Tameness and Complexity in QFTs and the Landscape

Thomas W. Grimm

Utrecht University

Based on work done in collaboration with

Mick van Vliet, Arno Hoefnagels, Jeroen Monnee, Damian van de Heisteeg, Lorenz Schlechter, Mike Douglas

Mathematicians: Benjamin Bakker, Christian Schnell, Jacob Tsimerman

Introduction

Complexity in Quantum Field Theories

- Given a Quantum Field Theory (QFT):

$$S^{(D)} = -\int d^D x \Big(K(\phi)(\partial\phi)^2 + V(\phi) + \ldots \Big)$$

Can we understand some of the properties of such a QFT, such as its vacuum structure, by using a measure of complexity associated to the QFT and its observables?

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Can we understand some of the properties of such a QFT, such as its vacuum structure, by using a measure of complexity associated to the QFT and its observables?

- Need to introduce a notion of complexity for a function and set

Geometric/logical complexity from tame geometry

roughly: amount of information that needs to be specified to encode set or function

Complexity in the space of QFTs

- Apply notion in the space of QFTs:

- Complexity generically infinite

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infinitely many free coefficients: generic analytic potential

$$V(\phi) = \sum_{n=1}^{\infty} a_n \phi^n$$

infinitely many vacua: $V(\phi) = \cos\phi + \cos\alpha\phi$ $\qquad$ $V(\phi) = \sin(1/\phi)$

α irrartional

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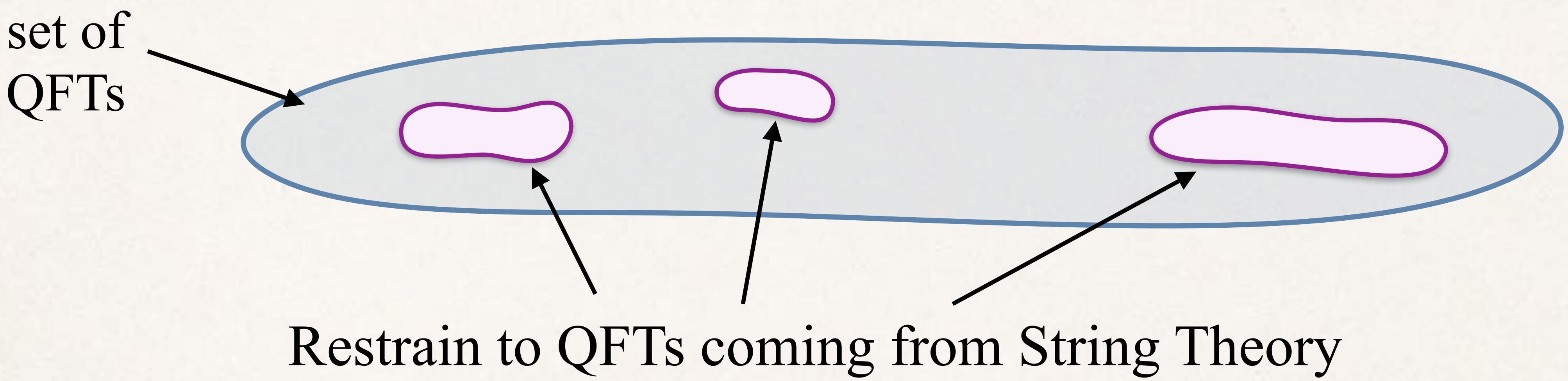

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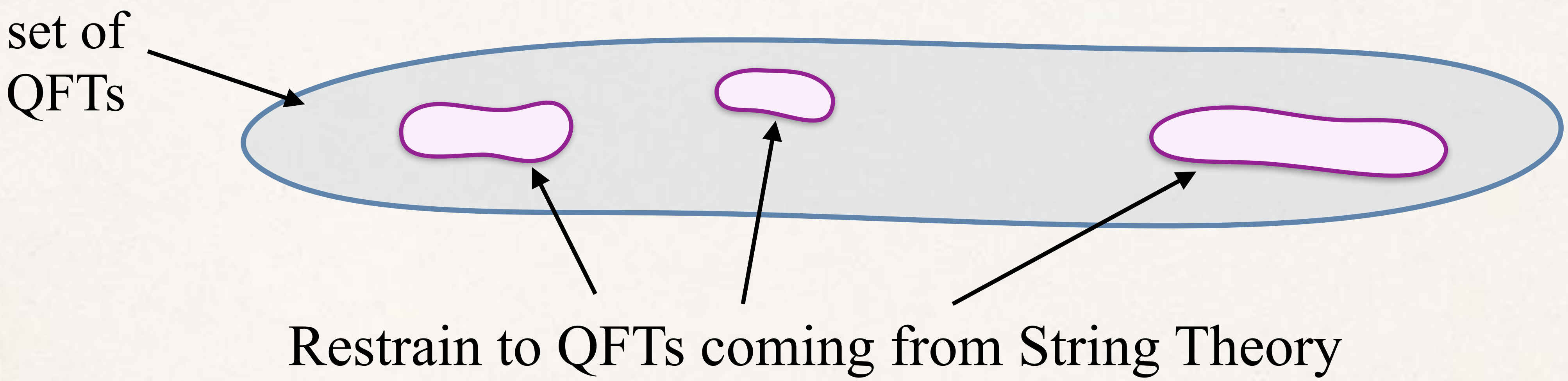

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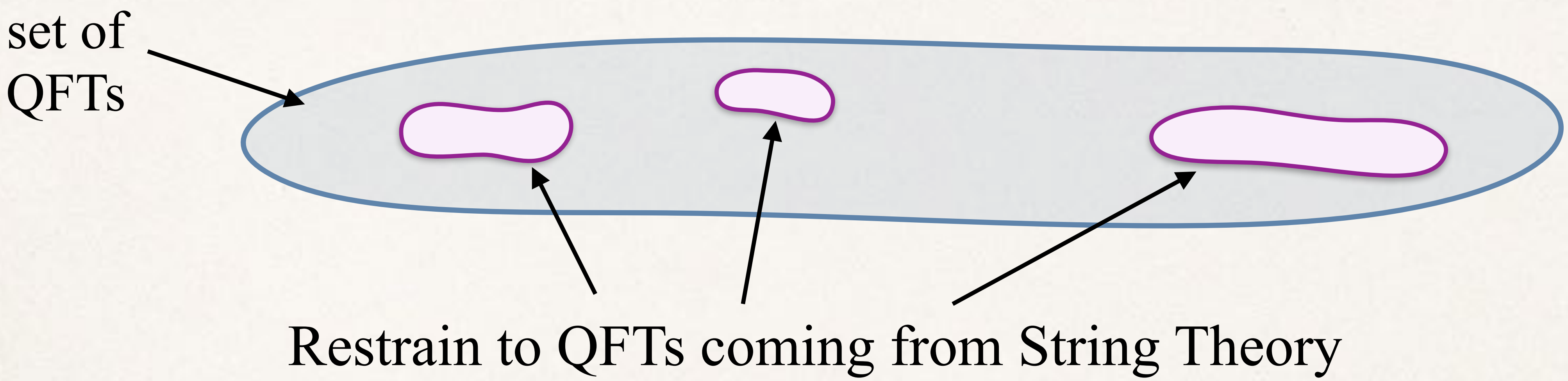

- Expect better finiteness properties
  - String theory has no continuous free parameters apart from ℓ_s
    $\rightarrow$ effective couplings determined by fields + discrete choices
    (topological data, fluxes,…)
  - Conjectures about finiteness of effective theories:
    [Douglas '05] [Vafa '05] [Acharya,Douglas '06]…[Hamada,Montero,Vafa,Valenzuela '21]…
    [Delgado,Heisteeg,Raman,Torres,Vafa '24]

    $\rightarrow$ central part of 'swampland program' talk by [H.-C. Kim]

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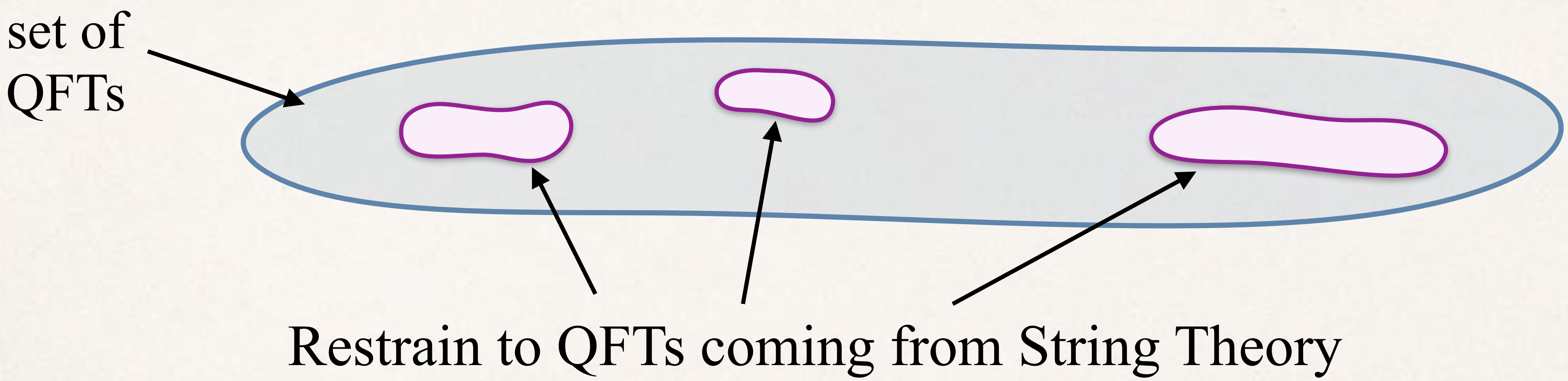

Propose unifying perspective:

Focus on tameness (o-minimality) as generalized finiteness principle

[TG '21][Douglas,TG,Schlechter '23]

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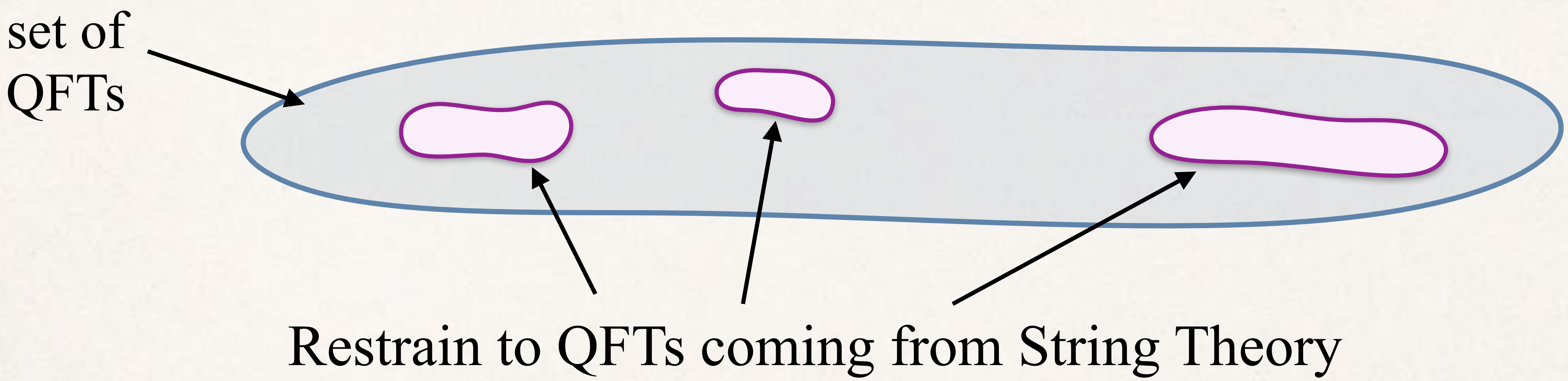

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[TG '21][Douglas,TG,Schlechter '23]

Use geometric complexity (sharp o-minimality) as quantitative measure within a QFT and on the space of QFTs.

[TG,Schlechter,van Vliet '23][TG,van Vliet '24]

Geometric Complexity

Intuition from polynomials

- Complexity for polynomials

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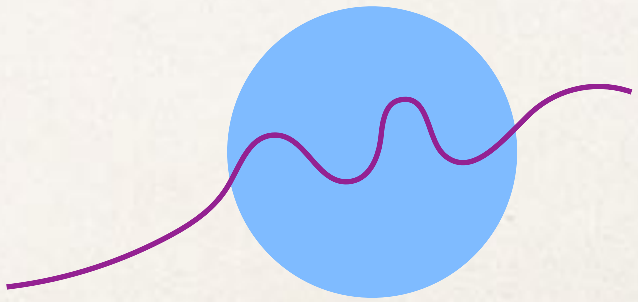
    $$\mathrm{Vol}(B^{n+1}(r) \cap A) \leq c(n)\, \mathcal{C}(F,D)\, r^n$$



see e.g. book [Yomdin,Comte]

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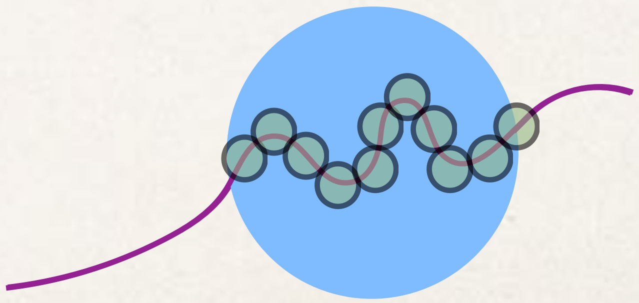
  - Number of ϵ-balls covering A



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  Necessary condition: Functions cannot be too wild!

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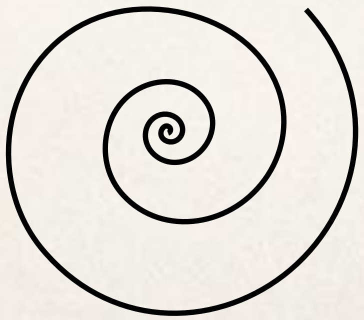

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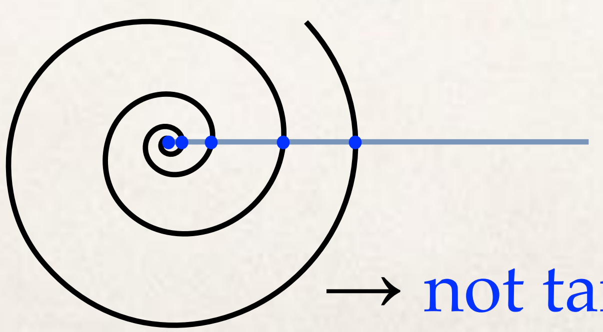

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$\rightarrow$ not tame: infinite intersections with real line

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- Last years: [Binyamini,Novikov '23] [Binyamini '24]

  General mathematical framework keeping track of finite information

  tame structures with a notion of complexity (sharp o-minimality)

Example class of tame functions

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→ bounds on zeros, volume etc.

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Pfaffian function: $g(x) = P(x_1, ..., x_n, f_1, f_2, ..., f_r)$ ⟶ (F, D)

While we need the more general framework in the following:
Can always think of (F, D) for Pfaffian setting.

On the Complexity of QFTs

Complexity of QFT Lagrangian

- Assign complexity to the Lagrangian

N scalar fields $$S^{(4)} = -\int d^4x \Big(\sum_{k=1}^{N} (\partial\phi_k)^2 + V(\phi_1, ..., \phi_N) \Big)$$

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- Generalizes to broader classes of QFTs [TG,van Vliet '24]

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- Perturbative QFT: showed that in QFT with finitely many particles and interactions all finite-loop amplitudes are tame functions of masses, external momenta, coupling constants

  [Douglas,TG,Schlechter '22]

  - Study their complexity and how it sees algebraic relations, symmetries

    [TG,Hoefnagels '24] [Britto,TG,Hoefnagels] in preparation

- Works for tree-level cosmological correlators which are Pfaffian functions

  [TG,Hoefnagels,van Vliet '24] using [Arkani-Hamed,Baumann,Hillman,Joyce,Lee,Pimentel '23]

Complexity in the String Landscape

- Two finiteness results -

String compactifications

- Consider String Theory, M-theory, or F-theory on compact manifold Y

  $\rightarrow$ effective four-dimensional theory

$$S^{(4)} = \int d^4x \sqrt{G} \Big(R - K_{kl}(\phi)(\partial\phi^k)(\partial\phi^l) - V(\phi) \Big) + \ldots$$

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- Concretely: Flux compactifications of Type IIB and F-theory

  reviews [Graña][Douglas,Kachru][Denef][McAllister,Quevedo]

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$n_i = \int_{C^i} G_4$ flux vector → integers

Finiteness in the landscape - **Result 1**

- Finiteness conjectures imply for Y (fixed topology):
  finite solutions to $D_{\phi^k} W = 0$ satisfying tadpole condition $\int_Y G_4 \wedge G_4 = \ell$
  (counting connected components = flat directions)

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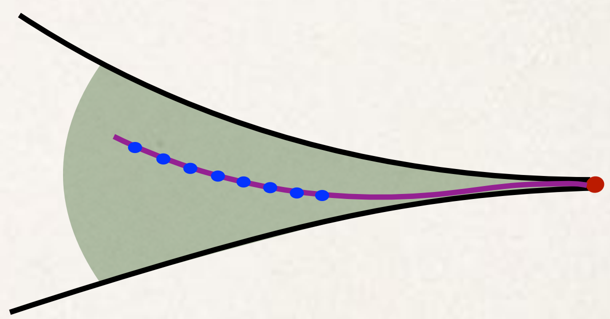
- Key complication: field space is non-compact
  $\rightarrow$ need to control every path to every boundary for every possible Y



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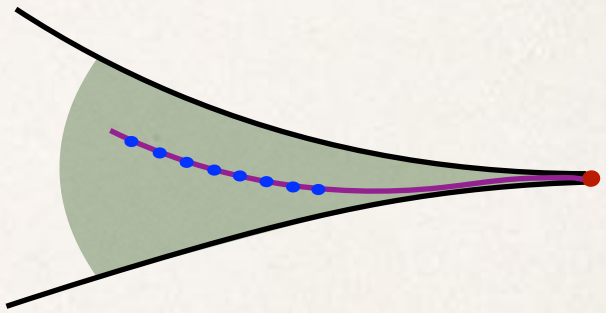

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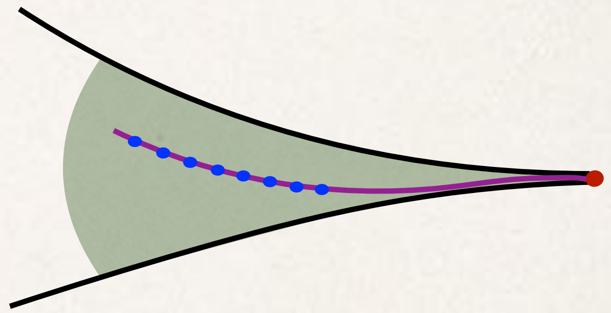

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$\rightarrow$ Flux vacuum landscape is tame set ! (for Y of fixed top.)

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  Conjecture complexity from flux density:

  $D = \text{poly}(\ell)$ $F = \mathcal{O}(h^{3,1}(Y))$ [TG,Monnee '23]

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- have essentially complete picture for flux vacua on Calabi-Yau fourfolds

"Arithmetic structure behind W=0 flux vacua" [TG,van de Heisteeg '24] using [Baldi,Klingler,Ullmo '21]

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  - however, it turns out that there is a more exciting structure

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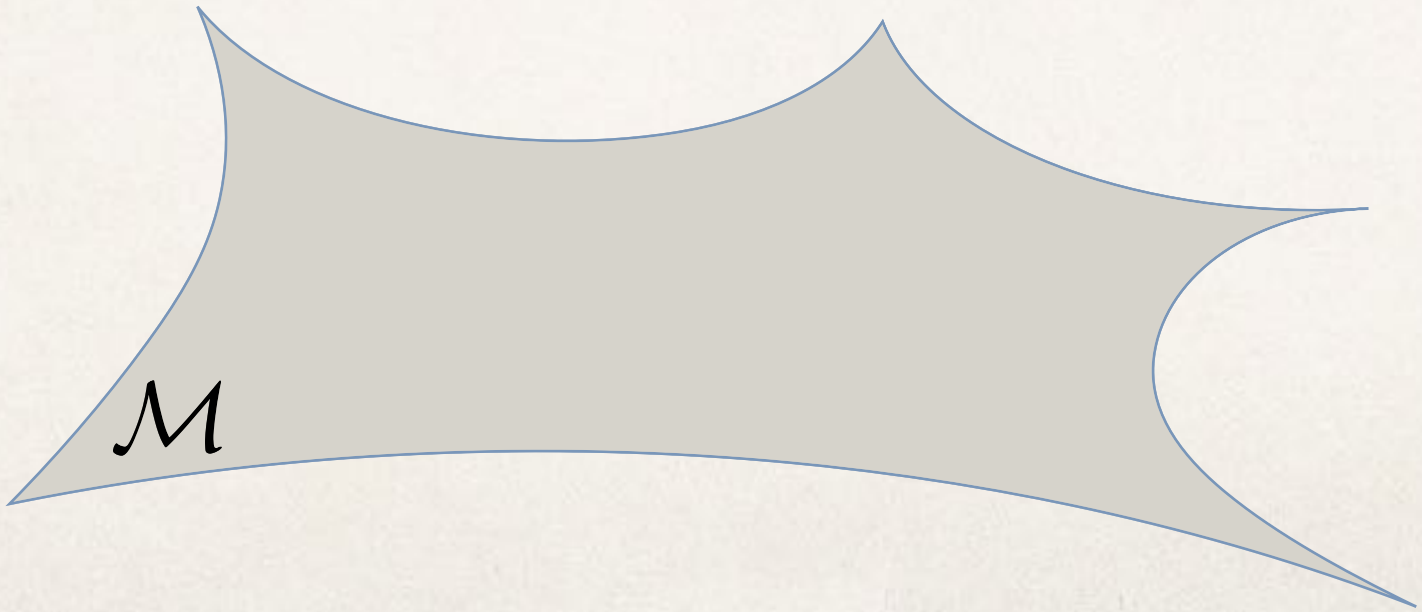
- General pattern in complex structure moduli space of Calabi-Yau fourfolds



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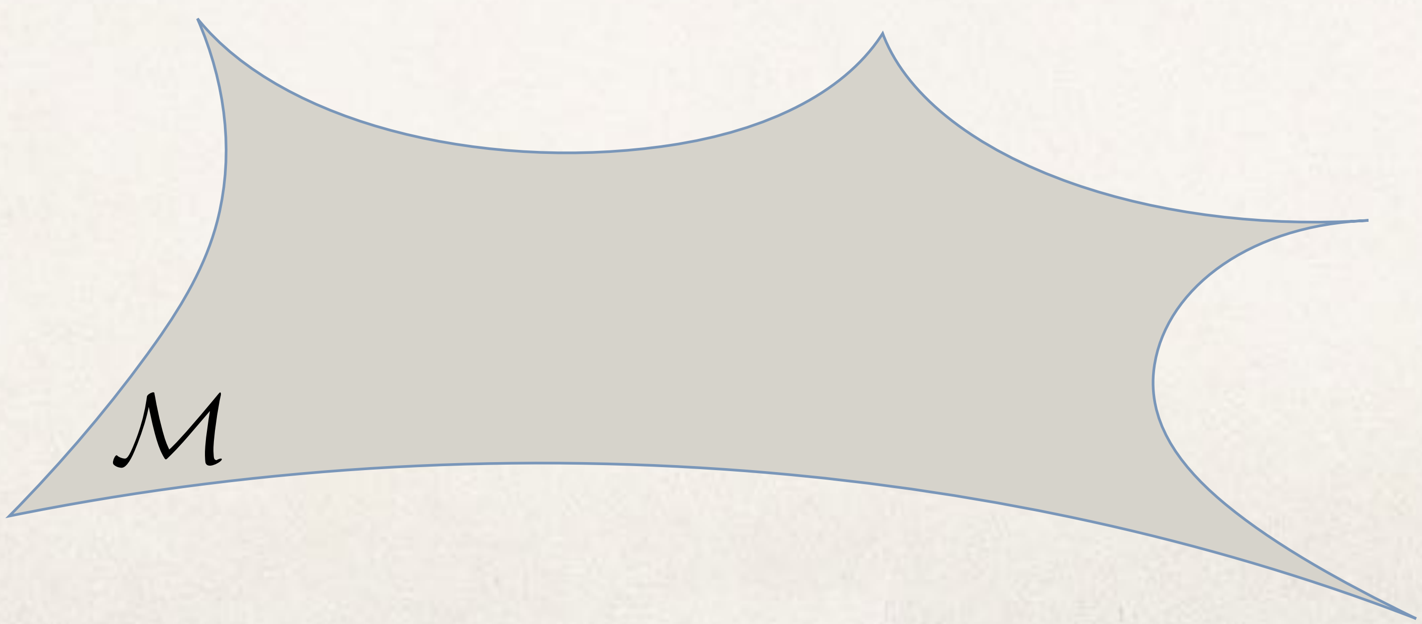

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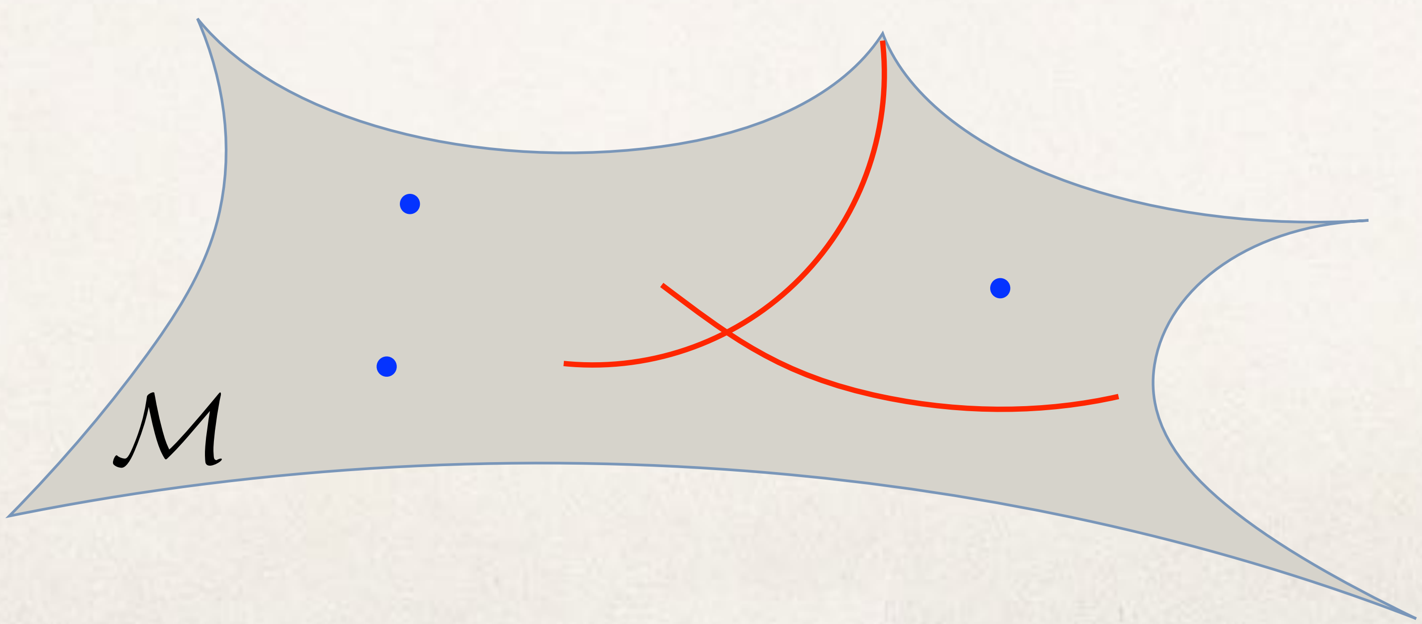

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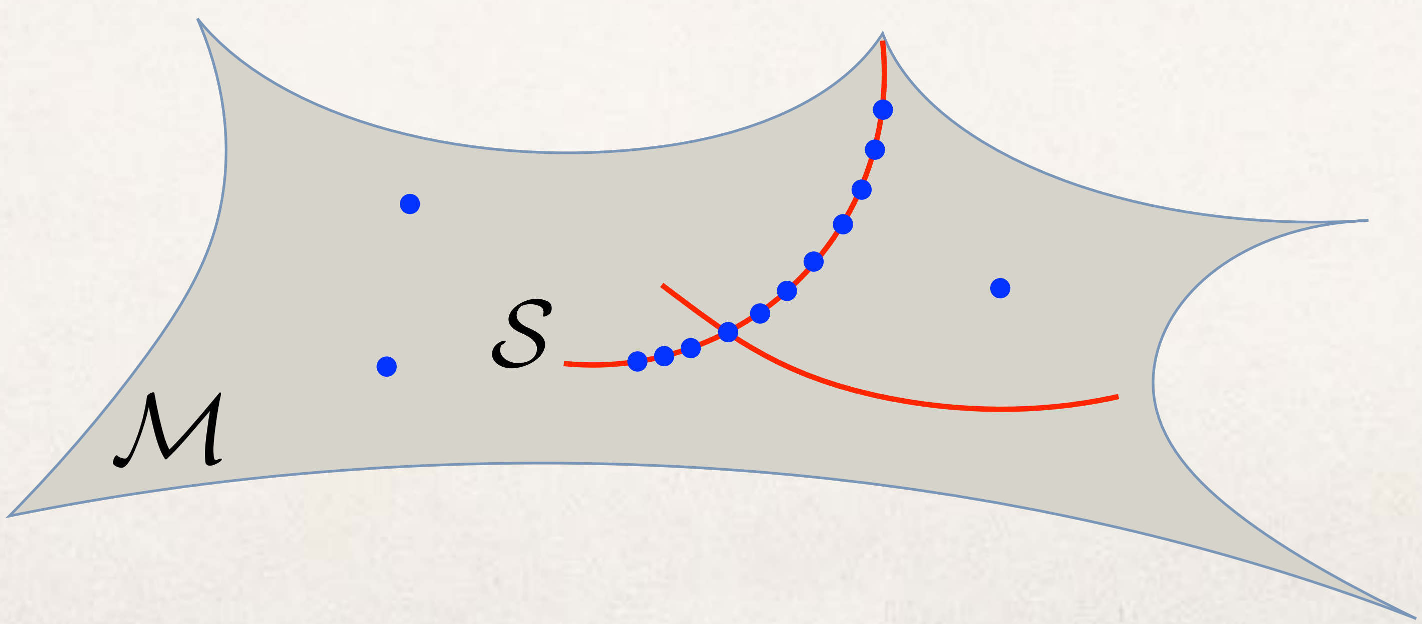
(2) on symmetry locus $\mathcal{S}$: part of the periods can become polynomial

$\rightarrow$ complexity reduction due to algebraic relations: $F =$ #variables

$\rightarrow$ flux vacua in $\mathcal{S}$ are dense (no tadpole bound)



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Exact F-theory flux vacua

- example: Hulek-Verrill fourfold

$$(X^1, \ldots, X^6) \in \quad \mathbb{T}^5 = \mathbb{P}^5 \backslash \{X_1 \cdots X_6 = 0\}$$

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- Periods: expanded around large complex structure [Jockers,Kotlewski,Kuusela '23]

$$(\Pi_i) = \begin{pmatrix} \Pi^0 \\ \Pi^I \\ \Pi_{IJ} \\ \Pi_I \\ \Pi_0 \end{pmatrix}$$

e.g.

$$\Pi^0 = \sum_{n_1,\ldots,n_6=0}^{\infty} \left(\frac{(n_1 + \ldots + n_6)!}{n_1! \cdots n_6!}\right)^2 (\phi^1)^{n_1} \cdots (\phi^6)^{n_6}$$

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highly complex, generically transcendental, periods

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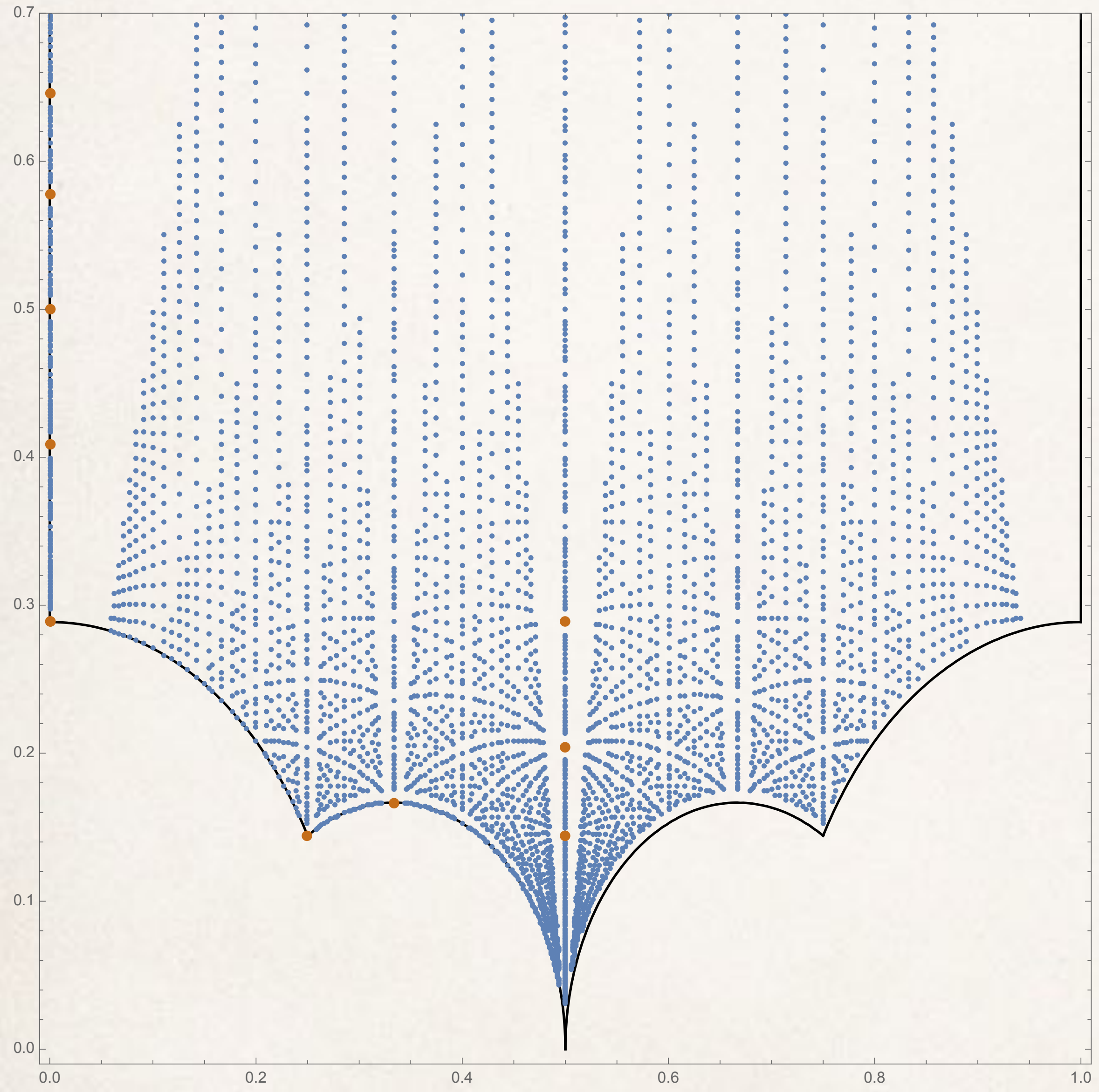

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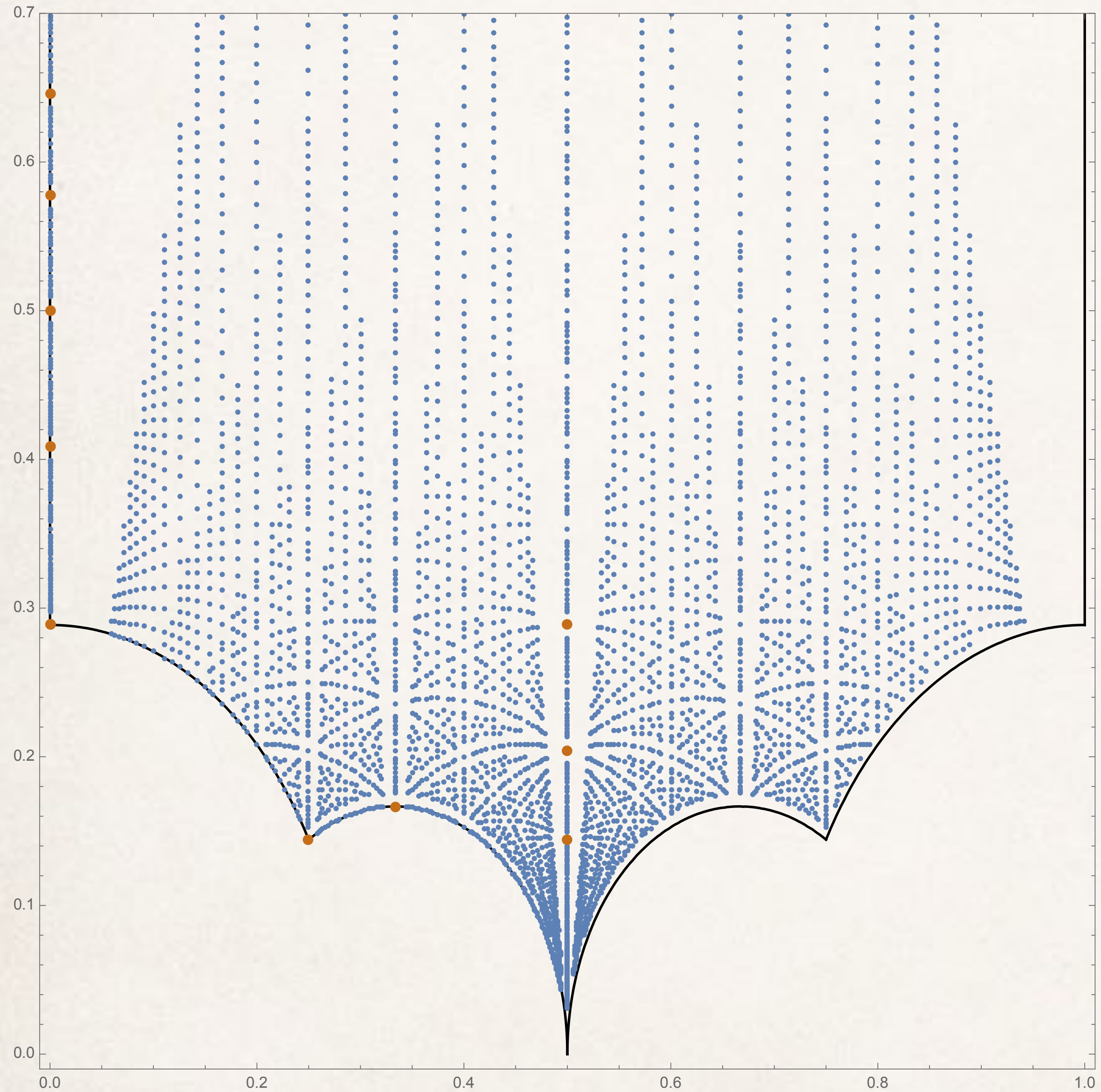

Calabi-Yau fourfolds containing 'attractive K3 surfaces' [Moore '98]

$\mathfrak{t} \in \mathbb{Q}(i\sqrt{D})$ $\quad D > 0$

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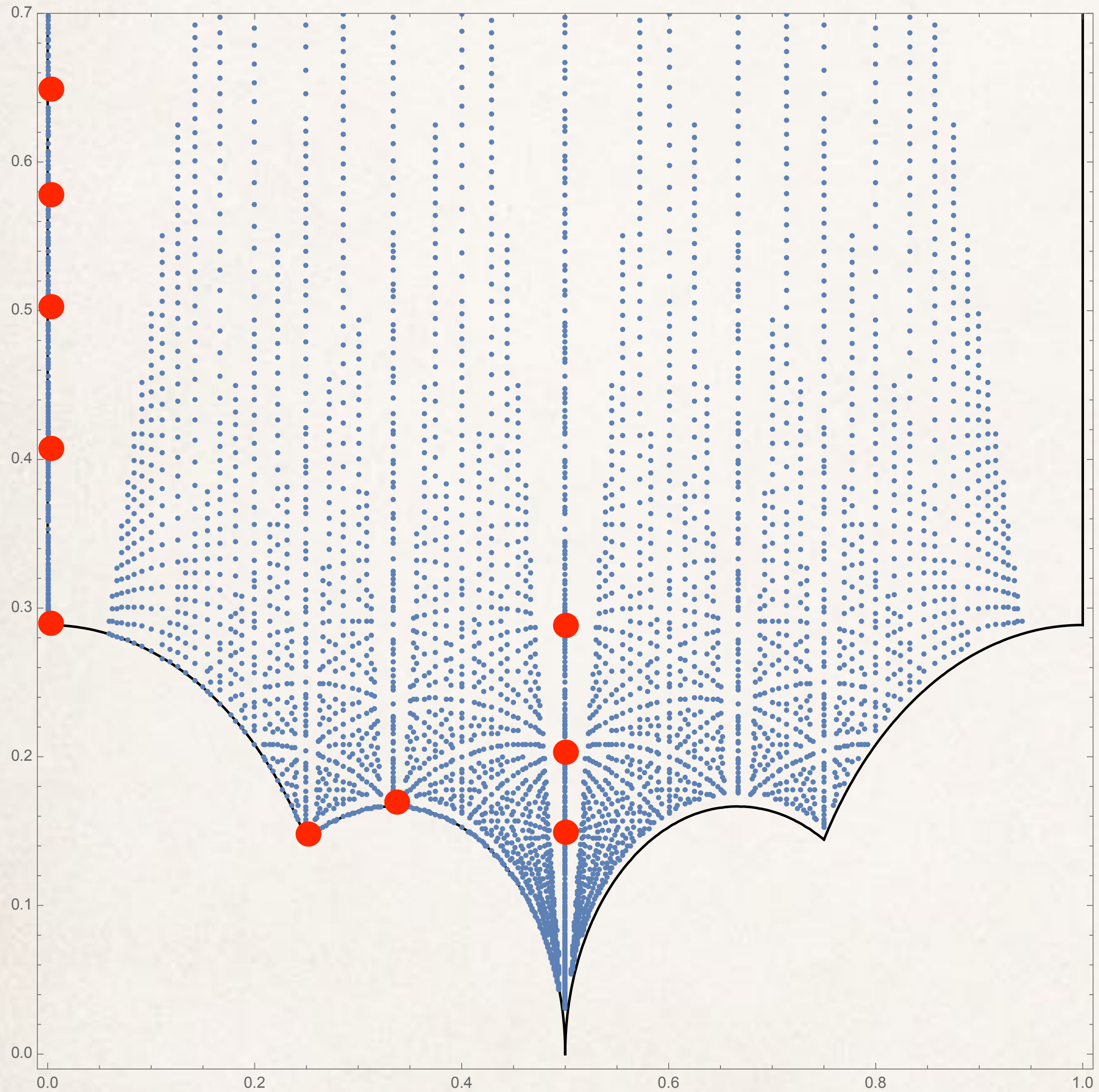

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First time: Exact flux vacua in Calabi-Yau fourfold [TG,van de Heisteeg '24]

(all complex structure moduli stabilized in tadpole bound)

Curse and Virtue of Complexity

Many vacua from high complexity

- Our universe within String Theory:

  $\rightarrow$ pick integer data such that couplings match observations (after moduli stabilization)

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A proposed realization: [Bousso,Polchinski '00] polynomial potential

$$V(n) = \Lambda_{\text{bare}} - \sum_{i,j} \mathcal{M}_{ij} n^i n^j$$

→ sufficiently many flux directions $F \gg 1$ → can get small Λ_*

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Growing number of arguments indicating: (1) need for revised approach to the landscape, (2) more positive assessment of its predictive power

Conclusions

- Introduced the notion of tameness, o-minimality, as generalized finiteness property → powerful tools from mathematics

- Refined o-minimal structures to allow for a definition of complexity → sharp o-minimality (#o-minimality)

- First ideas to assign complexity to a QFT (Lagrangian/observables)

- Two finiteness/tameness results: (1) DW=0, self-dual flux vacua
  (2) DW=0, W=0 landscape

  symmetry vs. transcendentality (presence of exp. corrections)

- Stressed that in String Theory we often solve transcendental problems over the integers → largely unexplored perspective
  (while common in amplitude/bootstrap community)

Thanks!

Some examples

- #complexity is minimal (*F*,*D*) needed to define the function

  see [TG,Schlechter,van Vliet '23] for definition and example applications in QFTs

- exponential function: e^{ax} $(F, D) = (2, 2)$

- fewnomials: $ax^{2d} + bx^d$ $(F, D) = (1, 2d)$

  alternative representation: $f_1 = x^d, \;\; f_2 = \frac{1}{x}$ $(F, D) = (3, 6)$

- trigonometric: $\cos(n\,x)$ on $[-\pi, \pi]$ $(F, D) = (3, 4 + n)$

Note: $x^2 = \sum_{n=0}^{\infty} a_n \cos(n\,x) \;\approx\; \sum_{n=0}^{N} a_n \cos(n\,x)$

N to infinity limit decreases complexity

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\+ work in progress