Schwinger's anomaly from qubits

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Chatterjee, Pace, SHS, arXiv: 2409.12220 (PRL)

Pace, Chatterjee, SHS, arXiv: 2412.18606

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Lattice chiral symmetry

- In recent years, we have seen dramatic advancements in our understanding of generalized global symmetries in continuum QFTs and in discretized lattice models, arising from an interdisciplinary collaboration among high energy physicists, condensed matter physicists, and mathematicians.

- The time is right to revisit an old topic: lattice chiral symmetries. Many existing proposals and constructions with their pros and cons.

- Given a QFT, can we realize its chiral global symmetries and their anomalies on the lattice?

Vector and axial symmetries

- Simplest setup: A free, massless Dirac fermion $\Psi = \binom{\Psi_L}{\Psi_R}$ in 1+1d:

$$\mathcal{L} = i\Psi_L^\dagger(\partial_t + \partial_x)\Psi_L + i\Psi_R^\dagger(\partial_t - \partial_x)\Psi_R$$

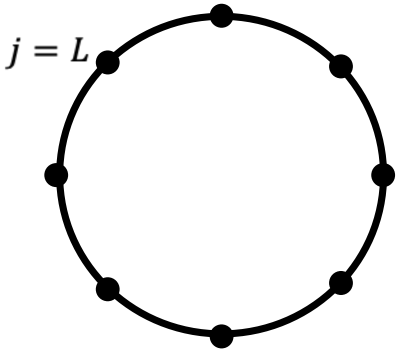
Ψ_L is a one-component, complex, left Weyl fermion, and similarly for Ψ_R.

vector $U(1)^{\mathcal{V}}$: $\Psi_L \to e^{-i\theta}\Psi_L \quad , \quad \Psi_R \to e^{-i\theta}\Psi_R$

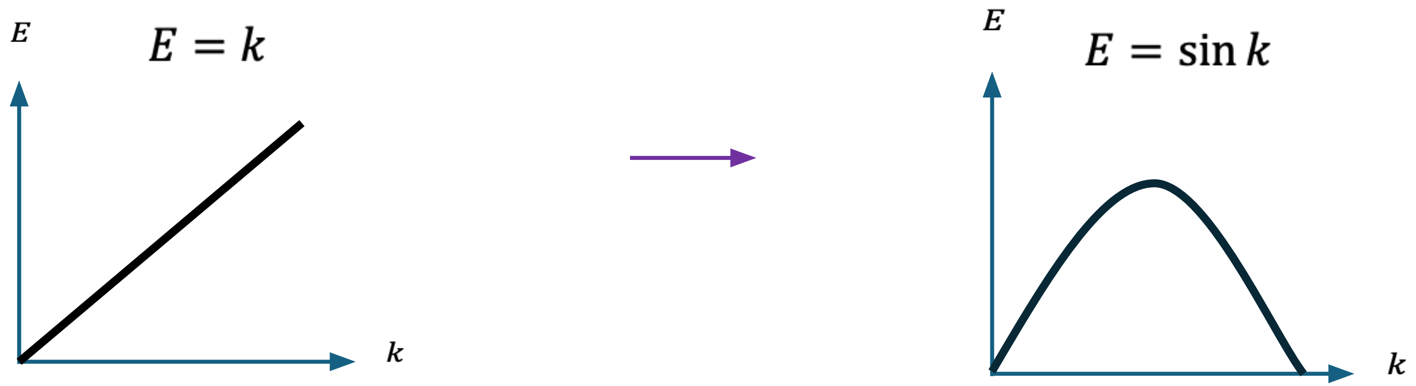
axial $U(1)^{\mathcal{A}}$: $\Psi_L \to e^{-i\theta}\Psi_L \quad , \quad \Psi_R \to e^{+i\theta}\Psi_R$

- The global form of the group is $[U(1)^{\mathcal{V}} \times U(1)^{\mathcal{A}}]/\mathbb{Z}_2$.

Lattice realization of a Dirac fermion



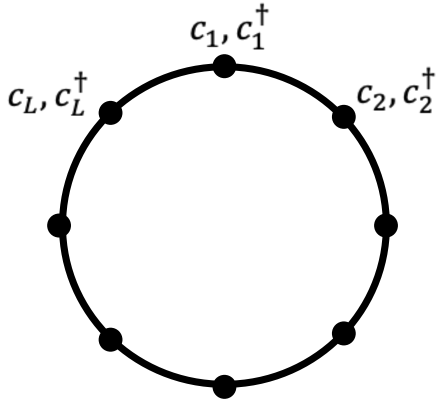
- We will work with the Hamiltonian lattice formalism.
- Time is continuous, but space is discrete. 1d spatial lattice of (even) L sites on a closed ring.
- Naïve lattice regularization:



- Buy one and get one for free: left and right fermions come in pairs

--- fermion doubling problem.

Lattice realization of a Dirac fermion



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- To avoid doubling, we place one complex fermion c_j on every lattice site j.

$$\{c_j, c_{j'}^\dagger\} = \delta_{jj'} \quad , \qquad \{c_j, c_{j'}\} = \{c_j^\dagger, c_{j'}^\dagger\} = 0$$

- One qubit per site. The Hilbert space is 2^L-dimensional.
- Staggered fermion Hamiltonian [Banks, Kogut, Susskind '75-'77]:

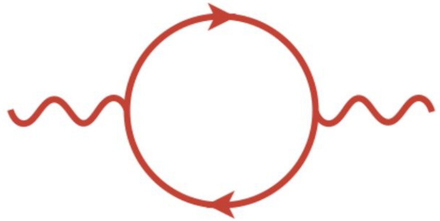
$$H = -i \sum_{j=1}^{L} (c_j^\dagger c_{j+1} + c_j c_{j+1}^\dagger)$$

- The continuum limit is <u>one</u> free, massless, non-chiral Dirac fermion.

U(1) symmetry on the lattice

- $$H = -i\sum_{j=1}^{L}(c_j^\dagger c_{j+1} + c_j c_{j+1}^\dagger)$$
- There is a manifest $U(1)^V$ symmetry, which flows to the vector symmetry of the continuum Dirac fermion theory.
$$U(1)^V:\ c_j \to e^{-i\theta} c_j$$
- Where is the axial $U(1)^A$ symmetry on the lattice?

Schwinger anomaly



- Common lore states that it's difficult to realize the axial U(1) symmetry on the lattice, because of an ('t Hooft) anomaly between the vector and axial global symmetries:

The axial symmetry is broken when the vector symmetry is gauged, and vice versa [Schwinger '59, Johnson '63].

$$\partial_\mu j^{A\mu} = \frac{1}{\pi} E$$

where $j^{A\mu}$ is the axial Noether current and E is the electric field.

- Perturbative anomalies are particularly challenging to realize on a lattice with finite-dim local Hilbert spaces (e.g., qubits).
- This anomaly has been realized with infinite-dim local Hilbert spaces [Gross-Klebanov '90, Sulejmanpasic-Gattringer '19, Gorantla-Lam-Seiberg-SHS '21, Cheng-Seiberg '22, Fazza-Sulejmanpasic '22 ...]

This talk [Chatterjee-Pace-SHS '24]

- $$H = -i \sum_{j=1}^{L} (c_j^\dagger c_{j+1} + c_j c_{j+1}^\dagger)$$
- We will discuss the realization of both the vector and the axial charges Q^V and Q^A in this lattice model.
- $[Q^V, H] = [Q^A, H] = 0$ and $Q^V \in \mathbb{Z}$, $Q^A \in \mathbb{Z}$.
- BUT,

$$[Q^V, Q^A] \neq 0$$

vector $U(1)^{\mathcal{V}}$: $\Psi_L \to e^{-i\theta}\Psi_L$, $\Psi_R \to e^{-i\theta}\Psi_R$
axial $U(1)^{\mathcal{A}}$: $\Psi_L \to e^{-i\theta}\Psi_L$, $\Psi_R \to e^{+i\theta}\Psi_R$

Charge conjugation symmetry

- The continuum vector and axial charges are related by

$$Q^A = \mathcal{C}^R Q^V (\mathcal{C}^R)^{-1}$$

- $\mathcal{C}^R$ is a charge conjugation that only acts on the right-movers.

$$\mathcal{C}^R \Psi_R (\mathcal{C}^R)^{-1} = \Psi_R^\dagger$$

- Since we already have the vector charge on the lattice, it suffices to find a lattice symmetry that flows to $\mathcal{C}^R$ in the continuum limit.

$$\{a_j, a_{j'}\} = \{b_j, b_{j'}\} = 2\delta_{jj'}$$

$$H = -2i \sum_{j=1}^{L} (a_j a_{j+1} + b_j b_{j+1})$$

Think real

- On the lattice, we decompose every complex fermion into two real fermions a_j, b_j:

Lattice: $c_j = a_j + ib_j$

$$\downarrow$$

Continuum: $\Psi_L = A_L + iB_L$

$$\Psi_R = A_R + iB_R$$

- $\mathcal{C}^R$ flips the sign of a single, right-moving Majorana-Weyl fermion:

$$\mathcal{C}^R A_R (\mathcal{C}^R)^{-1} = A_R \quad , \qquad \mathcal{C}^R B_R (\mathcal{C}^R)^{-1} = -B_R$$

- Which lattice symmetry only flips the sign of B_R in the continuum limit?

Lattice translation

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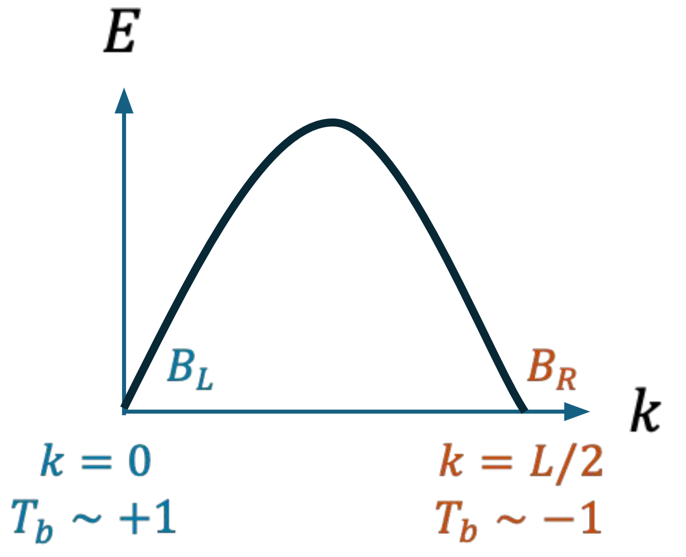

- Focusing on the b_j fermion, the states $|k\rangle$ near $k = 0$ and $k = L/2$ flow to B_L and B_R, respectively.
- Majorana lattice translation T_b for b_j:

$$T_b a_j T_b^{-1} = a_j, \qquad T_b b_j T_b^{-1} = b_{j+1}$$

$$T_b|k\rangle = e^{-2\pi i k/L}|k\rangle$$

- T_b flips the sign of the right-movers, but not the left-movers.

$$T_b \to \mathcal{C}^R$$

In contrast to

$$T a_j T^{-1} = a_{j+1}$$

$$T b_j T^{-1} = b_{j+1}$$

[..., Susskind '77]

Lattice axial charge

- To mimic the continuum relation $Q^A = \mathcal{C}^R Q^V (\mathcal{C}^R)^{-1}$, we define the following operator on the lattice:

$$Q^A = T_b Q^V T_b^{-1} = \frac{i}{2} \sum_j a_j b_{j+1}$$

where $Q^V = \frac{i}{2} \sum_j a_j b_j$.

- Q^A flows to the continuum axial charge.

Lattice axial charge

- $$Q^A = \frac{i}{2}\sum_j a_j b_{j+1}$$

1. It commutes with the Hamiltonian, $[Q^A, H] = 0$.
2. It is quantized, i.e., it has integer eigenvalues, $Q^A \in \mathbb{Z}$.
3. It is a sum of local terms. Lattice Noether current.
4. It acts locally on the fermions:

$$e^{i\theta Q^A} a_j e^{-i\theta Q^A} = \cos\theta\, a_j + \sin\theta\;\; b_{j+1}$$
$$e^{i\theta Q^A} b_j e^{-i\theta Q^A} = \cos\theta\, b_j - \sin\theta\;\; a_{j-1}$$

But...

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$$[Q^V, Q^A] \neq 0 \text{ on the lattice}$$

$$[\mathcal{Q}^V, \mathcal{Q}^A] = 0 \text{ in the continuum}$$

Non-abelian algebra $[Q^V, Q^A] \neq 0$

- Q^V, Q^A generate the Onsager algebra [1944]:

$$[Q_n, Q_m] = iG_{m-n} \ , \qquad [G_n, G_m] = 0$$
$$[Q_n, G_m] = 2i(Q_{n-m} - Q_{n+m})$$

where $Q_n = \frac{i}{2}\sum_j a_j b_{j+n}$, $G_n = \frac{i}{2}\sum_j (a_j a_{j+n} - b_j b_{j+n})$ and $Q_0 = Q^V, Q_1 = Q^A$.

- $\lim_{L\to\infty} G_n = 0$, so $[Q^V, Q^A] \to 0$ in the continuum limit.

A tale of two charges

- Quantized charge

  [Thacker '94, Horvath-Thacker '98]

  $$Q^A = \frac{i}{2}\sum_j a_j b_{j+1}$$

  $$[Q^A, Q^V] \neq 0 \quad , \qquad Q^A \in \mathbb{Z}$$

- Unquantized charge

  [Banks-Susskind-Kogut '76]

  $$\tilde{Q}^A = \frac{i}{4}\sum_j (a_j b_{j+1} - b_j a_{j+1})$$

  $$[\tilde{Q}^A, Q^V] = 0 \quad , \qquad \tilde{Q}^A \notin \mathbb{Z}$$

- Both Q^A and $\tilde{Q}^A$ flow to the same axial charge in the continuum limit.
- We do not have a lattice axial charge that is both (1) quantized and (2) commutes with the vector charge.

Obstruction to a gapped phase

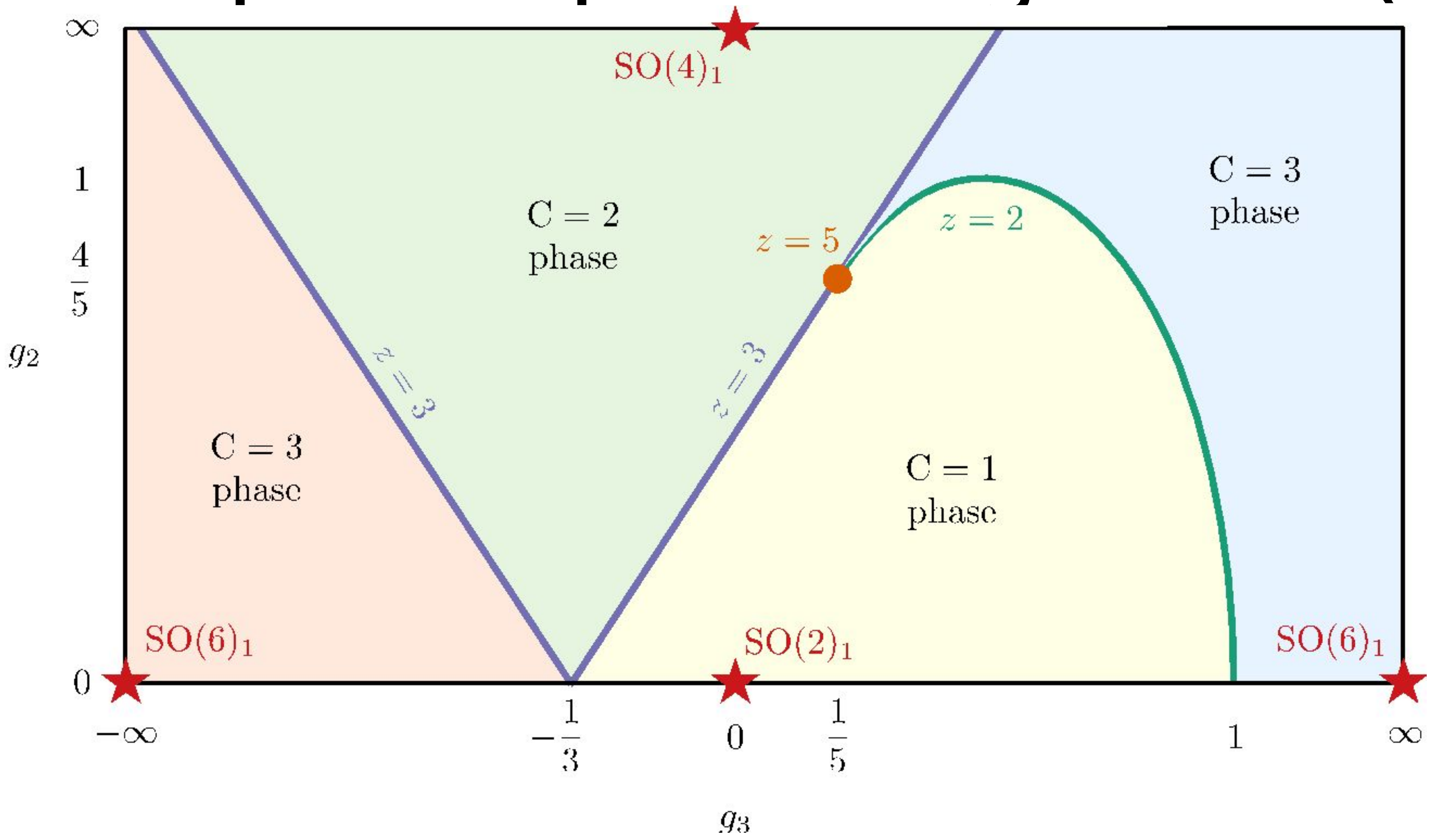
[Chatterjee-Pace-SHS '24]

- What are the consequences of these lattice symmetries Q^V, Q^A?
- We show that

Any Hamiltonian that commutes with the lattice vector and axial symmetries Q^V, Q^A is necessarily gapless.

- It is reminiscent of the consequence of perturbative anomalies in continuum field theory.

Gapless phases preserving both U(1)s



- A 2-dim slice of the phase diagram preserving both U(1)s. The entire phase diagram is gapless.
- C stands for the number of Dirac fermion modes. Stars are the relativistic CFT points.

Winding symmetry in the XX spin chain
[Pace-Chatterjee-SHS '24]

- Bosonization maps the fermion Hamiltonian to the XX model:

$$H_{XX} = \sum_j (X_j X_{j+1} + Y_j Y_{j+1})$$

which flows to the c=1 compact boson CFT at radius $R = \sqrt{2}$ corresponding to the "Dirac point."

- Exact lattice momentum Q^M and winding Q^W symmetries [Vernier-O'Brien-Fendley '18, Yuan '21, Popkov-Zhang-Gohmann-Klumper '23].
- Exact T-duality associated with a non-invertible symmetry operator D:

$$DQ^M = 2Q^W D$$

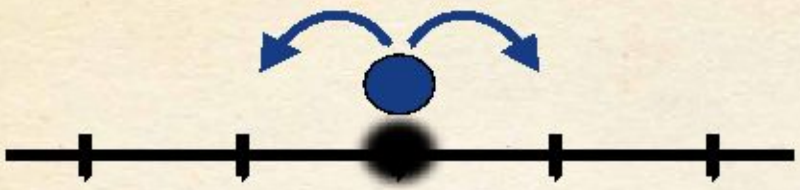
Summary

- $$H = -i\sum_{j=1}^{L}\left(c_j^\dagger c_{j+1} + c_j c_{j+1}^\dagger\right)$$
- In addition to the manifest vector $U(1)^V$ symmetry $c_j \to e^{-i\theta} c_j$, this lattice Hamiltonian has an exact axial $U(1)^A$ symmetry.
- However, the lattice vector and axial charges Q^V, Q^A don't commute, and generate the Onsager algebra.
- The Schwinger anomaly of the continuum free Dirac fermion field theory emanates from the non-abelian Onsager algebra on the lattice. Novel anomaly matching mechanism between UV and IR [John, Xie, Meng, and Maissam's talks].
- The lattice symmetries Q^V, Q^A force the Hamiltonian to be gapless.

Outlook

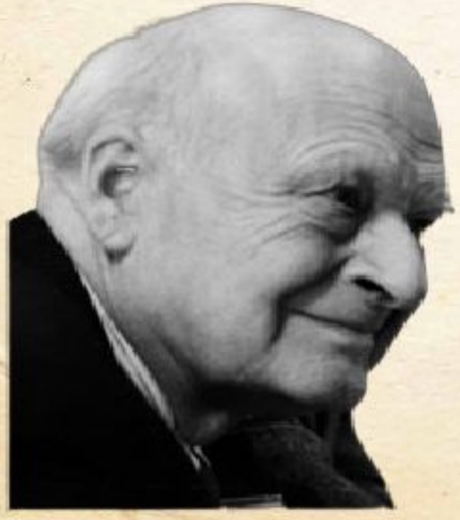
- Relation to the Nielsen-Ninomiya theorem.
- Can we realize more general chiral symmetries on the lattice? Non-abelian chiral symmetries?
- Generalized symmetries in lattice QED and QCD. Relation to the mass shift formula [Dempsey-Klebanov-Pufu-Zan ‘22].
- Generalization to higher spacetime dimensions?

Lattice fermions



Onsager's algebra

$[Q^V, Q^A] \neq 0$



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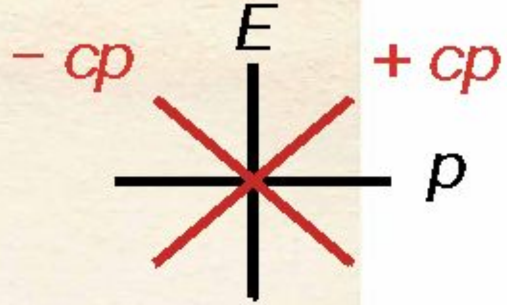
Dirac fermion



Schwinger's anomaly