Surfaceology for string/particle amplitudes

Song He (Institute of Theoretical Physics, CAS)

based on works with Nima Arkani-Hamed, Qu Cao, Jin Dong (see poster), Carolina Figueiredo

2312.16282, 2401.05483, 2401.00041, 2408.11891, …

+ with Qu Cao, Jin Dong, Canxin Shi, Fan Zhu, 2406.03838, 2412.19629, …

Strings 2025 **(NYU Abu Dhabi)**

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Combinatorial Geometries ("polytopes") underlying scattering amplitudes & beyond

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- Amplituhedron: all-loop integrands of N=4 SYM in momentum twistor space [Arkani-Hamed, Trnka '13 + Thomas '17;…]
- ABJM amplituhedron: all-loop integrands of ABJM (reduced from SYM) [SH, Kuo, Li, Zhang '22; SH, Huang, Kuo, '23,…]
- Correlahedron for half-BPS correlator in SYM [Eden, Heslop, Mason; SH, Huang, Kuo, '24…] —> energy correctors [SH et al '24]

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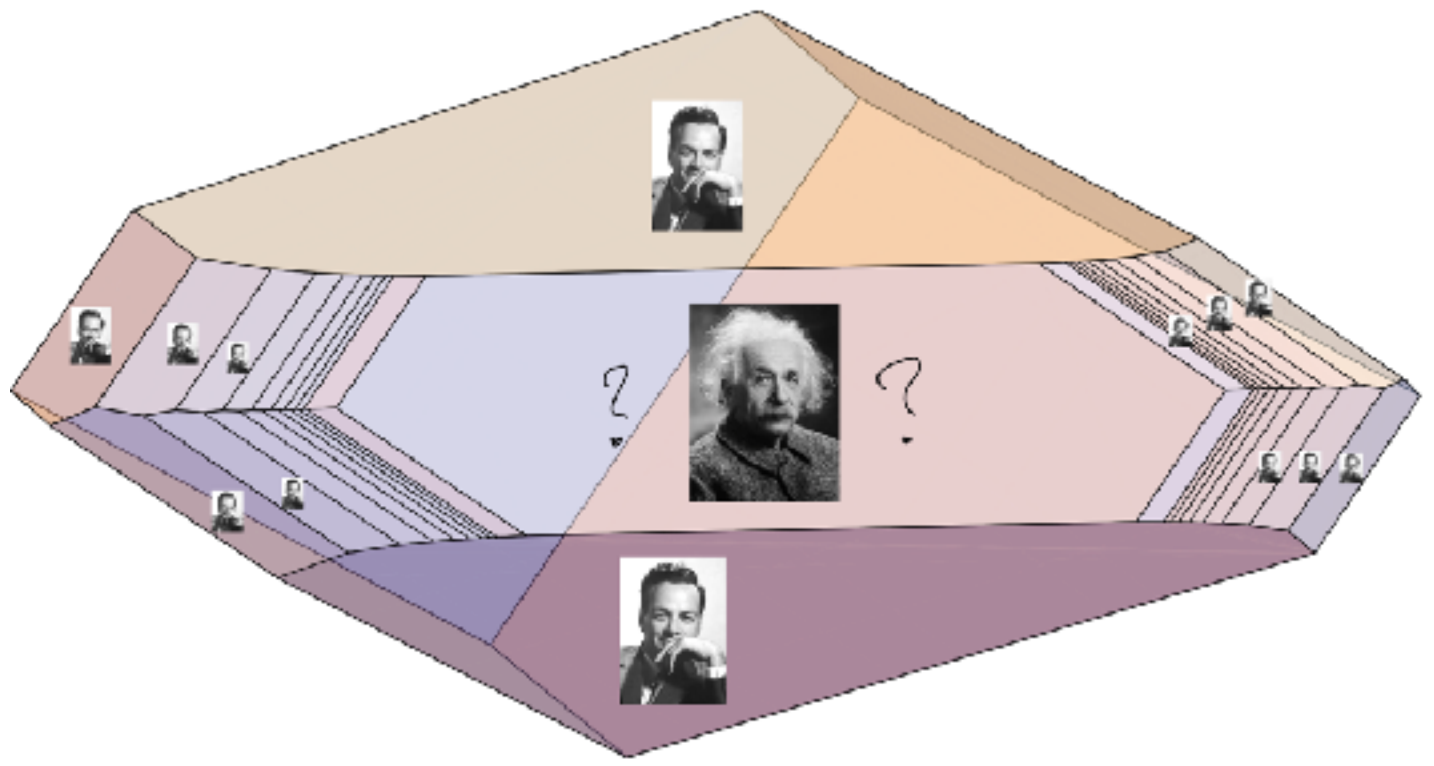
- kinematic associahedra (Tr ϕ^3 tree amps) & (Deligne-Mumford) worldsheet associahedra [Arkani-Hamed, Bai, SH, Yan, '17; …]
- surfacehedra + curve-integrals on surfaces => all-order Tr ϕ^3 amplitudes + (bosonic) string [Arkani-Hamed et al, 20-24,…]
- cosmological polytopes (each graph) [Arkani-Hamed et al '17,…]—> cosmohedra for wavefunction/correlator in Tr ϕ^3 [2412.19881]
- wider context: tropical geometries for Feynman integrals etc. positive geometries in dS/AdS (Mellin) amps? [c.f. w. Cao, Li, Tang '24]

Curve integral: from toy model to Real World

A new way of thinking about particle/string amplitudes based on combinatorial & geometric objects without reference to Feynman diagrams/worldsheet physics



Curve integrals on surfaces: "counting problem" => "u" variables for bosonic string to all orders = stringy completion of (simplest colored scalars) Tr ϕ^3 amps (="surfacehera") [ABHY '17; Arkani-Hamed, Frost, Salvatori, Plamondon, Thomas: 2309.15913, 2311.09284 …]

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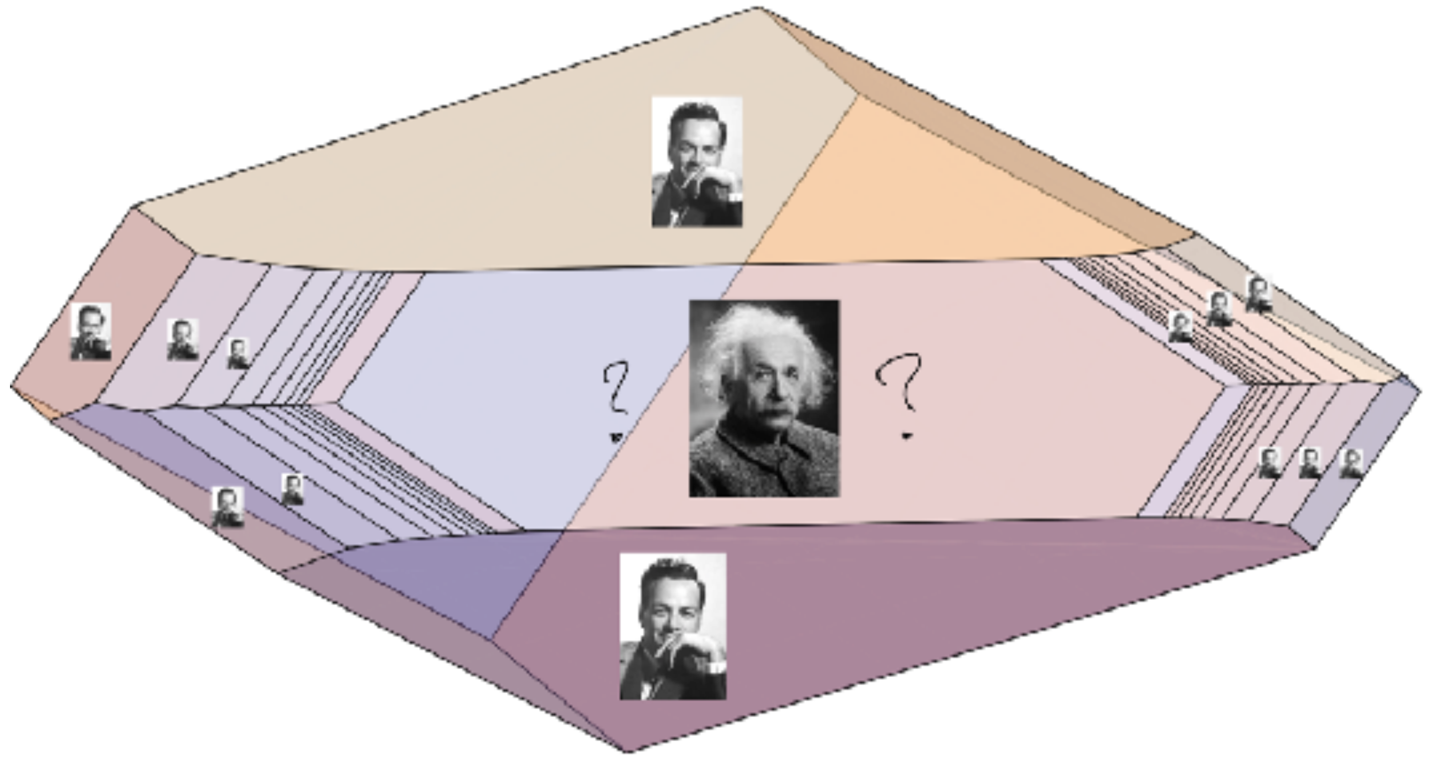

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- Manifest (tree) factorization (finite α', no blowup): massive residues, asymptotics etc. [c.f. Arkani-Hamed, Figueiredo, Remmen 2412.20639]
- All-loop cuts ($\alpha' \to 0$) from surfaces => all-loop recursion for Tr ϕ^3+ any colored-scalars [c.f. Arkani-Hamed, Frost, Salvatori 2412. 21027]
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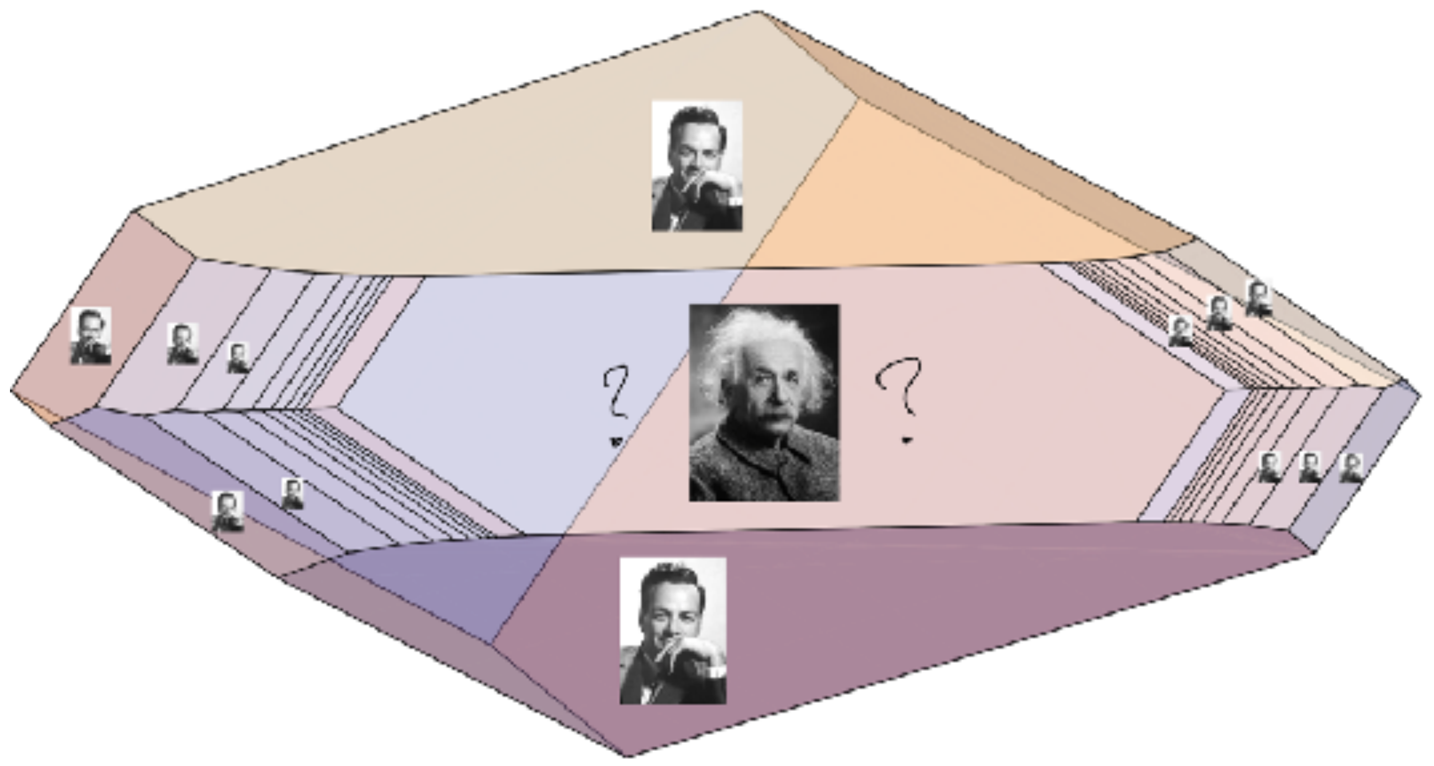

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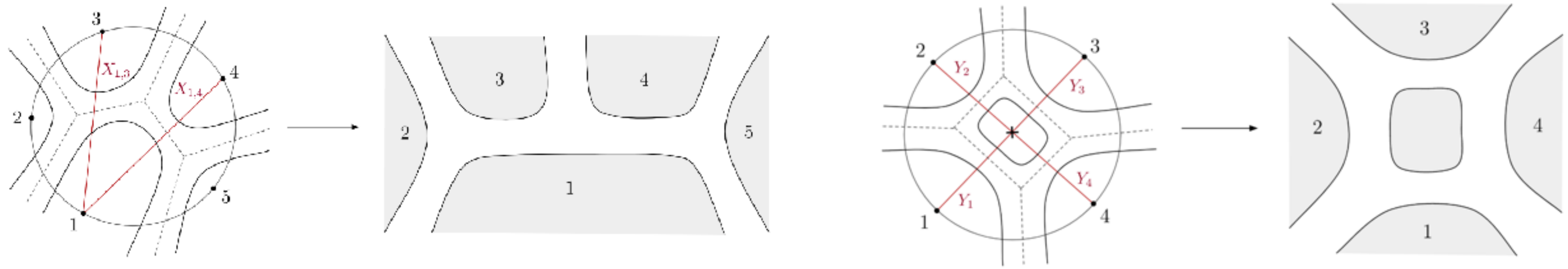
Surprisingly, this toy model contains realistic theories: amplitudes of pions & gluons from stringy Tr ϕ^3 by kinematic shifts!

- The unity of ϕ^3, pions & gluons (tree) => all-loop non linear sigma model (+ mixed amps) $\subset \mathrm{Tr}\ \phi^3$
- Reveals novel features of all these theories, e.g. hidden patterns of zeros & factorization/splits near zeros
- "Combinatorial origin of Yang-Mills": scalar-scaffolded gluons => all-loop YM in stringy Tr ϕ^3

Tr ϕ^3 amplitudes [Arkani-Hamed, Bai, SH, Yan, ’17; Arkani-Hamed, Frost, Salvatori, Plamondon, Thomas, ’23,…]

$$\mathcal{L}_{\mathrm{Tr}(\phi^3)} = \mathrm{Tr}(\partial\phi)^2 + g\,\mathrm{Tr}(\phi^3),$$

ϕ : N by N matrix -> fat graphs, genus expansion (only planar graphs for $N \to \infty$)



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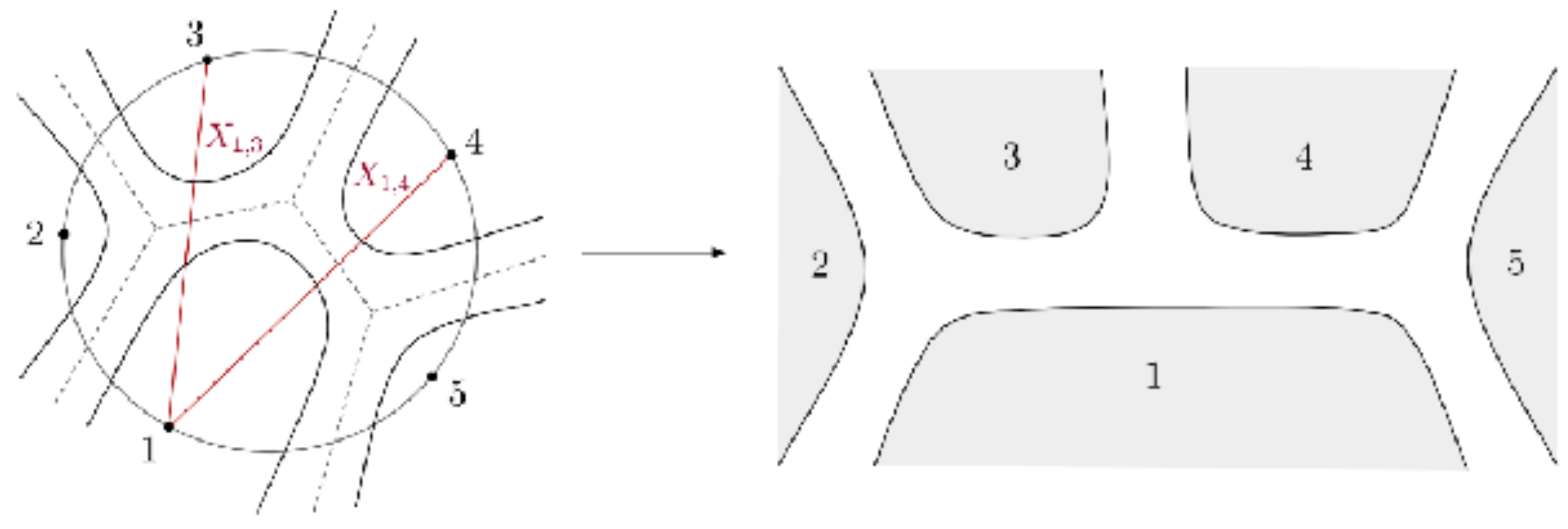
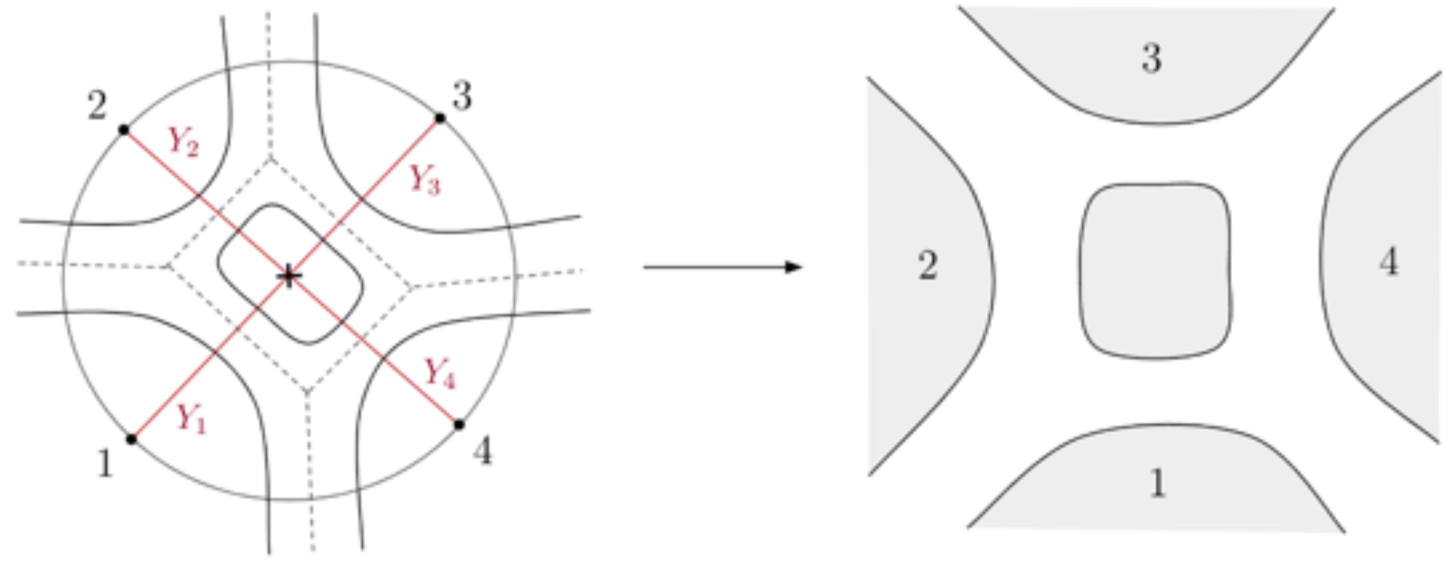
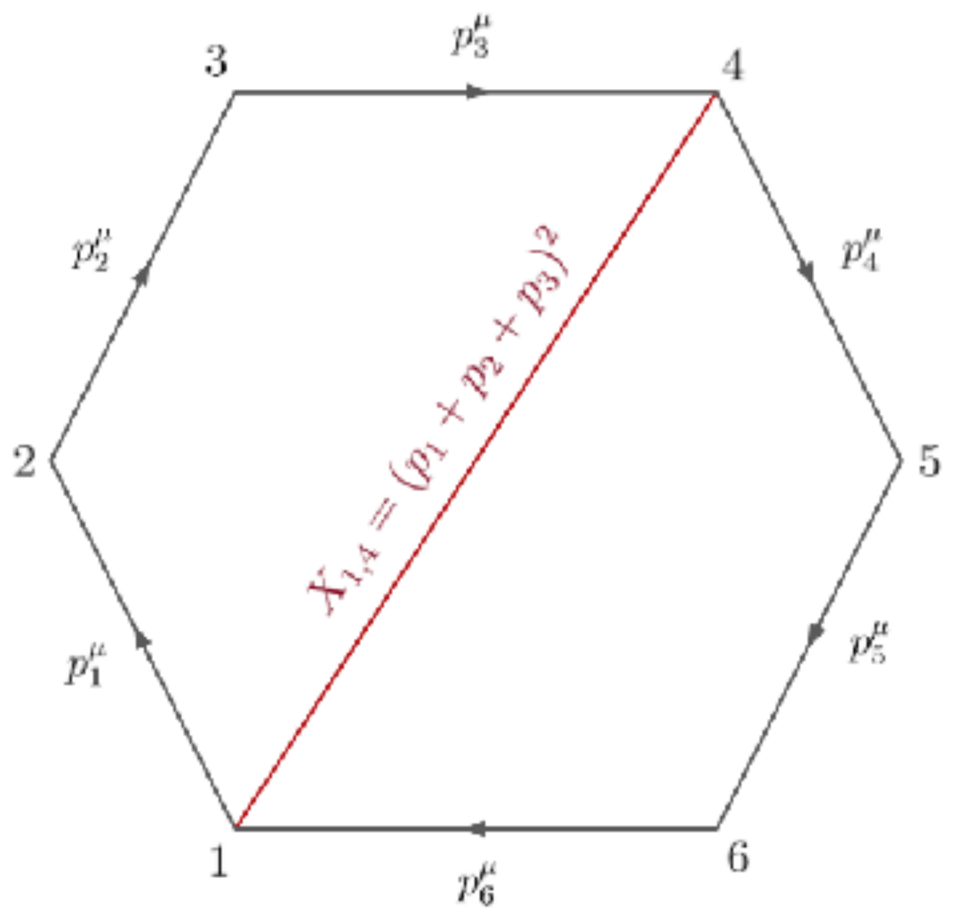




kinematics: e.g. tree amps (planar)

$$X_{i,j} = (p_i + \cdots + p_{j-1})^2.$$



$$c_{i,j} := -2p_i \cdot p_j = X_{i,j} + X_{i+1,j+1} - X_{i+1,j} - X_{i,j+1}\,,$$

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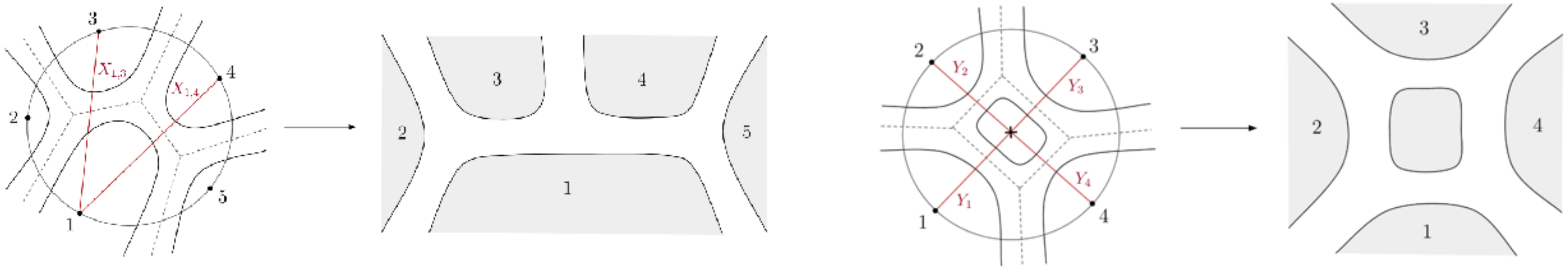
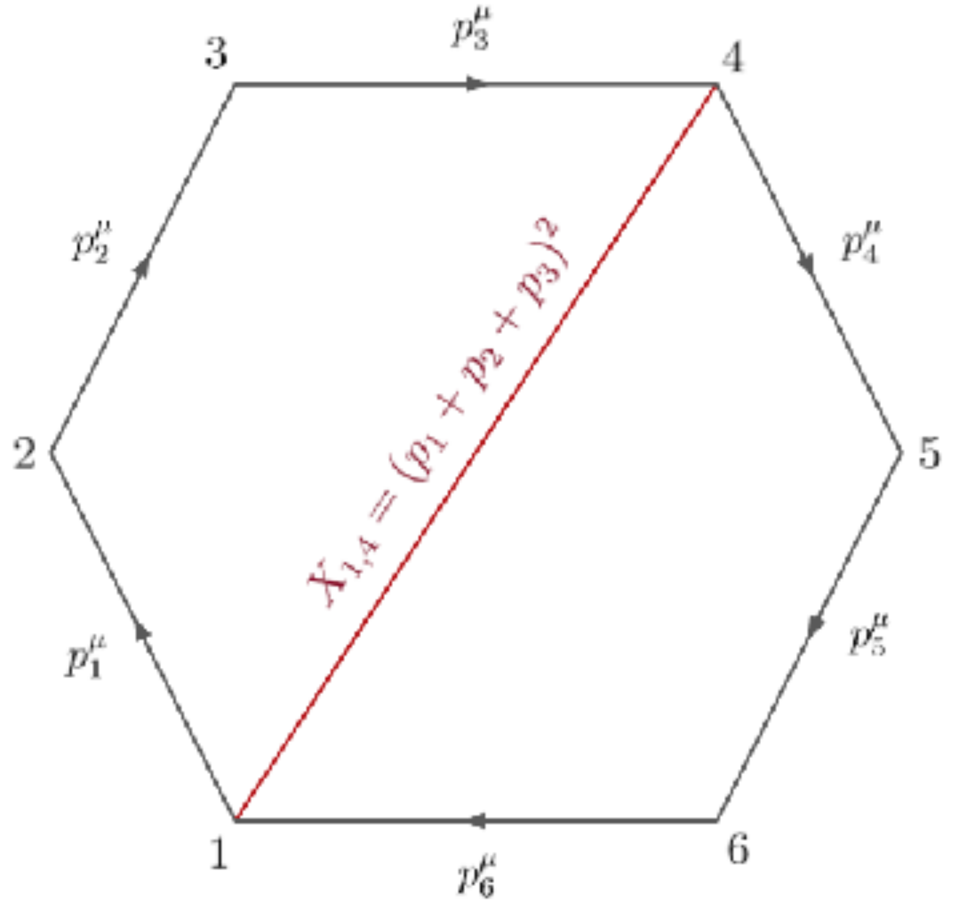
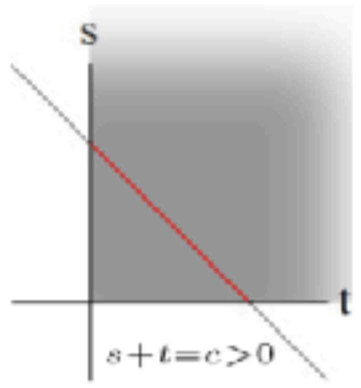




tree amp = sum over n-gon triangulations = canonical form of associahedron

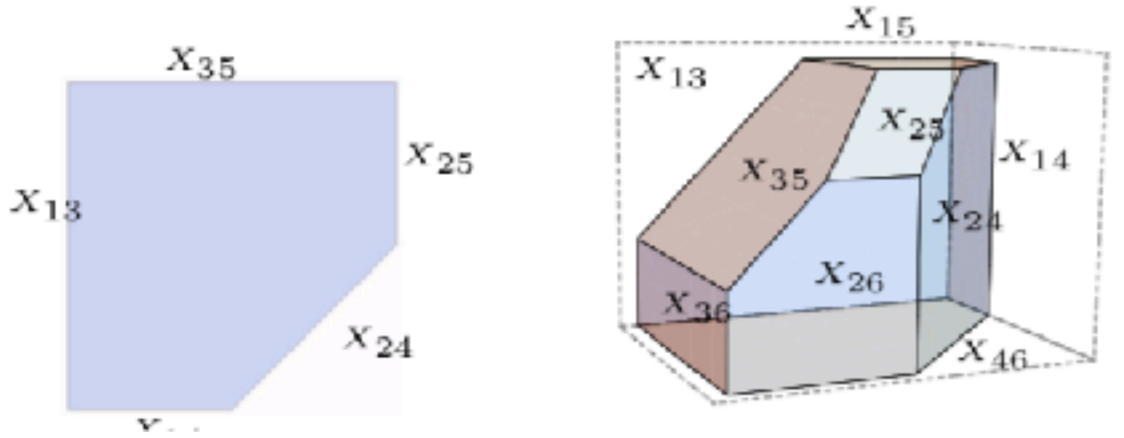
$$\mathcal{A}_4 = \frac{1}{X_{13}} + \frac{1}{X_{24}}\,,$$



$e.g.\ \mathcal{A}_1 = \{s>0, t>0\} \cap \{-u = \mathrm{const} > 0\}$

$\mathcal{A}_2 = \{X_{13}, \cdots, X_{25} > 0\} \cap \{-s_{13} = c_{13}, -s_{14} = c_{14}, -s_{24} = c_{24}\}$

$$\mathcal{A}_5 = \frac{1}{X_{1,3}X_{1,4}} + \frac{1}{X_{2,4}X_{2,5}} + \frac{1}{X_{1,3}X_{3,5}} + \frac{1}{X_{1,4}X_{2,4}} + \frac{1}{X_{2,5}X_{3,5}}.$$



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Disk integral = stringy Tr ϕ^3 amp [c.f. Arkani-Hamed, SH, Lam, 19]

a la Veneziano-Koba-Nielsen; Z-theory [Carrasco, Mafra, Schlotterer 16',…]

$$\mathcal{I}_n^{\mathrm{Tr}\,\phi^3}(1,2,\ldots,n)=\int_{D(1\ldots n)}\frac{\mathrm{d}z_1\ldots\mathrm{d}z_n}{\mathrm{vol}\;\mathrm{SL}(2,\mathbb{R})}\underbrace{\frac{1}{z_{1,2}z_{2,3}\ldots z_{n,1}}}_{\mathrm{PT}(1,2,\ldots,n)}\times\underbrace{\prod_{i<j}z_{i,j}^{2\alpha' p_i\cdot p_j}}_{\text{Koba-Nielsen factor}}$$

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u variables: $u_{a,b} := \frac{z_{a,b-1}z_{a-1,b}}{z_{a,b}z_{a-1,b-1}}$ [Koba, Nielsen, '69; ...]

=> $\int \frac{d^{n-3}z}{z_{12}\cdots z_{n1}} \prod_{a<b} u_{a,b}^{\alpha' X_{a,b}}$ w. positive parametrization $y_{I=1,\cdots,n-3}$ => $\mathcal{I}_n^{\mathrm{Tr}\phi^3} = \int_0^\infty \prod_I \frac{dy_I}{y_I} \prod_C u_C(y)^{\alpha' X_C}$ curve-integral on the disk!

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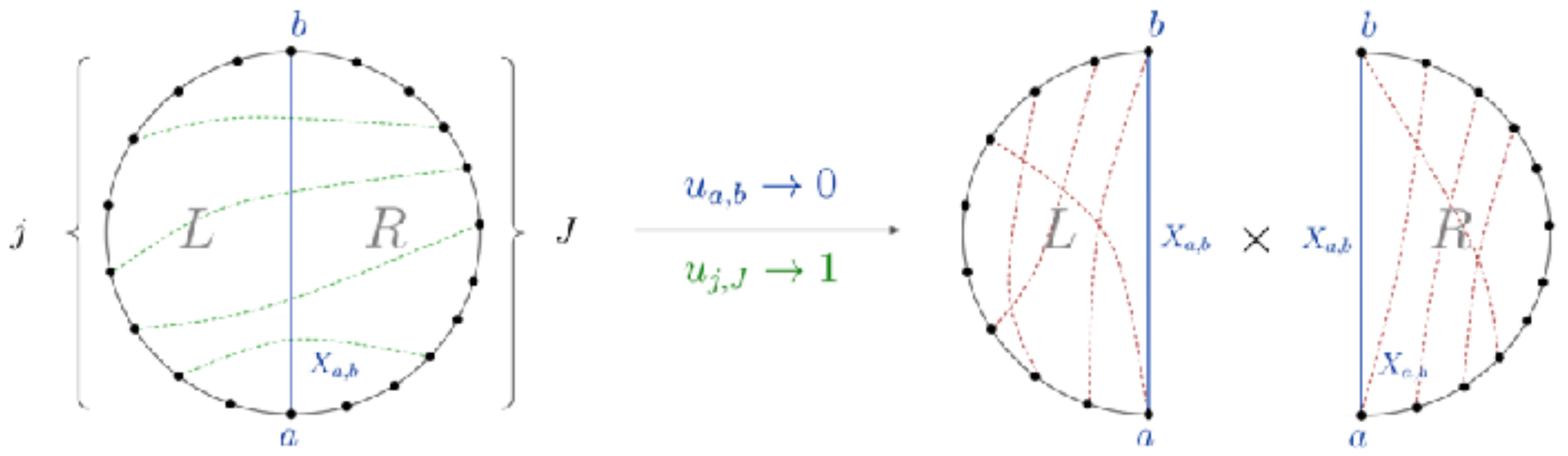

e.g. n=4: $\int_0^\infty \frac{dy}{y} \left(\frac{y}{1+y}\right)^{\alpha' X_{1,3}} \left(\frac{1}{1+y}\right)^{\alpha' X_{2,4}} = \frac{\Gamma(\alpha' X_{1,3})\Gamma(\alpha' X_{2,4})}{\Gamma(\alpha'(X_{1,3}+X_{2,4}))}$

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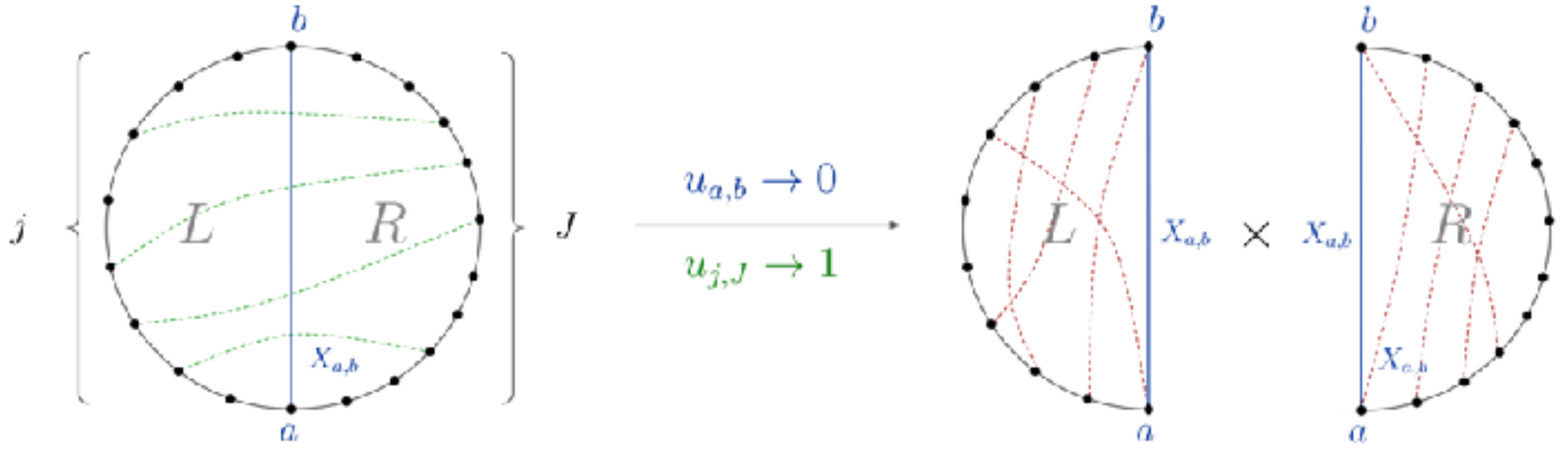

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stringy extension of associahedra [Arkani-Hamed, SH, Lam, 19; …] -> similar "u" variables (curves)+ positive para. (triangulation) for any surface!
-> combinatorial formulation of (bosonic) strings: hyperbolic geometry (positive) vs. Riemann surfaces (complex); easier to see $\alpha' \to 0$ limit

Zeros & splits of amplitudes [ACDFH 2312; see D'Adda, Sciuto, D'Auria, Gliozzi, 71]

Zeros of Veneziano amp: by setting $\alpha' c_{1,3} = -n$, $\mathcal{I}_4^{\mathrm{Tr}(\phi^3)} \to \sum_{k=0}^{n} \underbrace{\int_{\mathbb{R}_{>0}} \frac{dy_{1,3}}{y_{1,3}} y_{1,3}^{\alpha' X_{1,3}+k}}_{=0} = 0.$

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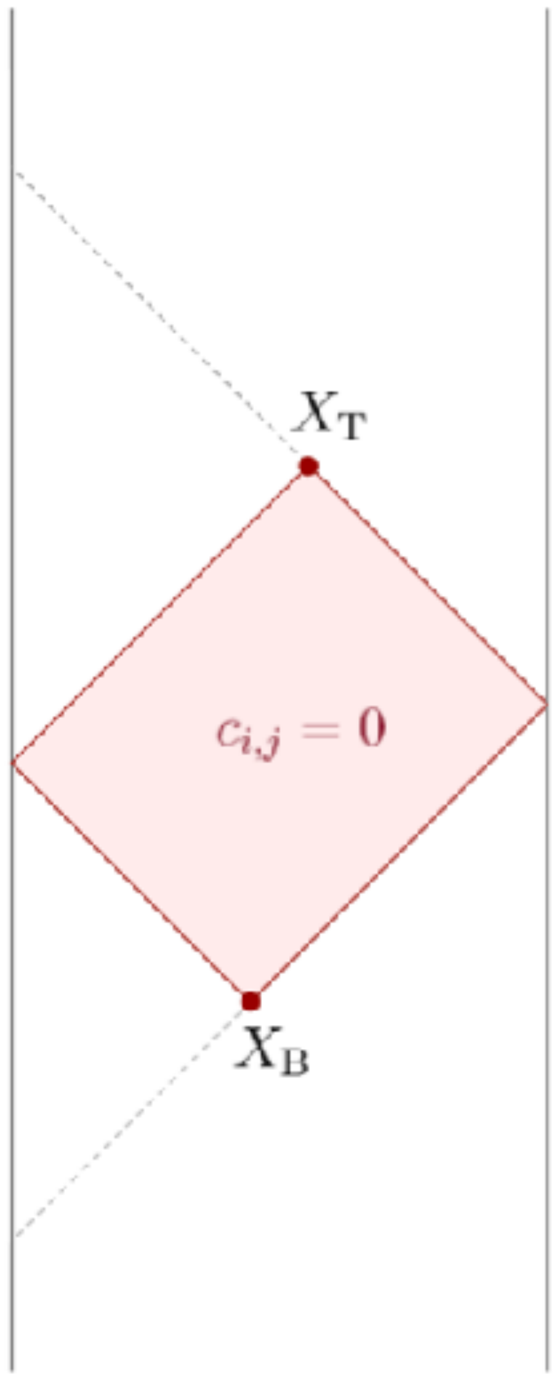
$$\mathcal{I}_n^{\mathrm{Tr}\,\phi^3} \to \sum_{k_{a_1,b_1},\ldots,k_{a_N,b_N}=0}^{n_{a_1,b_1},\ldots,n_{a_N,b_N}} (\text{remaining integrals}) \times \underbrace{\int_{\mathbb{R}_{>0}} \frac{dy_{1,i}}{y_{1,i}} y_{1,i}^{\alpha' X_{1,i}+k_{a_1,b_1}+\cdots+k_{a_N,b_N}}}_{=0} = 0$$

$c_{i,j} = -n_{ij}$, $\quad 1 \leq i < a-1,\ a \leq j < n$

n(n-3)/2 infinite families of zeros



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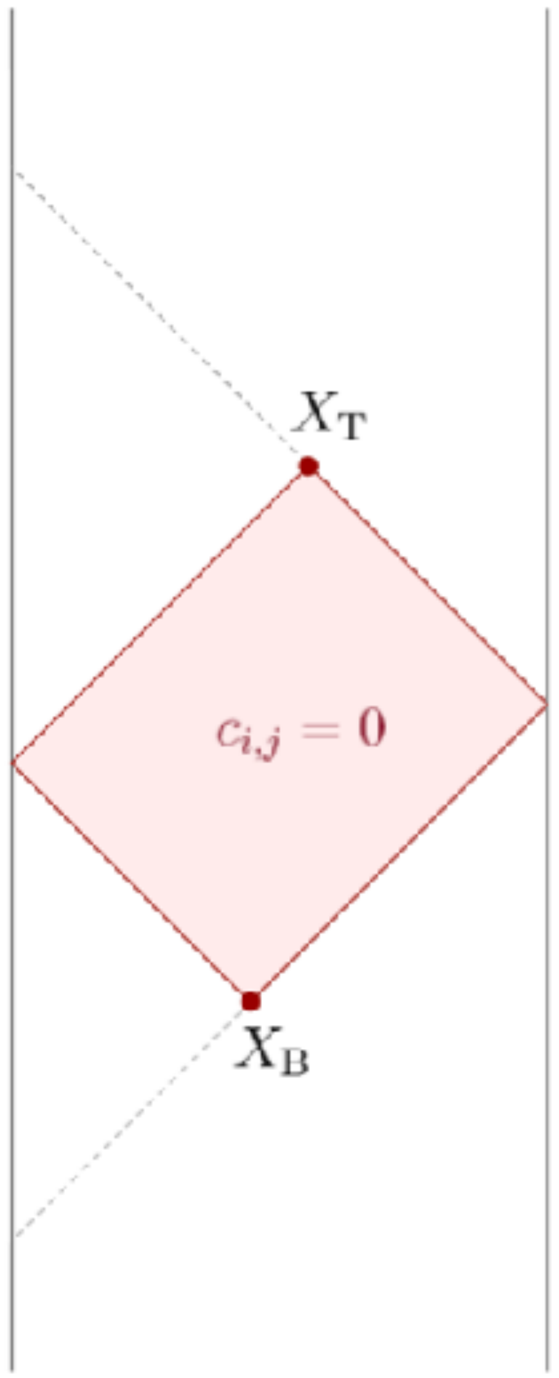

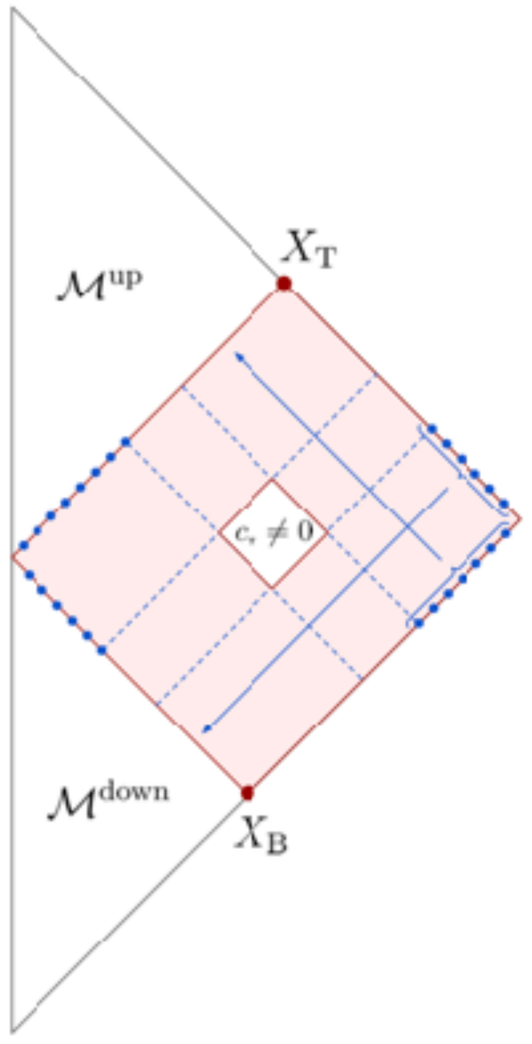

$$\mathcal{I}_n^{\mathrm{Tr}\,\phi^3} \to \mathcal{I}_i^{\mathrm{down},\mathrm{Tr}\,\phi^3} \times \mathcal{I}_{n-i+2}^{\mathrm{up},\mathrm{Tr}\,\phi^3} \times \mathcal{I}_4^{\mathrm{Tr}\,\phi^3}(\alpha' X_{1,i}, \alpha'(c_{km} - X_{1,i})).$$

$$X_{l,i} \to X_{l,i} + X_{1,i} = X_{l,n}, \text{for} \quad l = 2,\ldots,k.$$

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shifted kinematics —> currents (with an off-shell leg)

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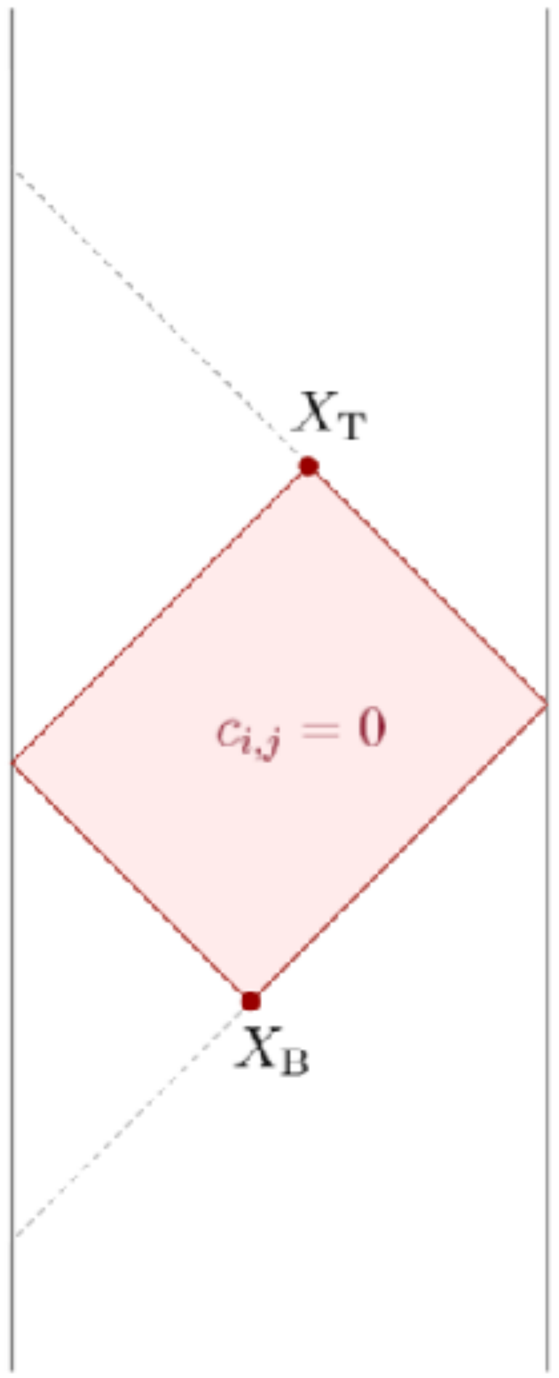

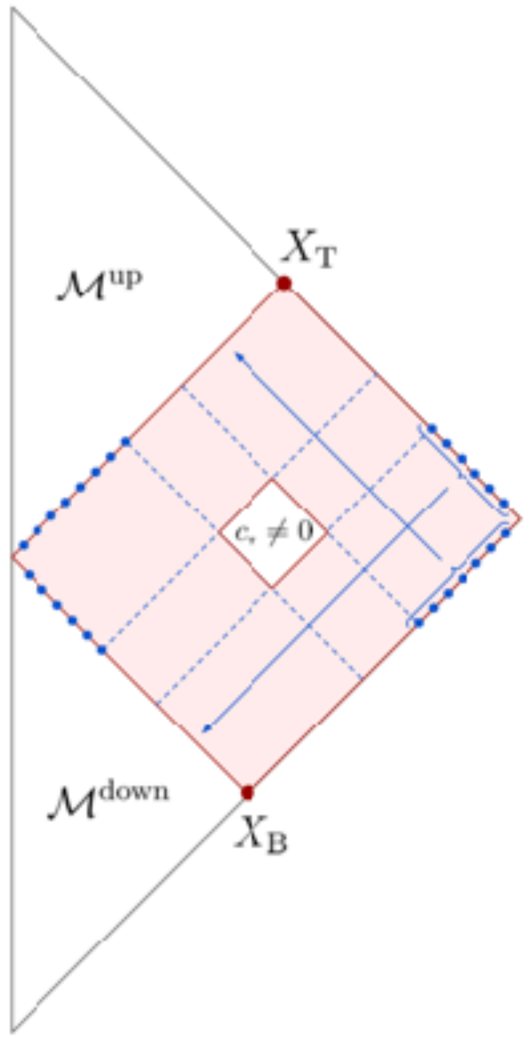

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$$X_{i-1,j} \to X_{i-1,j} - X_{i-1,n} = X_{1,j}, \text{for} \quad j = m, \ldots, n-1.$$

$$\mathcal{M}_n(c_\star \neq 0) = \left(\frac{1}{X_{\mathrm{B}}} + \frac{1}{X_{\mathrm{T}}}\right) \times \mathcal{M}^{\mathrm{down}} \times \mathcal{M}^{\mathrm{up}}.$$

shifted kinematics —> currents (with an off-shell leg)

$$X = X_B + \tilde{X}. \qquad \tilde{X} = X_{\mathrm{T}} + X,$$

Deformed to the real world [ACDFH 2312]

$$\mathcal{I}_{2n}^{\delta} = \int_{\mathbb{R}_{>0}^{2n-3}} \prod_{I=1}^{2n-3} \frac{dy_I}{y_I} \prod_{(a,b)} u_{a,b}^{\alpha' X_{a,b}} \left(\frac{\prod_{(e,e)} u_{e,e}}{\prod_{(o,o)} u_{o,o}} \right)^{\alpha'\delta} , \qquad \mathcal{I}_{2n}^{\delta} = \mathcal{I}_{2n}^{\mathrm{Tr}\,\phi^3}[\alpha' X_{e,e} \to \alpha'(X_{e,e} + \delta), \alpha' X_{o,o} \to \alpha'(X_{o,o} - \delta)] \,.$$

Deformed to the real world [ACDFH 2312]

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$\alpha'\delta = \pm 1,\ (\pm 2, \ldots)$

$$\mathcal{L}_{\mathrm{YMS}} = -\mathrm{Tr}\left(\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}D^\mu\phi^I D_\mu\phi^I - \frac{g_{\mathrm{YM}}^2}{4}\sum_{I\neq J}[\phi^I, \phi^J]^2\right)$$

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Tr ϕ^3 2n-pt string amps => 2n-scalar amp in NLSM or in YM + scalar: same function @ different pts!

A universal splitting of string/particle amps [w. Cao, Dong, Shi, Zhu, '24]

A new behavior for a wide class of string/particle tree amp => factorizes into two off-shell currents (total #=n+3)

Universally hold for scalars (ϕ^3, NLSM, special Galileon, YMS/DBI) + gluons/gravitons in bosonic & superstring; implies & extends splittings near zeros etc. to all these theories [see also J. Trnka et al; L. Rodina; Y. Zhang;…]

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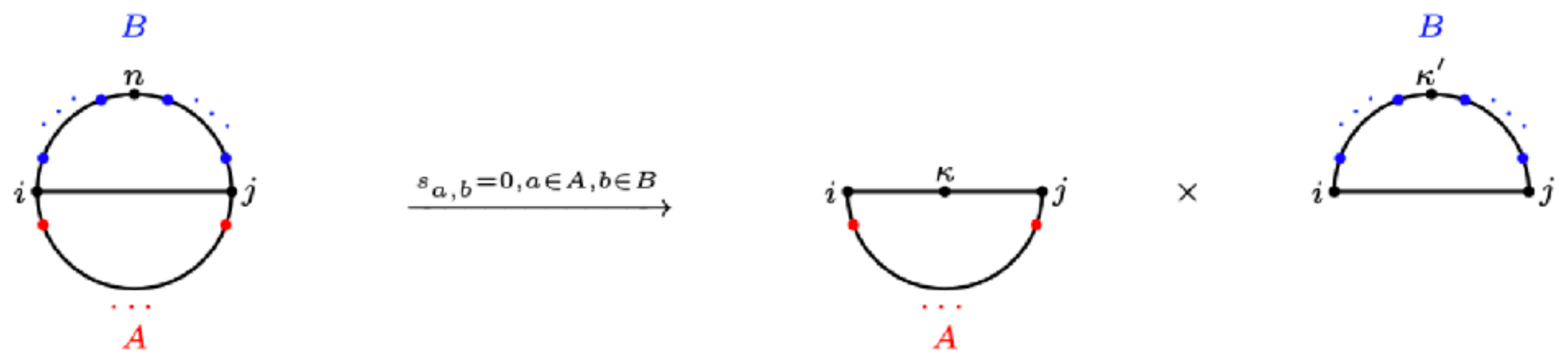

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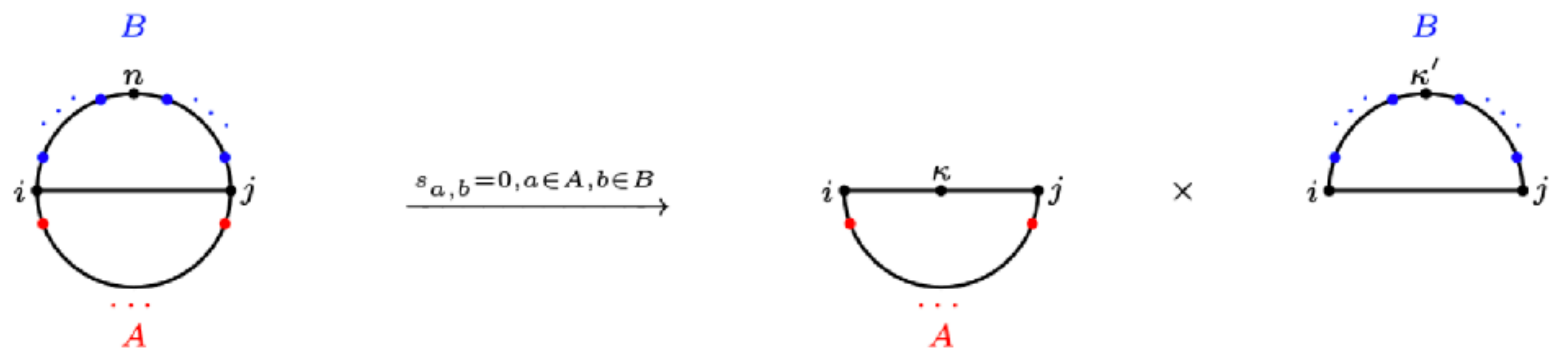

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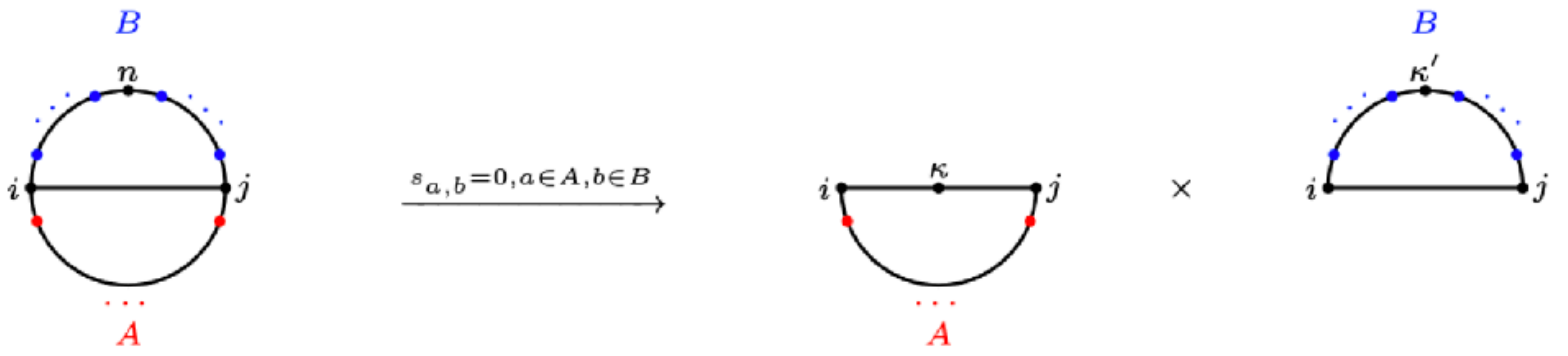

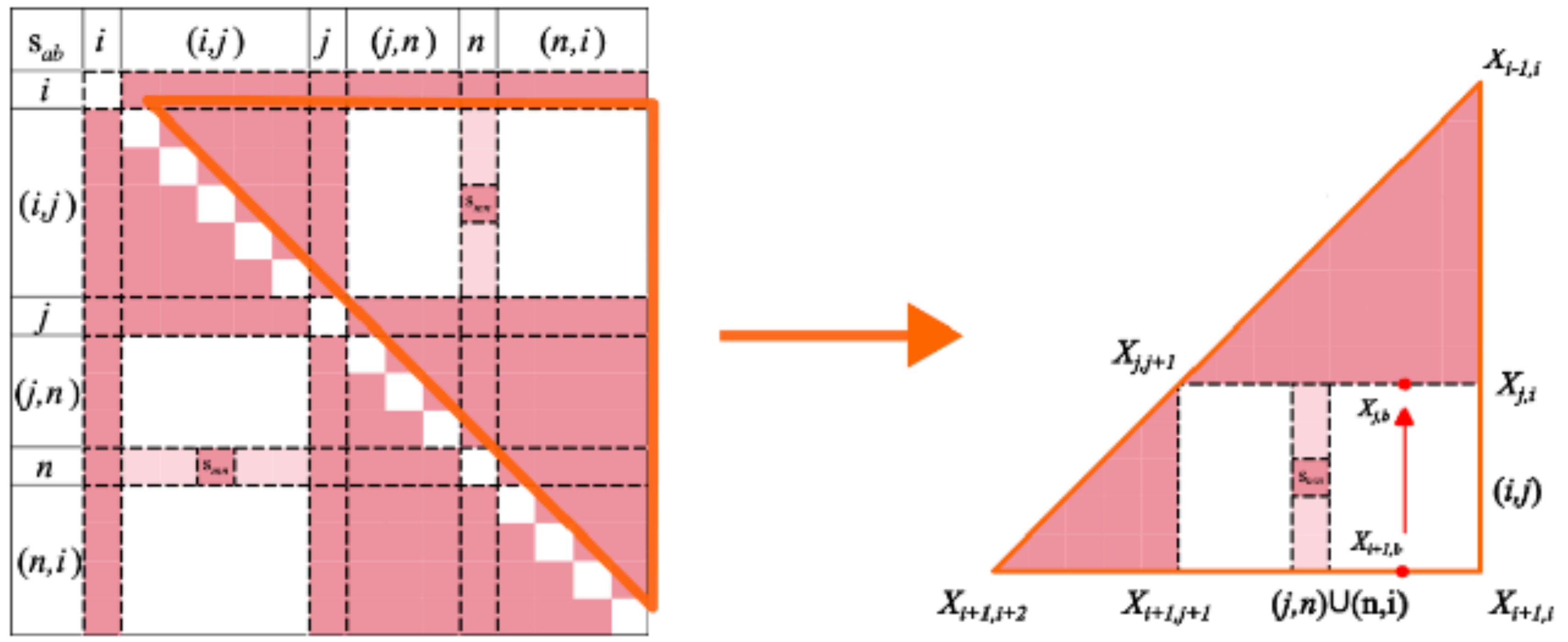

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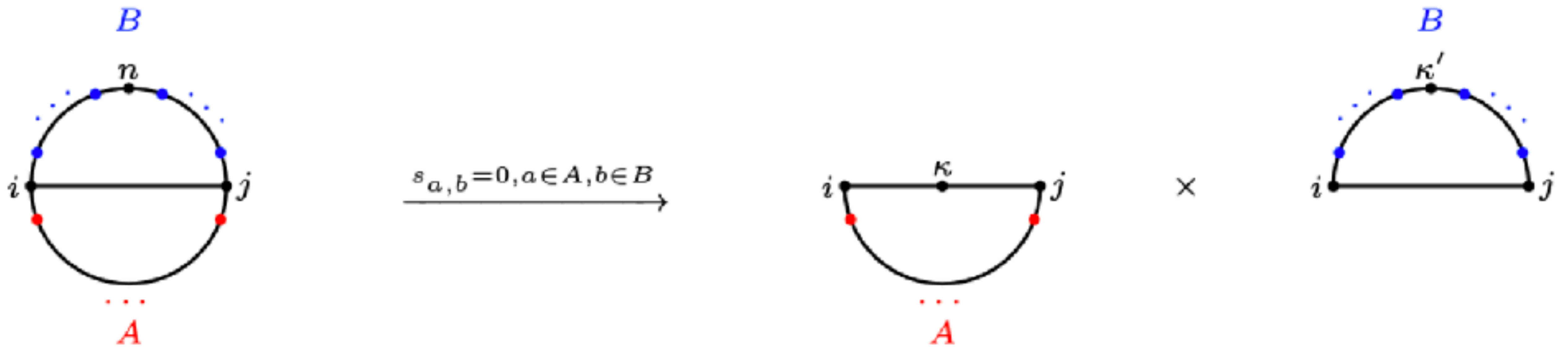

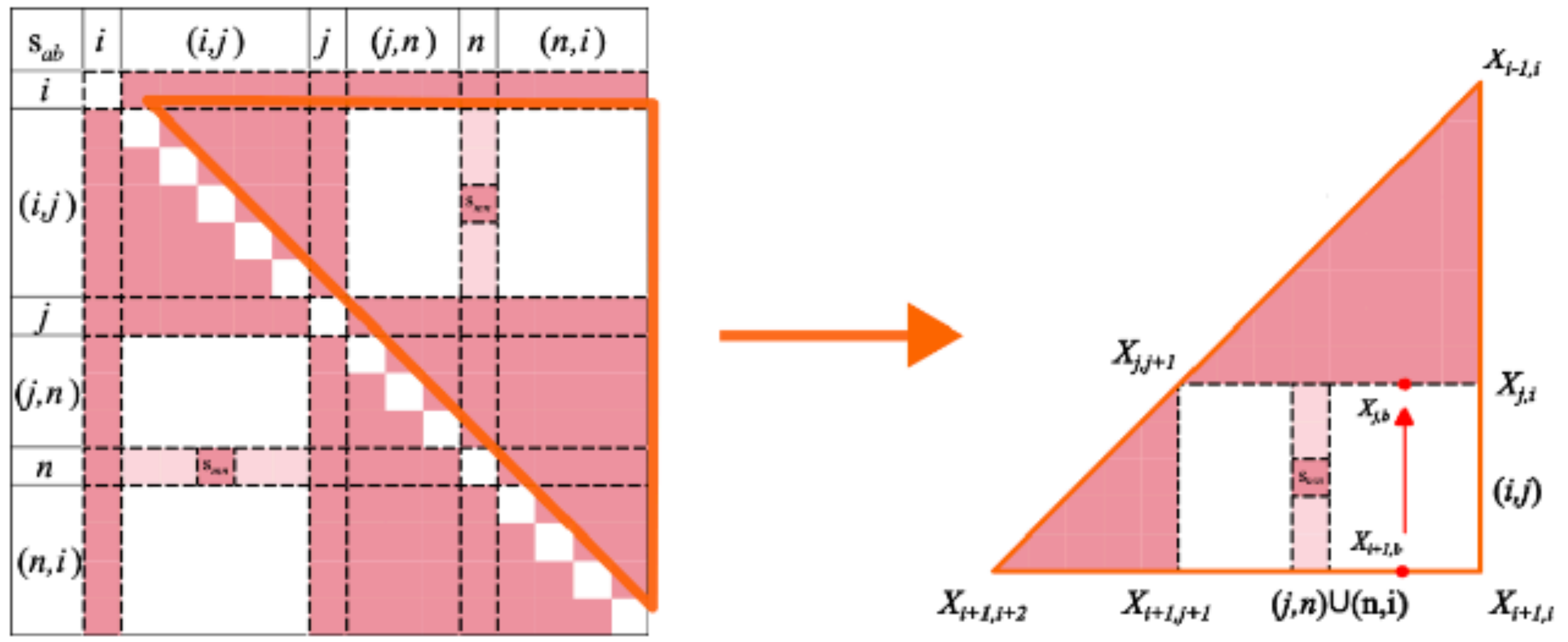

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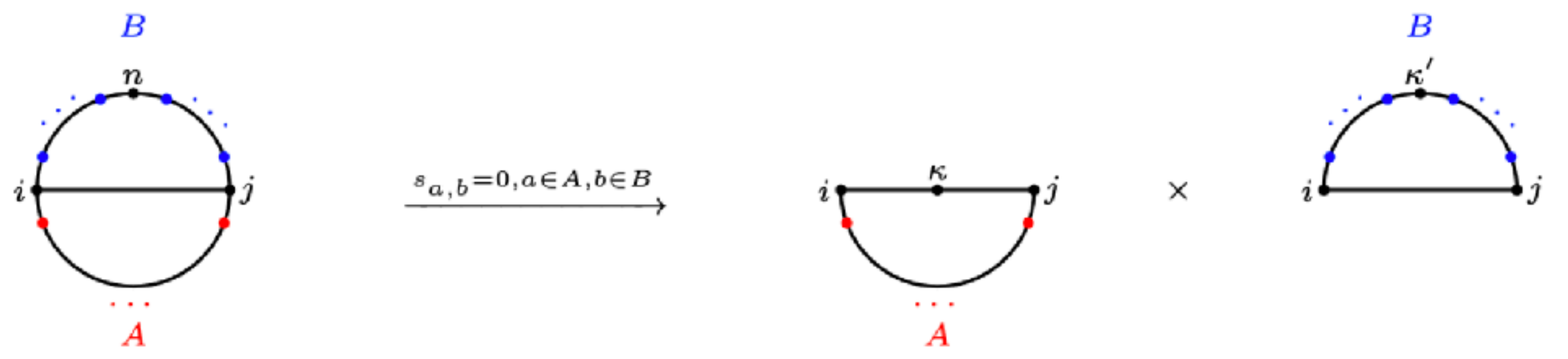

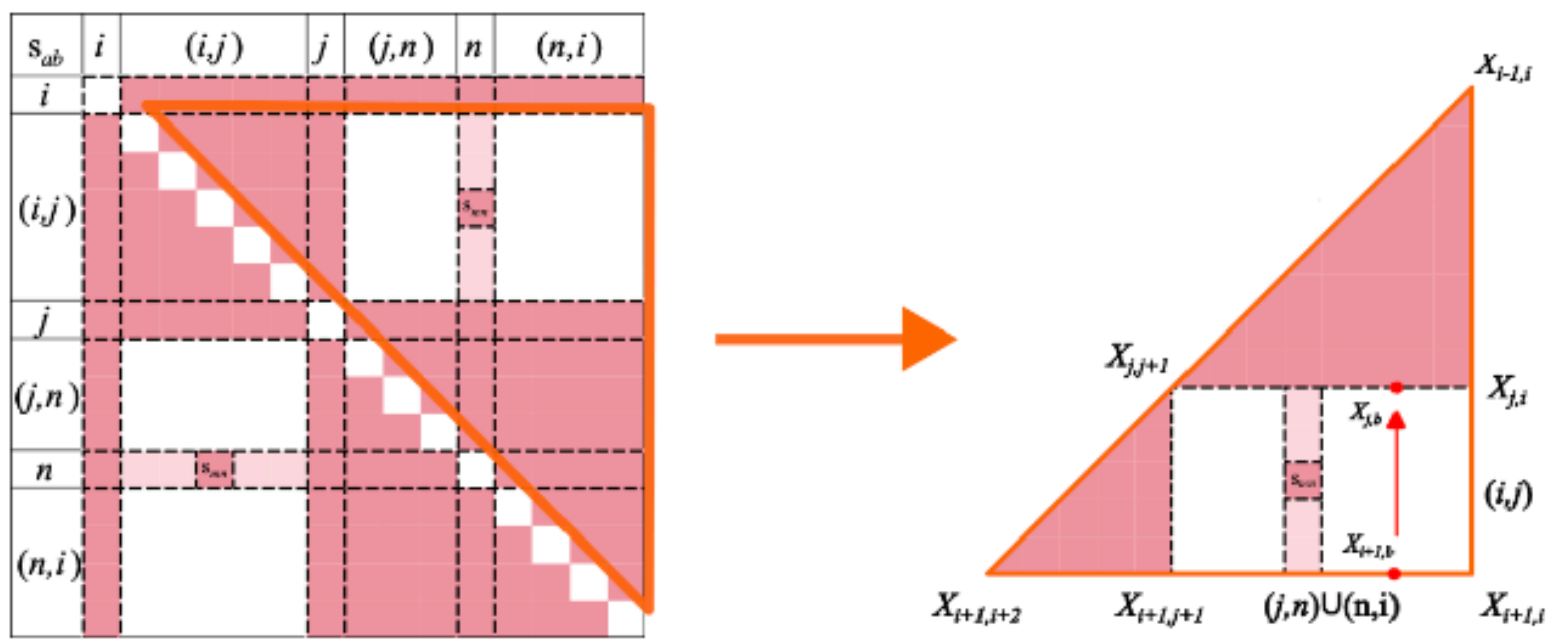

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Very nicely, extends to all loops for stringy ϕ^3 & NLSM etc. from "gluing surfaces" [Arkani-Hamed, Figueiredo, '24]

Splitting & soft theorems [w. Cao, Dong, Shi, Zhu, '24]

$$\epsilon_a \cdot \epsilon_b = \epsilon_a \cdot \epsilon_{i,j,k} = 0$$
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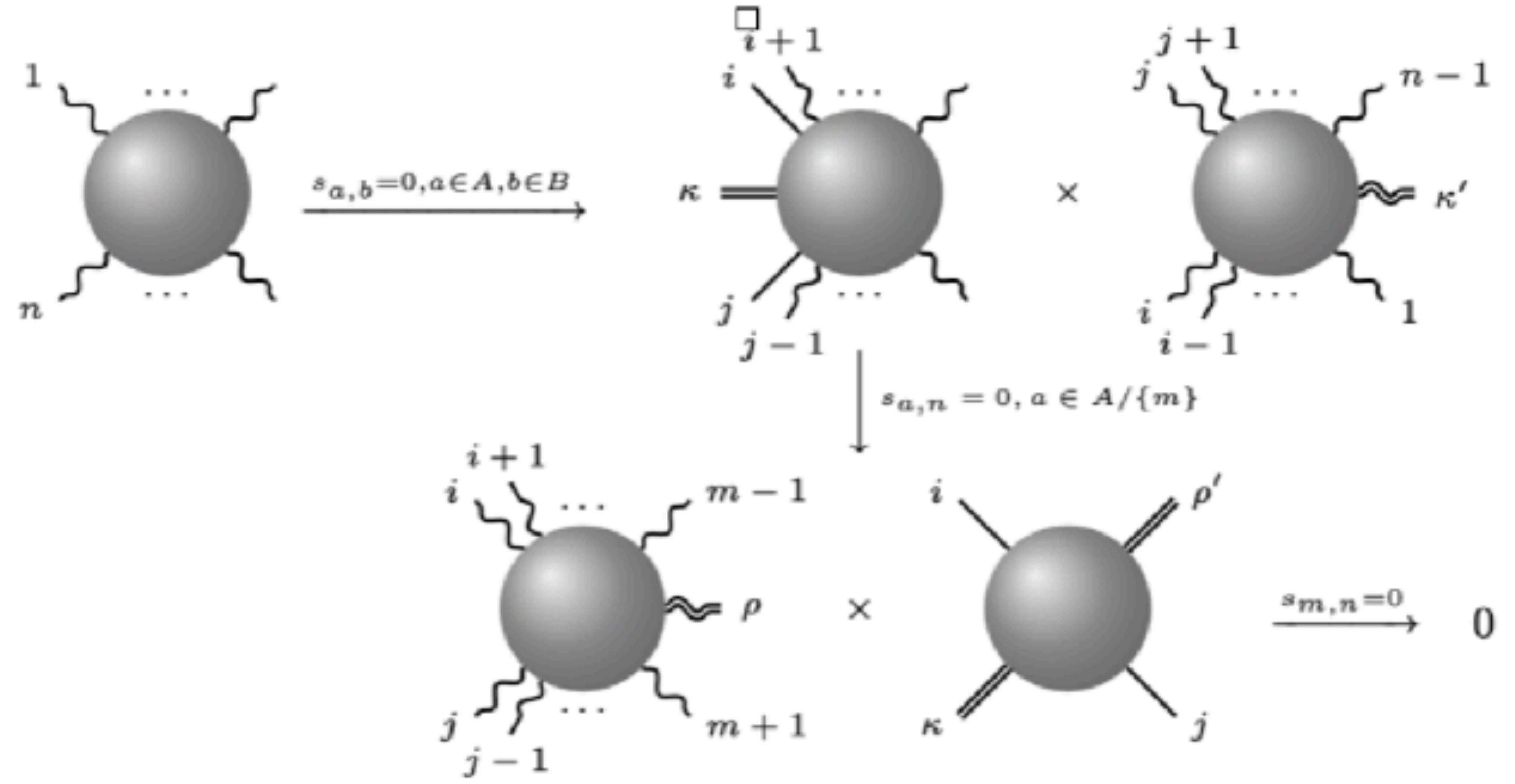

same for $\tilde{\epsilon}$: $$\mathcal{M}_n^{\text{closed}} \to \mathcal{J}(i^\phi, A, j^\phi; \kappa^\phi) \times \mathcal{J}(j, B, i; \kappa')_{\mu\nu} \epsilon_n^\mu \tilde{\epsilon}_n^\nu$$

swap a, b for $\tilde{\epsilon}$: $$\mathcal{M}_n^{\text{closed}} \to \mathcal{J}(i^g, A, j^g; \kappa^g)_\mu \epsilon_n^\mu \times \mathcal{J}(j^g, B, i^g; \kappa'^g)_\nu \tilde{\epsilon}_n^\nu .$$

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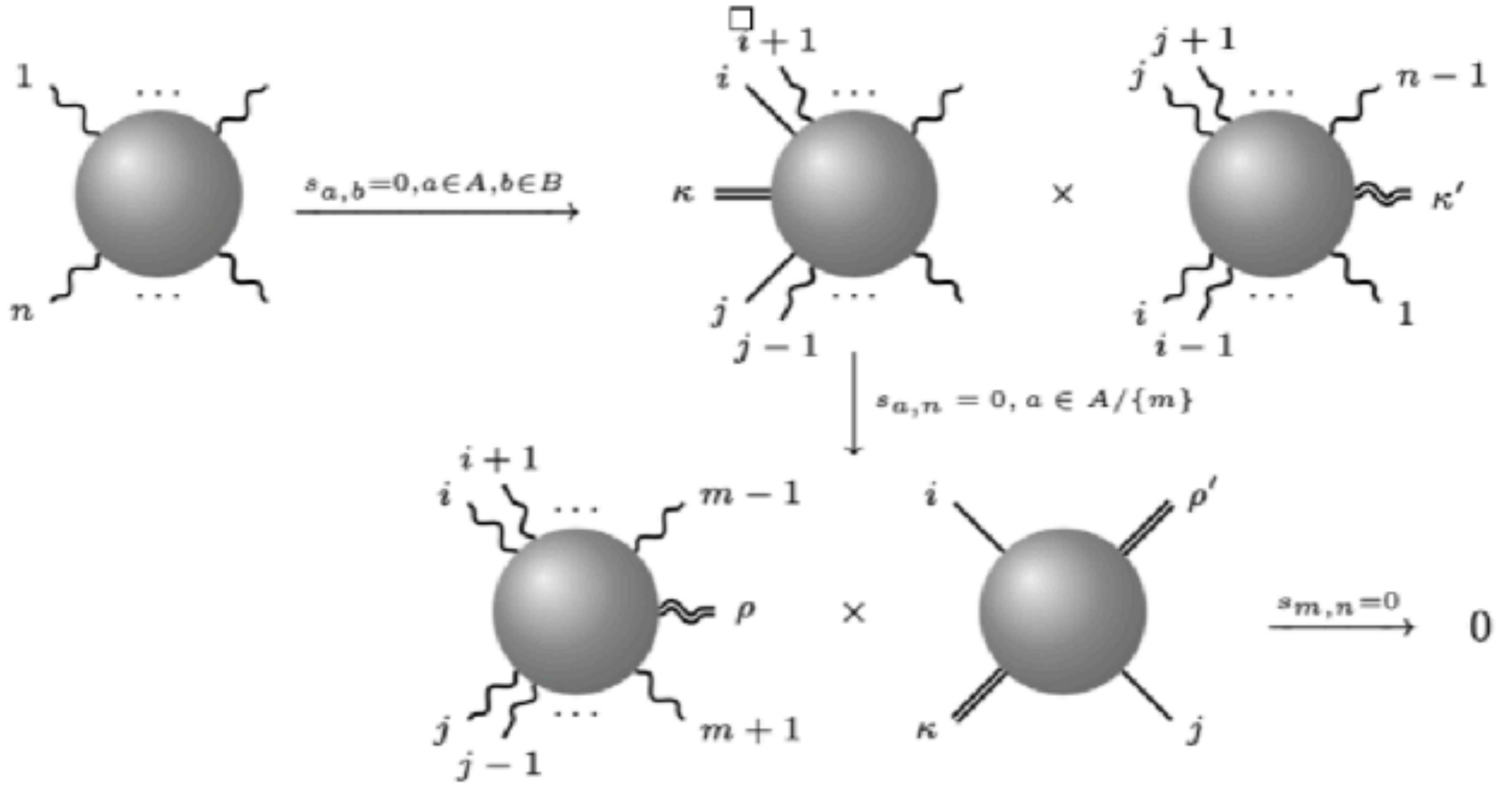

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simplest "skinny" splits => Weinberg's soft theorems for gluons/gravitons (string amp) + subleading [c.f. Cachazo, Strominger]

NLSM, DBI, sGal: enhanced Adler's zeros

general splits => multi-soft theorems

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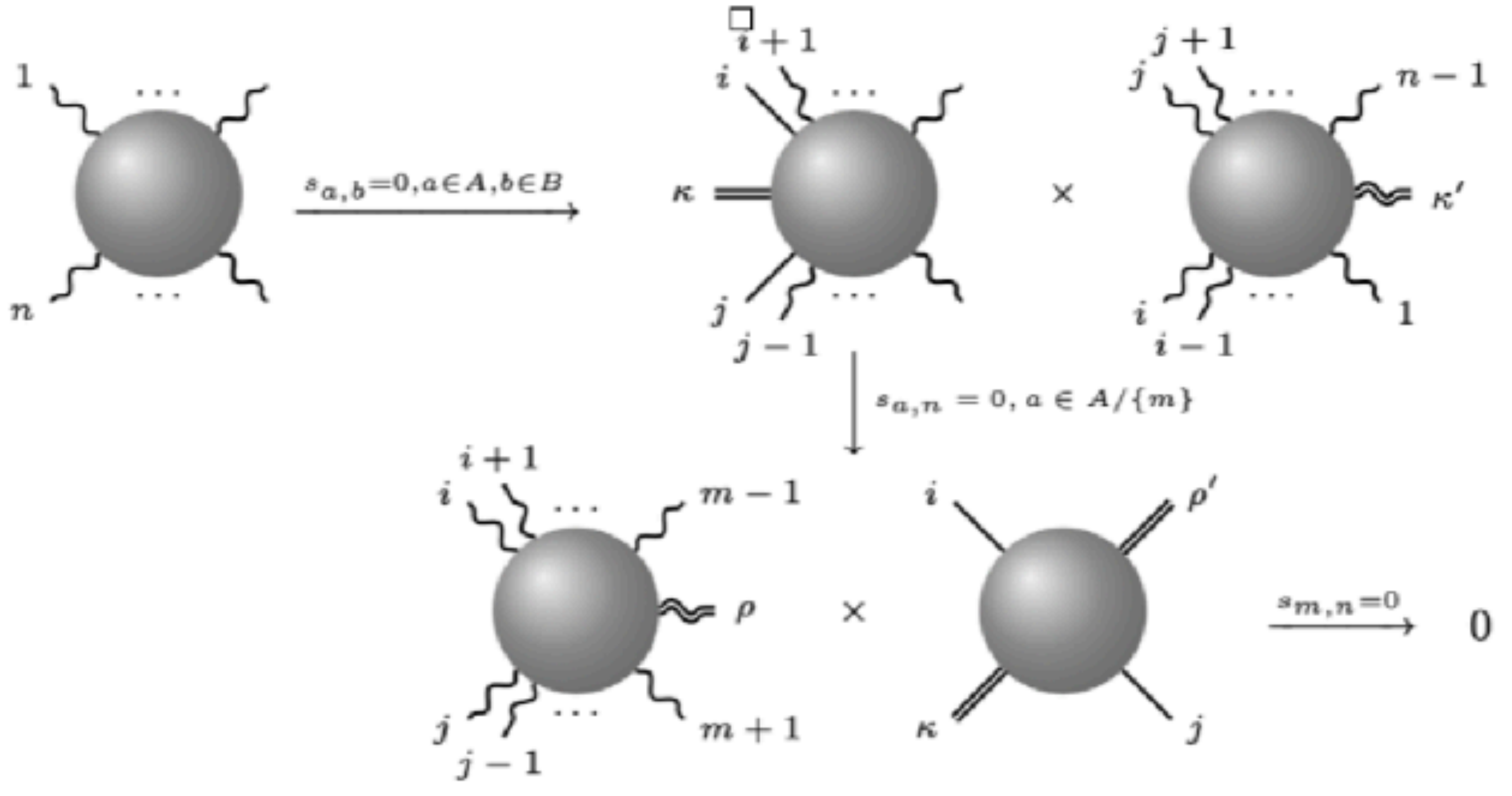

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the "skinny" case: $A = \{a\}$: $$\mathcal{J}^{\text{mixed}}(i^\phi, a, j^\phi; \kappa^\phi) = \epsilon_a \cdot p_i B(s_{i,a}, s_{j,a}+1) - \epsilon_a \cdot p_j B(s_{i,a}+1, s_{j,a})$$

soft gluon limit: (n-1)-pt current-> amplitude

sum over choices of k (i,j fixed to be adjacent to a):

$$\sum_{k \neq i,j,a} \mathcal{J}^{\text{mixed}} \times \mathcal{J}_{n-1} \to \left(\frac{\epsilon_a \cdot p_i}{p_a \cdot p_i} - \frac{\epsilon_a \cdot p_j}{p_a \cdot p_j} \right) \times \mathcal{M}_{n-1}^{\text{YM}},$$

similarly for gravity: sum over i, j ,k

$$\sum_{k,i,j \neq a} \mathcal{J}^{\text{mixed}} \times \mathcal{J}_{n-1} \to \left(\sum_{b \neq a} \frac{\epsilon_a \cdot p_b \tilde{\epsilon}_a \cdot p_b}{p_a \cdot p_b} \right) \mathcal{M}_{n-1}^{\text{GR}}$$

All-loop NLSM contained in Tr ϕ^3 [ACDFH 2401]

$$\mathcal{I}_{2n}^{\delta} = \int_{\mathbb{R}_{>0}^{2n-3}} \prod_{I=1}^{2n-3} \frac{dy_I}{y_I} \prod_{(e,e)} u_{e,e}^{\alpha'(X_{e,e}+\delta)} \times \prod_{(o,o)} u_{o,o}^{\alpha'(X_{o,o}-\delta)} \times \prod_{(o,e)} u_{o,e}^{\alpha' X_{o,e}}$$

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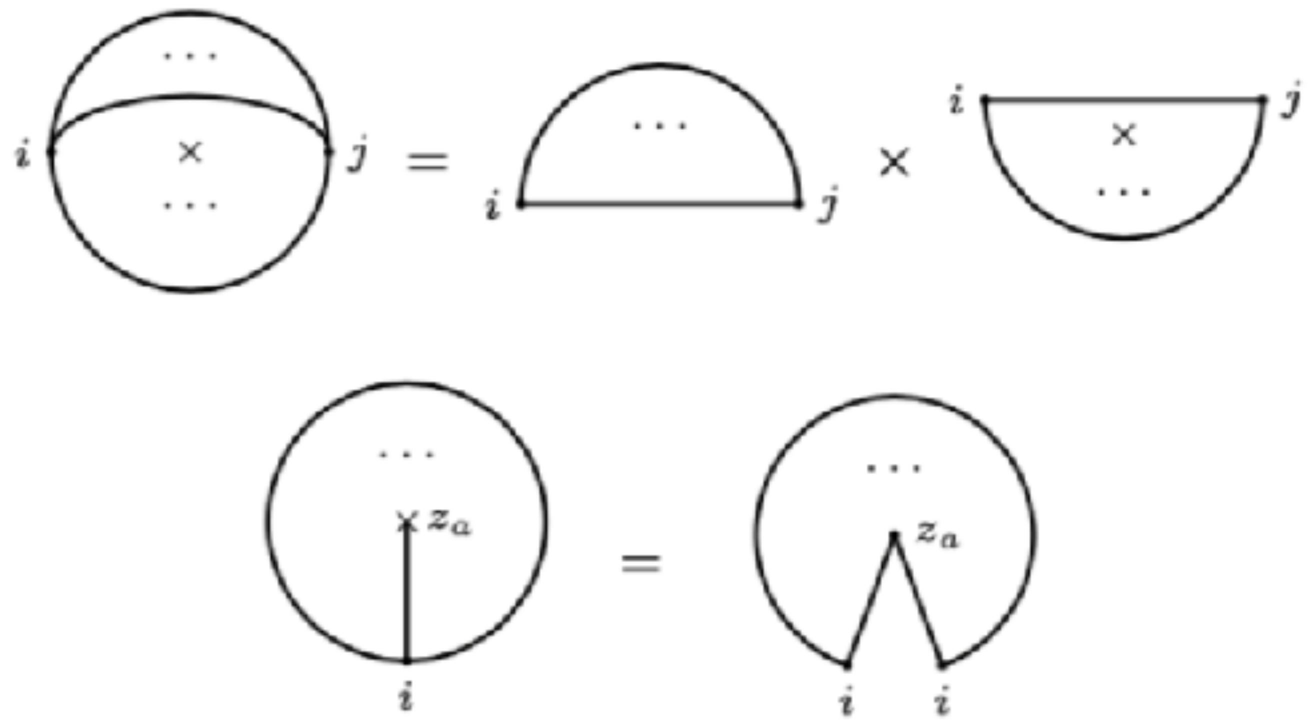
Same shift works for planar integrand of NLSM: $X_{e,e} \to X_{e,e} + \delta\,, \quad X_{o,o} \to X_{o,o} - \delta$ (inc. loop punctures)

$$\lim_{\delta\to\infty} \sum_{z_{a=1,\cdots,L}\ \mathrm{even/odd}}^{2^L} (\delta)^{n+2L-2} A^{\delta}_{n,L} = A^{\mathrm{NLSM}}_{n,L}\,.$$



All-loop NLSM contained in Tr ϕ^3 [ACDFH 2401]

$$\mathcal{I}_{2n}^{\delta} = \int_{\mathbb{R}_{>0}^{2n-3}} \prod_{I=1}^{2n-3} \frac{dy_I}{y_I} \prod_{(e,e)} u_{e,e}^{\alpha'(X_{e,e}+\delta)} \times \prod_{(o,o)} u_{o,o}^{\alpha'(X_{o,o}-\delta)} \times \prod_{(o,e)} u_{o,e}^{\alpha' X_{o,e}}$$
$$\to \mathcal{A}_{2n}^{\mathrm{Tr}\,\phi^3}(X_{e,e} \to X_{e,e}+\delta, X_{o,o} \to X_{o,o}-\delta),$$

Field-theory directly take $\delta \to \infty$:

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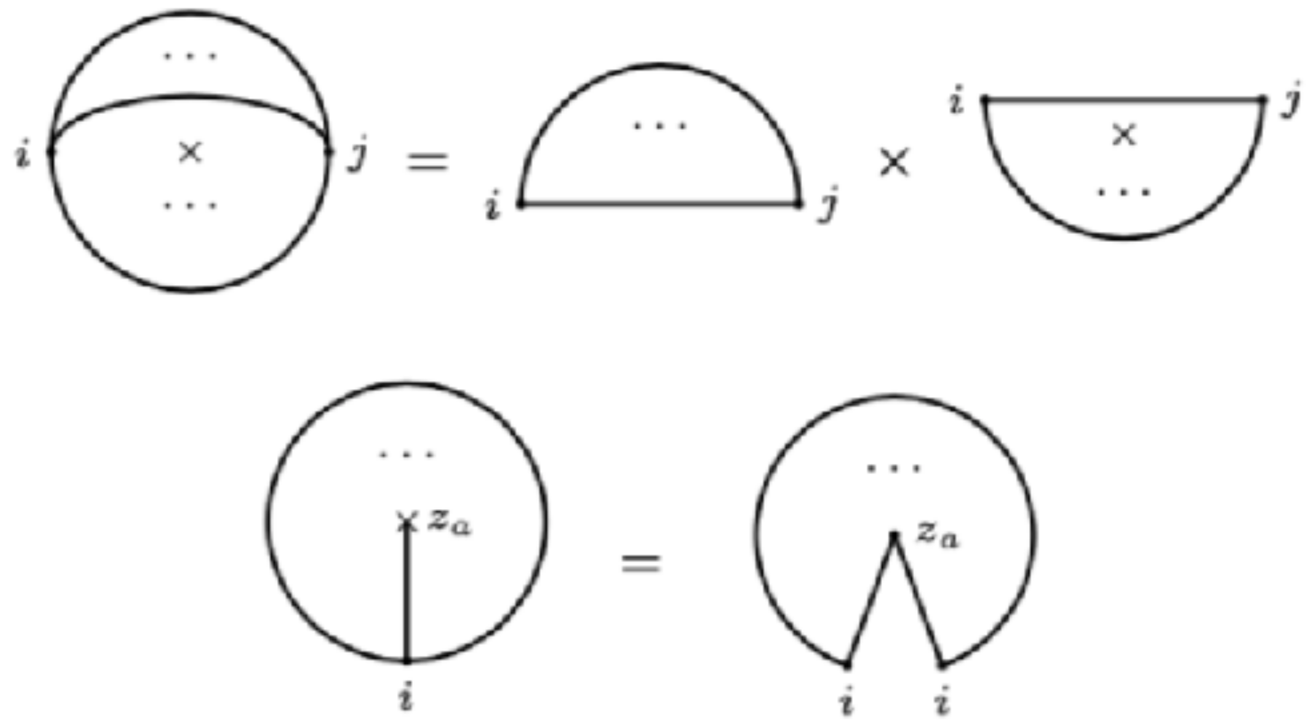

"Adler zero" manifest: soft limit -> scaleless integrals! very practical, e.g. 4-loop 4-pt NLSM integrand;
Lagrangian origin of NLSM $\subset \mathrm{Tr}\phi^3$ [Arkani-Hamed, Figueiredo, '24] => any quantities e.g. off-shell, cosmological, …

Scalar-scaffolded gluons [ACDFH, 2312]

$\alpha'\delta = 1$ gives 2n-scalar stringy YMS amp = 2n-scalar in bosonic string!

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$$\mathcal{A}_n^{\text{tree}}(1,2,\ldots,2n) = \int \frac{\mathrm{d}^{2n} z_i}{\mathrm{SL}(2,\mathbb{R})} \left(\prod_{i<j} z_{i,j}^{2\alpha' p_i \cdot p_j} \right) \exp\left(\sum_{i \neq j} 2\frac{\epsilon_i \cdot \epsilon_j}{z_{i,j}^2} - \frac{\sqrt{\alpha'}\epsilon_i \cdot p_j}{z_{i,j}} \right) \Bigg|_{\text{multi-linear in } \epsilon_i},$$

$$p_i \cdot \epsilon_j = 0, \quad \forall\ (i,j) \in (1, \ldots, 2n),$$

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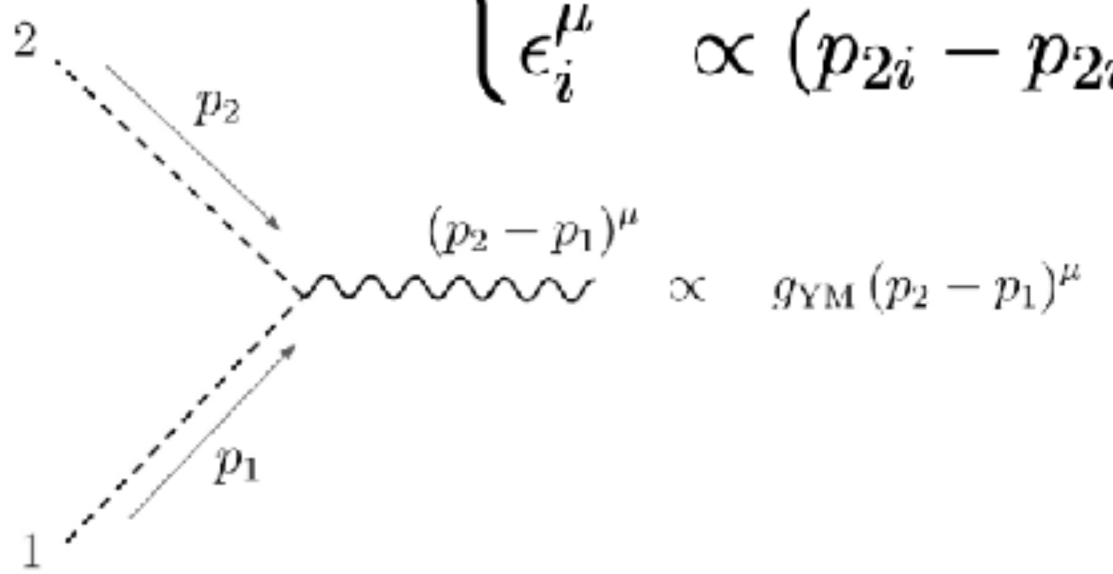

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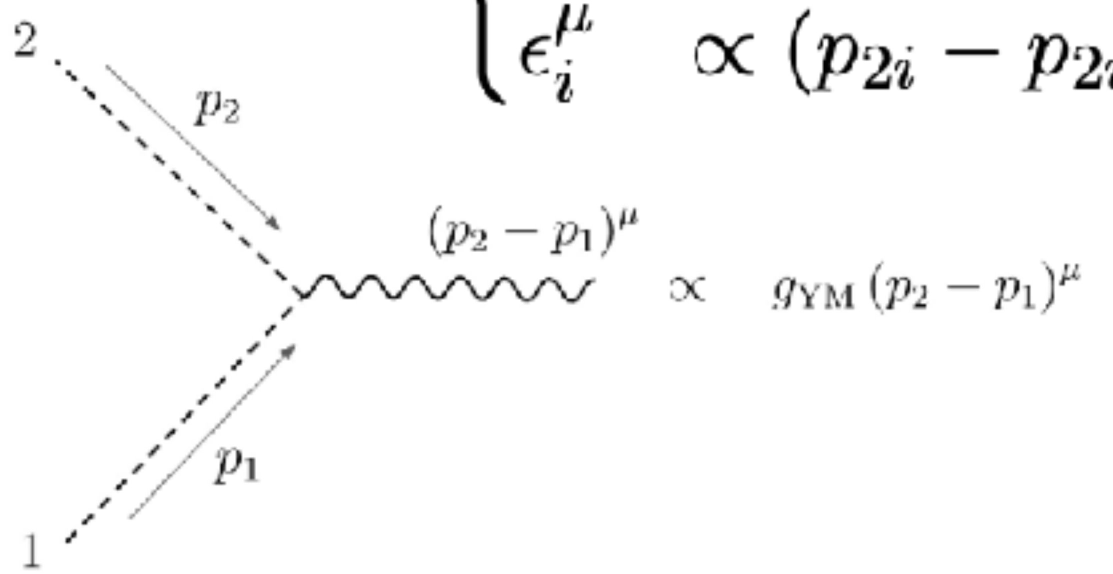

$$\mathcal{A}_{2n}(1,2,\ldots,2n) \xrightarrow{\text{special kinematics}} \int \frac{\mathrm{d}^{2n} z_i}{\mathrm{SL}(2,\mathbb{R})} \prod_{i<j} z_{i,j}^{2\alpha' p_i \cdot p_j} \frac{1}{z_{1,2}^2 z_{3,4}^2 z_{5,6}^2 \cdots z_{2n-1,2n}^2}$$

$$= \underbrace{\int \frac{\mathrm{d}^{2n} z_i}{\mathrm{SL}(2,\mathbb{R})} \frac{1}{z_{1,2} z_{2,3} z_{3,4} \cdots z_{2n,1}} \prod_{i<j} z_{i,j}^{2\alpha' p_i \cdot p_j}}_{\text{Stringy Tr}\,\phi^3} \frac{z_{2,3} z_{4,5} z_{6,7} \cdots z_{2n,1}}{z_{1,2} z_{3,4} z_{5,6} \cdots z_{2n-1,2n}} = \left(\prod u_{e,e} / \prod u_{o,o}\right) \quad (\alpha'\delta = 1)$$

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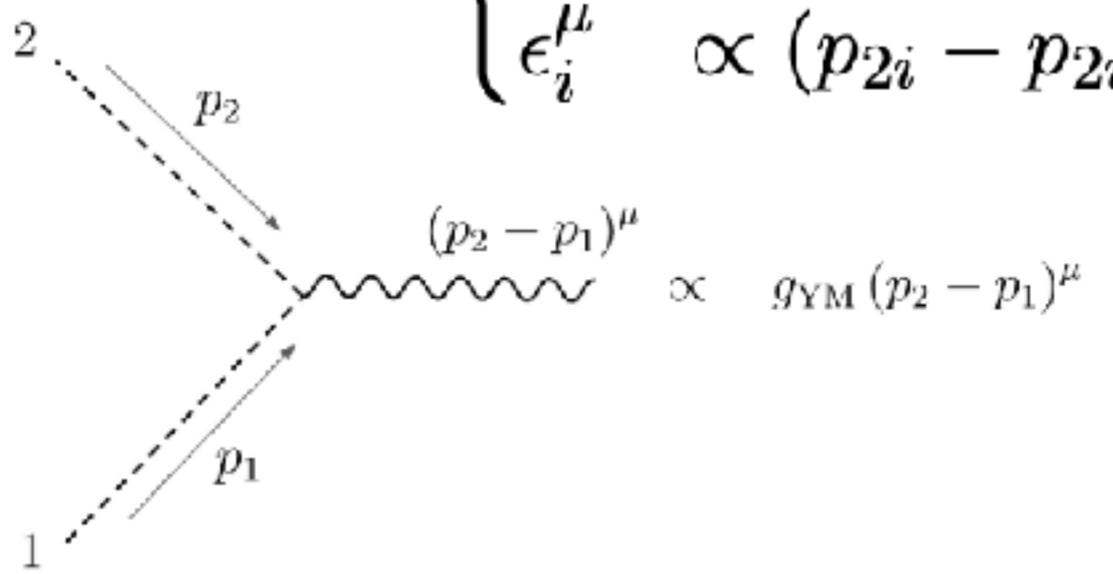

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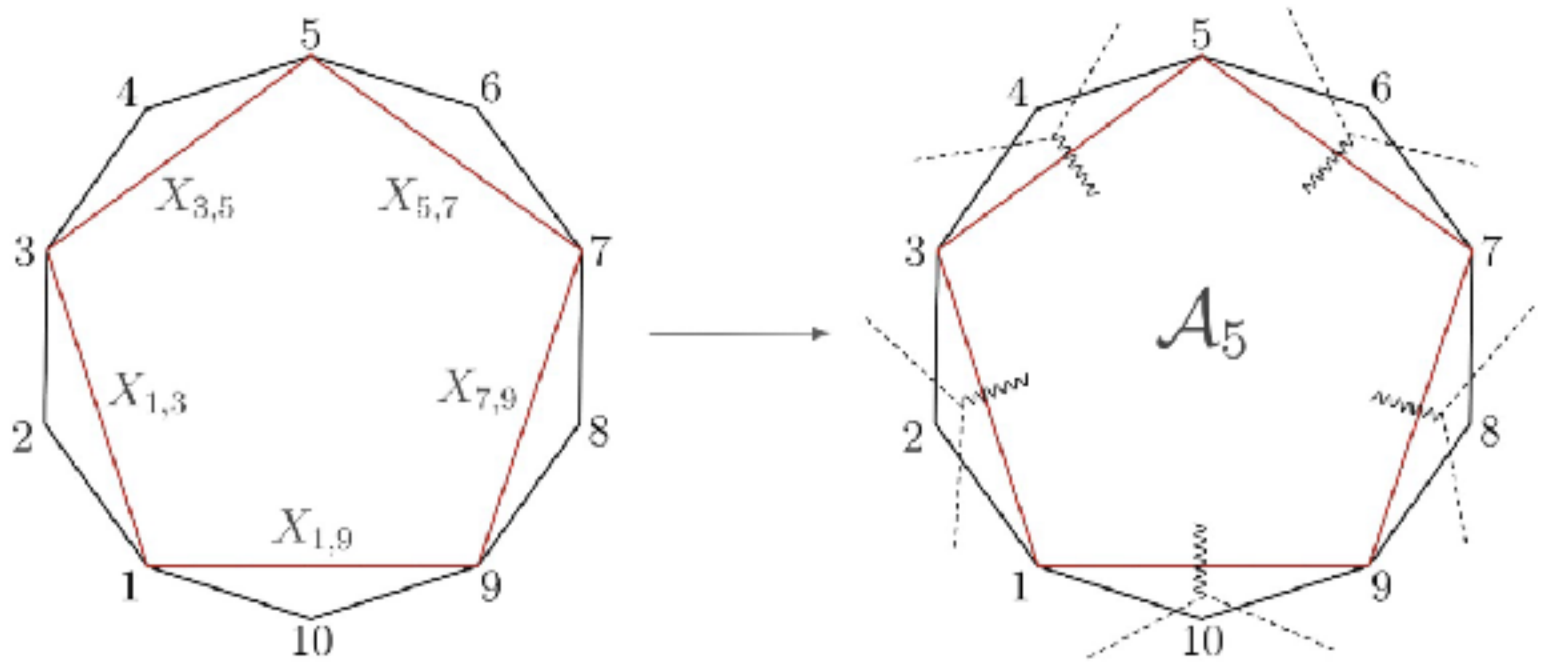
taking n ``scaffolding residues" $s_{1,2} = s_{3,4} = \cdots = 0$ => n-gluon bosonic string amps (in 2n-scalar language)

$$\mathcal{I}_{2n}^{\delta} = \int_{\mathbb{R}_{>0}^{2n-3}} \underbrace{\prod_{i=1}^{n} \frac{dy_{2i-1,2i+1}}{y_{2i-1,2i+1}^2} \prod_{I \in \mathcal{T}'} \frac{dy_I}{y_I^2} \prod_{(a,b)} u_{a,b}^{\alpha' X_{a,b}}}_{\Omega_{2n}},$$

$$\mathcal{I}^{\delta}_{2n} = \int_{\mathbb{R}^{2n-3}_{>0}} \underbrace{\prod_{i=1}^{n} \frac{dy_{2i-1,2i+1}}{y^2_{2i-1,2i+1}} \prod_{I\in\mathcal{T}'} \frac{dy_I}{y_I^2} \prod_{(a,b)} u_{a,b}^{\alpha' X_{a,b}}}_{\Omega_{2n}},$$

$$\mathcal{I}^{\text{gluon}}_{n} = \int_{\mathbb{R}^{n-3}_{>0}} \text{Res}_{y_{1,3}=0}\left(\text{Res}_{y_{3,5}=0}\left(\ldots\left(\text{Res}_{y_{1,2n-1}=0}\left(\Omega_{2n}\right)\right)\ldots\right)\right)\Big|_{X_{2i-1,2i+1}=0}$$

$$X_{1,3} = X_{3,5} = \cdots = X_{1,2n-1} = 0$$



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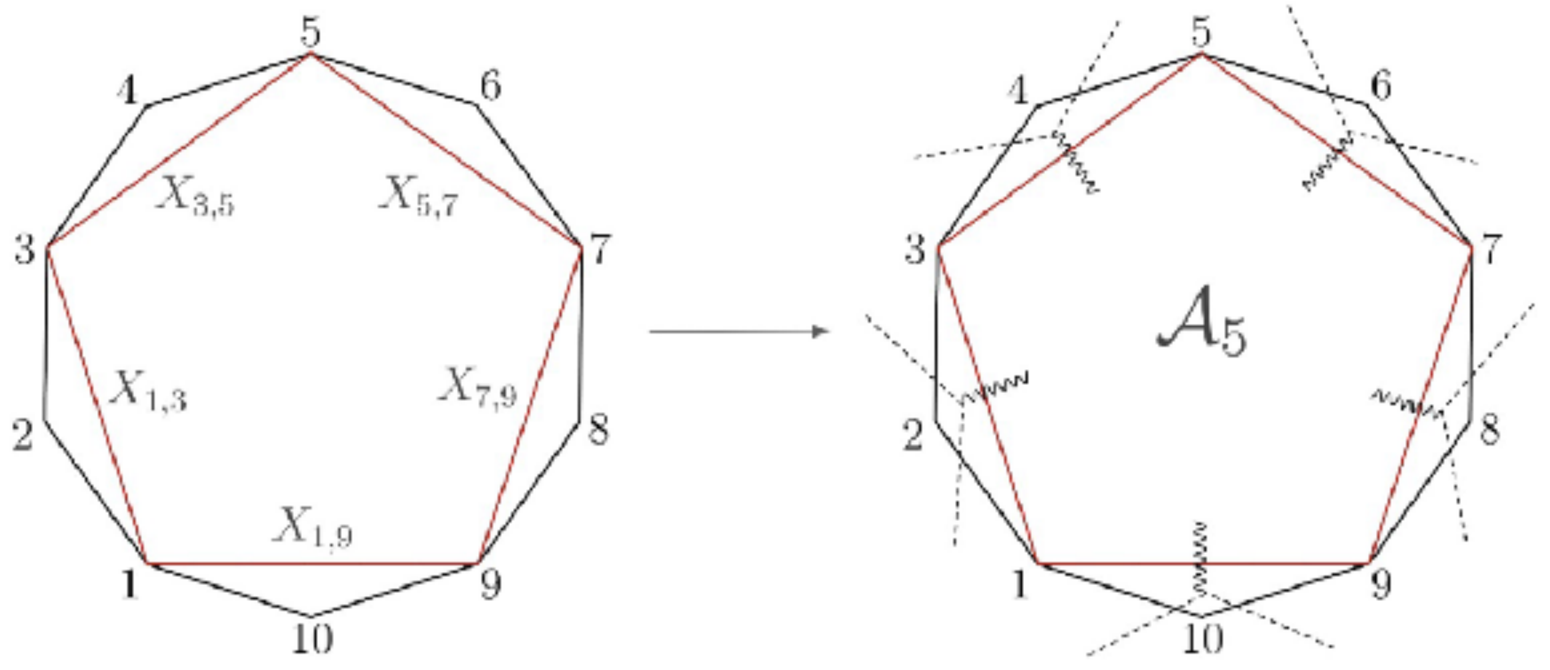

$$A_3^{\text{gluon}} = \alpha'^2 \left(c_{1,3}c_{1,5} + c_{1,3}c_{2,5} + c_{1,3}c_{3,5} + c_{1,4}c_{3,5} + c_{1,5}c_{3,5} + c_{1,5}c_{3,6}\right) \quad + \\ - \alpha'^3 \left(X_{1,4}X_{2,5}X_{3,6}\right)$$

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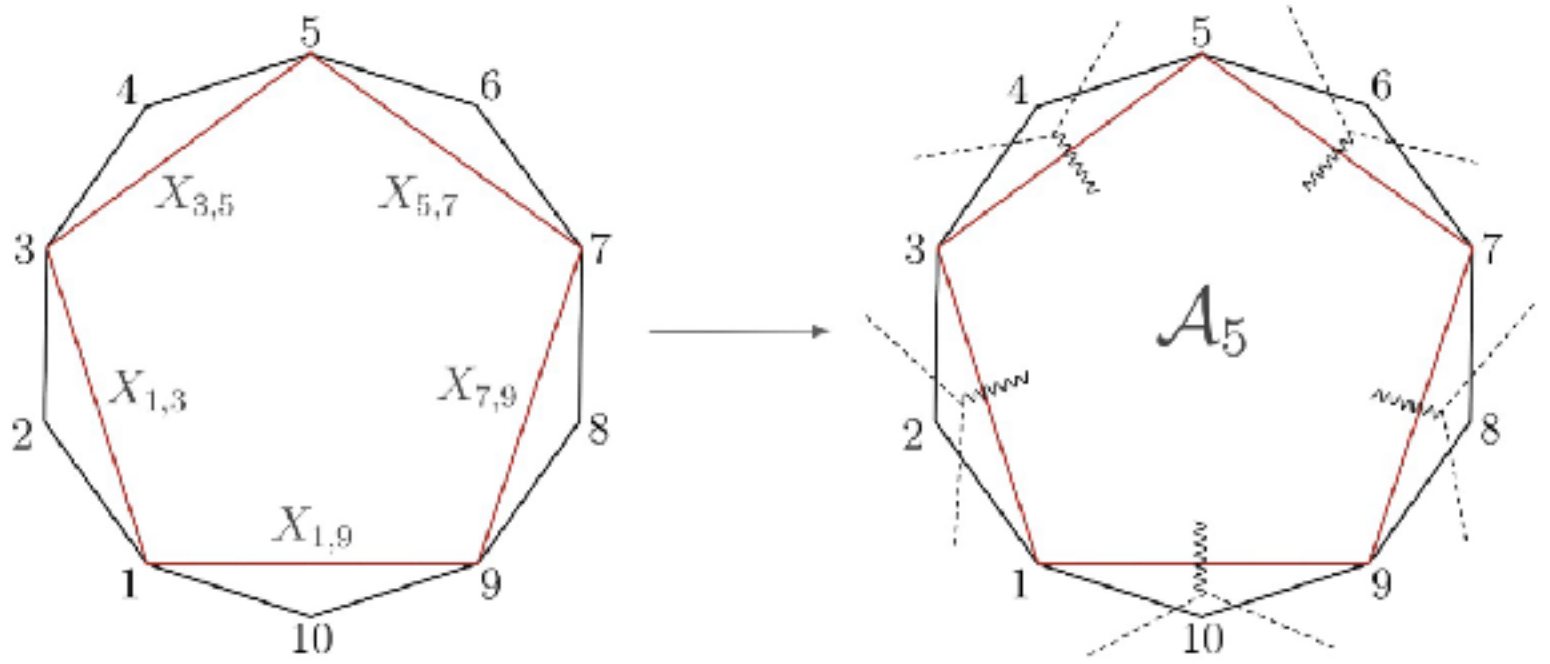

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surfaceology -> gauge invariance + gluon factorization (in X variables)

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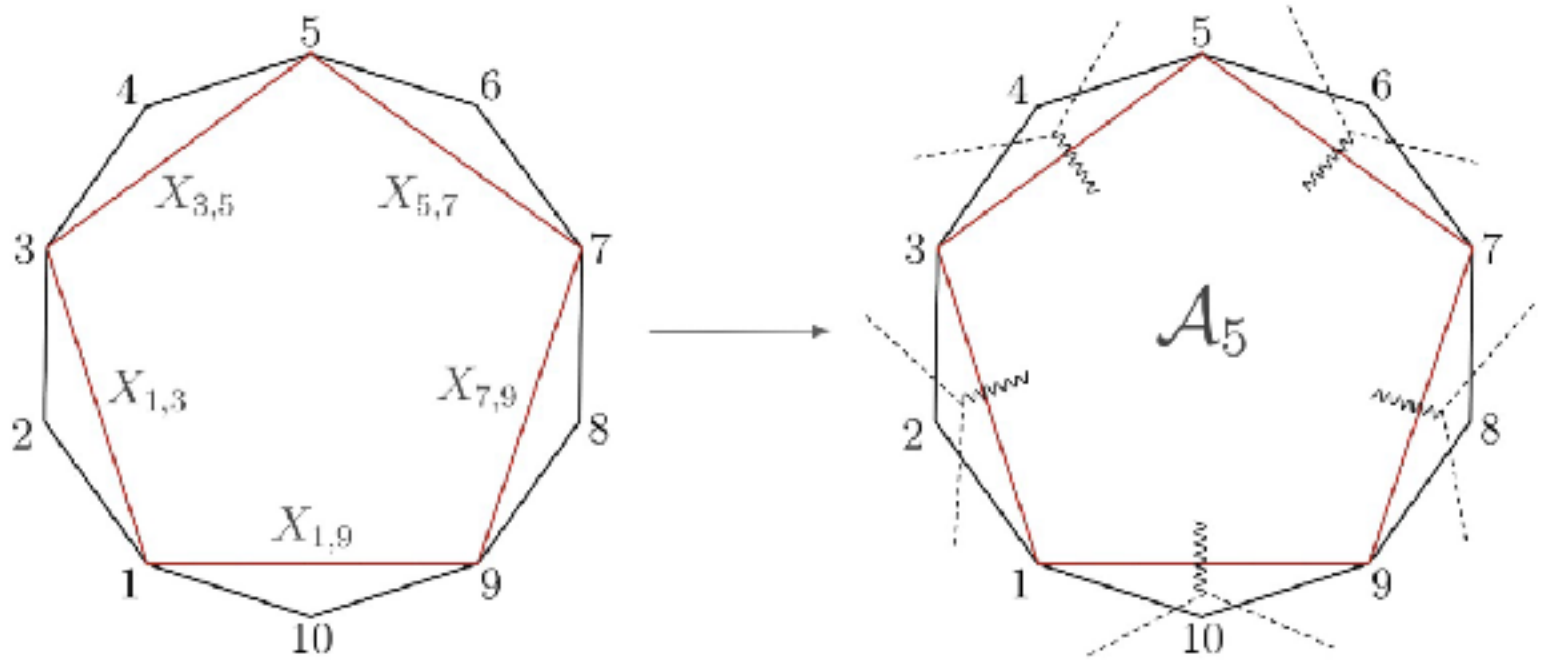

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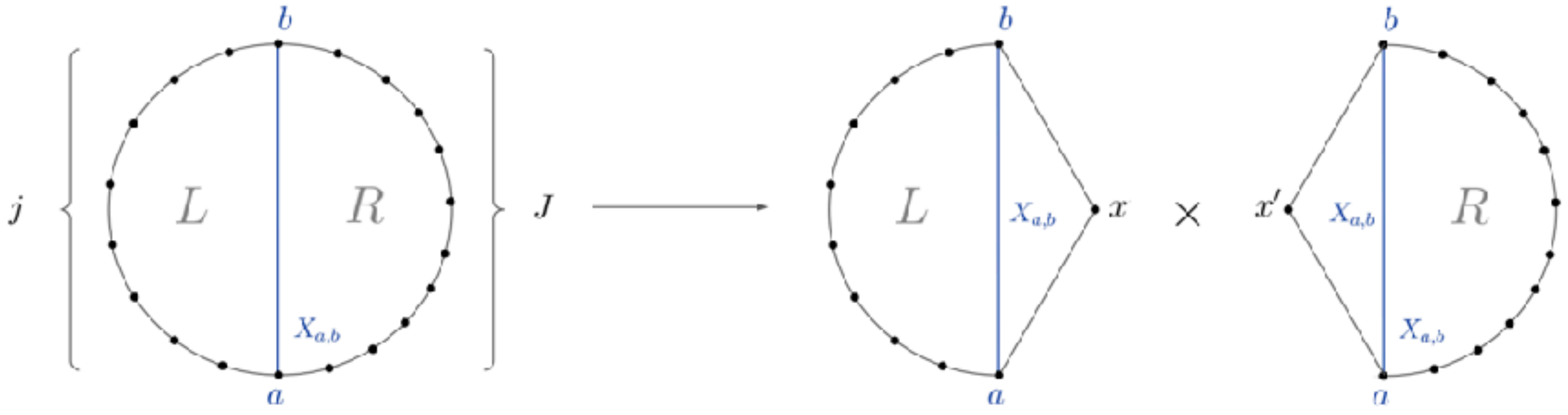

$$\text{Res}_{X_{a,b}=0}\,\mathcal{A}_n = -\sum_{j,J}\left(X_{j,J} - X_{j,b} - X_{a,J}\right) \underbrace{\prod_{X_L\in\mathcal{L}_j} \tilde{u}_{X_L}}_{Q_j^L} \cdot \underbrace{\prod_{X_R\in\mathcal{R}_J} \tilde{u}_{X_R}}_{Q_J^R}$$

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All-loop YM in stringy Tr ϕ^3 [ACDFH]

Surfaceology => generalize tree (disk) to loops (higher-genus surfaces):

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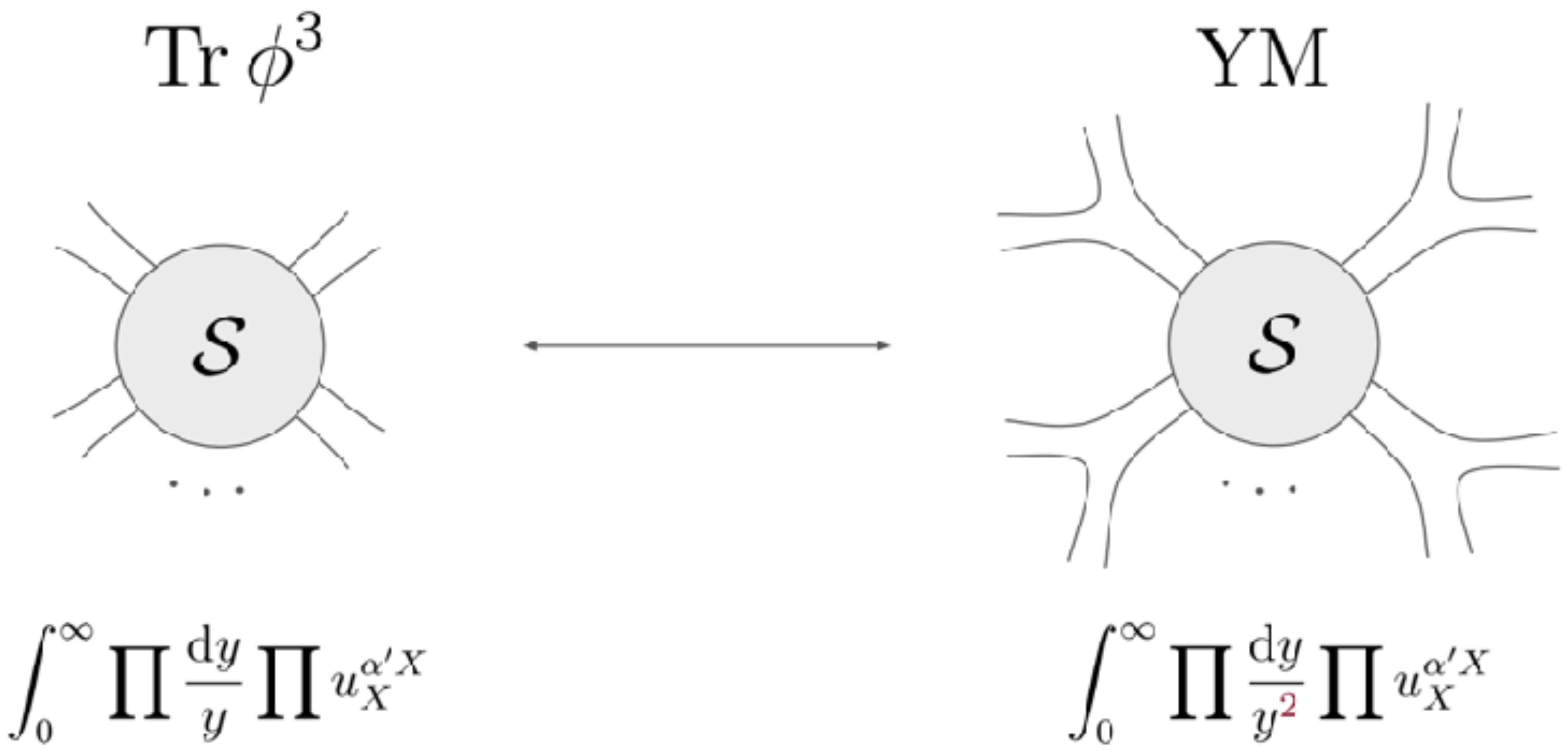

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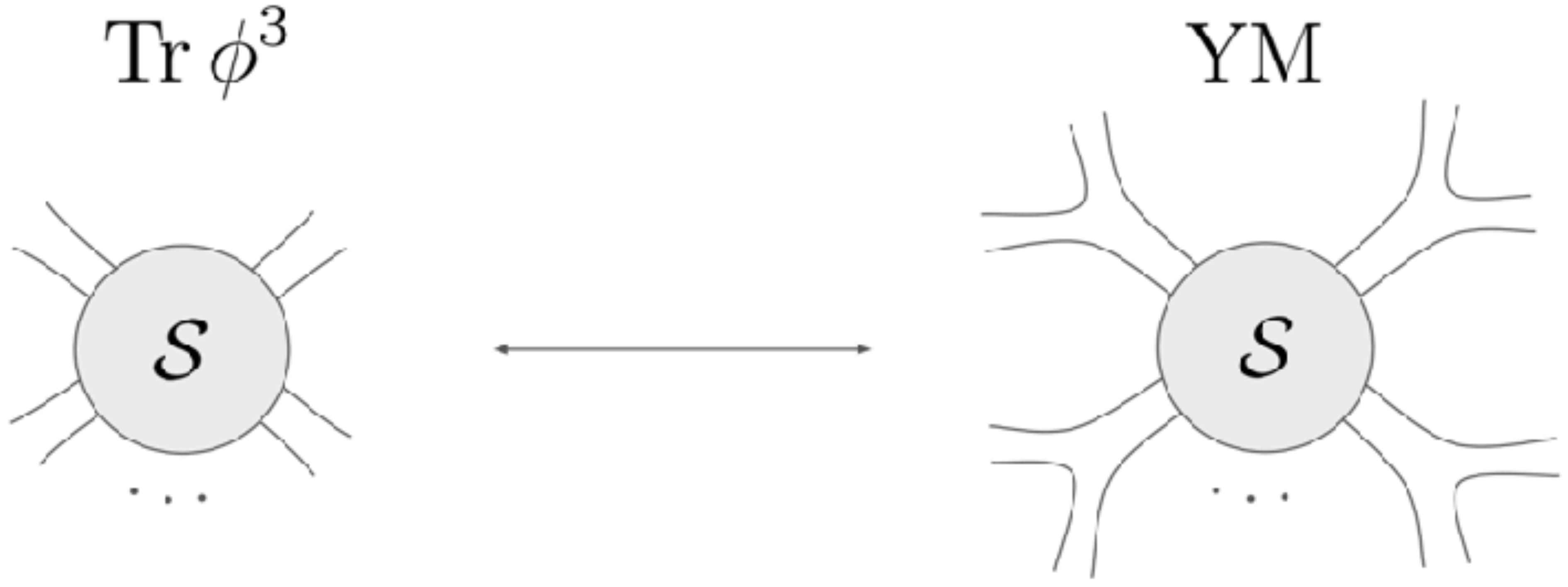

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e.g. one-loop w. self-intersecting curves & closed curve Δ

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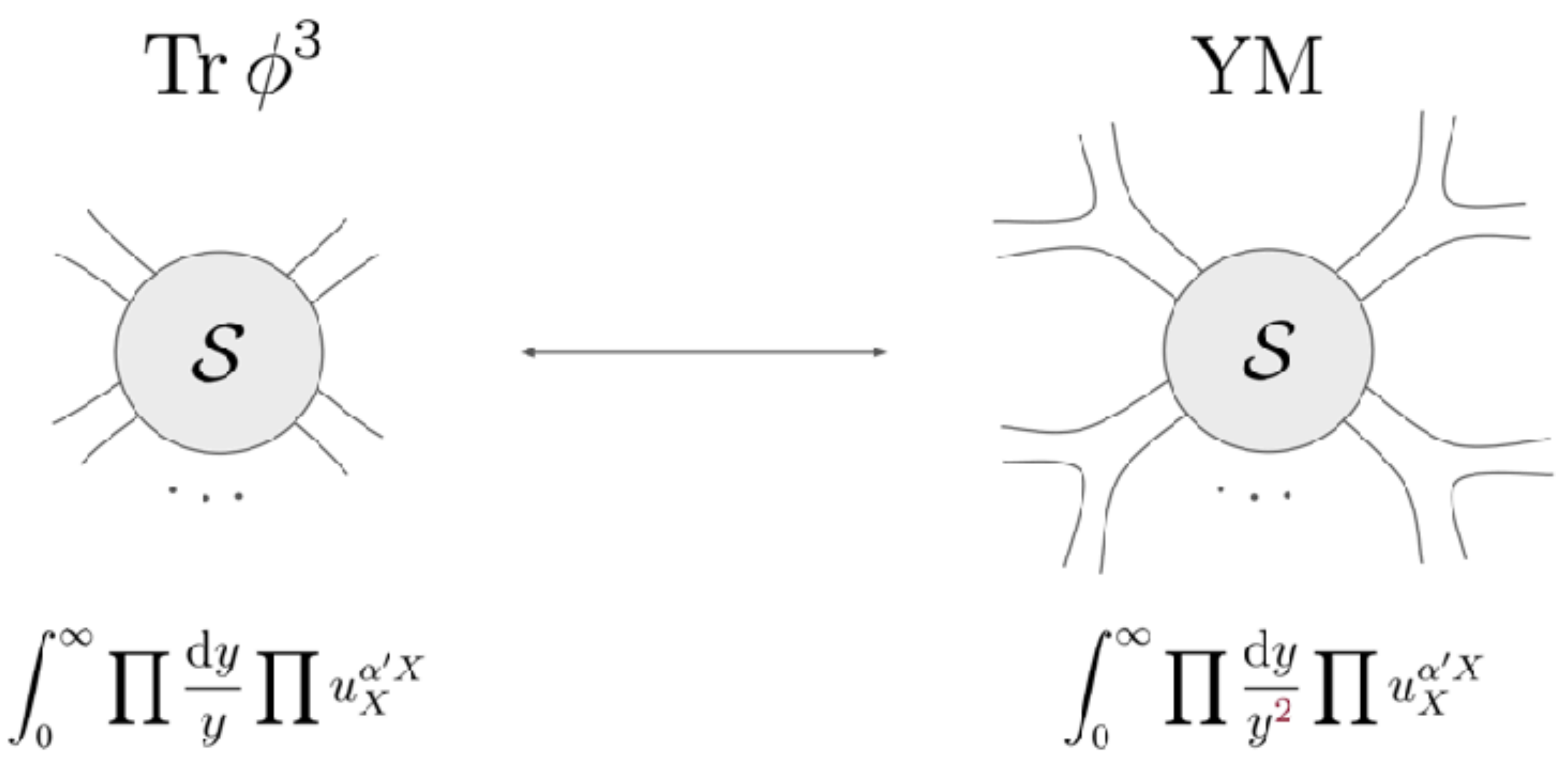

- extend the surface notion of gauge invariance + factorization/cuts
- proof for all-loop leading singularities (max. residue) with $\Delta = 1 - D$:

residue of $\int \prod \frac{dy}{y^2} \prod u^X$ = gluing of 3pt YM+ F^3 (in X space)

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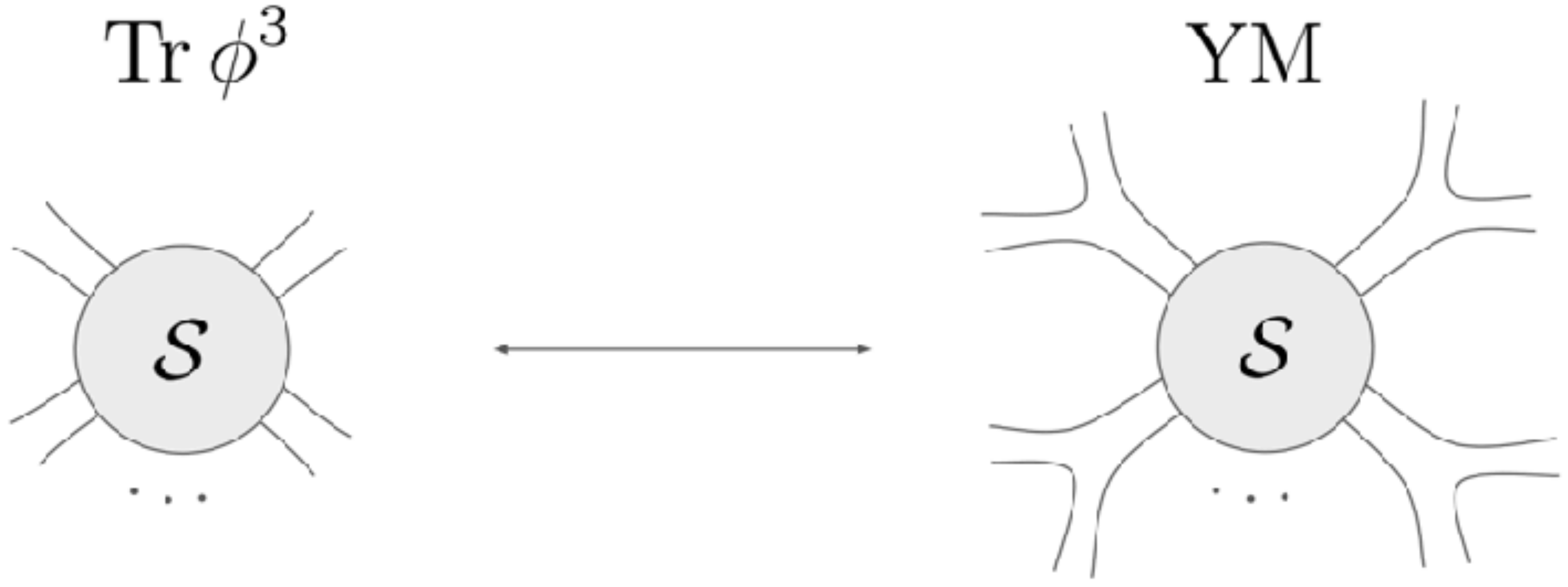

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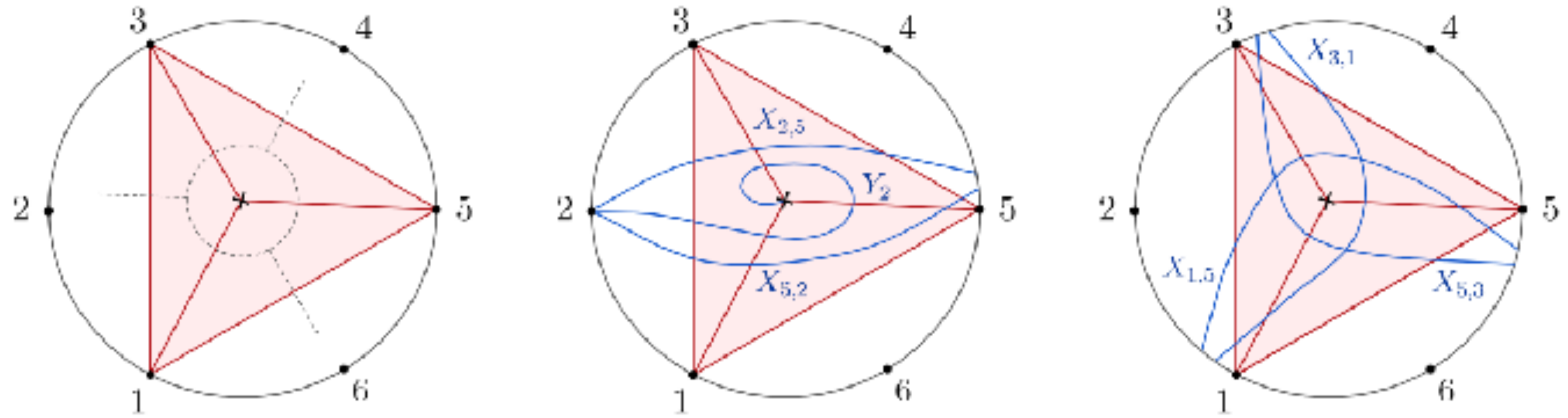

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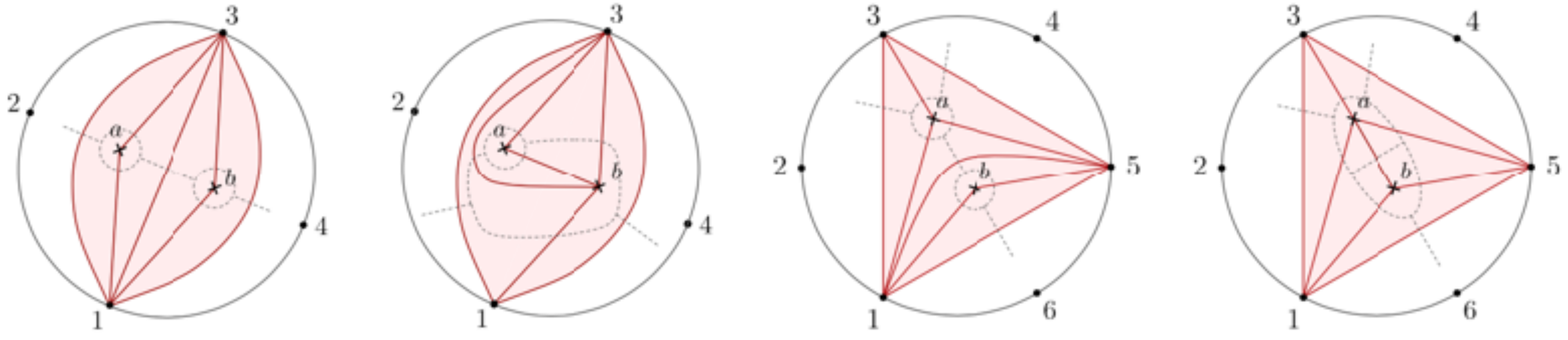

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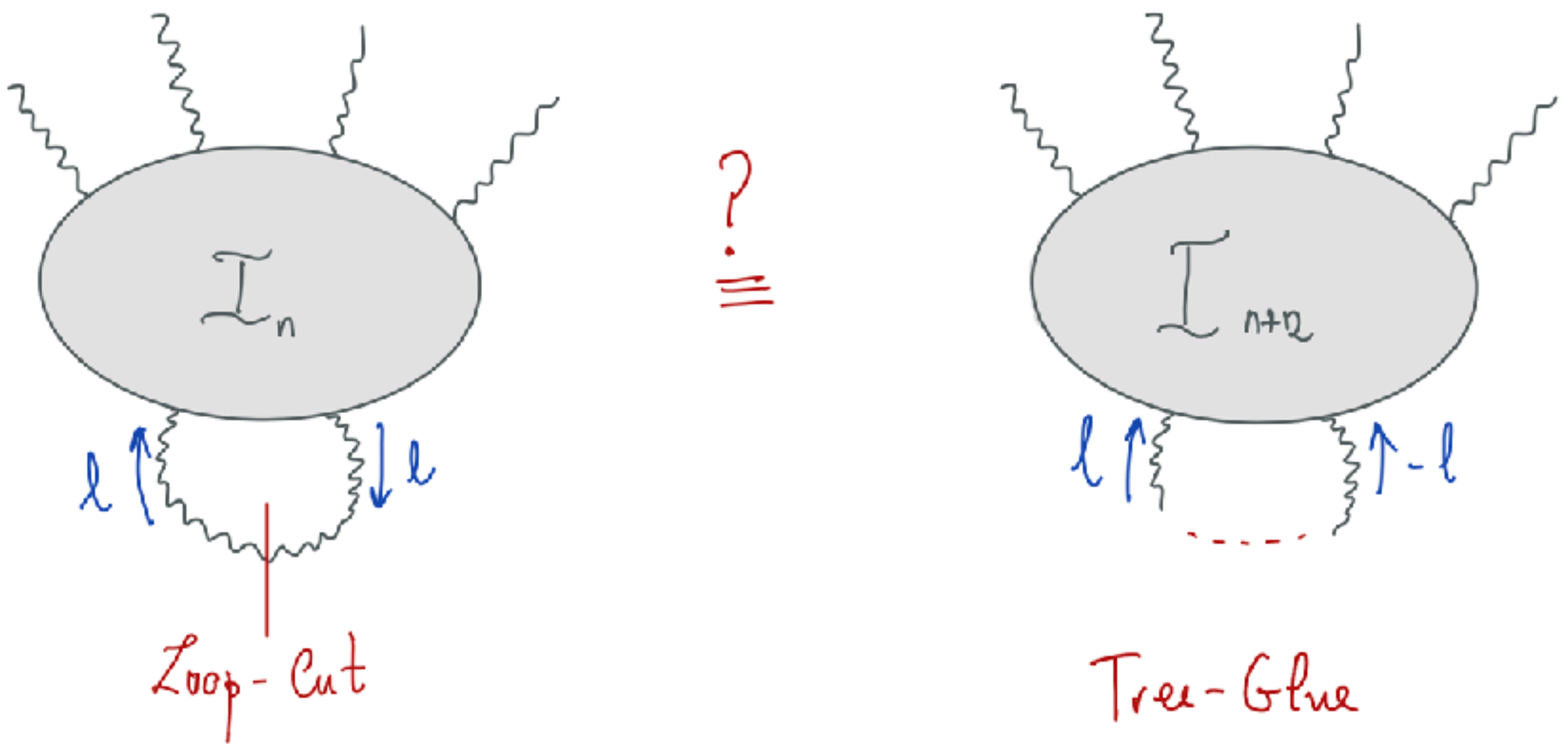
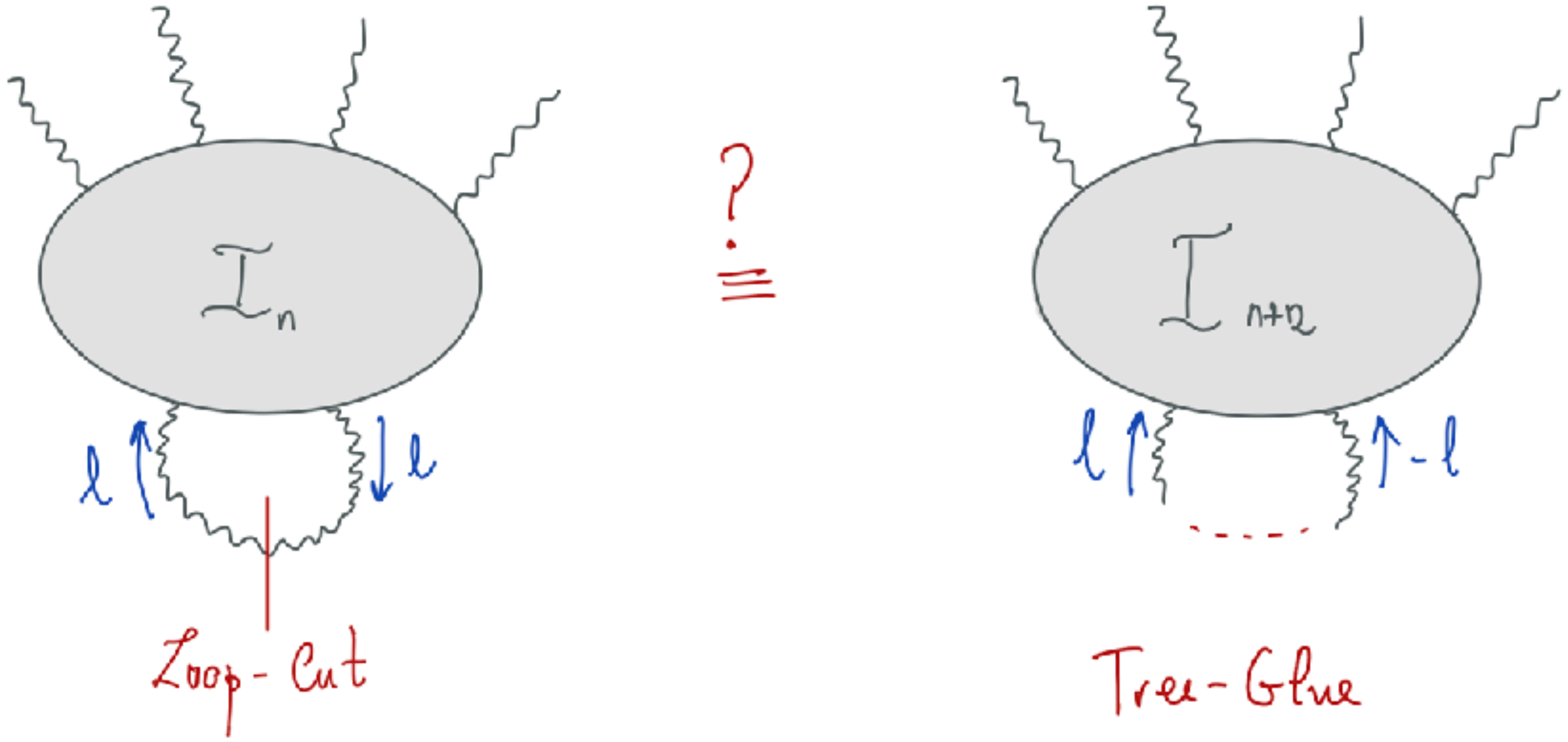


naively divergent => “the integrand”
(e.g. Adler zero, gauge inv.) ill defined!

no issues for scalars, but for gluons 1/0 !
(cancels in super-Yang-Mills)

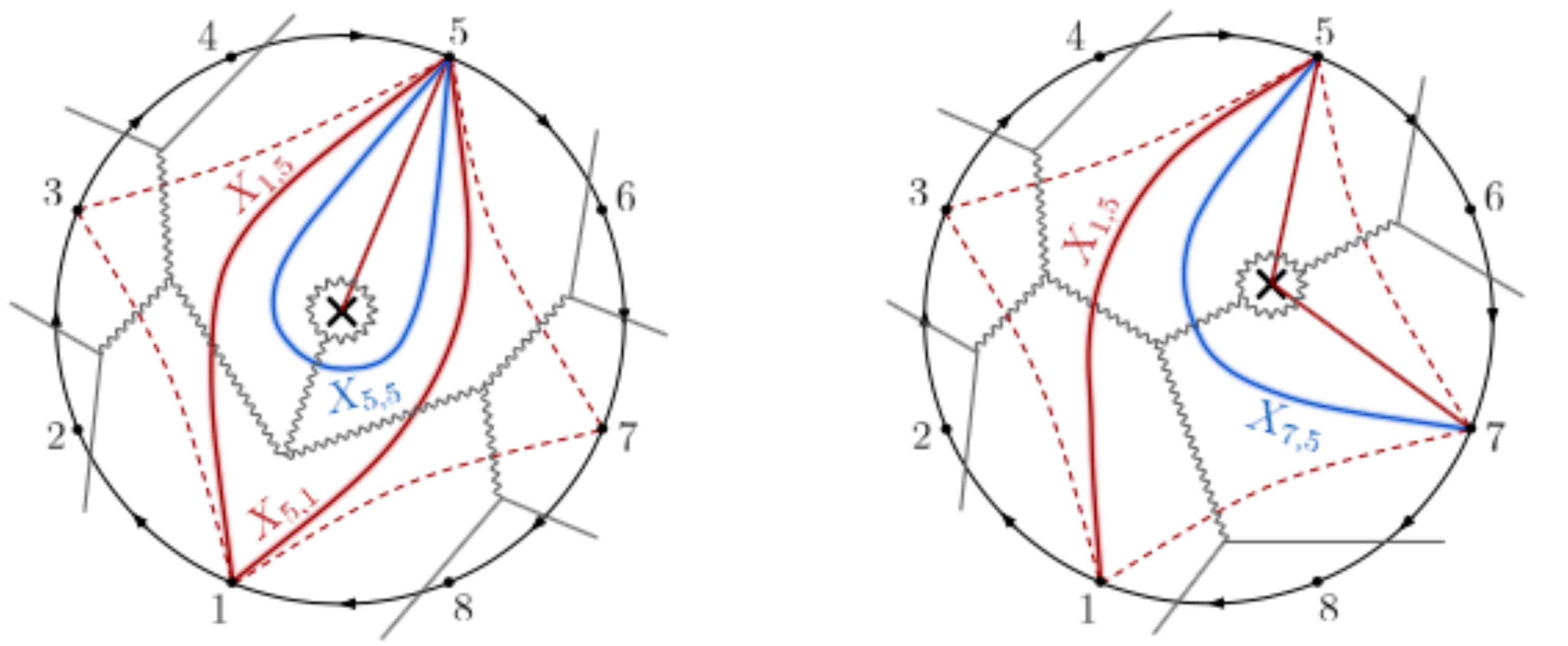
How to determine “the integrand” of YM?

Similar to tree factorizations, all we need are cuts: e.g. 1-loop single-cut = forward limit (gluing tree)



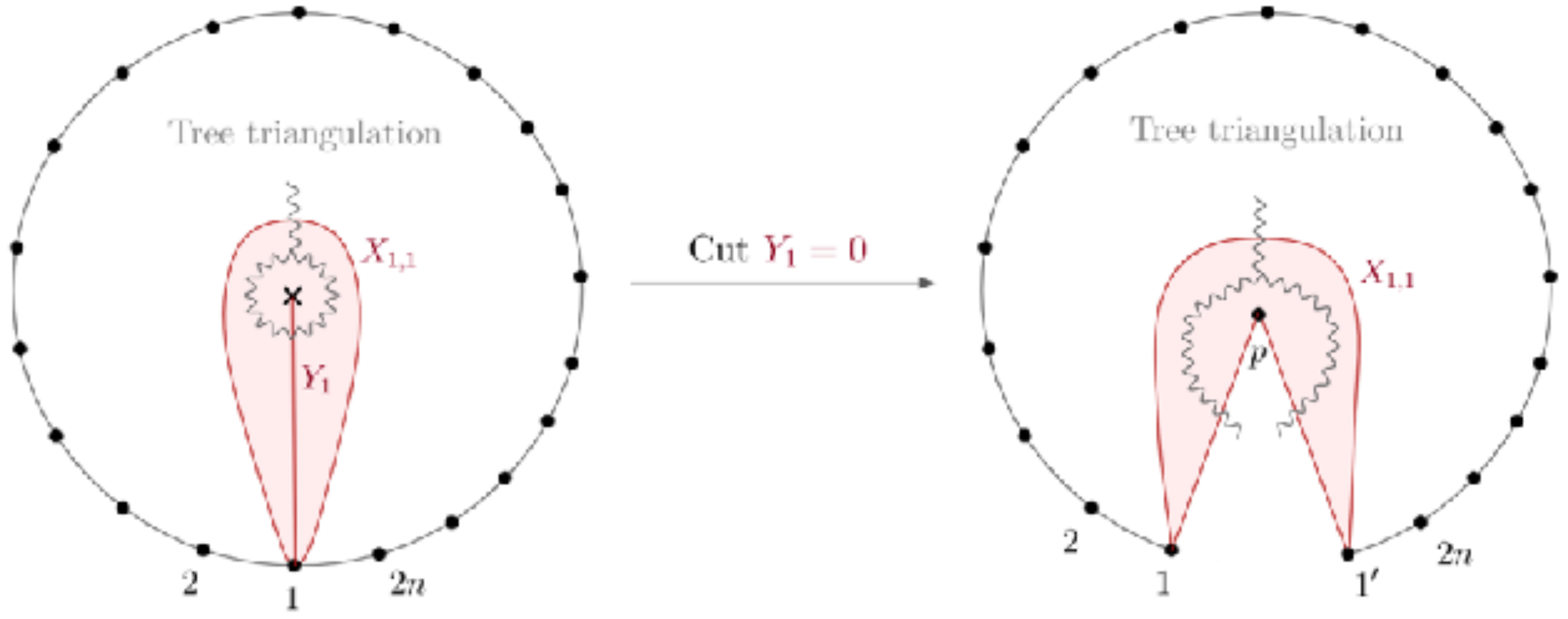
naively divergent => “the integrand”
(e.g. Adler zero, gauge inv.) ill defined!

no issues for scalars, but for gluons 1/0 !
(cancels in super-Yang-Mills)



surface provides a natural way out: curves without standard momentum (e.g. tadpoles) => “the integrand”

“doubling” variables: similar to Lorentzian -> complex in 4d tree kinematics



Loop recursions in YM [ACDFH,2408, in progress]

Surface makes it clear "the integrand" to all loops exist (beyond planar limit)
=> the notion of surface gauge invariance+ cuts

Gauge-Invariance and Linearity (in gluon 1):

$$\mathcal{I}_n^{\mathcal{S}} = \sum_{j\neq 2}\left[(X_{2,j}-X_{1,j})\frac{\partial \mathcal{I}_n^{\mathcal{S}}}{\partial X_{2,j}} + (X_{j,2}-X_{j,1})\frac{\partial \mathcal{I}_n^{\mathcal{S}}}{\partial X_{j,2}}\right] + X_{2n,1}\times\left[\frac{\partial \mathcal{I}_n^{\mathcal{S}}}{\partial X_{2n,2}} + \frac{\partial \mathcal{I}_n^{\mathcal{S}}}{\partial X_{2n,1}}\bigg|_{2\to 1}\right]$$

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Loop integrands reconstructed from "residues" e.g. single-cut
=> forward-limit recursions for "the integrand" @ 1-loop & higher

$$\operatorname*{Res}_{X_{a,p}=0}(\mathcal{I}_n^{\mathcal{S}}) = \sum_{j,k}\left[(X_{j,p}+X_{k,p}-X_{k,j})\frac{\partial^2\mathcal{A}_{n+2}^{\mathcal{S}}}{\partial X_{j,x}\partial X_{k,x'}}\right]\Bigg|_{a'\to a} - d\frac{\partial \mathcal{A}_{n+2}^{\mathcal{S}}}{\partial X_{x',x}}; \quad j,k\in(a,a+1,\cdots,a-1,a'),$$

$$\mathcal{I}_n^{\mathcal{S}} = \sum_i \frac{\mathrm{Res}_{X_{i,p}=0}\left(\mathcal{I}_n^{\mathcal{S}}(X_{j,p}\to X_{j,p}-X_{i,p})\right)}{X_{i,p}}\,.$$

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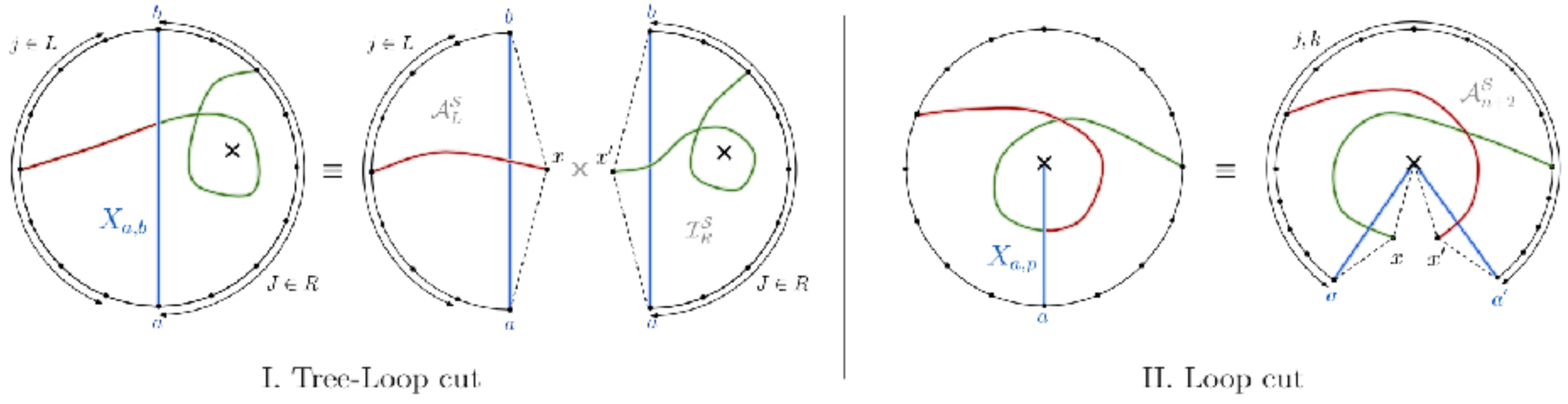

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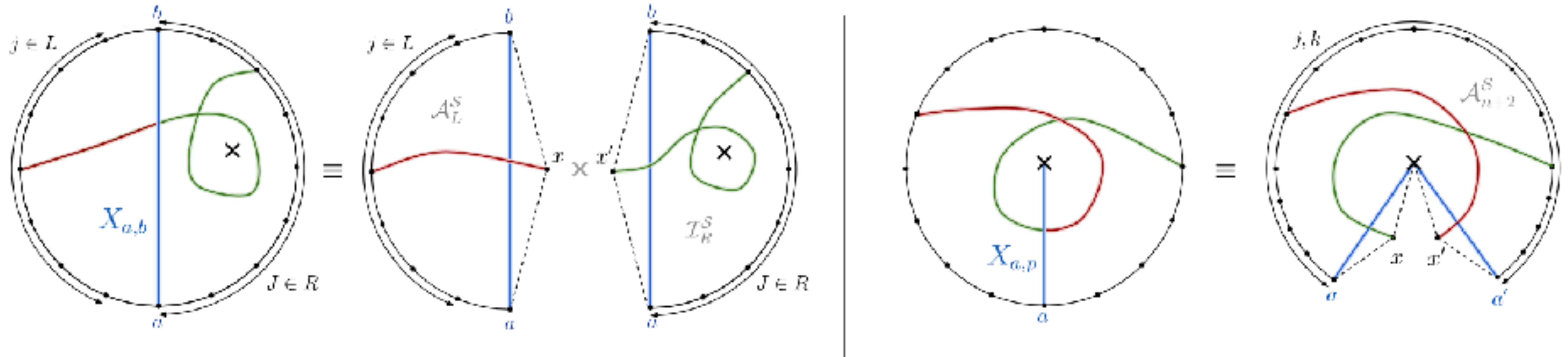

I. Tree-Loop cut

II. Loop cut

Discard scaleless integrals => physical integrands up to 2-loop 6-pt (-> 3-loop 4-pt in progress) [w. Cao, Dong, Zhu]

—> correct amplitudes after loop integration (in D dim), e.g. 1-loop helicity amps up to n=5; all-plus to all n?

enormous simplifications when reducing to 4d spinor-helicity: new results for higher loops?

Fermions, general gauge theories & SYM [w. Cao, Dong, Zhu, 2412]

How to include matters ([c.f. De et al, 2406.04411] for Yukawa): fermions in the loop? nice structure obtained via worldsheet [w. Edison et al '20, '22]

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"universal expansion" of gluon tree + 2 gluons/fermions/scalars
Forward Limit (surface) => 1-loop gluon amps in gauge theories

$$\mathcal{A}^{\rm YM}_{n+2}(-^g,+^g)=\sum_{m,\alpha\in S_m}\epsilon_-\cdot f_{\alpha_1}\cdot f_{\alpha_2}\cdots f_{\alpha_m}\cdot\epsilon_+\times\mathcal{A}^{\rm mixed}(+,\alpha,-)$$

$$\mathcal{A}^{\rm gauge}_{n+2}(-^f,+^f)=\sum_{m,\alpha\in S_m}\bar{\chi}_-\not{f}_{\alpha_1}\not{f}_{\alpha_2}\cdots\not{f}_{\alpha_m}\xi_+\times\mathcal{A}^{\rm mixed}(+,\alpha,-)$$

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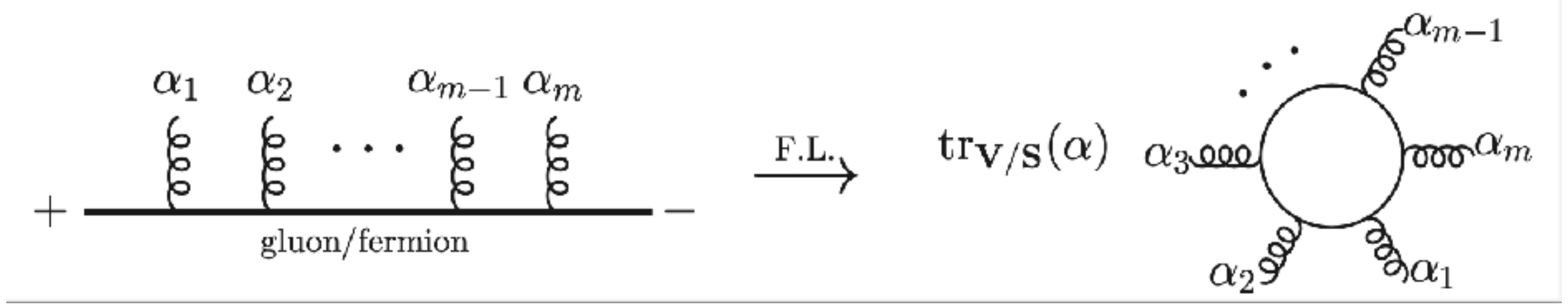

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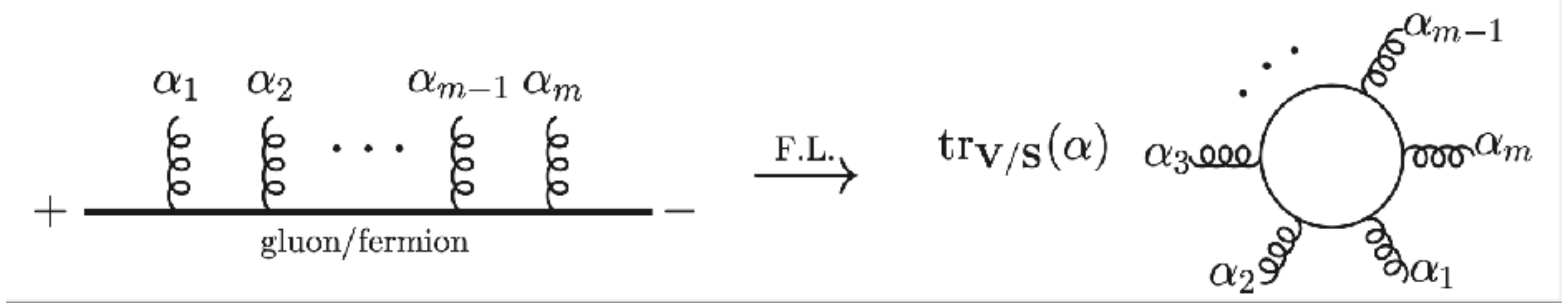

already new formula/relations for n-gluon amps: pure YM 1-loop = sum of mixed scalar-loop amps with Lorentz traces

$$\mathcal{A}^{\rm YM}_n=\sum_{m,\alpha}{\rm tr}_{\rm V}(f_{\alpha_1}\cdots f_{\alpha_m})\mathcal{A}^{\rm scalar-loop}_{\alpha}$$

$$n=2:\quad (D-2)\mathcal{A}_{\emptyset}+{\rm tr}_{\rm V}(f_1f_2)\mathcal{A}_{1,2},$$

$$n=3:\quad (D-2)\mathcal{A}_{\emptyset}+[{\rm tr}_{\rm V}(f_1f_2)\mathcal{A}_{1,2}+2\ \text{perms}]+{\rm tr}_{\rm V}(f_1f_2f_3)\mathcal{A}_{1,2,3}.$$

Almost identical for gen. gauge theory! any multiplet in the loop = sum of mixed scalar-loop amps with “vector/spinor trace”

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(m=0: $\mathscr{T}_\varnothing := (D-2)\mathbf{n}_v + \mathbf{n}_s - 2^{(D-2)/2}\frac{\mathbf{n}_f}{2}$, counts # of on-shell d.o.f)

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universal 1-loop objects: scalar loop with tree blobs of m scalars (in α ordering) + (n-m) gluons attached

most important (m=0): $\mathscr{A}_\varnothing$ as the coefficient of “D-2” (large D limit), simply attaching n gluons to scalar loop

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effective SUSY Ward identities! In particular, huge simplifications for SYM, e.g. w, maximal SUSY

$\mathscr{T}_{m<4} = 0$ since it is proportional to $\mathbf{n}_v - 2^{D/2-5}\mathbf{n}_f = 0$ (D=10, $\mathbf{n}_f = \mathbf{n}_v, \mathbf{n}_s = 0$, or D=4, $\mathbf{n}_f = 8\mathbf{n}_v, \mathbf{n}_s = 6\mathbf{n}_v$)

"no triangle/bubble/tadpole" + correct power-counting ℓ^{m-4} for the m-gon!

e.g. m=4: t_8 tensor for box numerator, $\mathrm{tr}_V - \frac{1}{2}\mathrm{tr}_S|_{D=10} = \frac{1}{2}[\mathrm{tr}_V(1,2,3,4) - \frac{1}{4}\mathrm{tr}_V(1,2)\mathrm{tr}_V(3,4) + \mathrm{cyc.}]$

Towards superstring?

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Scaffolded n-gluon in superstring: $\mathcal{I}_{2n} = \int d^{2n-3}z d^{2n-2}\theta \prod_{i<j} |z_i - z_j + \theta_i\theta_j|^{2\alpha' p_i \cdot p_j} \times \frac{1}{(12)(34)\cdots(2n-12n)}$

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u variables => sum over pairings as “corrections” to bosonic string: $\prod \frac{dy}{y^2}(1 + ss\prod u + \ldots + s\ldots s\prod u)$

e.g. n=3: $\mathcal{I}_6 = \int \prod \frac{dy}{y^2} u^X \left(-1 + \frac{\alpha'^2 s_{1,4}s_{2,3}u_{2,4}}{u_{1,3}u_{3,5}u_{3,6}} + \frac{\alpha'^2 s_{1,3}s_{2,4}}{u_{1,3}u_{3,5}u_{3,6}} \right)$

$$\propto \underbrace{\frac{-1}{z_{1,2}^2 z_{3,4}^2 z_{5,6}^2}}_{\mathrm{YM}+F^3} + \underbrace{\frac{\alpha'^2 s_{1,4}s_{2,3}}{z_{1,2}z_{2,3}z_{3,4}z_{4,1}z_{5,6}^2} + \frac{\alpha'^2 s_{1,3}s_{2,4}}{z_{1,2}z_{2,4}z_{4,3}z_{3,1}z_{5,6}^2}}_{-F^3}$$

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Q: surface origin of worldsheet SUSY? e.g. manifest cancellation of F^3 in (tree) LS

Q: how spacetime SUSY appears? Could we see SUSY cancellations already for loop-level LS?

Outlook

Surfaceology: stringy ϕ^3 from curve-integral => all-loop amps of pions, gluons etc. (real world)

- Hidden zeros + splits: physical implications for trees & loops (scalars, YM, even gravity)?
- All-loop recursion for YM: "tropical" formula? form factors + correlators? practically usable form for integrations?
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Thank you!

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e.g. level-1, residue @ $X_{1,3} = -1 \to \frac{\partial}{\partial y_{1,3}} : \int \frac{dy_{1,3}}{y_{1,3}^2} (\tilde{u}_{1,3})^{X_{1,3}} \prod_{i,j} u_{i,j}^{X_{i,j}} \Longrightarrow X_{1,3}\frac{\partial \log \tilde{u}_{1,3}}{y_{1,3}} + \sum_{j>3}\left(X_{2,j}\frac{\partial log u_{2,j}}{\partial y_{1,3}} + X_{1,j}\frac{\partial \log u_{1,j}}{\partial y_{1,3}} \right)$

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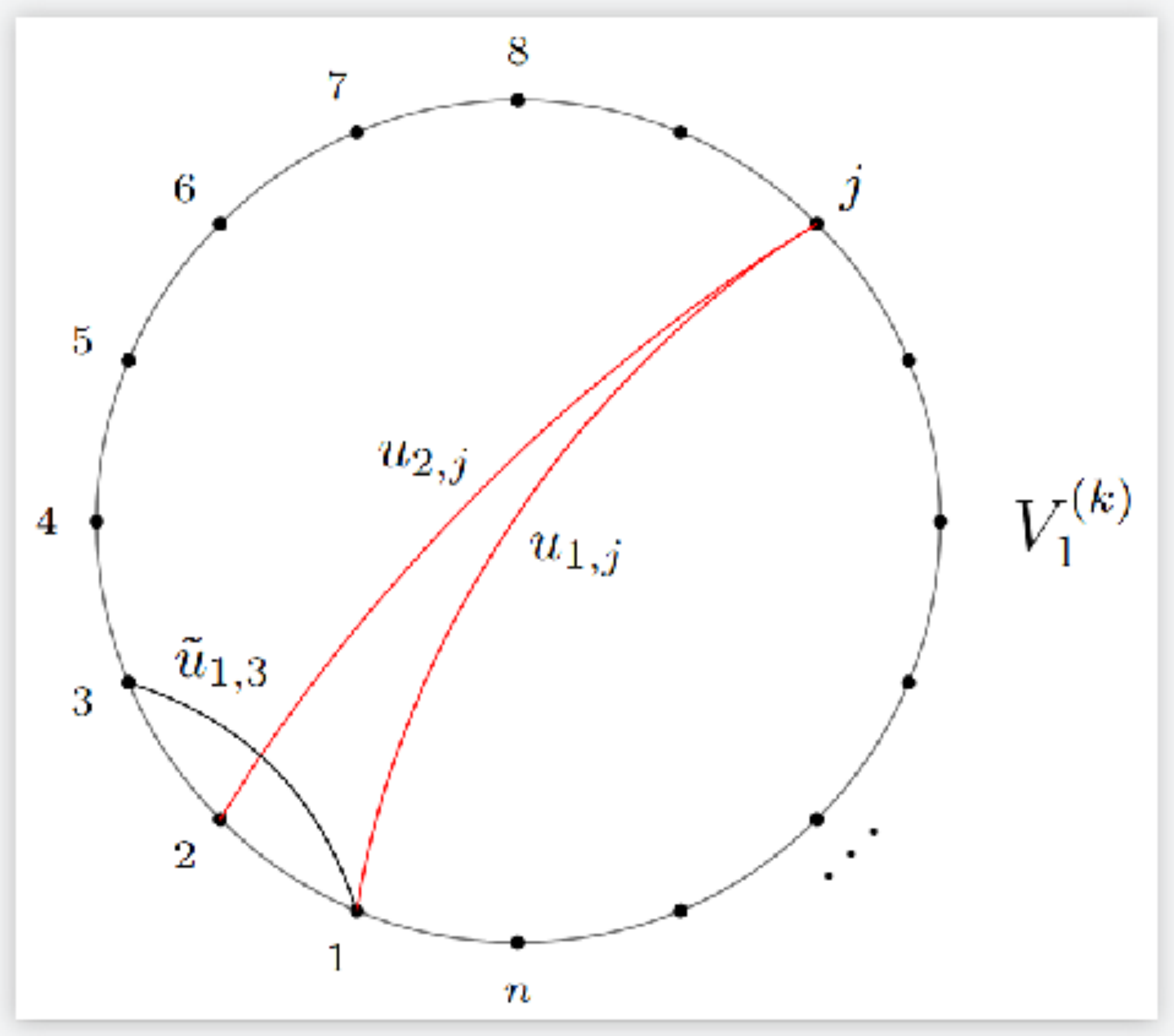

same for higher levels, e.g. residue @ $X_{1,3} = -2 \to \frac{\partial^2}{\partial y_{1,3}^2}$:

$$V_1^{(2)} := \left(V_1^{(1)} \right)^2 + \sum_j \epsilon_1 \cdot x_{3,j} \frac{\partial^2 \log u_{2,j}}{\partial y_{1,3}^2}$$ (massive spin-2, 1, 0)

"vertex operators" for any k: $V_1^{(k)} \supset \left(V_1^{(1)} \right)^k, \cdots, \sum_j \epsilon_1 \cdot x_{3,j} \frac{\partial^k \log u_{2,j}}{\partial y_{1,3}^k}$

more massive string states e.g. $X_{1,3} = X_{5,7} = -1 \to V_1^{(1)} V_3^{(1)} + \partial_{y_{5,7}} V_1^{(1)} \implies \underbrace{(X_{1,5} + X_{2,6} - X_{1,6} - X_{2,6})}_{c_{1,5}} \frac{\partial^2 log u_{2,6}}{\partial y_{1,3} \partial y_{5,7}} \equiv W_{1,3}^{(1,1)}$

Just like scaffolded gluons: 2n-gon massless => massive states with $X_{1,3} = -n_1,\ X_{3,5} = -n_2,\ X_{5,7} = -n_3, \dots$

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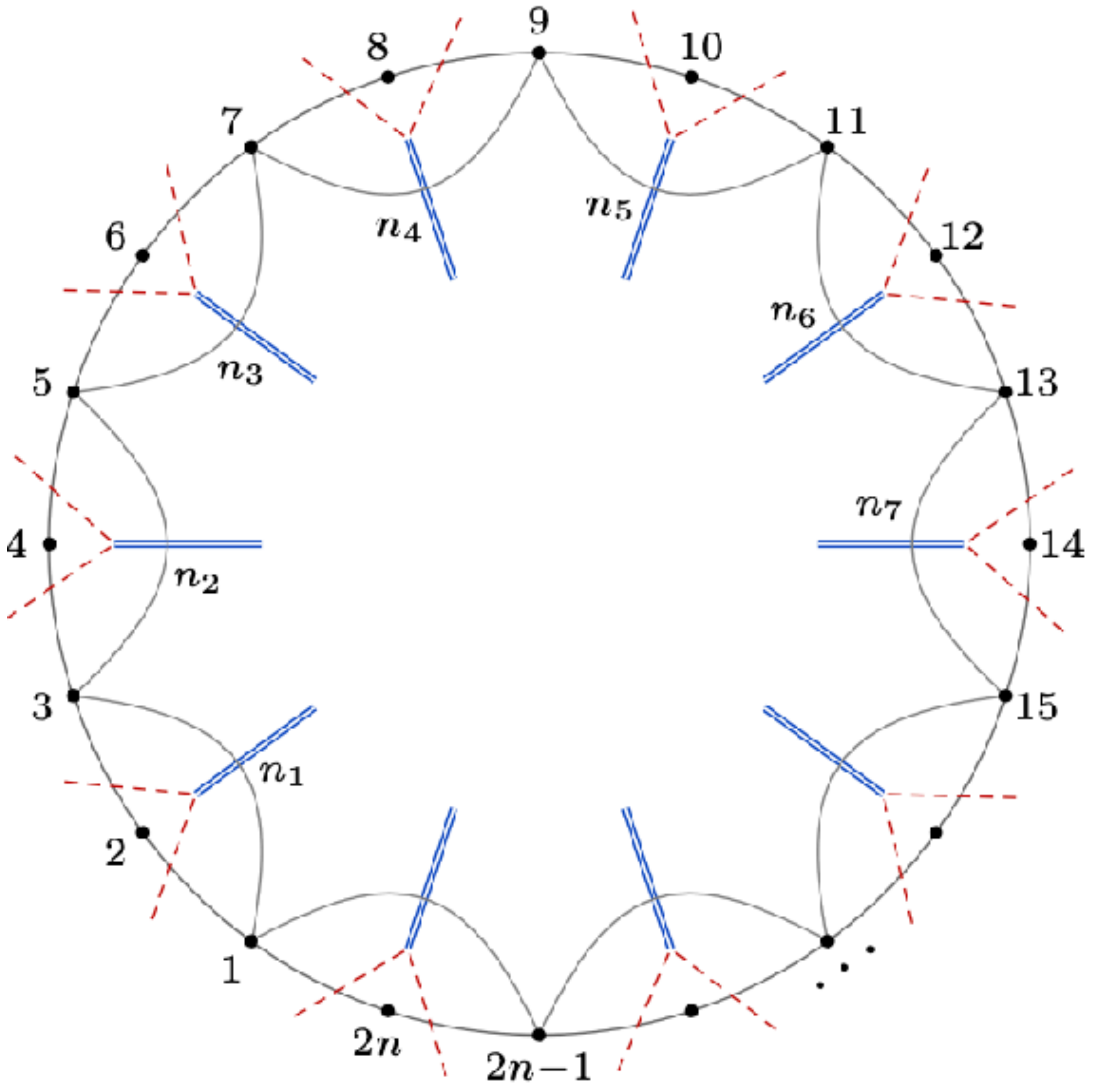

A sum of products of $V_a^{(m_a)}, W_{a,b}^{(m_a,m_b)}$ with total degree $n_1, n_2, \cdots,$

e.g. $\mathscr{A}_3^{(1,1,1)} = V_1^{(1)} V_2^{(1)} V_3^{(1)} + \left(W_{1,2}^{(1,1)} V_3^{(1)} + \text{cyc}\,. \right)$

explicit 4-pt (any level): $\mathscr{A}_4^{(n_1,n_2,n_3,n_4)} = \underbrace{\left(\sum \prod \tilde{V}\ \tilde{W} \right)}_{\text{polynomial in X}} \times \dfrac{\Gamma(X_{1,5})\Gamma(X_{3,7})}{\Gamma(X_{1,5} + X_{3,7})}$

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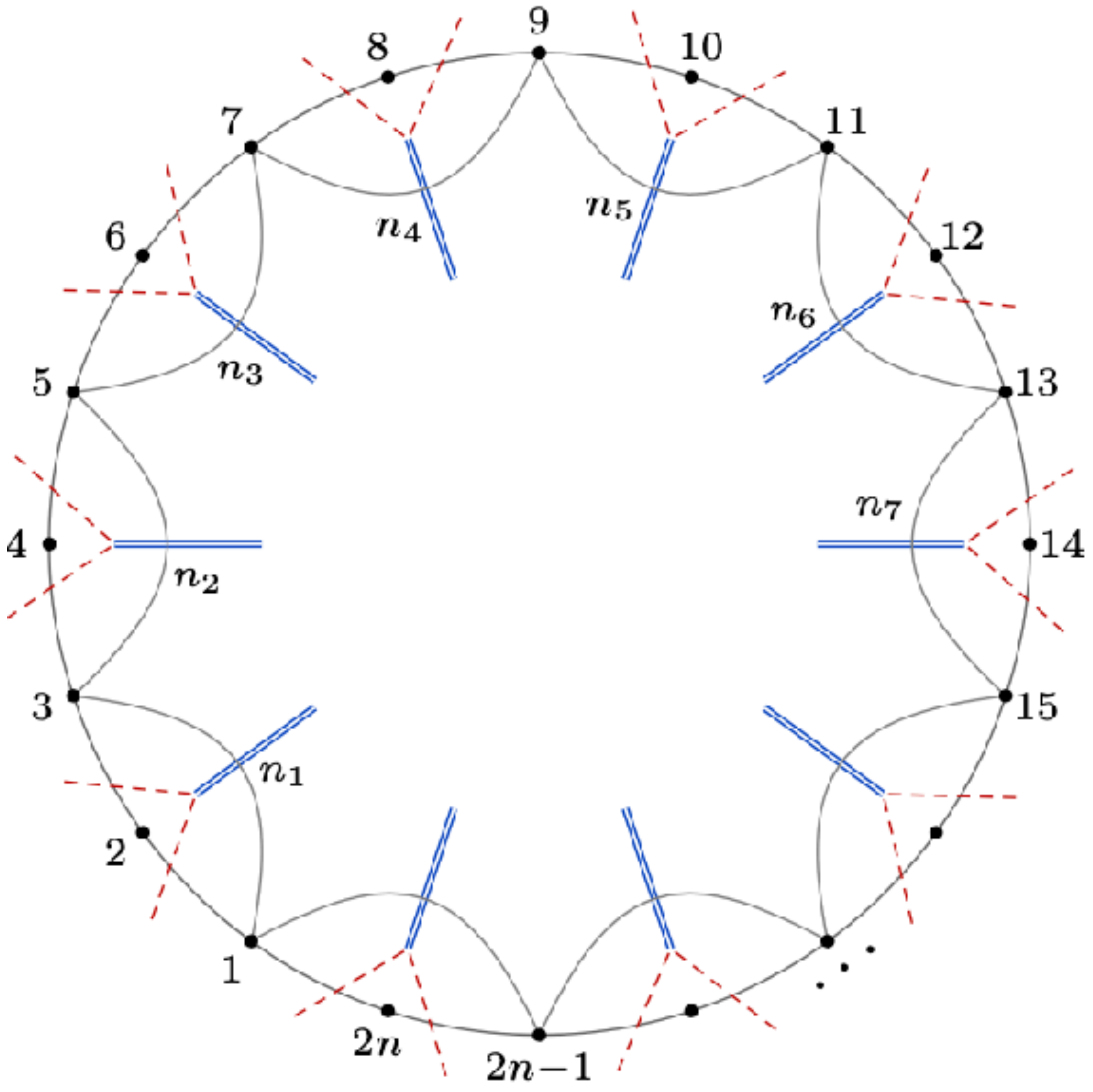

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compare residues with worldsheet calculation: leading trajectories

Q: can we produce & label all string states in this way?

Massive factorizations: open & closed

explicit formula for massive factorizations: gluing of massive states @ $X_{a,b} = -k \to$ k-th derivatives @ $u_{a,b} = y_{a,b}\tilde{u}_{a,b} = 0$

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$Q_j^L := \frac{\partial \mathcal{A}^L(a, \cdots, b, x)}{\partial X_{x,j}}, \quad Q_J^R := \frac{\partial \mathcal{A}^R(b, \cdots, a, y)}{\partial X_{y,J}}$; obvious generalization to k-th level!

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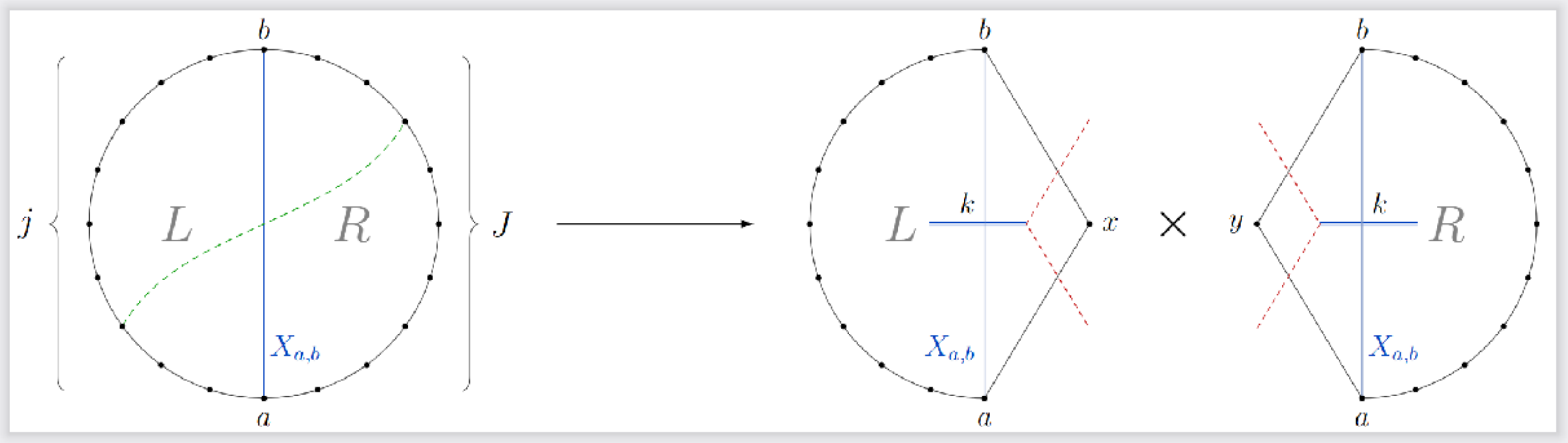

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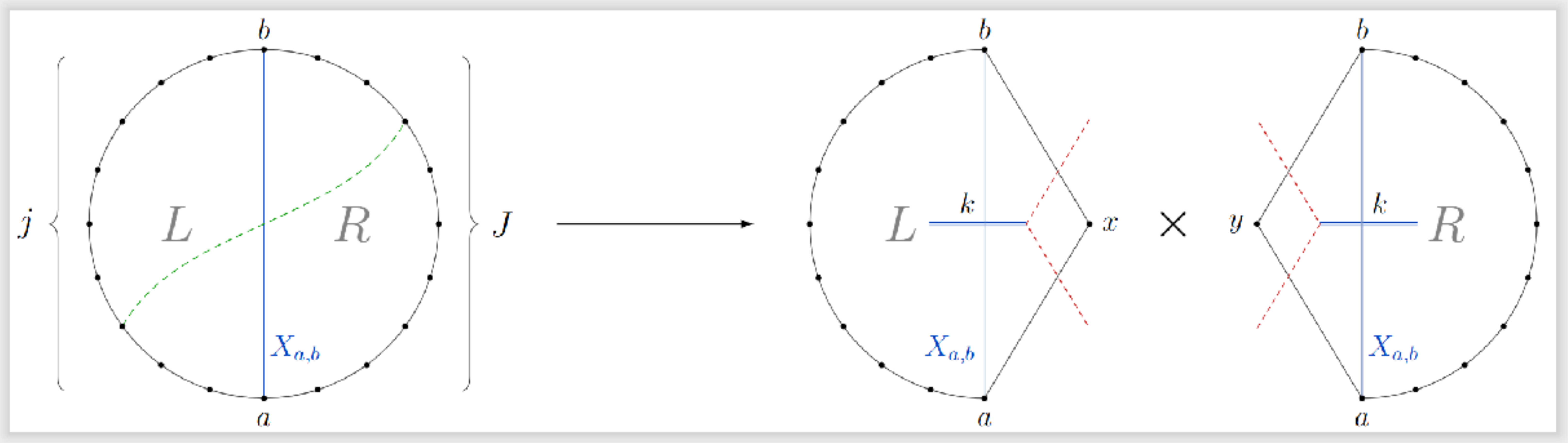

Even more interesting for closed strings: already $\int_{\mathbb{C}} \prod |\frac{dy}{y}|^2$ => detect massive poles for all possible orderings!

e.g. non-planar poles starts @ $s_{1,3} = -1$ for $n \geq 4$, $s_{1,3,5} = -2$ for $n \geq 6$, $s_{1,3,5,7} = -3$ for $n \geq 8$, same factorization formula