

If we know the distribution of the middle x eigenvalues, we can immediately solve for the initial velocities in the middle.

We can then evolve forward in both chambers and find the h distribution at the end-points where the operators are.

For example, for a Wigner semi-circle distribution we get



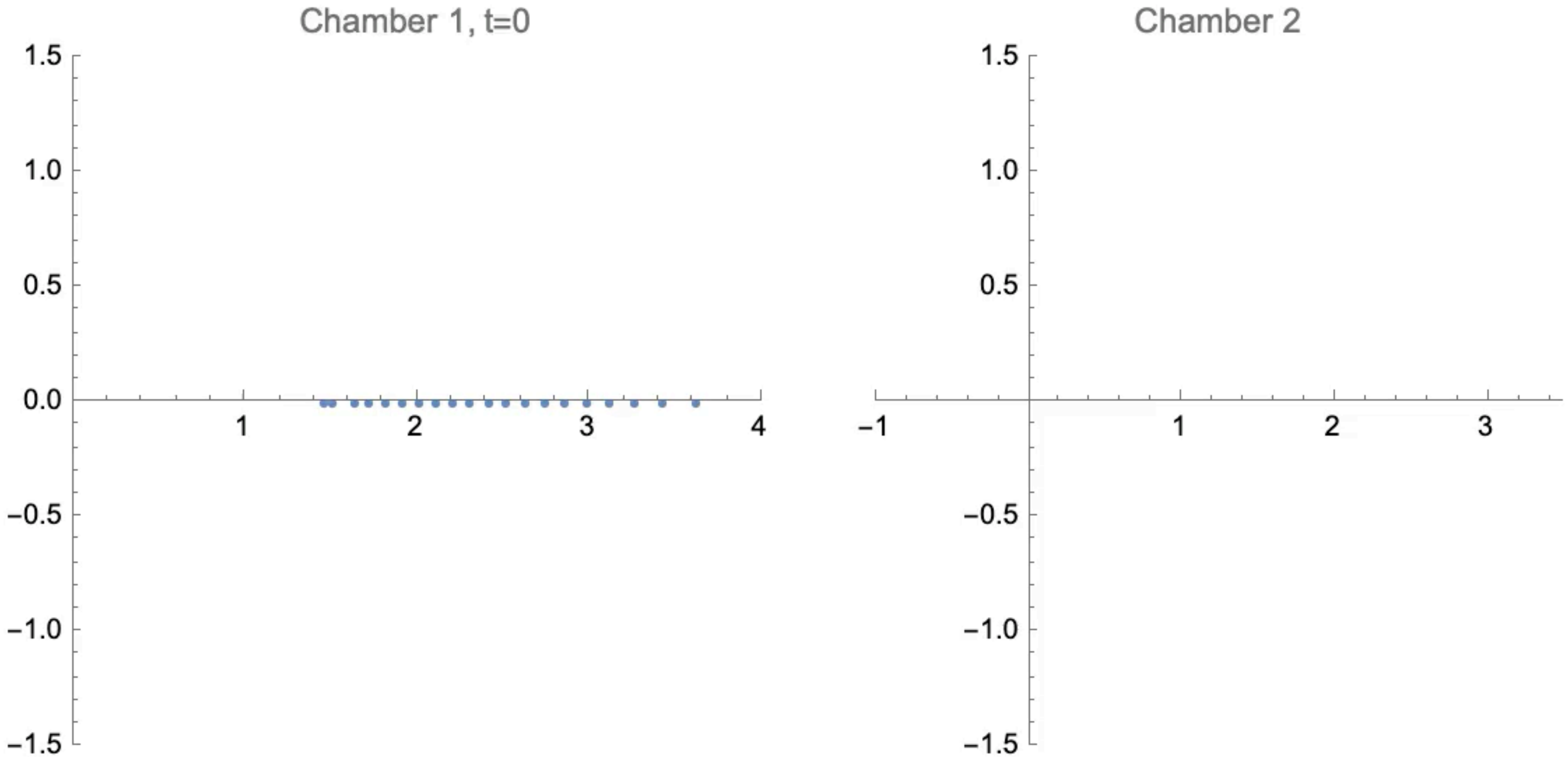
Trajectories of the N fluid bits $x_i(t)$ with follow the classical rational Colegero-Moser integrable model with equations of motion

$$\frac{d^2 x_i}{dt^2} + \frac{2}{N^2}\sum_{j\neq i}\frac{1}{(x_i - x_j)^3} = 0$$

[Calogero, 1981. Polychronakos 2006 for a nice recent review]

The real problem

- Of course, the real problem is ***not*** to start from a density of x in the middle. That is what we call the easy local problem.
- The real problem is about starting with **three densities for h outside**. That problem — which we call the hard global problem — is harder but can also be solved*. Example of final flow for three equal Trapezia YT's:



* The trick is to start with a flow under good control (say the one in the previous slide) and use adiabatic deformation to arrive at any desired flow. As long as the topology is the same (number of cuts etc) this works great. That the global problem was much harder was well recognized in the literature. The great lecture notes by Govind Menon https://www.dam.brown.edu/people/menon/talks/cmsa.pdf nicely illustrate the challenges:

[6] As of Feb. 2016, I have not been able to use this scheme to solve the BVP! It appears that what one needs is a multiple shooting method.

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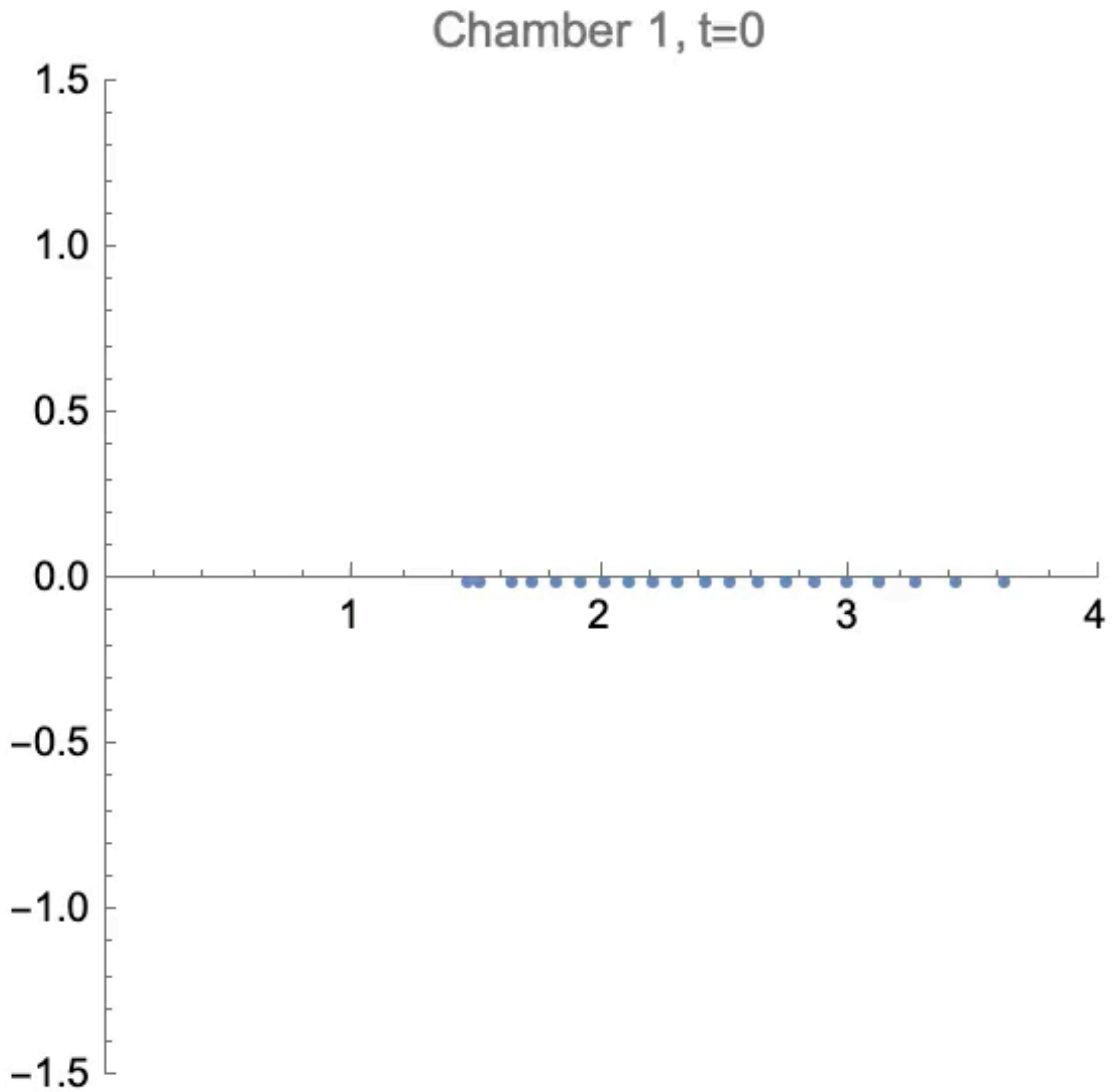

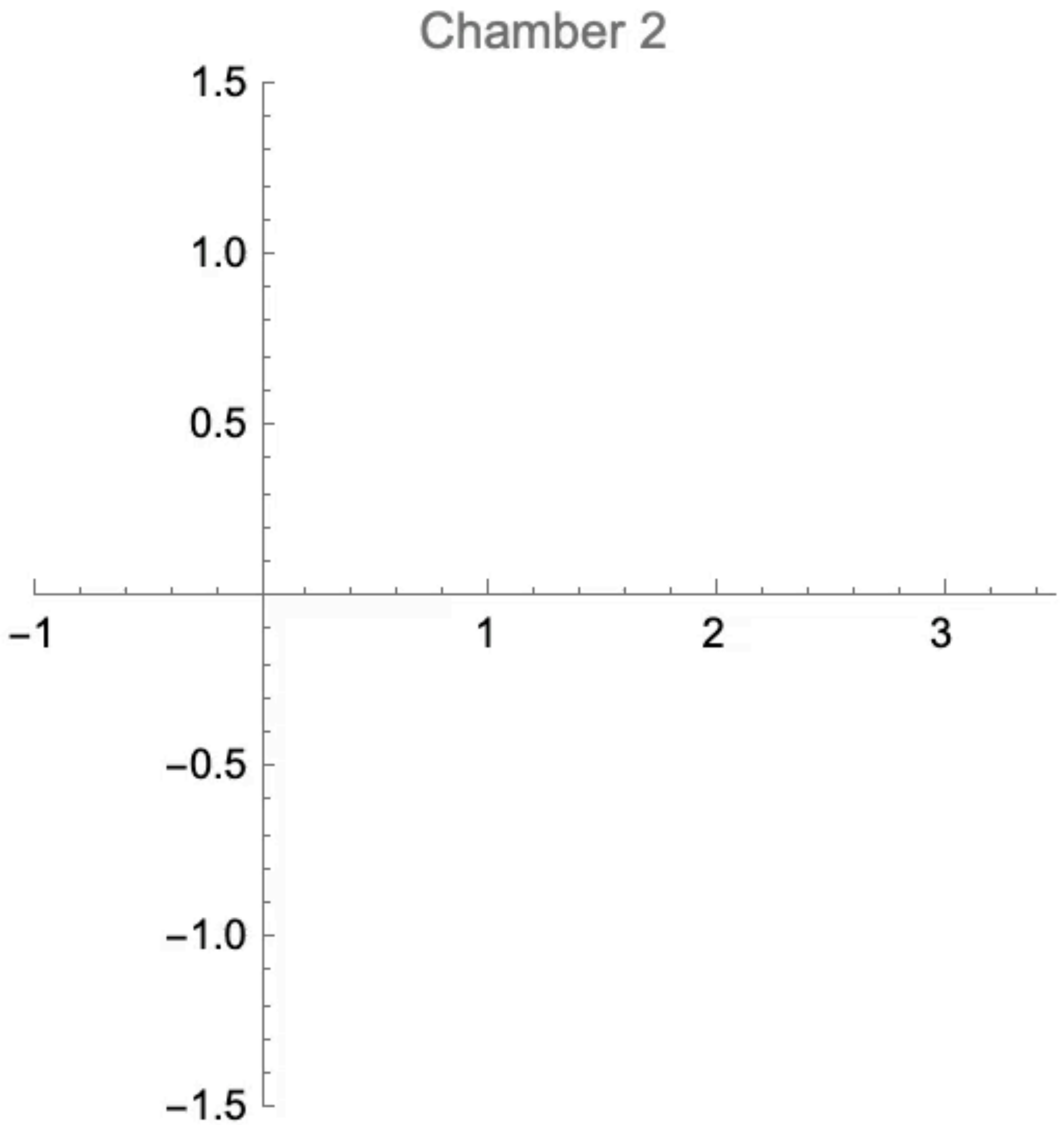

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Conclusions

- Structure constants of huge BPS operators are given by an *integrable* 1+1 system:
  A one dimensional inviscid fluid [Riemann-Hopf system] or its known integrable discretization [the Colegero-Moser rational model]

  It is unexpected to find integrability is this highly non-planar regime. Charges: $Q_{n,m} \equiv \oint dx \int_0^{v(x,t)+i\pi\rho(x,t)} dz \; (x-tz)^n (x+(1-t)z)^m$

  We can now compute structure constants of generic (i.e. non-extremal*) huge correlators. E.g. $\log C_{123}^{\text{three triangle YTs}} \simeq 0.4465 N^2$

- From our Gauge theory combinatorics computation, we have a prediction for SUGRA: The action for the merging of three Lin-Lunin-Maldacena geometries should be given by a bunch of 1+1 d fluids. [We stress that knowing the LLM 2pt function is far from telling us what the 3pt geometry is.]

  (We could try to compute the pre-factor and find out how to reproduce it from quantum gravity as well.)

- Different operators are described by different initial densities and will give different correlators. This is different from the sort of words we would say about BHs. There we expect lots of universality and the details of the micro-states should not matter so much.

- How to go in this BH direction? A small first step could be to generalize our 1/2-BPS to 1/4-BPS.

Thanks

* Extremal correlators are much easier to study and there are exact results about those. Their physics is a bit different (they are often effectively given by a bunch of two-point functions so that higher point correlators no longer exponentiate for instance)