THE HOLOGRAPHIC COVARIANT ENTROPY BOUND

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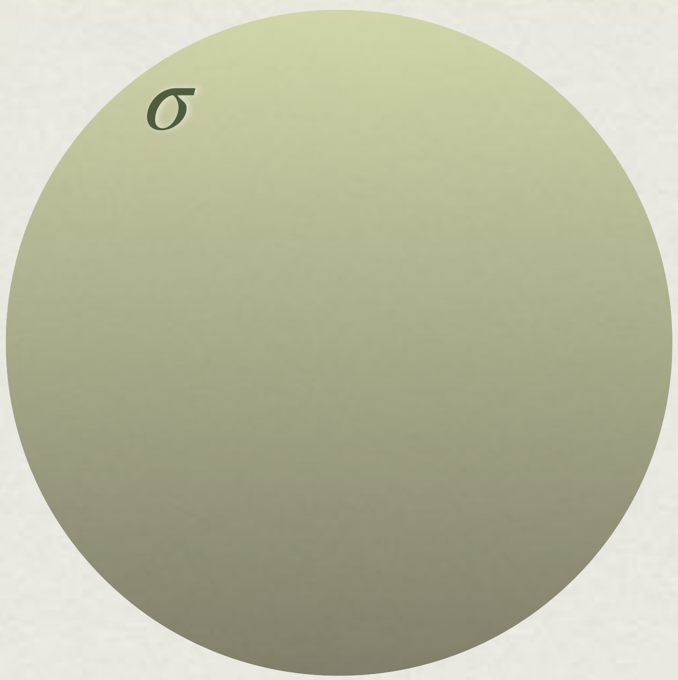
Chennai Mathematical Institute

Based on 2407.16769 and 250x.xxxxx with Aron C. Wall

WHAT'S THE CONNECTION BETWEEN ENTROPY AND AREA?



A cartoon explanation:

$\frac{A(\sigma)}{4G_N}$ = The amount of entropy you can throw into a surface σ before it collapses into a black hole.

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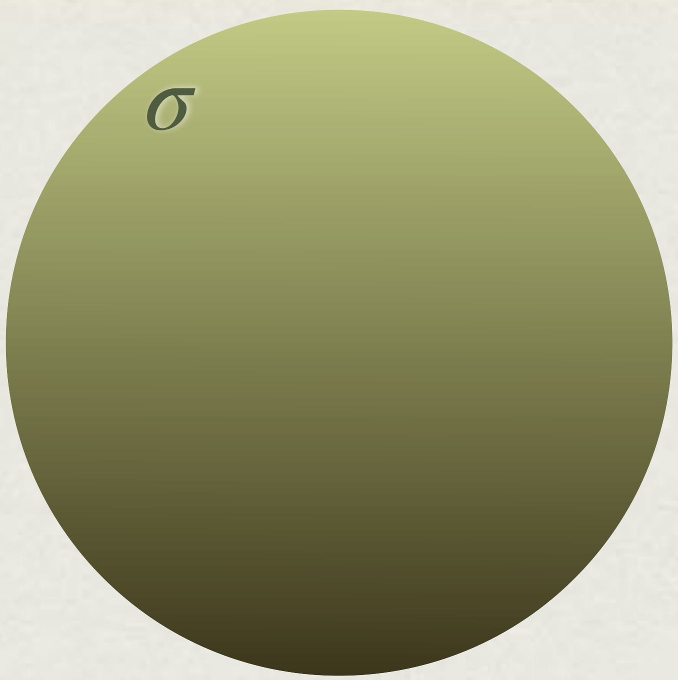

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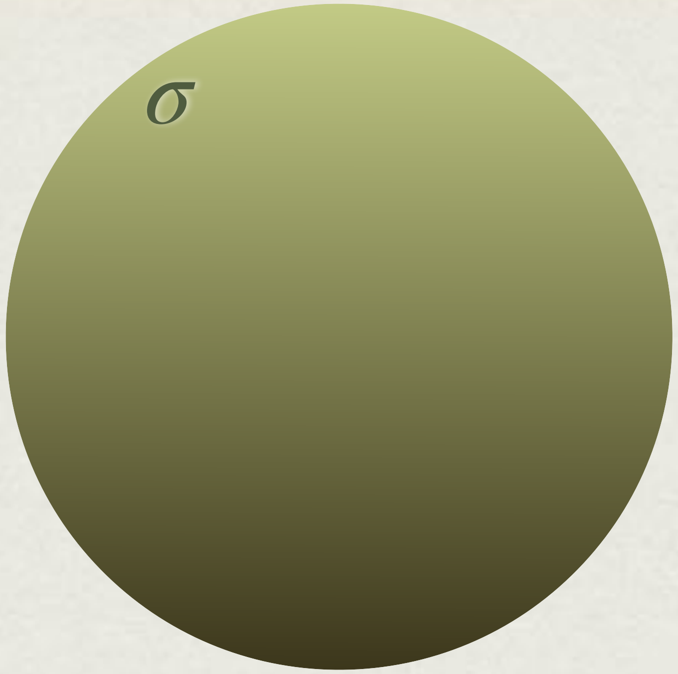


A cartoon explanation:

$\frac{A(\sigma)}{4G_N}$ = The amount of entropy you can throw into a surface σ before it collapses into a black hole.

Two important things:

- Area is an ***upper bound*** on entropy inside σ.
- It is an upper bound ***given that*** σ

  a. has area A (and is spherically symmetric).

  b. is outside a black hole.

THERMODYNAMICS FROM MAXENT

This sort of thing is familiar:

- Thermodynamic entropy is a function of the macrostate: it is the maximum entropy among microstates with the right macroscopic properties.
- If we are **given** only that $\operatorname{tr} \rho H = E$,

$$\text{thermal entropy} = \max_{\rho | \operatorname{tr} \rho H = E} S(\rho) = S_{\text{vN}} \left(\frac{e^{-\beta(E) H}}{Z(\beta(E))} \right).$$

- If we are **also given** that energy has small fluctuations, we get instead the microcanonical entropy.

THIS TALK

Today I want to explain a MaxEnt question that provides a (hopefully) new perspective on gravitational entropy.

1. Bulk: Maximise entropy **over spacetimes**.

   Similar to outer entropy (Engelhardt-Wall-Bousso-Nomura-Remmen-Wang) but a little more general.

2. “Boundary”: Maximise entropy **over states** in a strange field theory that lives on a Cauchy slice.

   Extends work by Caputa-Kruthoff-Parrikar

These are the same maximisation, related by **Cauchy slice holography**.

Araujo-Regado—Khan—Wall 2022

Comments on **holographic tensor networks** and **black hole information**.

The Bulk Story

ALGORITHM TO MAKE NEW ENTROPY BOUNDS

To generalise the cartoon, we decompose into three steps:

Definition

1. Define a $(D-2)$-dim surface σ by some local properties.

   Cartoon: intrinsic metric.

2. For any D-dim on-shell spacetime $\mathscr{M} \supset \sigma$, define some (sensible) "entropy of $\mathscr{M}$" $S(\mathscr{M}, \sigma)$.

   Cartoon: entropy of matter inside σ.

3. Upper-bound this entropy,

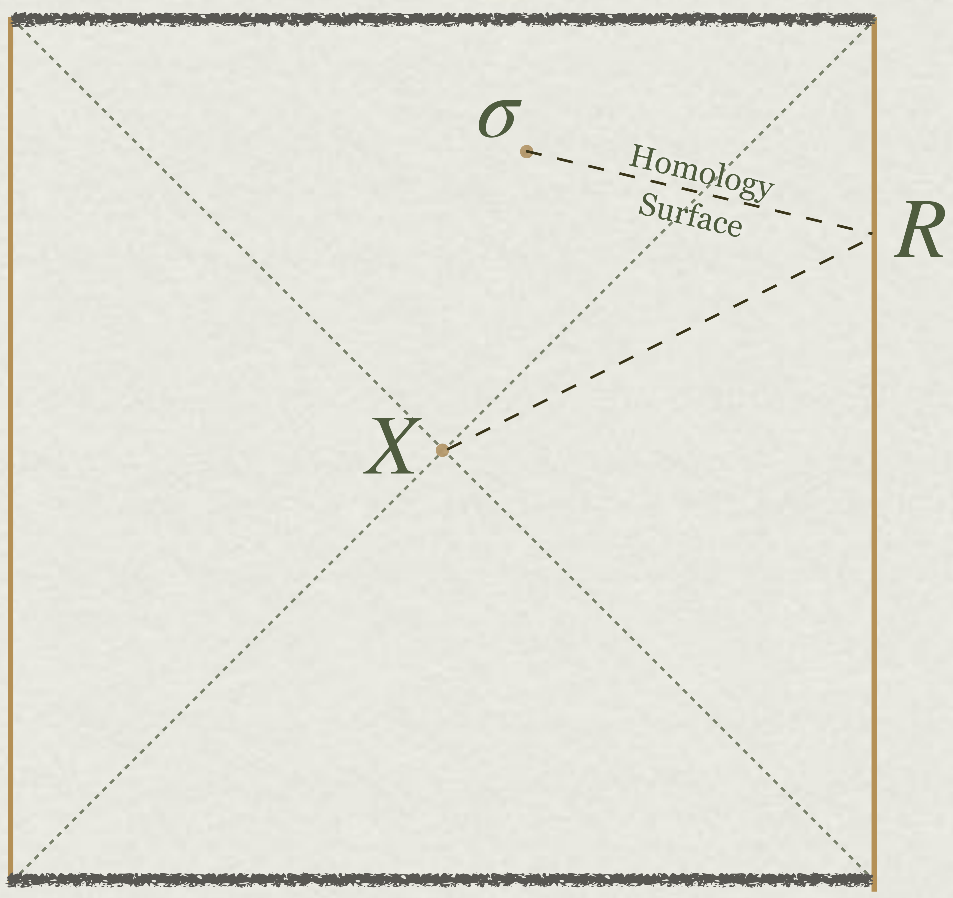
Result

$$S_{\mathrm{bd}}(\sigma) \equiv \sup_{\mathscr{M} \supset \sigma} S(\mathscr{M}, \sigma)\,.$$

OUR CHOICE OF ENTROPY (STEP 2)

- We will stick to asymptotically AdS spacetimes, and take σ to be topologically a sphere.

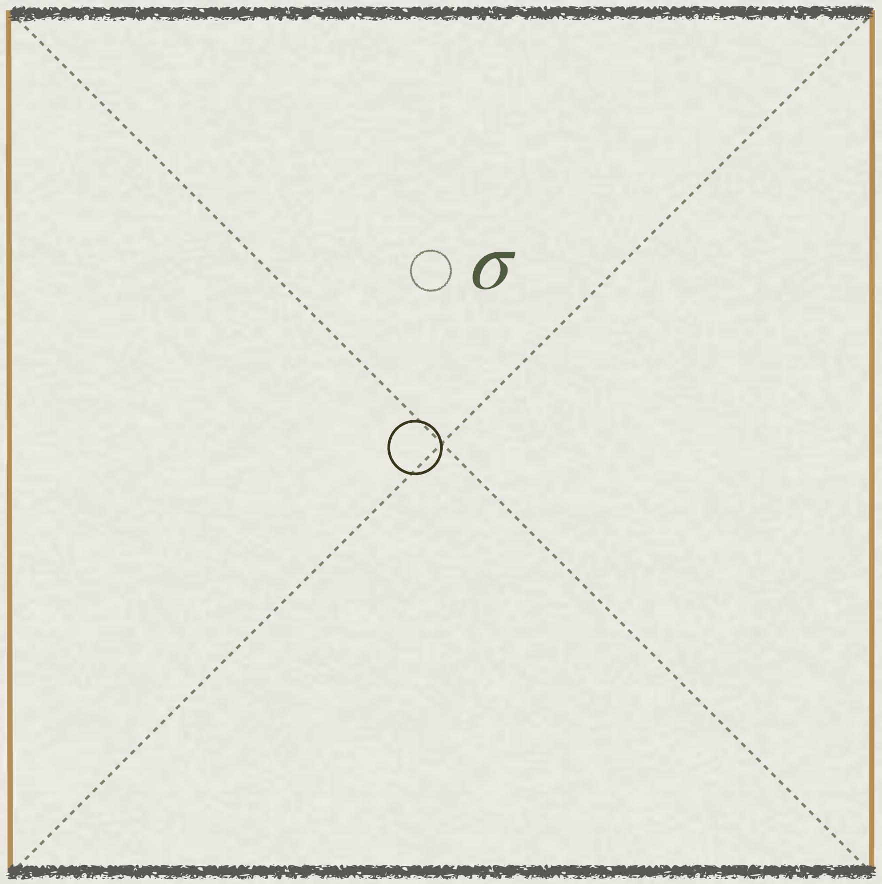
- $S(\mathscr{M}, \sigma)$ is the area of the minimal extremal surface in the same homology class as σ.



$$S(\mathscr{M}, \sigma) = S_E(R) = \min_{X \sim R \sim \sigma} \operatorname{ext} \frac{A(X)}{4G_N}$$

STEP 1 ATTEMPT 1: AREA IS NOT ENOUGH

- Suppose we define σ just by its intrinsic metric $\gamma_{\mu\nu}$.
- The upper bound is ∞!
- Not very useful.):



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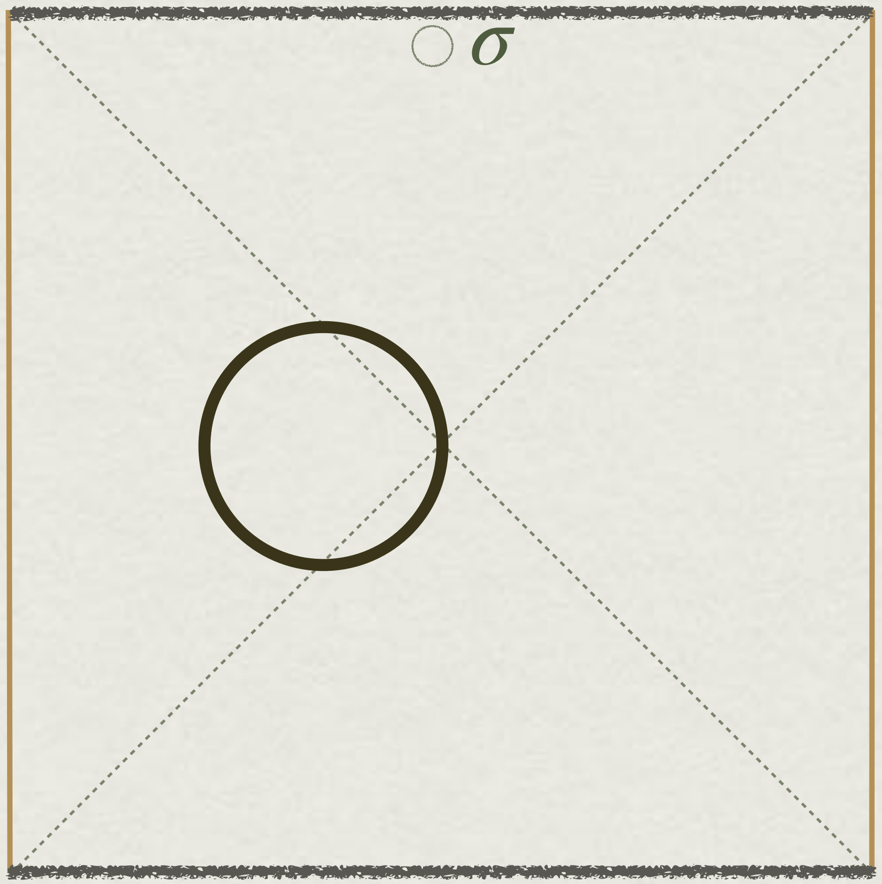
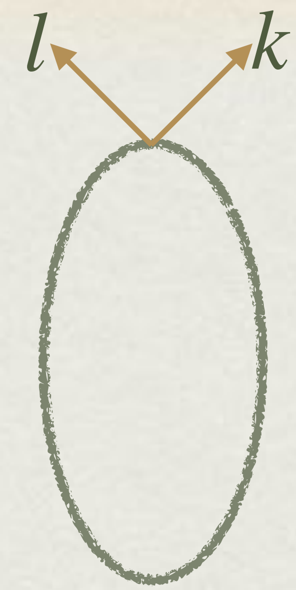
Flow of spacetimes



BETTER DEFINITION FOR σ



- Define a surface by its intrinsic metric *as well as* its extrinsic curvature.

- Since σ is codim-2, there are two orthogonal null normals k, l; $k^2 = l^2 = 0, k \cdot l = -1$.

- This defines two extrinsic curvatures $K_{(\alpha)\mu\nu} = (\nabla n_{(\alpha)})_{\mu\nu}$, $\alpha = k, l$.
  Expansion $\theta_{(\alpha)} = K_{(\alpha)\mu}{}^{\mu}$.

- σ is specified by its intrinsic metric $\gamma_{\mu\nu}$ and the two extrinsic curvatures $K^{(\alpha)}_{\mu\nu}$
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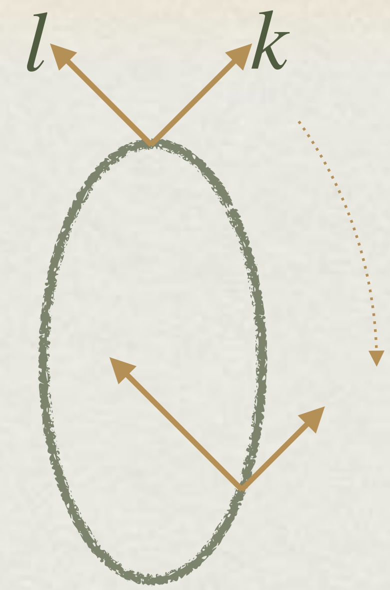

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There's also something called the "twist," which is a gauge field for local boosts around σ. I will ignore this; see paper for more details.

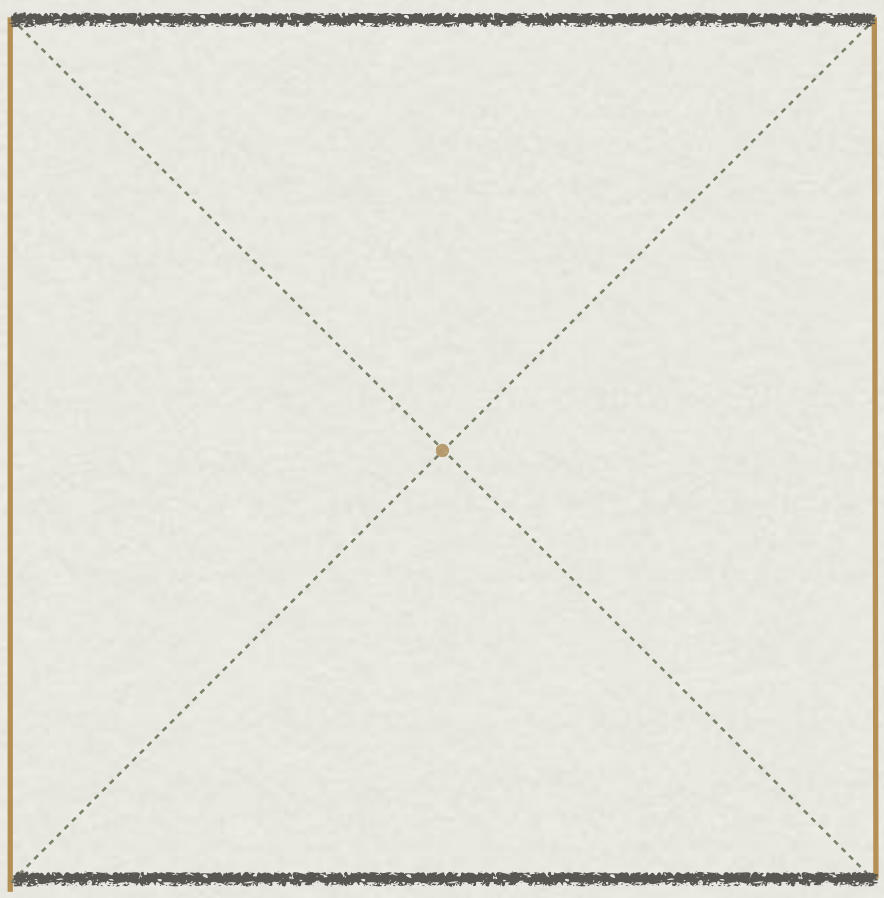
- σ is specified by its intrinsic metric $\gamma_{\mu\nu}$ and the two extrinsic curvatures $K^{(\alpha)}_{\mu\nu}$ up to coordinate transformations.

RESULT FOR 3D GR

- Take σ to agree with a rotationally symmetric surface in non-rotating BTZ black hole.

- We find

$$S_{\mathrm{bd}}(\sigma) = \frac{A(\sigma)}{4G_N}\sqrt{1 + 2\theta_{(k)}\theta_{(l)}}.$$



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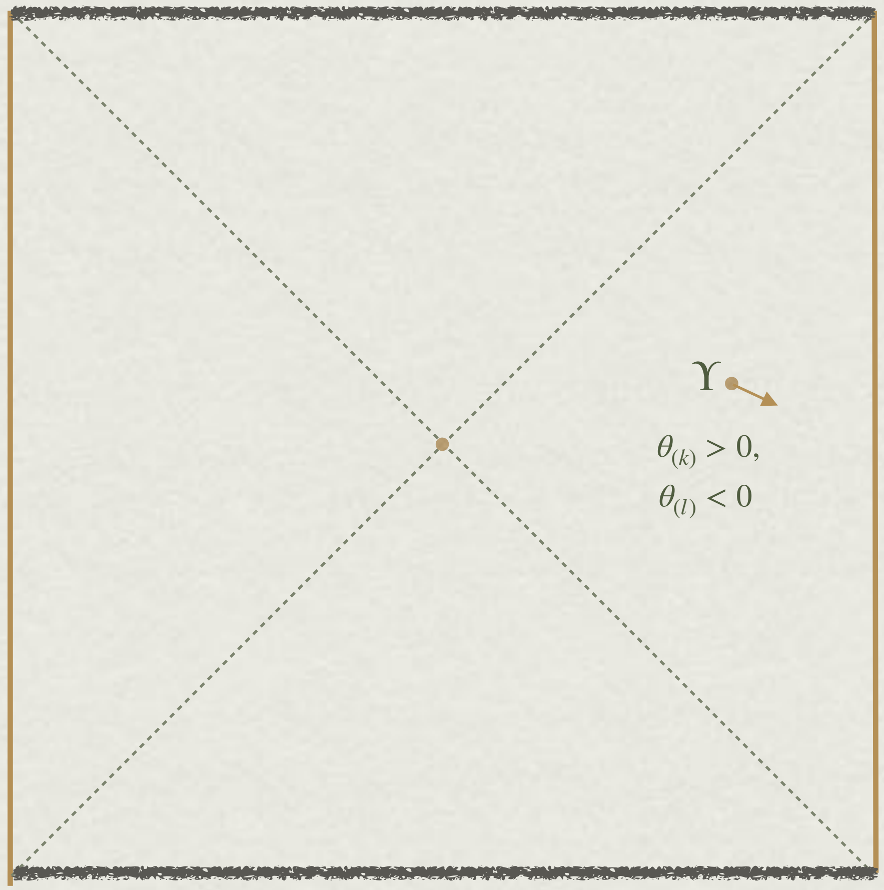

Normal surface Υ: $S_{\rm bd} < A/4G_N$

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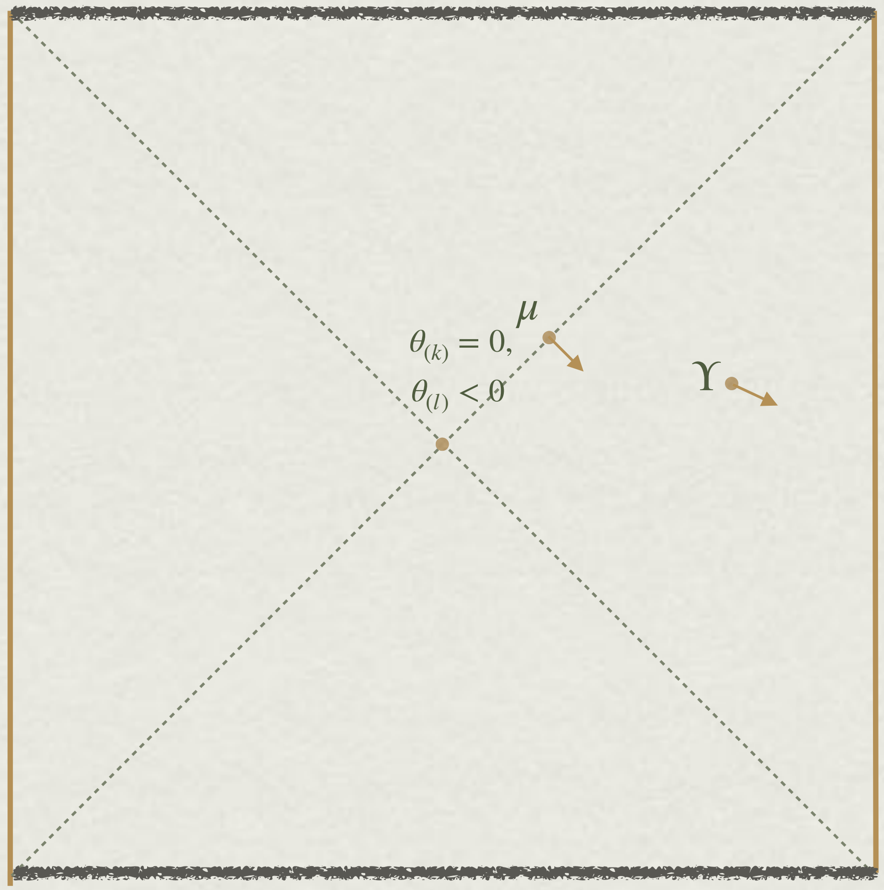

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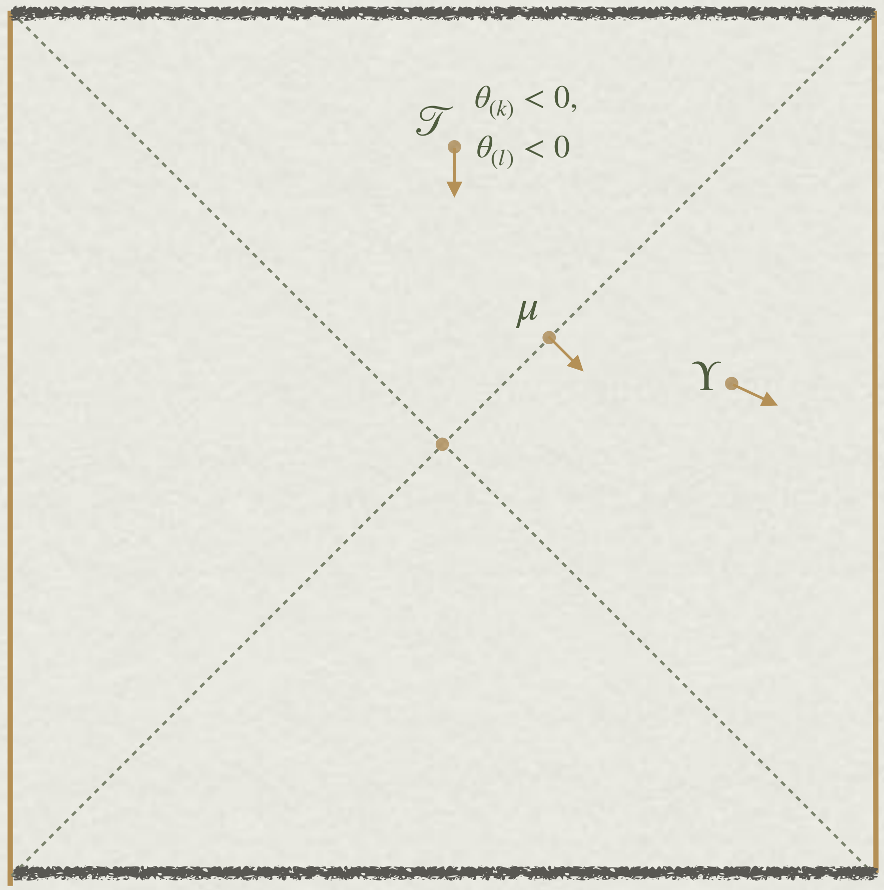

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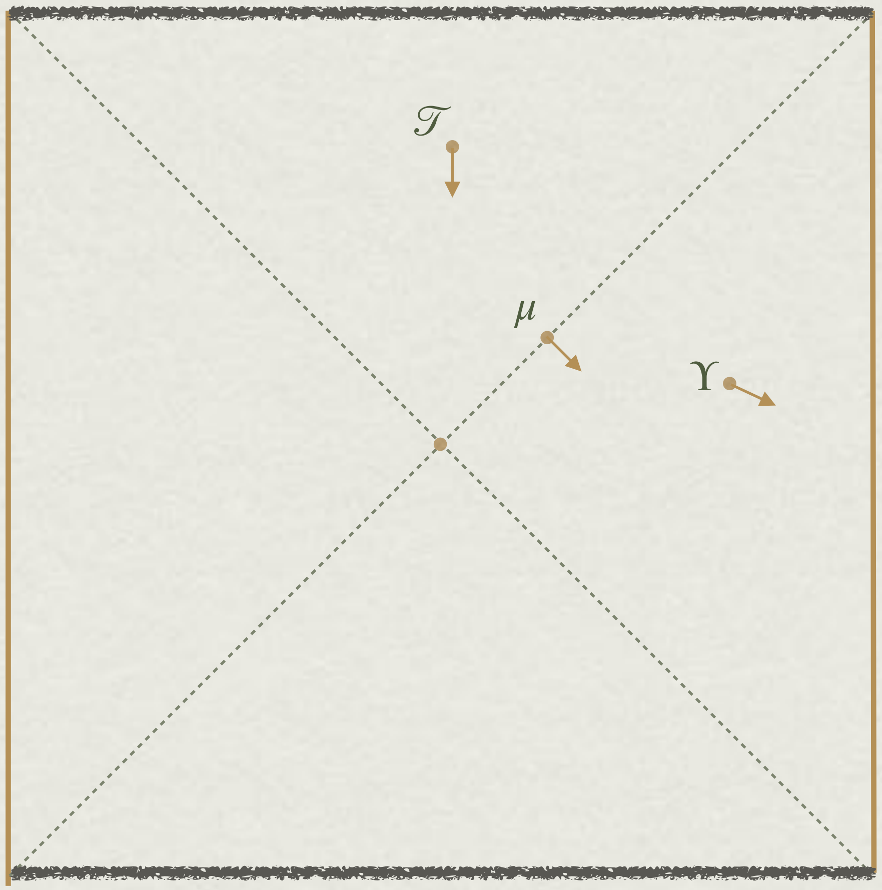
$$S_{\text{bd}}(\sigma) = \frac{A(\sigma)}{4G_N}\sqrt{1 + 2\theta_{(k)}\theta_{(l)}}.$$

- More generally,
$$S_{\text{bd}}(\sigma) = F\left(Pe^{i\oint_\sigma \mathbb{A}}\right),$$
where $\mathbb{A}$ is an $SO(2,2)$ gauge field.

Ammon-Castro-Iqbal-Llabrés 2013-18
McGough-Verlinde 2013
Mertens-Simón-Wong 2022
+ Lin 2021, Hung's talk
Akers-RMS-Wei 2024



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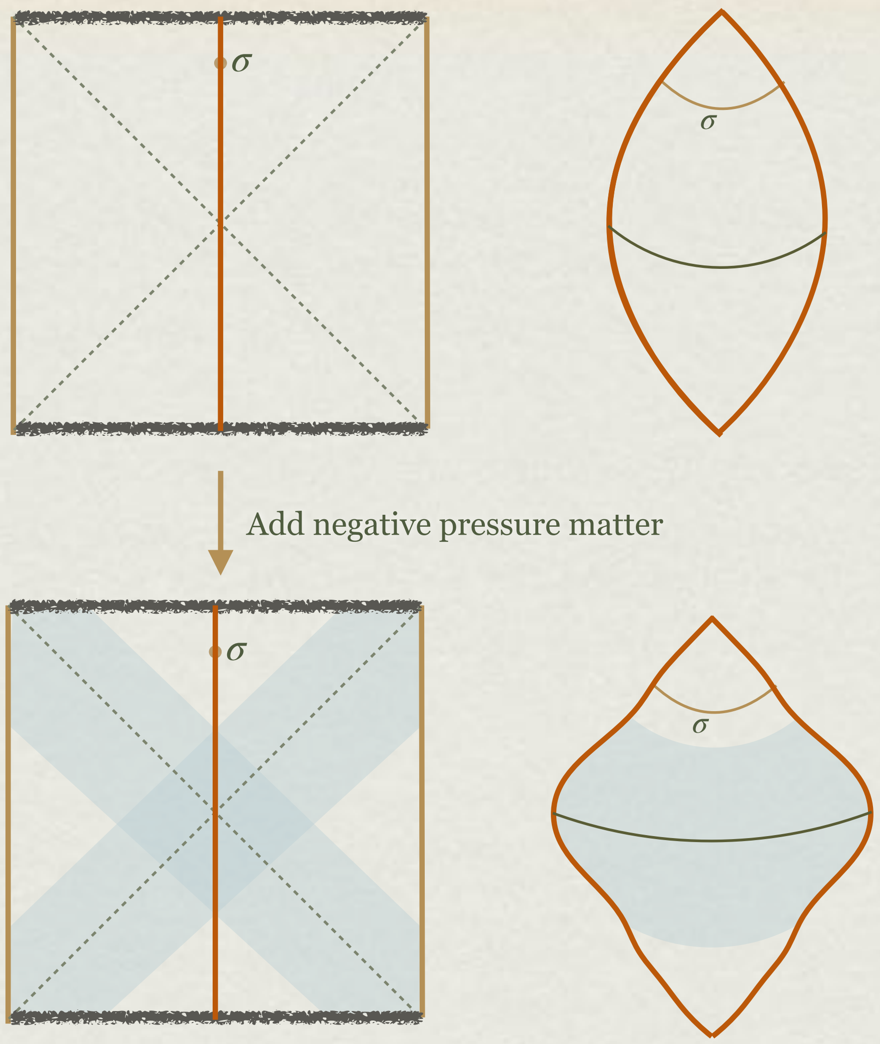

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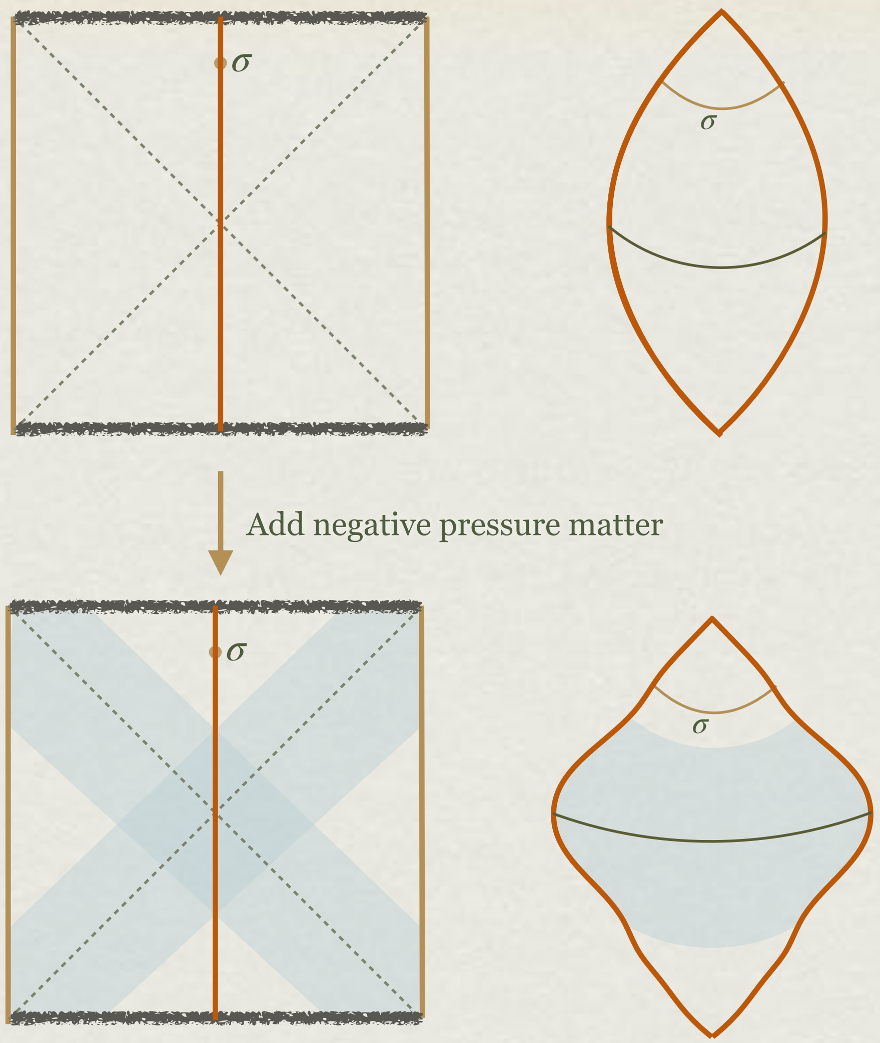
- We find that a surface inside a black hole does not constrain the area of the black hole at all, so



$$S_{\text{bd}}(\sigma) \geq \begin{cases} S_{\text{bd}}^{(GR)} & \theta_{(k)}\theta_{(l)} \leq 0 \text{ (normal/marginal/extremal surface)} \\ \infty & \theta_{(k)}\theta_{(l)} > 0 \text{ (anti-/trapped surface)} \end{cases}$$

WHY DOES THIS WORK?

Why does local data on σ know *anything* about areas far away?

Because gravity has **constraints**: a set of initial data on a Cauchy slice Σ, the metric $g_{\mu\nu}$ and conjugate momenta $\pi_{\mu\nu} \propto K_{\mu\nu} - g_{\mu\nu}K^{\rho}_{\rho}$,
is physical only if it satisfies certain relations.

$$\text{Hamiltonian constraint}: \quad H = \det K - \frac{1}{2}R(\Sigma) + \Lambda = 0$$

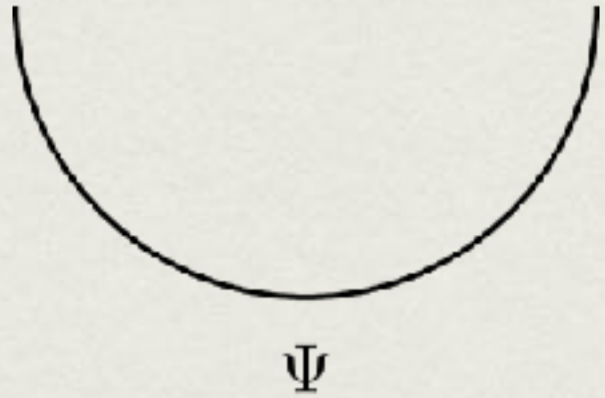
$$\text{Momentum constraint}: \quad P_{\mu} = \nabla^{\mu}\pi_{\mu\nu} = 0$$

We find the bounds by solving these constraints.

The Cauchy Slice Story

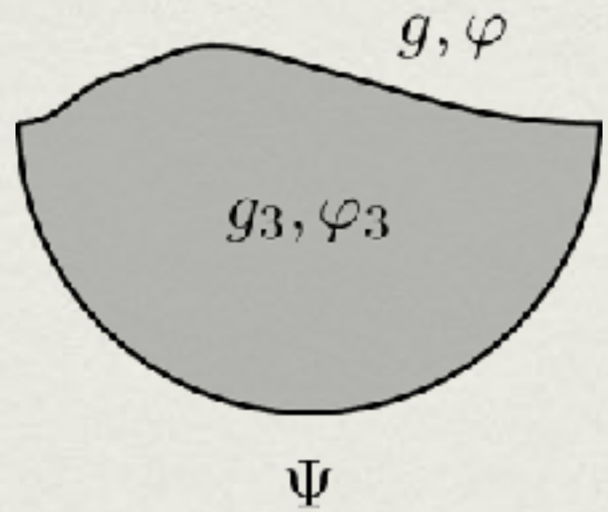
WHEELER-DEWITT WAVEFUNCTION

$|\Psi\rangle =$

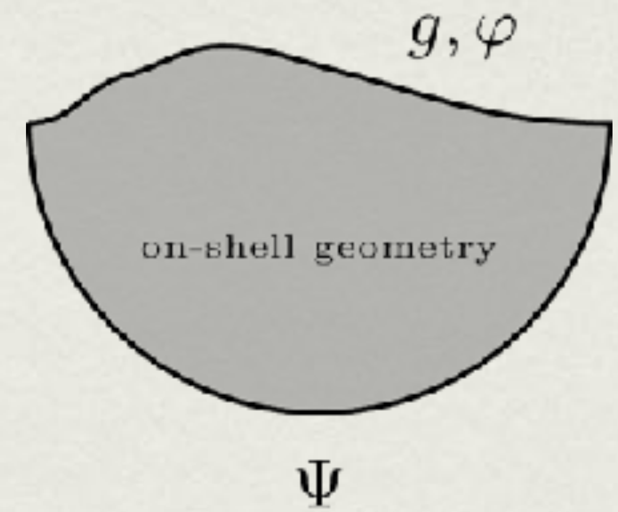


$\in \mathscr{H}_{\text{CFT}}^{\otimes 2}$

$$\Psi_{\text{WdW}}[g,\varphi] \equiv \int Dg_3 D\varphi_3$$



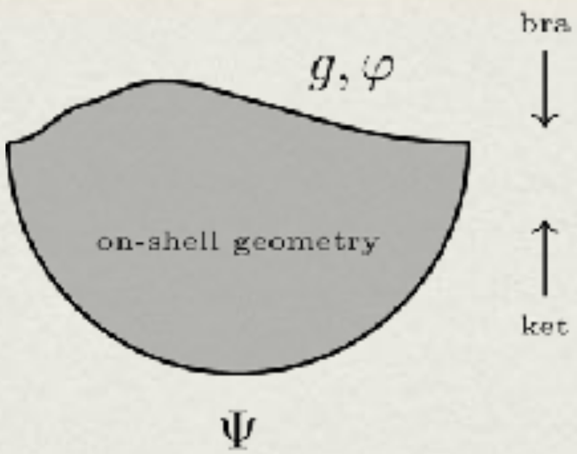
$\approx$



Diff-invariance $\Longrightarrow$ $\hat{H}\Psi_{\text{WdW}} = \hat{P}_\mu \Psi_{\text{WdW}} = 0.$

CAUCHY SLICE HOLOGRAPHY

$\langle g, \varphi \,|\, \Psi \rangle =$



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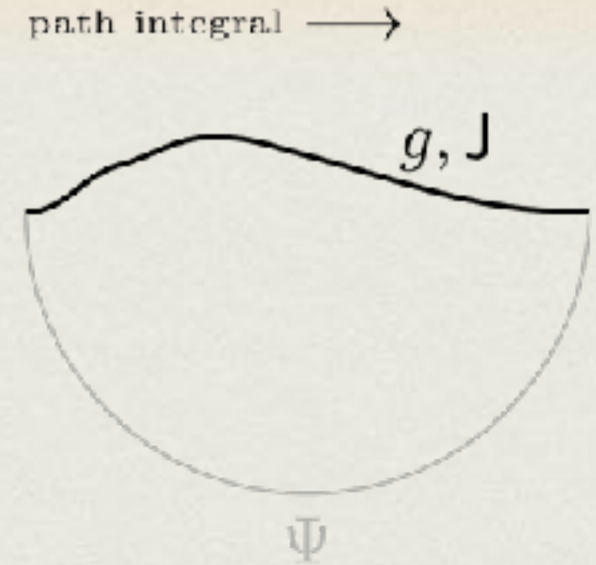

in some
putative field theory
that lives on
the Cauchy slice

$$= \sum_{h_{L,R},\bar{h}_{L,R}} \underbrace{\langle h_{L,R}, \bar{h}_{L,R} \,|\, \Psi \rangle}_{\text{Bd wavefunction}} \langle h_R \bar{h}_R \,|\, \underbrace{\mathbb{T}[g, \mathsf{J} = \varphi]}_{\text{Transition matrix with sources, computed by path integral on the Cauchy slice}} \,|\, h_L, \bar{h}_L \rangle$$

Bd wavefunction

Transition matrix with sources,
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Constraints become relations among stress tensor expectation values

$$\langle \det T - R/2 + \Lambda_3 \rangle = \langle \nabla_\mu T^{\mu\nu} \rangle = 0.$$

This is known as a ***$T\bar{T}$*-deformed theory**.

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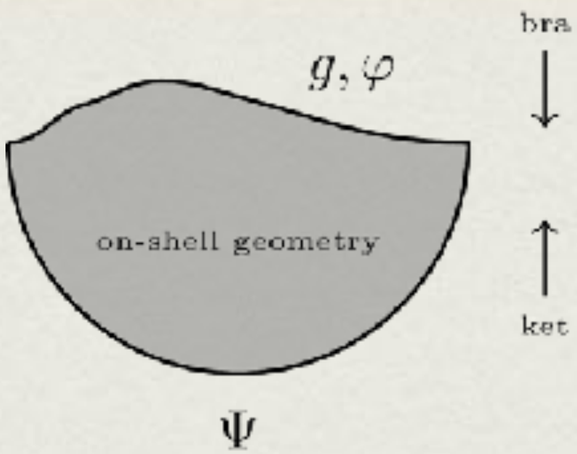

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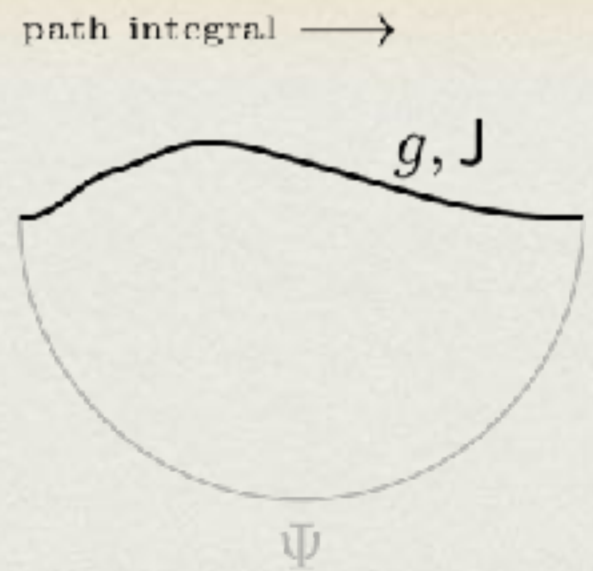

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Introduced more carefully in Torroba's talk

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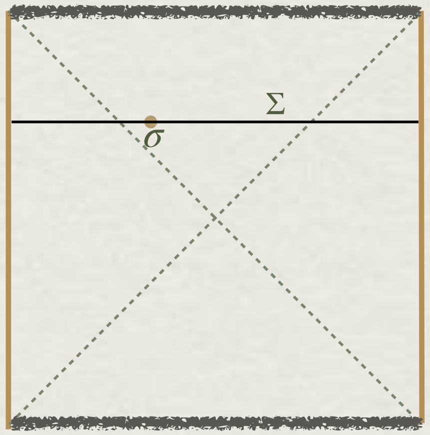

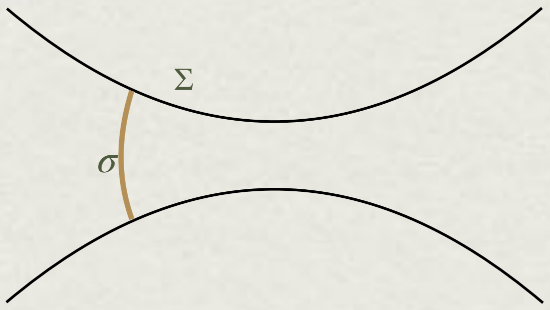

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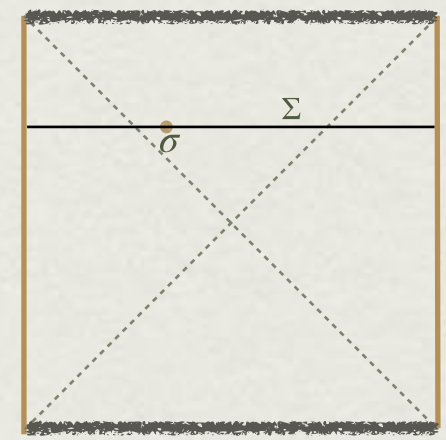

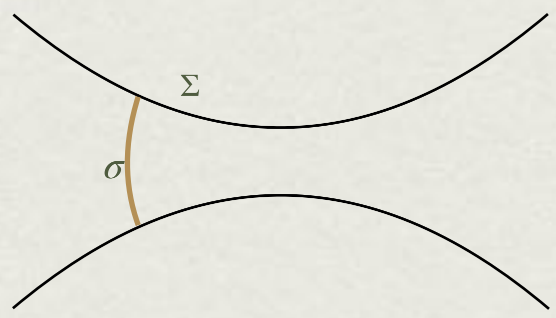

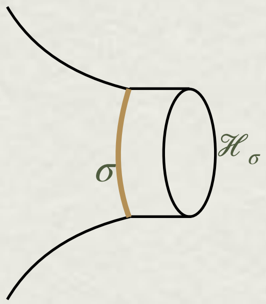

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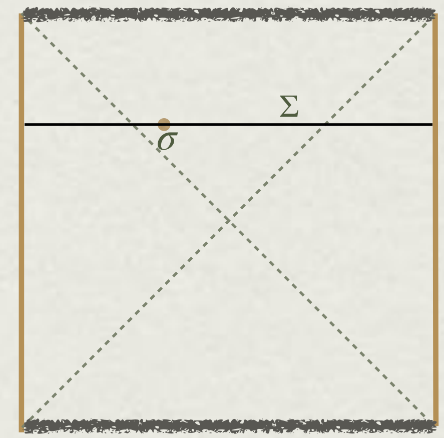

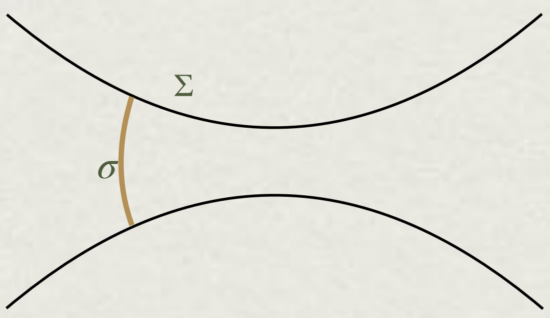

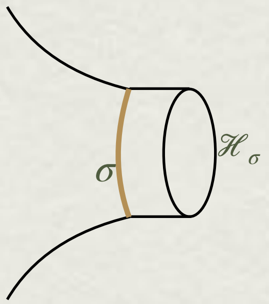

$$E(\sigma) = \frac{A(\sigma)}{8\pi G_N}\sqrt{-2\theta_{(k)}\theta_{(l)}}$$

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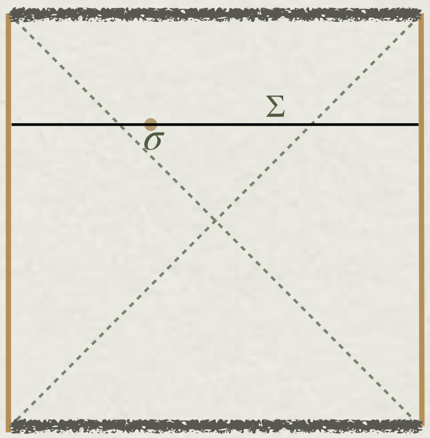

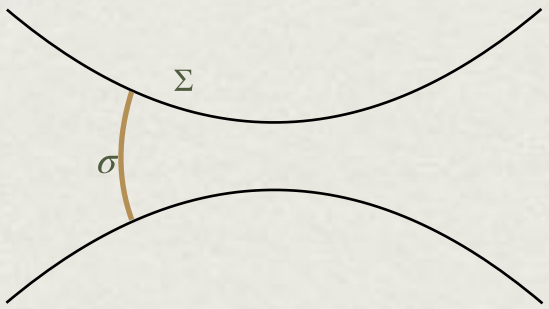

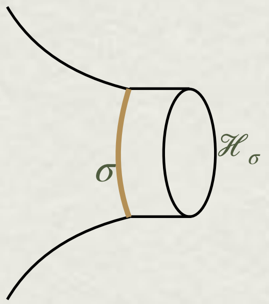

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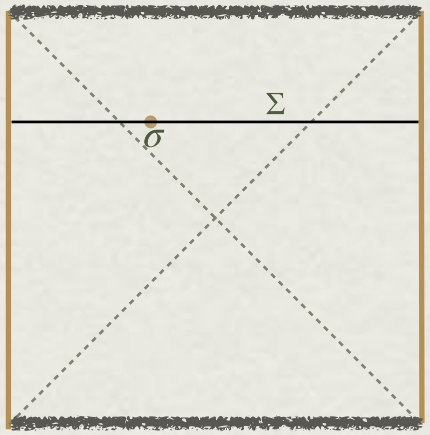
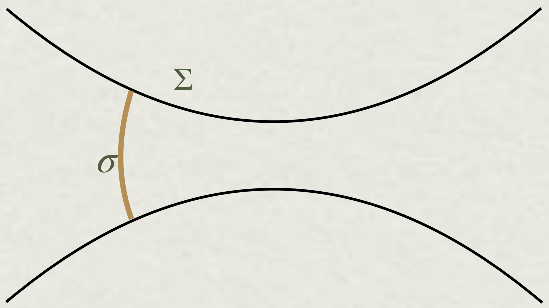
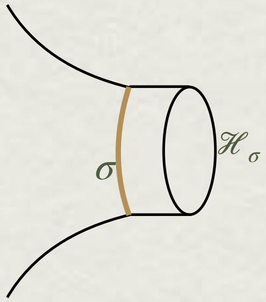
- $K_{(\alpha)\mu\nu} \rightarrow$ energy in this Hilbert space.
- Spacetime is classical $\rightarrow$ energy has small fluctuations.
- MaxEnt then gives μcanonical entropy, which agrees with previous formulas for GR w/out matter.







$$\mathscr{H}_\sigma \quad E(\sigma) = \frac{A(\sigma)}{8\pi G_N}\sqrt{-2\theta_{(k)}\theta_{(l)}}$$

$$S_{\rm bd}(\sigma) = S_{\rm Cardy}\left(h = \bar{h} = \frac{A^2\pi^2}{64G_N} - G_N E^2\right)$$
$$= \frac{A(\sigma)}{4G_N}\sqrt{1 + 2\theta_{(k)}\theta_{(l)}}$$

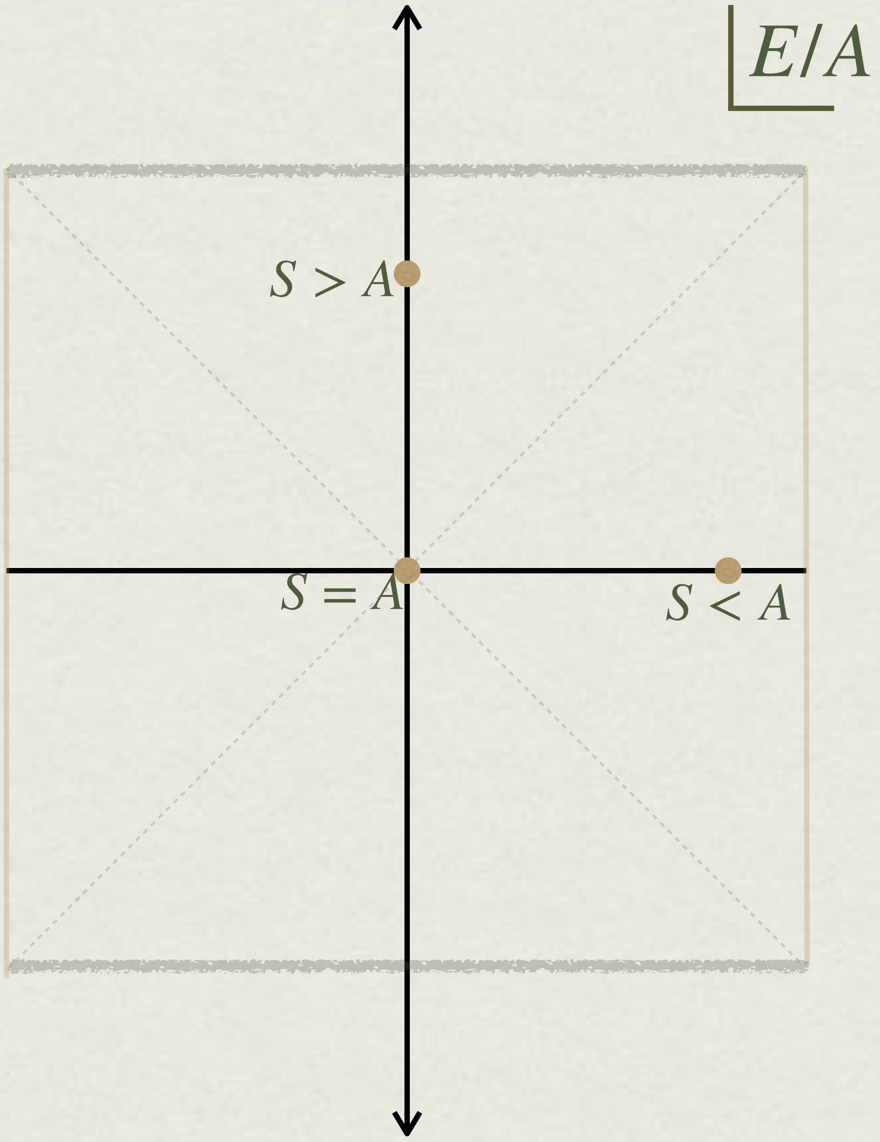
A RADICAL SPECTRUM

$$E = \frac{A}{8\pi G_N}\sqrt{-2\theta_{(k)}\theta_{(l)}}$$

$S < A \leftrightarrow$ real non-zero energy

$S = A \leftrightarrow$ zero energy

$S > A \leftrightarrow$ imaginary energy



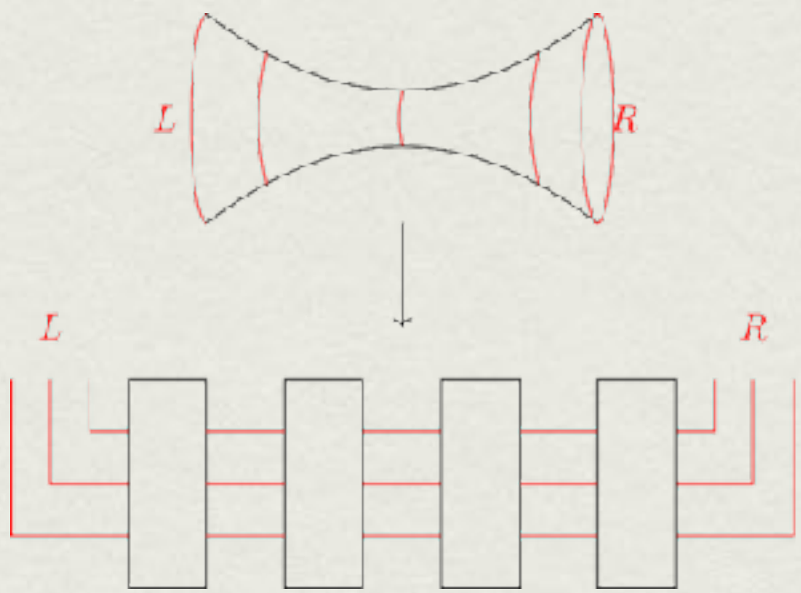
Tensor Networks

TENSOR NETWORK

Can create TN by discretising the path integral.

Log Bond dimension = S_{bd}.

Pure GR



cf. Chandra-Hartman 2023

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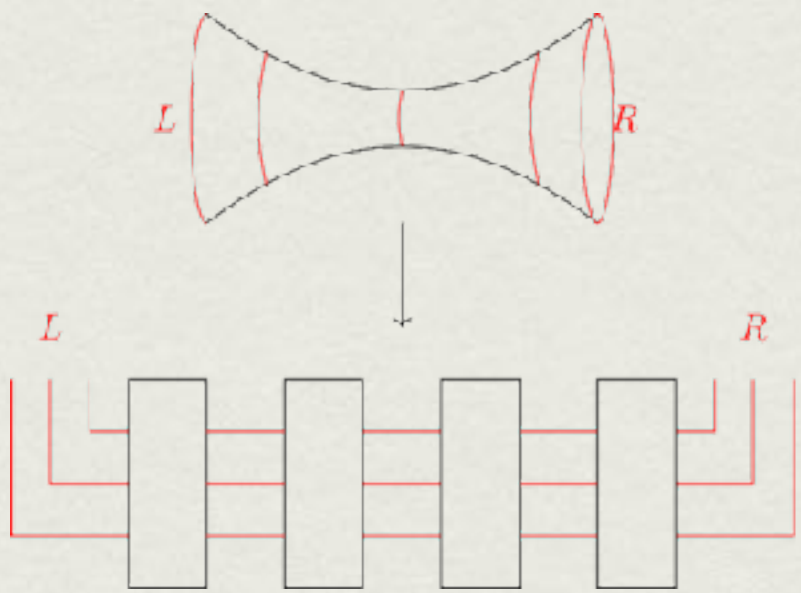
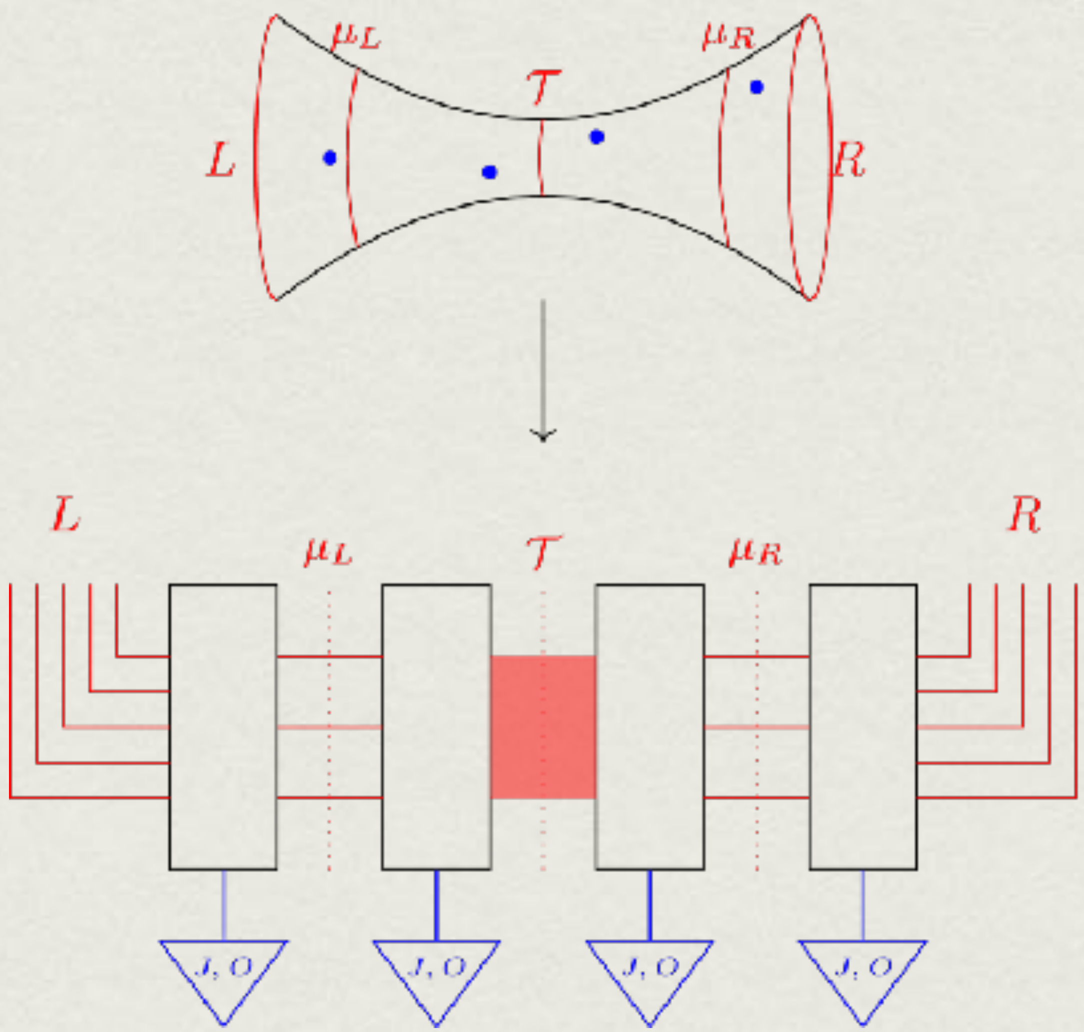


With Classical Matter



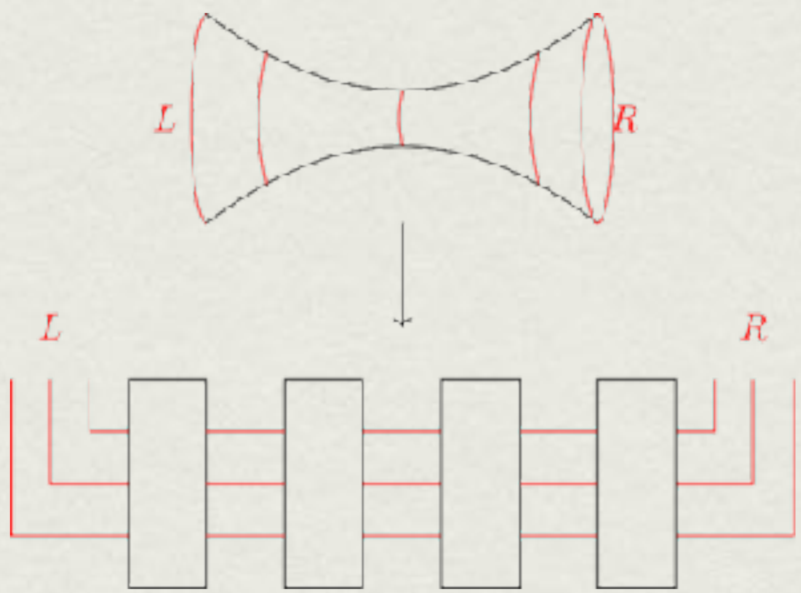
cf. Chandra-Hartman 2023

TENSOR NETWORK

Can create TN by discretising the path integral.

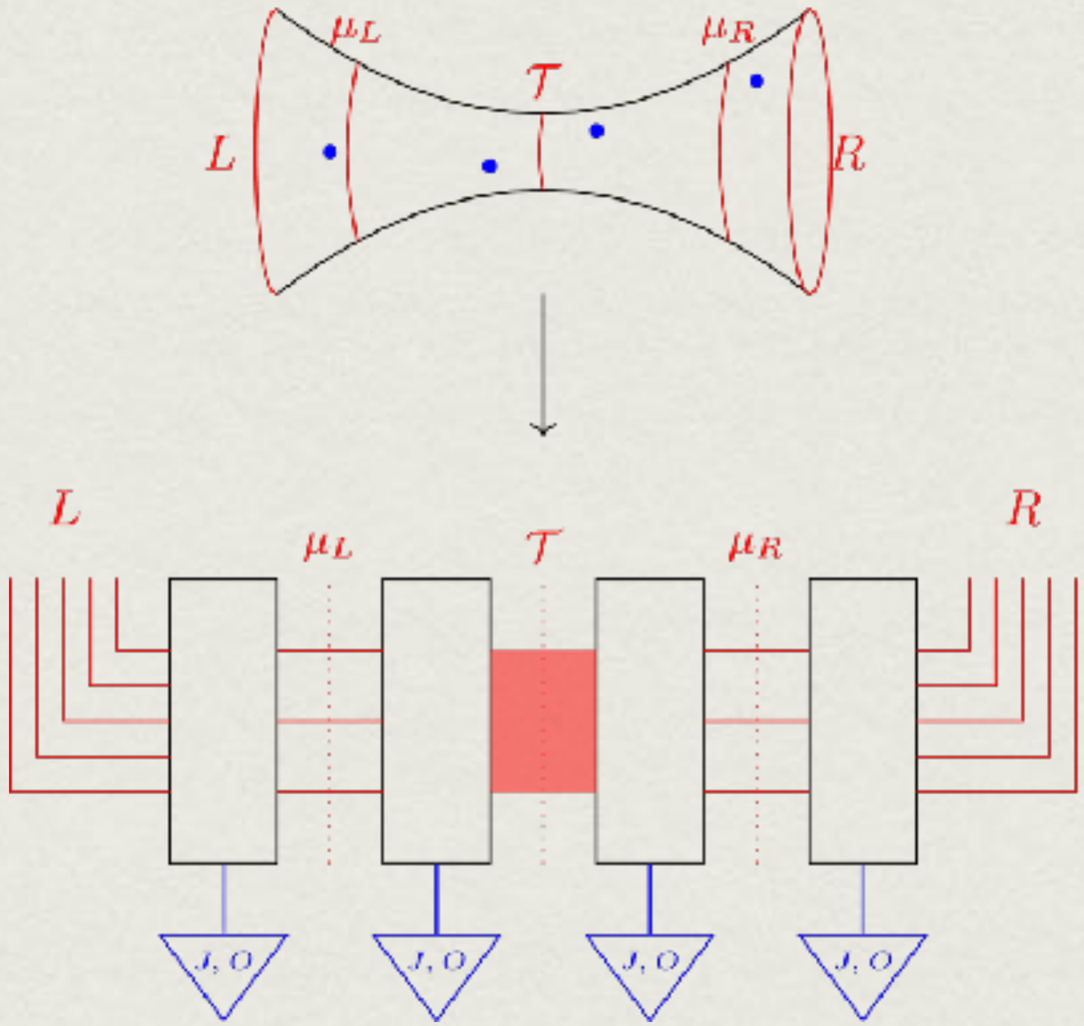
Log Bond dimension = S_{bd}.

Pure GR



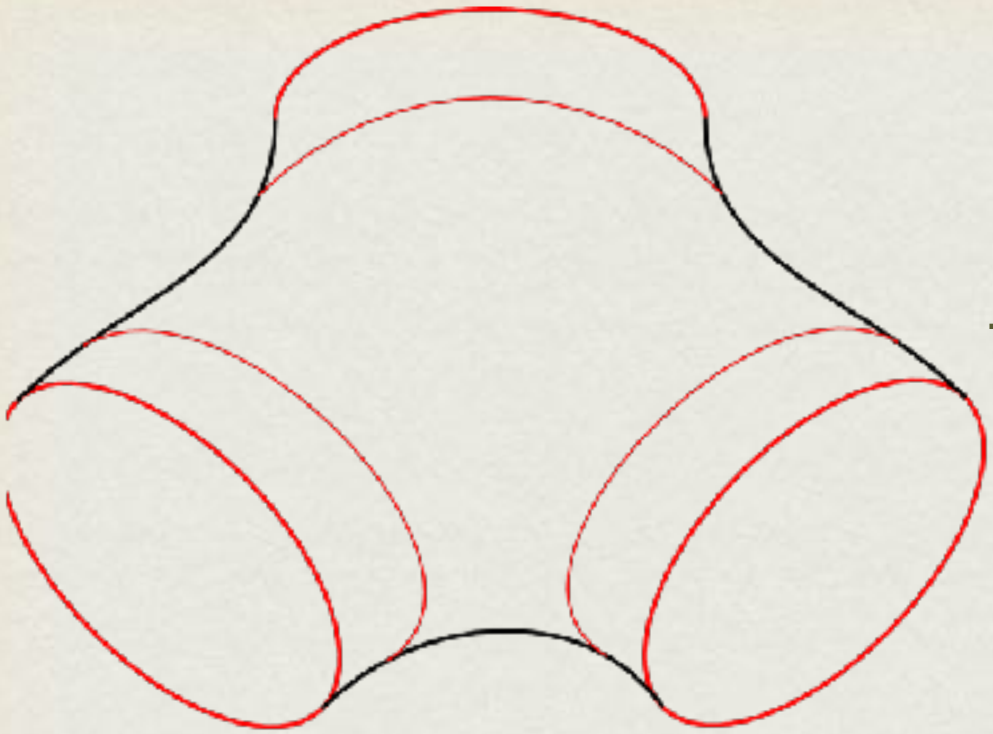
Works for any Cauchy slice (unlike other TNs).

With Classical Matter



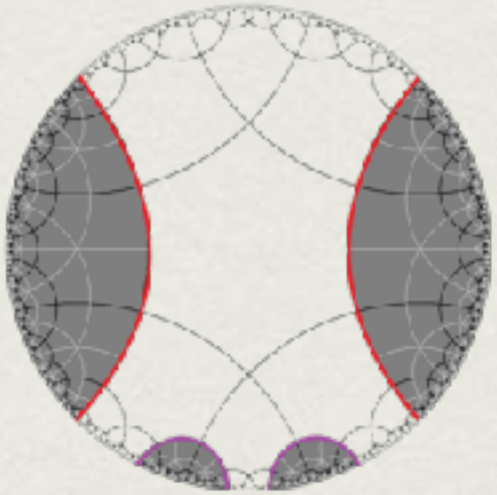
cf. Chandra-Hartman 2023

MULTIPARTITE TENSOR NETWORKS



Usual story →

Peach-Ross 2017

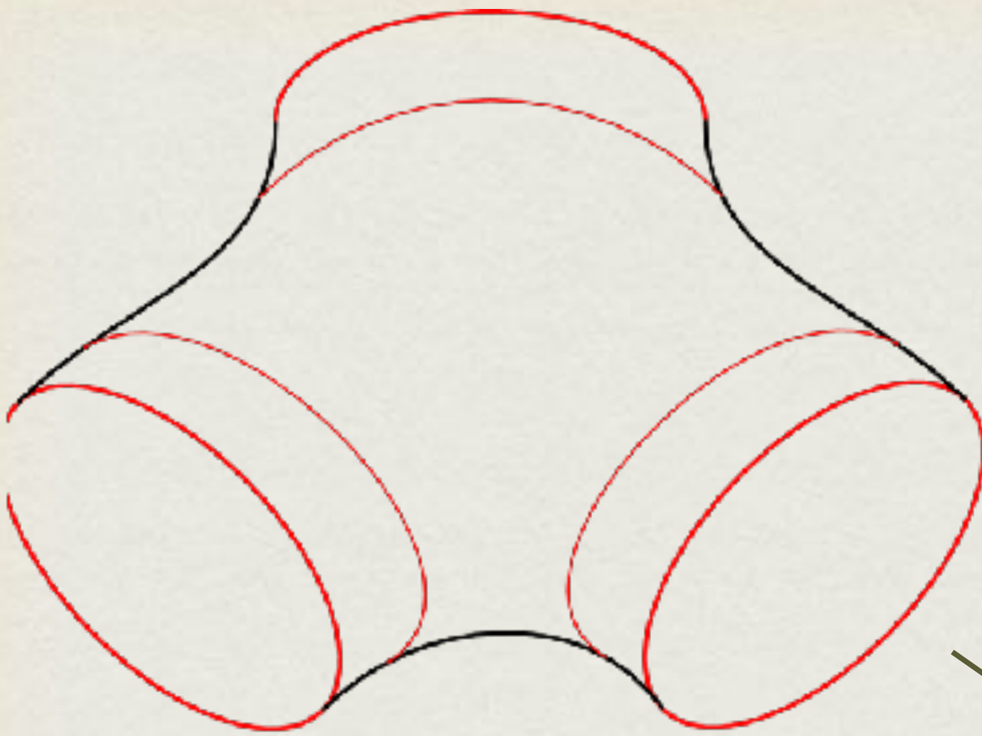


This requires breaking up each surface into pieces.

But: $S_{\text{bd}}(\sigma)$ is not a local function on σ,

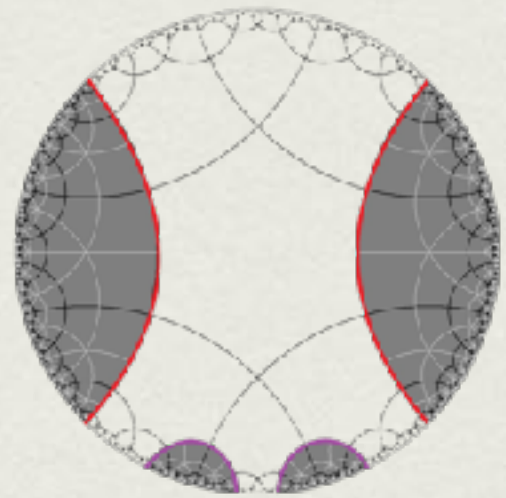
so we can't do this.

MULTIPARTITE TENSOR NETWORKS



Usual story

Peach-Ross 2017



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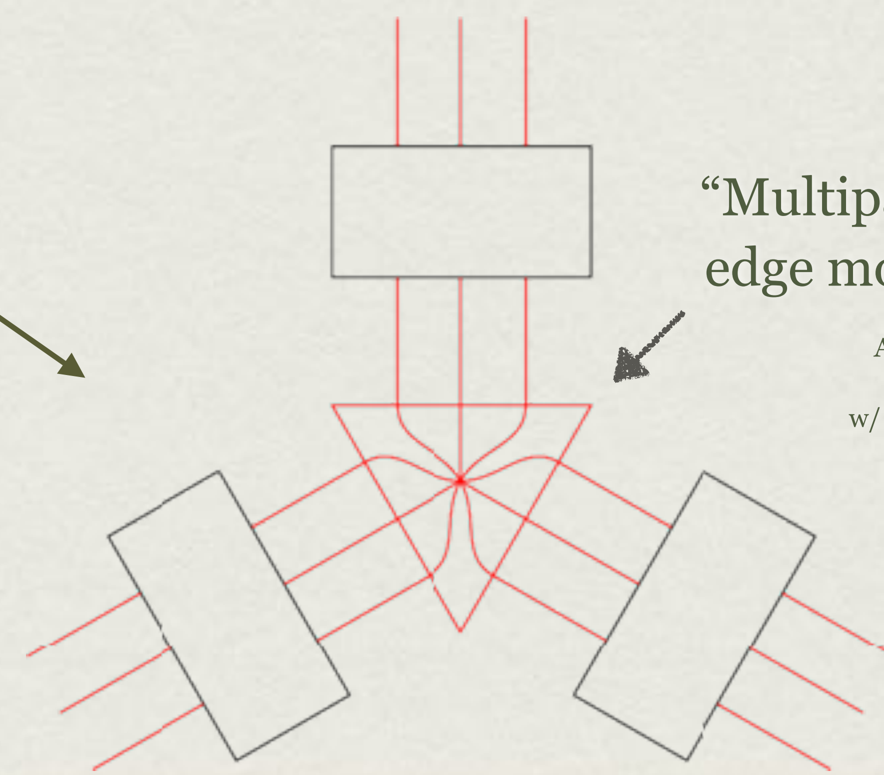
Our story

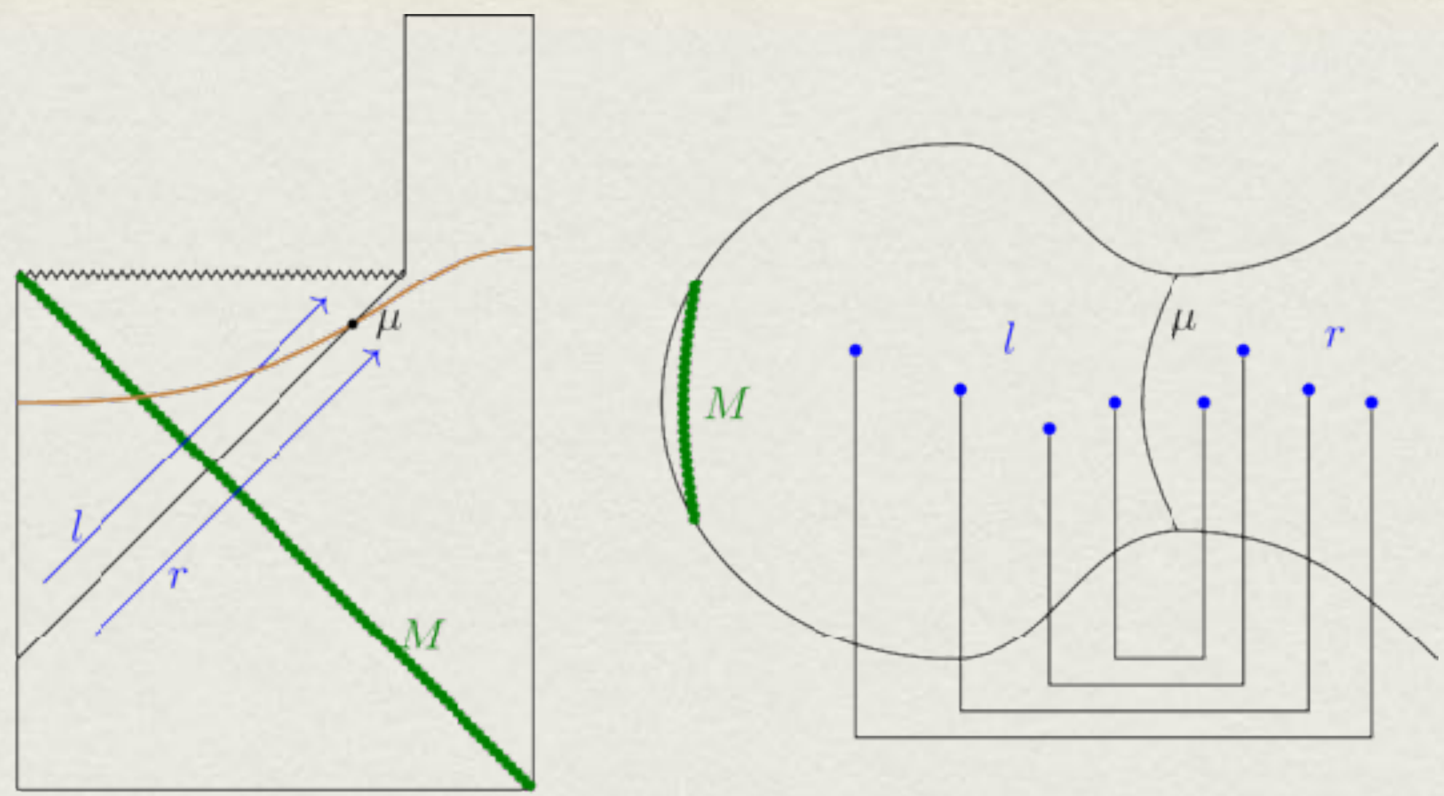


"Multipartite edge modes"

Akers-RMS-Wei 2024
Related discussions w/ Manish Ramchander

Black Hole Information

POSTSELECTION AT THE APPARENT HORIZON



At late times, $S(lM) > S_{\text{bd}}(\mu)$, so there is postselection at μ. Corroborates that bulk-boundary map is non-isometric **without using Euclidean wormholes**.

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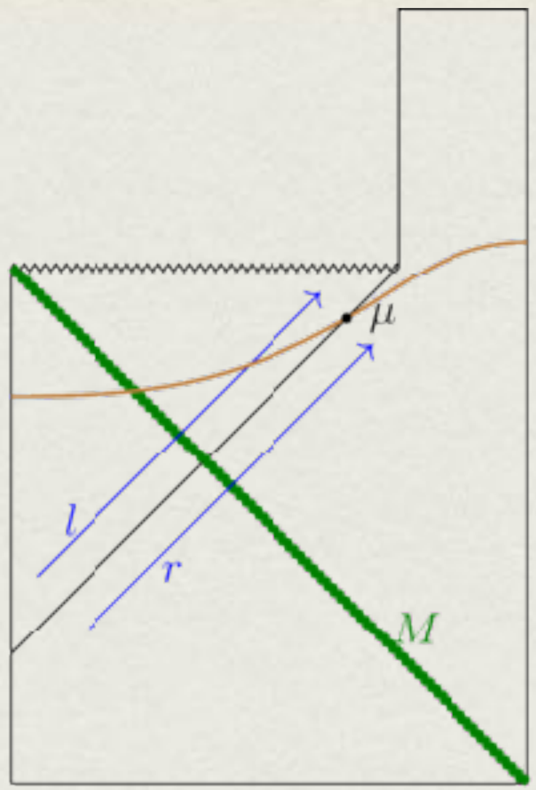

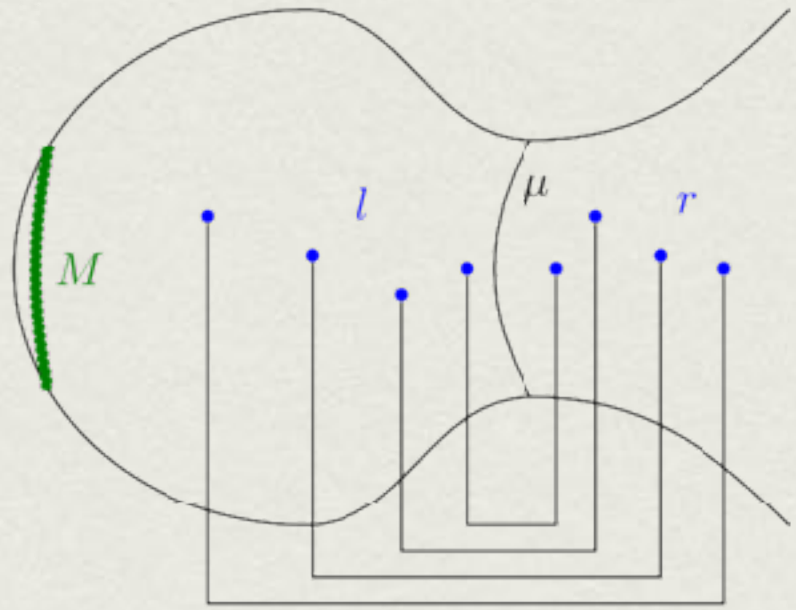

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Corroborates that bulk-boundary map is non-isometric **without using Euclidean wormholes**.

Brown-Gharibyan-Penington-Susskind 2019
PHEVA 2022
See also Chandra-Hartman 2023 for a similar story.

Conclusions

SUMMARY

- Explained a bulk MaxEnt question that genuinely upper-bounds UV entropy from data on a codimension-two surface.
- Found the entropy bound in some simple cases.
- Entropy can be bigger than area! Hilbert space dimension of a bulk surface is not constrained by area; area turns up only in a microcanonical entropy.
- There is a dual question in a Cauchy slice theory.
- Obtained TNs for arbitrary Cauchy slices.
- Can argue for non-isometry without using Euclidean wormholes.

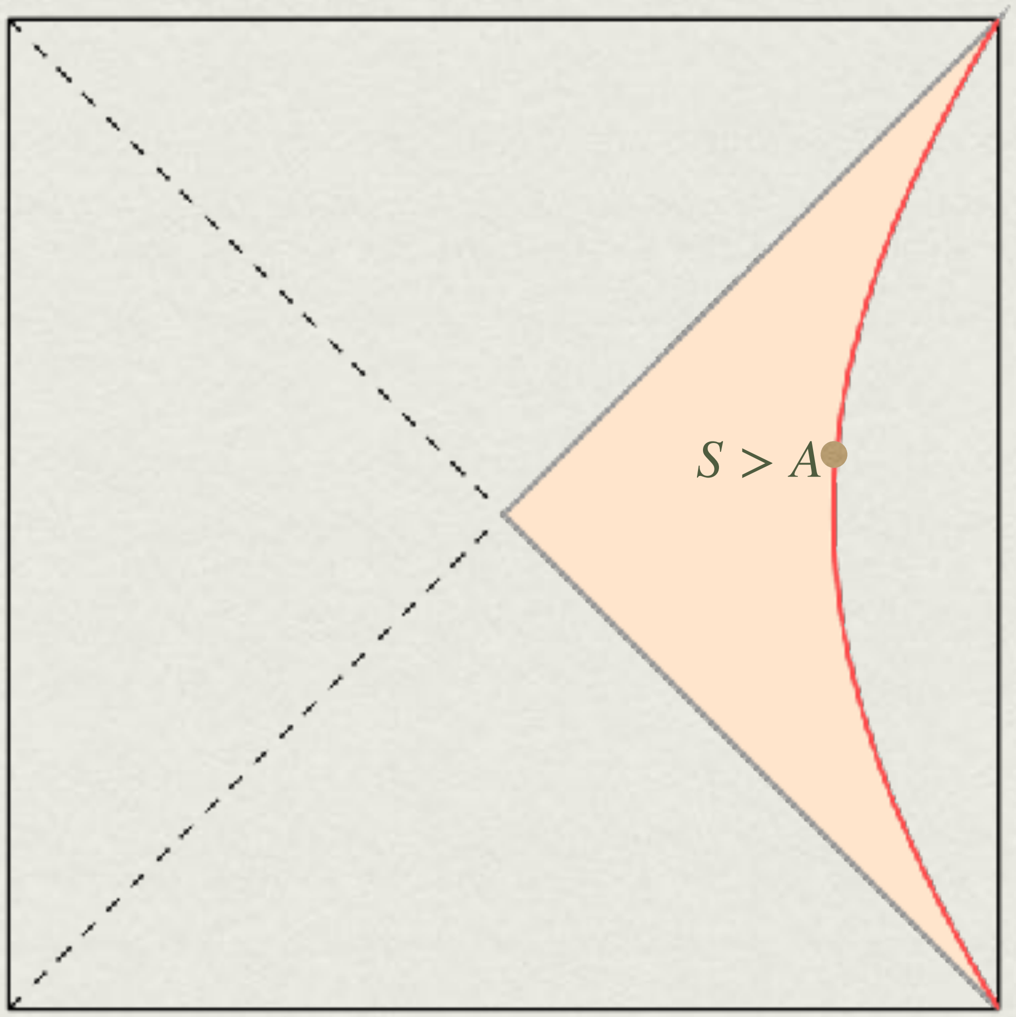
AN ASIDE ON DS

- In dS, entropy is bigger than area for normal surfaces.

Batra-Coleman-de Luca-Mazenc-Shyam
-Silverstein-RMS 2021-4
Torroba's talk

- This means that we can put holographic screens far away from the horizon while still having the correct entropy. Important for any "observer holography" ideas.

$$S_{\rm bd}(\sigma) = \frac{A}{4G_N}\sqrt{1 - 2\theta_{(k)}\theta_{(l)}}$$



OPEN QUESTIONS

- Higher dimensions without rotational symmetry?
- Understand infinitely big Python's lunch.
- Derive holographic map in more detail.
- Is there a boundary interpretation of different Cauchy slices?
  Subfactor theory might be useful.
- Use this to factorise the bulk Hilbert space?

OPEN QUESTIONS

- Higher dimensions without rotational symmetry?

  Bousso-Nomura-Remmen 2019

  (Higher dimensions *with* rotational symmetry dealt with in the paper)

- Understand infinitely big Python's lunch.

- Derive holographic map in more detail.

- Is there a boundary interpretation of different Cauchy slices?
  Subfactor theory might be useful.

  Casini-Huerta-Magán-Pontello 2019
  Magán's talk

- Use this to factorise the bulk Hilbert space?

  Wall 2021