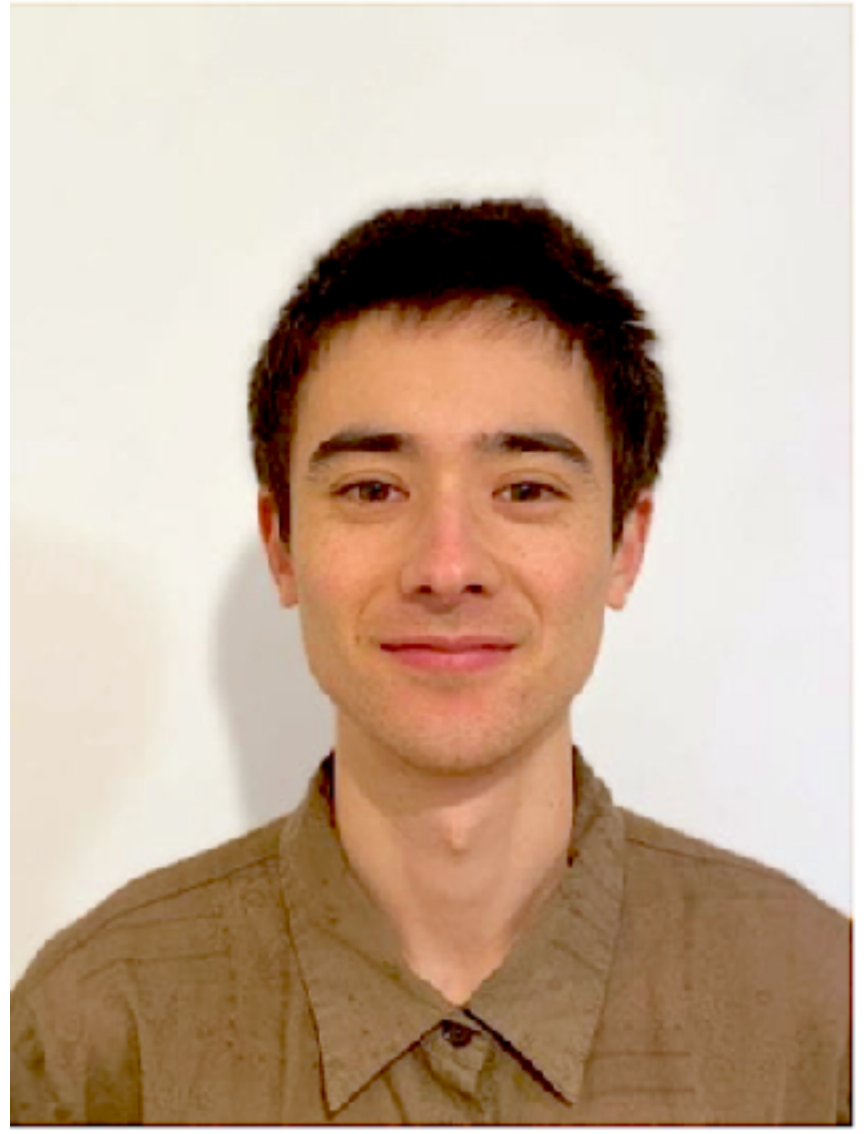
Abelian Chern-Simons Theory on the Lattice

Tin Sulejmanpasic (Durham U)
in collaboration with Theo Jacobson (UCLA)
(and Samson Chan (Durham U))

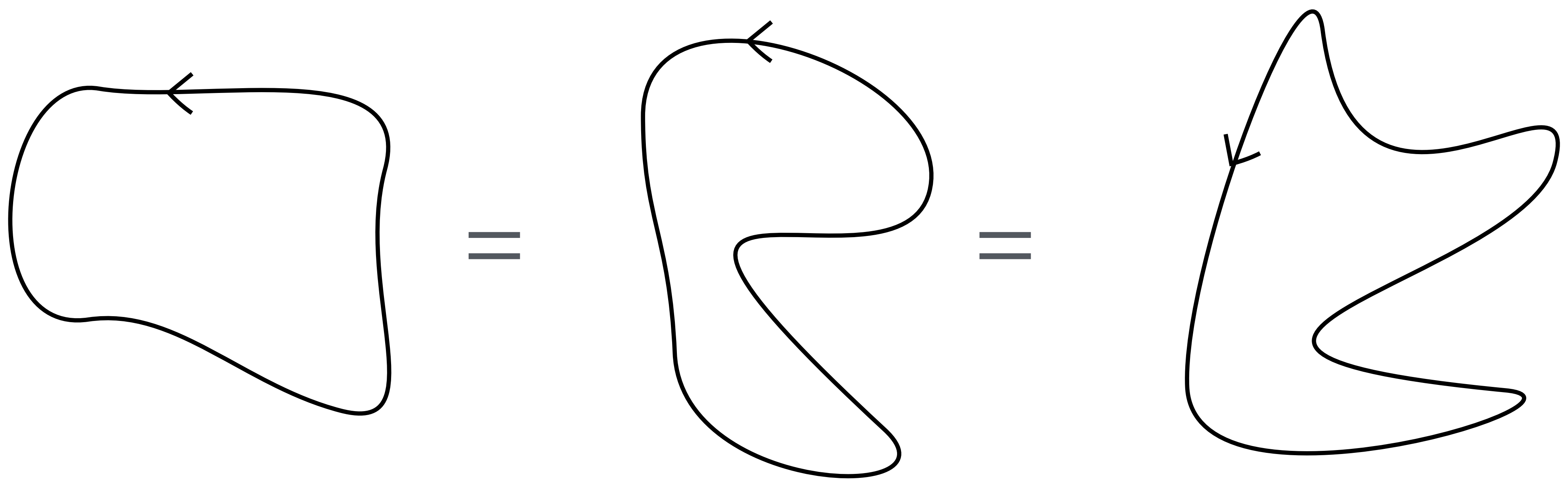


Based on:
2303.06160, 2401.09597 and work in progress

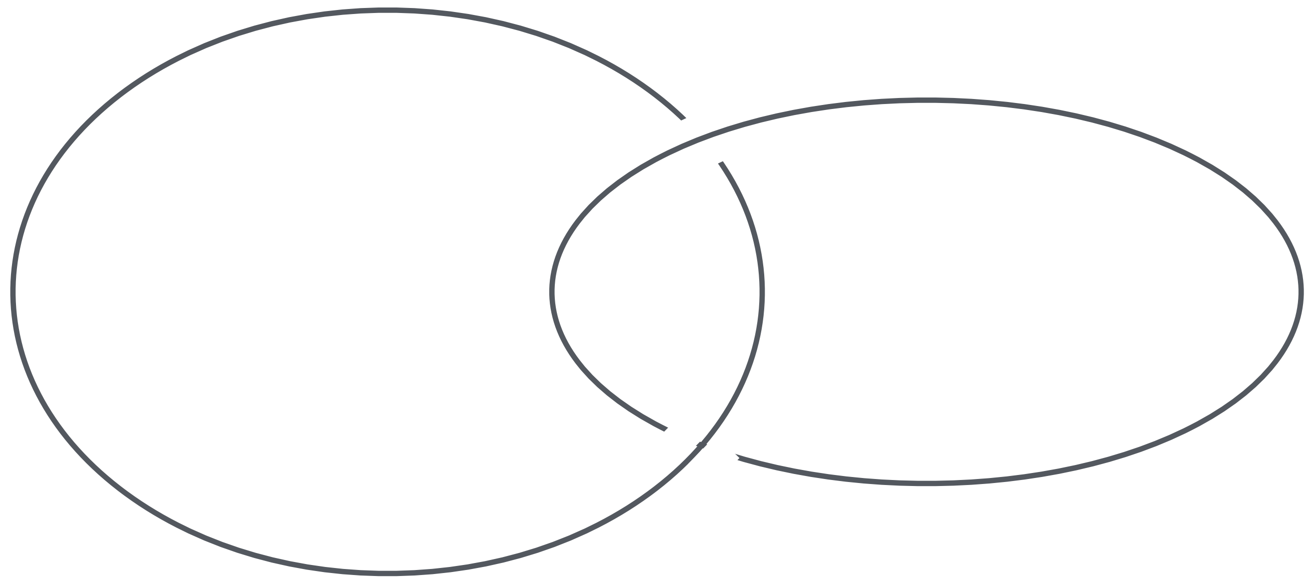
Strings 2024, Abu Dhabi

Chern-Simons theory

$$S = \frac{k}{4\pi}\int A \wedge dA$$



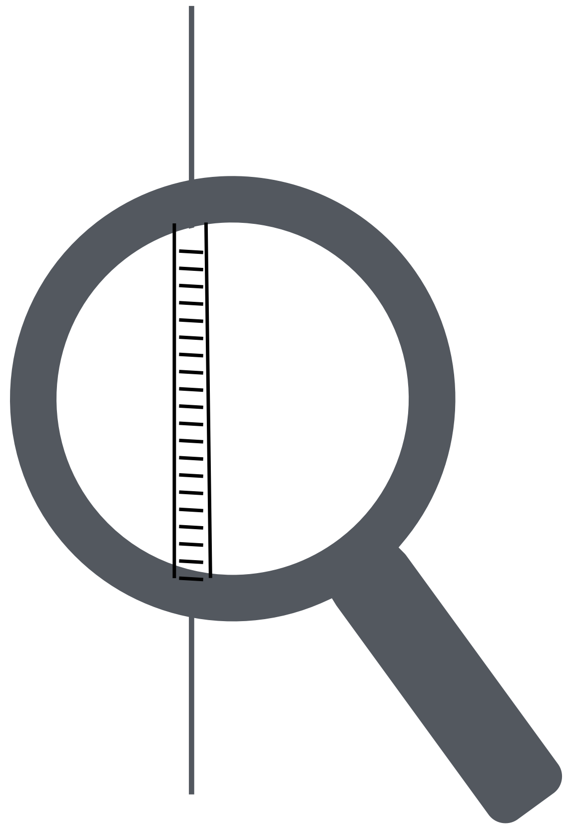
- Wilson loops are topological



$$= e^{\frac{2\pi i}{k}}$$

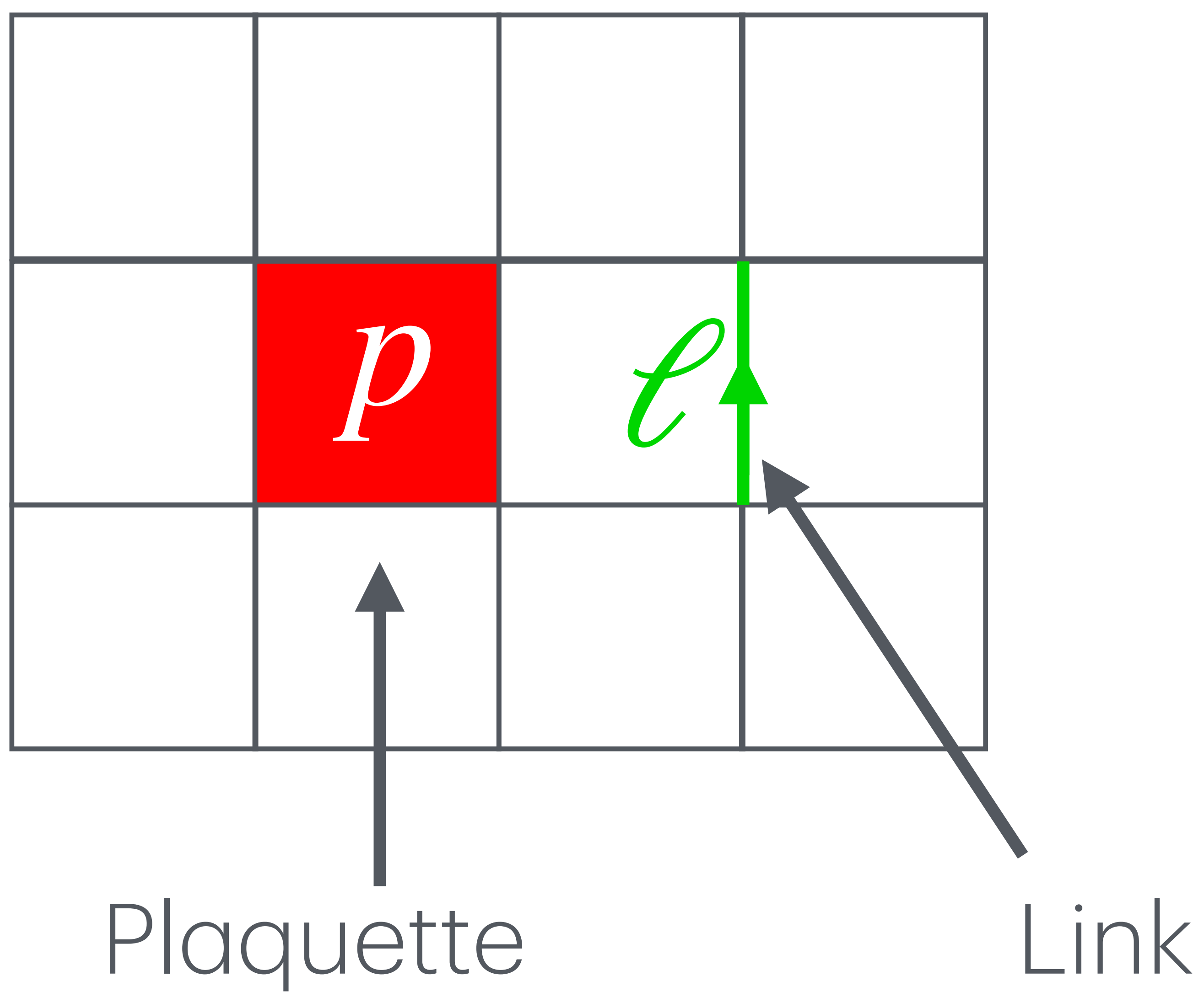
- A $\mathbb{Z}_k$ phase determined by linking

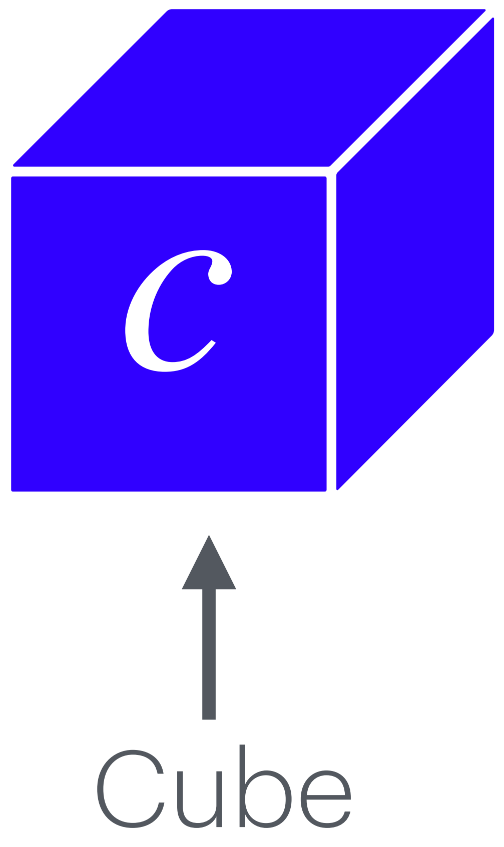
$$= e^{\frac{2\pi i}{2k}}$$



- Self-linking $\mathbb{Z}_{2k}$ phase

U(1) Lattice Gauge Theory

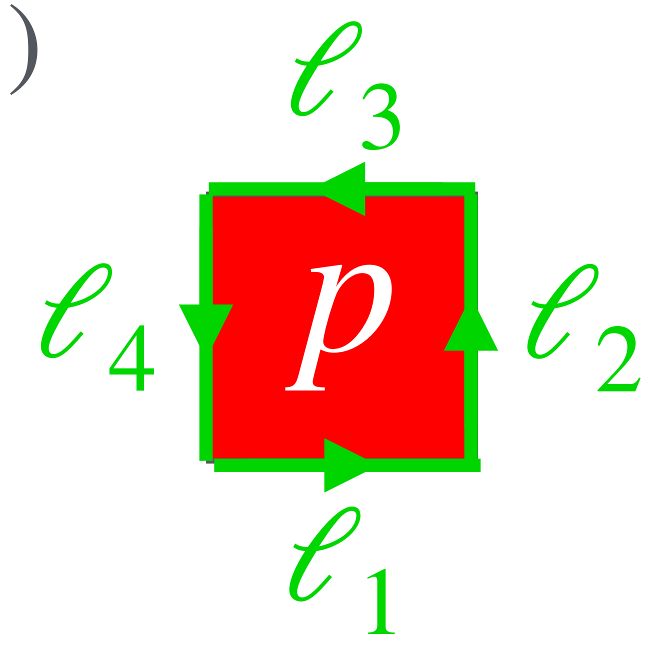




U_l phases on links

$$S = -\frac{\beta}{2}\sum_p (V_p + V_p^*)$$

$$V_p = U_{l_1}U_{l_2}U_{l_3}U_{l_4}$$



A problem: lattice is full of holes, and dynamical monopole defects can naturally occur

Solution: modified Villain formalism

The (Modified) Villain Action

$$U_l = e^{iA_l}$$

$$S = -\frac{\beta}{2}\sum_p (V_p + V_p^*) = -\beta \sum_p \cos(dA)_p \approx \sum_p \frac{\beta}{2}((dA)_p + 2\pi n_p)^2$$

when $\beta \gg 1$

$$V_p = e^{i(dA)_p}$$

← an exterior derivative on the lattice

$$A_\ell \to A_\ell + 2\pi k_\ell$$
$$n_p \to n_p - (dk)$$

Discrete g.s.

- $(dn)_c \neq 0$ is interpreted as a monopole in the cube c

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-Imposing the constraint $dn = 0$ projects out monopoles

$$\sum_p \frac{\beta}{2}((dA)_p + 2\pi n_p)^2 + i\sum_c \varphi_{\star c}(dn)_c$$

[TS, Gattringer '19]
[Gorantla, Lam, Seiberg, Shao, '21]

- This leads to a lot of interesting stuff
  - electric magnetic duality
  - ability to couple electric and magnetic matter simulationously
  - interracting non-supersymmetric self-dual theories and maybe have potential non-SUSY Arygires-Douglas fixed points furnishin UV completion of electromagnetism [Anasova/Gattringer/Iqbal]
  - non-invertible symmetries on the lattice [Cordova/Ohmori, Shao et al]
  - application to compact scalars [Gross/Klebanov '90, TS Gattringer '19, Gorantla/Seiberg/Shao '21]
  - application to fracton physics [Gorantla/Seiberg/Shao/Lam..., Fazza/TS...]

Discretizing the pure CS theory

$$A_\mu \partial_\nu A_\rho \epsilon^{\mu\nu\rho} \to \begin{cases} A_{x,\mu}(dA)_{x,\rho}\epsilon^{\mu\nu\rho} \\ \\ \end{cases}$$

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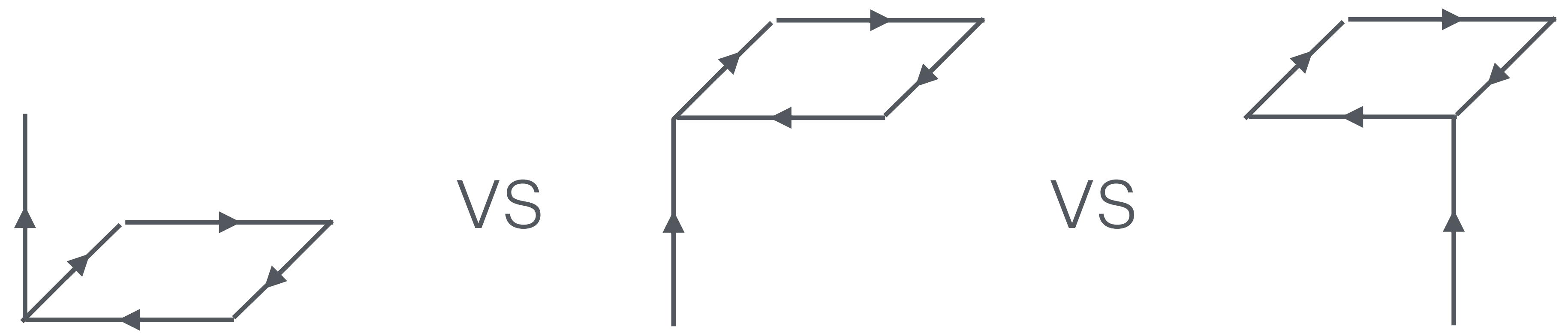

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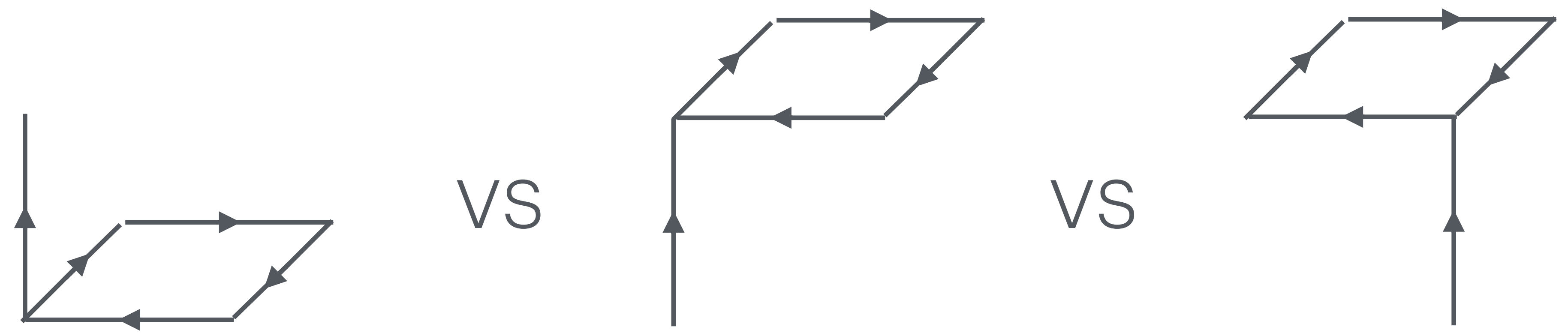

Of course basic requirement is that the discretization obeys the gauge invariance

$$A_l \to A_l + (d\lambda)_l$$

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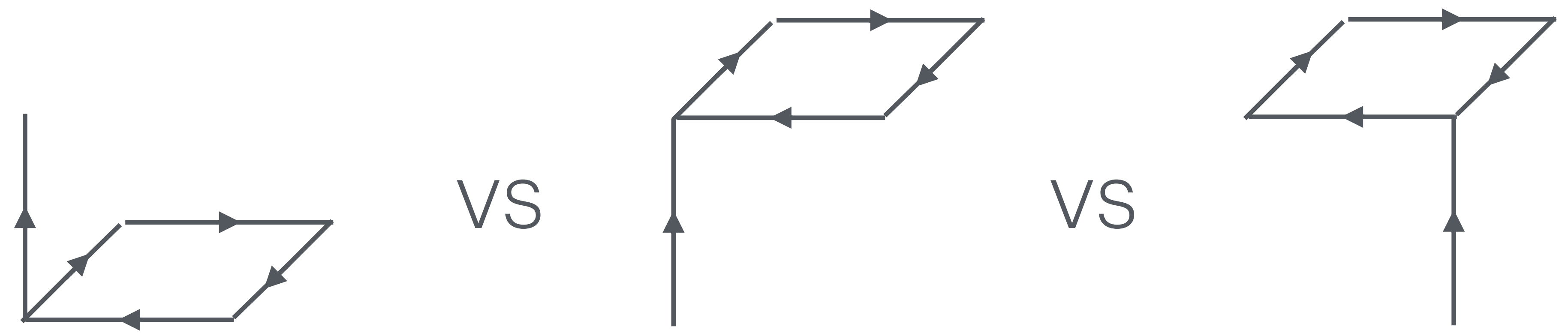

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Consider instead a setup with A field on link and B field on plaquettes of the dual lattice

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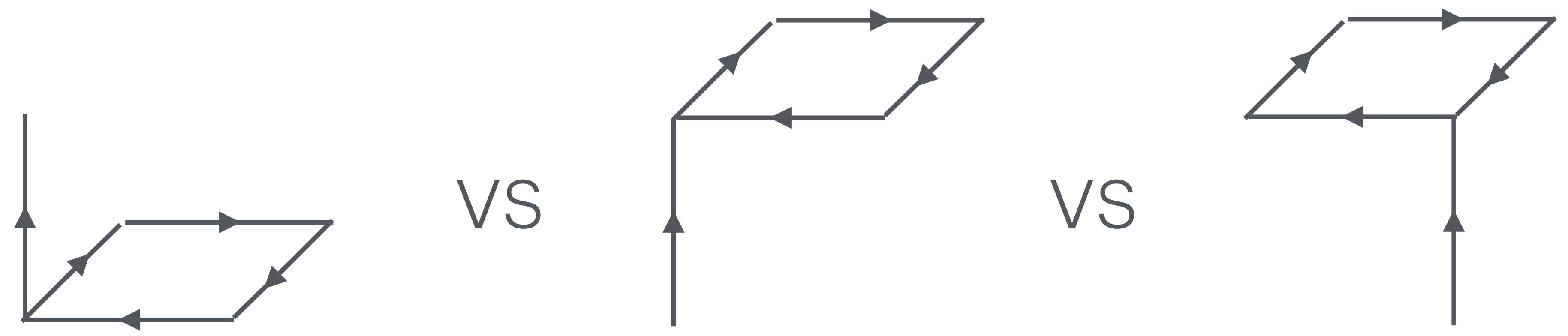

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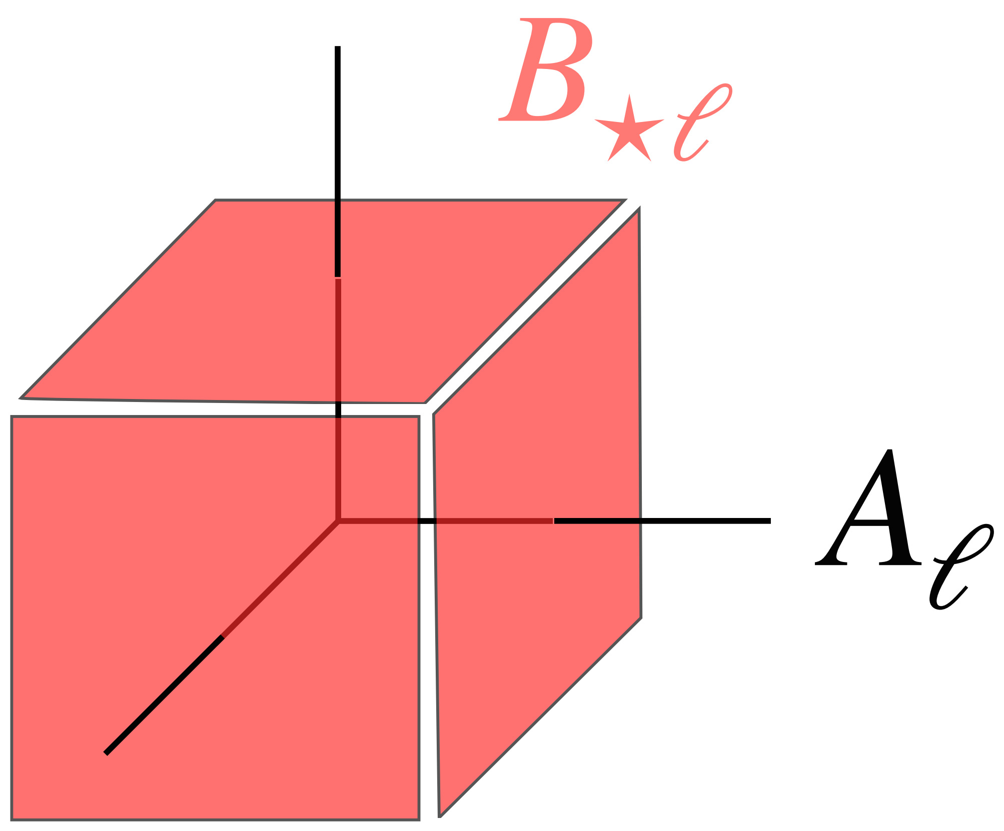

$$\sum_\ell A_\ell B_{\star\ell} \rightarrow \sum_\ell A_\ell B_{\star\ell} + \sum_\ell (d\lambda)_\ell B_{\star\ell}$$

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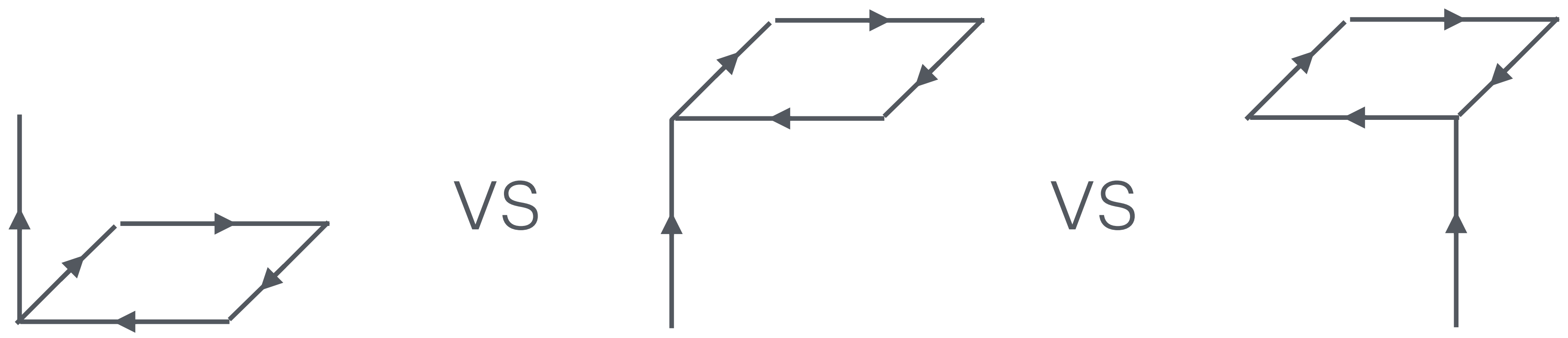

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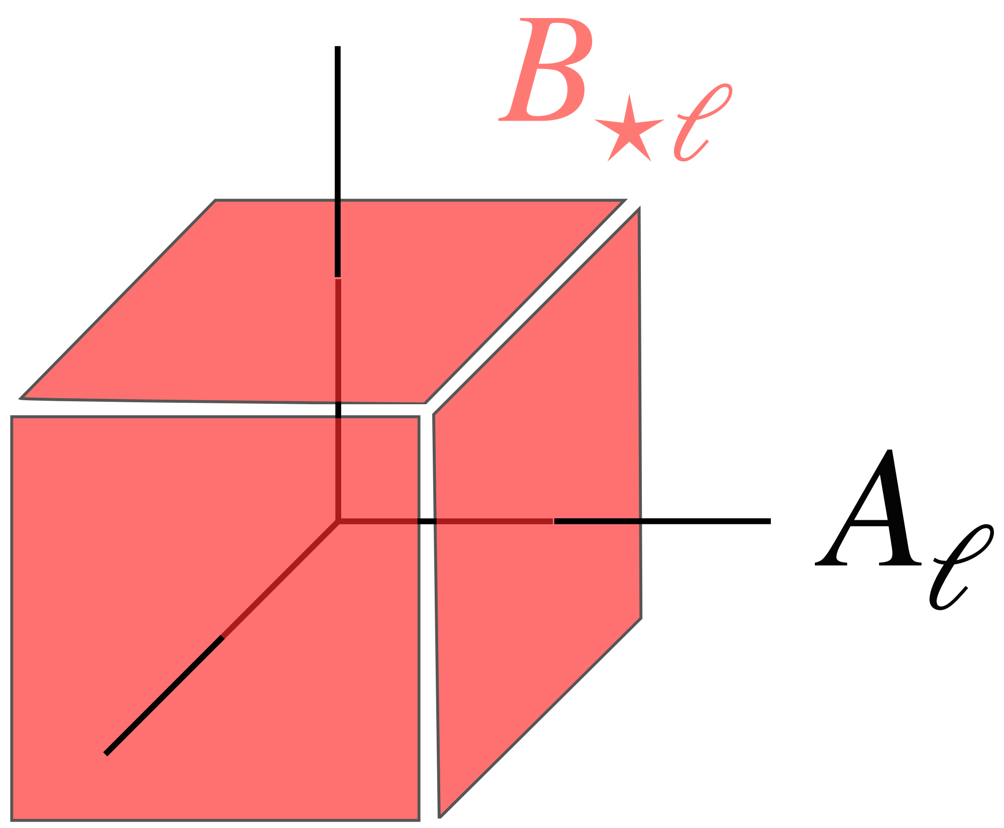

$$\sum_\ell A_\ell B_{\star\ell} \rightarrow \sum_\ell A_\ell B_{\star\ell} + \sum_x \lambda_x (dB)_{\star x}$$

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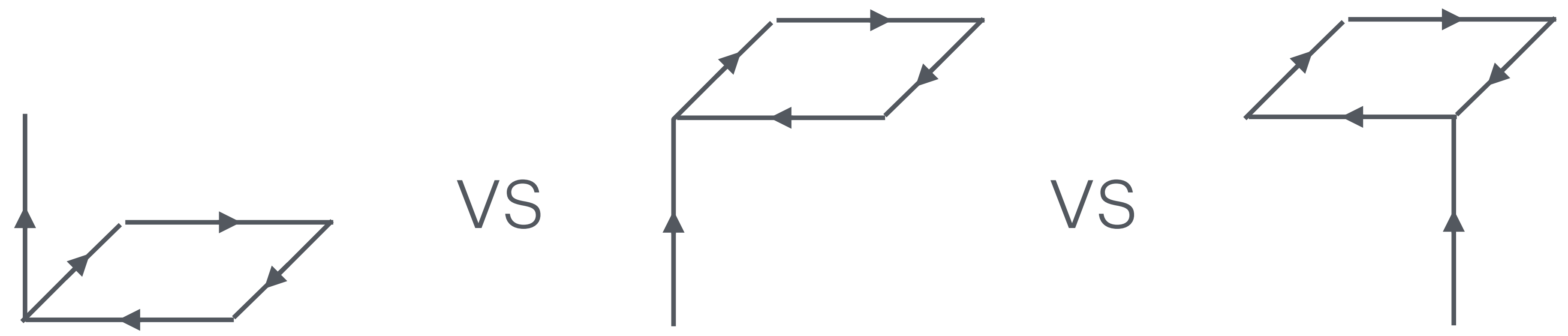

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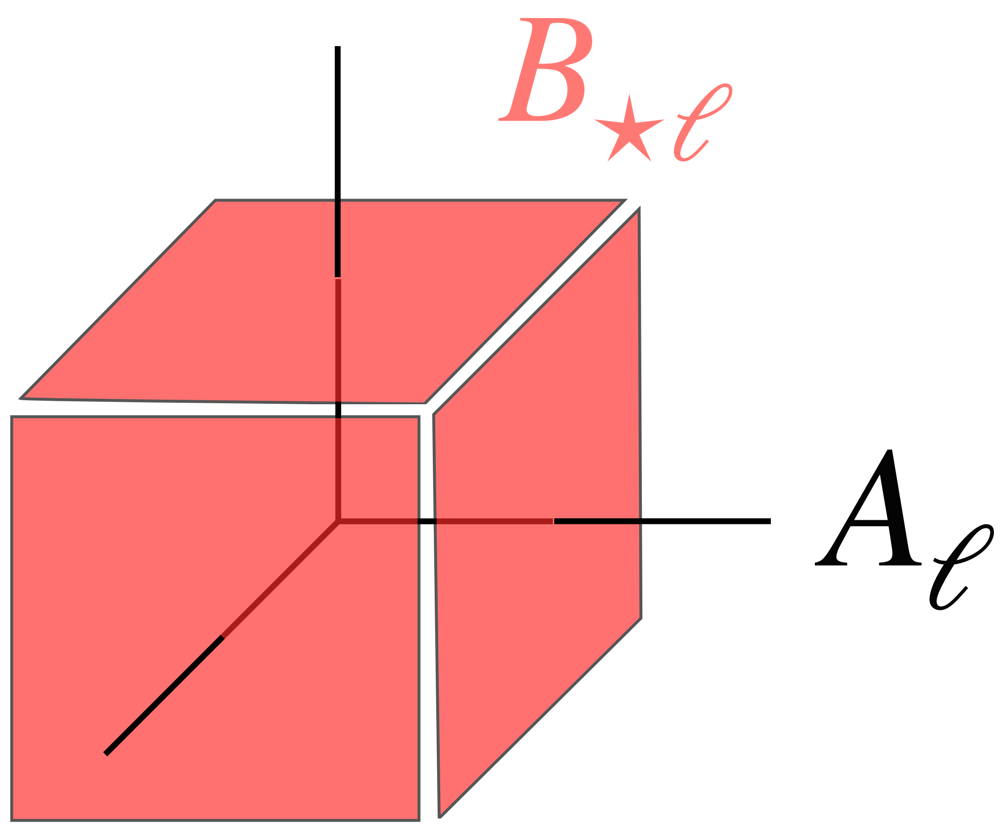

$$\sum_\ell A_\ell B_{\star\ell} \to \sum_\ell A_\ell B_{\star\ell} + \underbrace{\sum_x \lambda_x (dB)_{\star x}}_{=0 \text{ if } dB=0}$$

$$A_\ell \to A_\ell + (d\lambda)_\ell$$

A cup product

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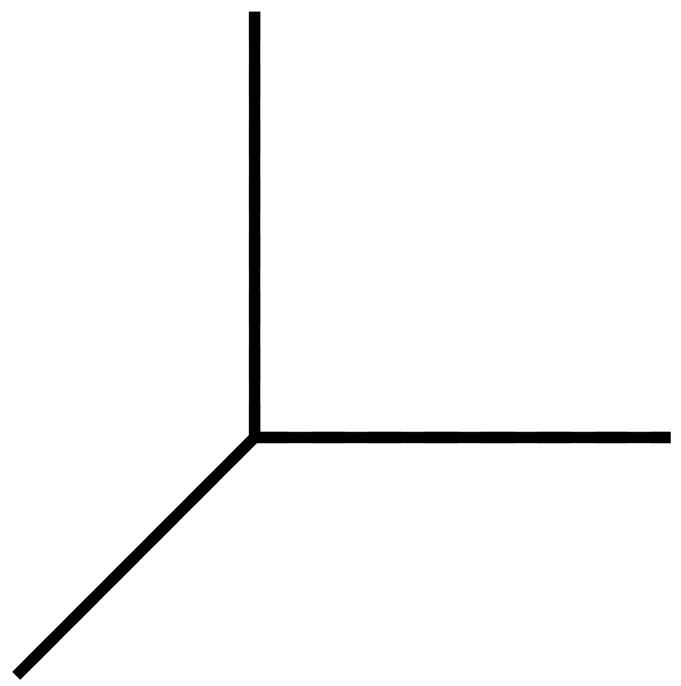

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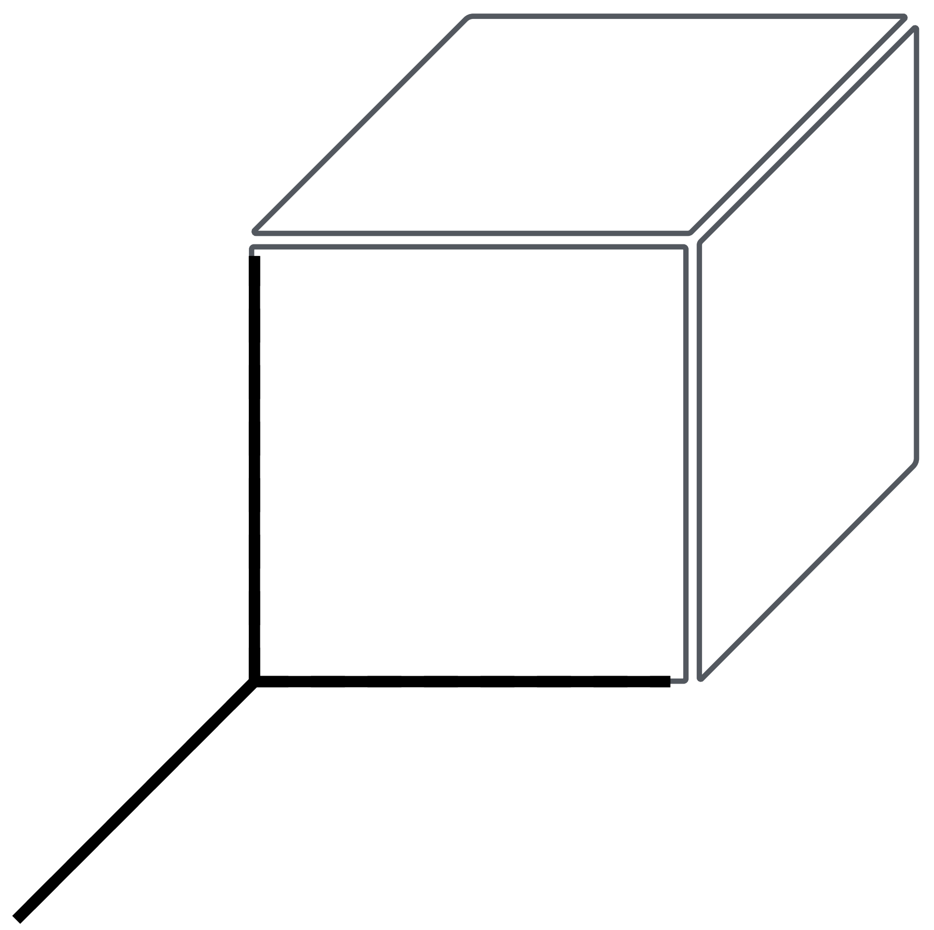

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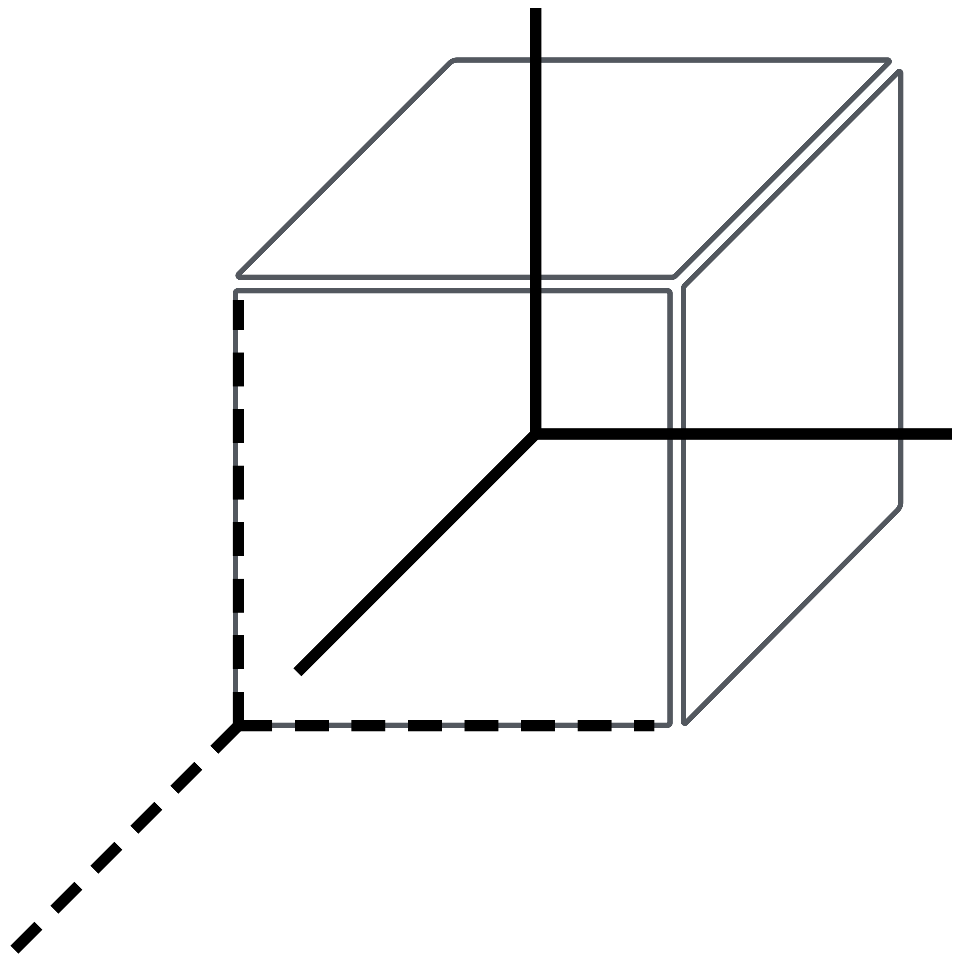

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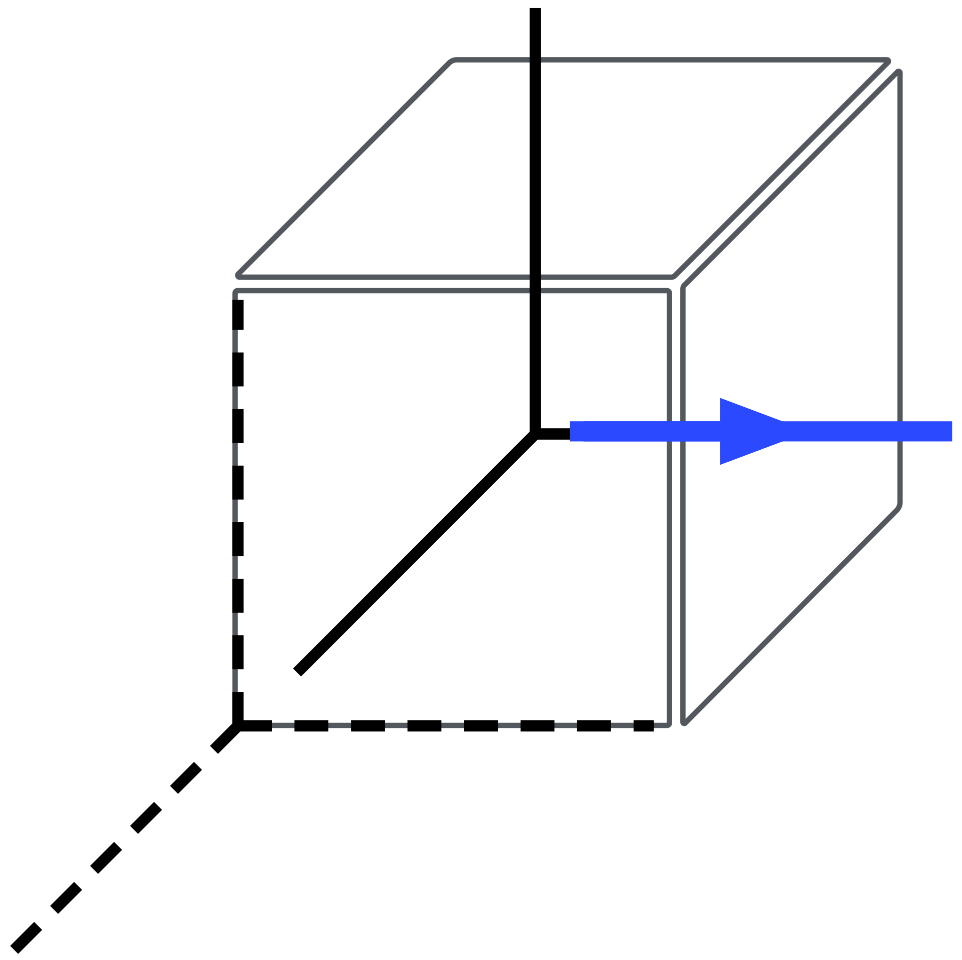

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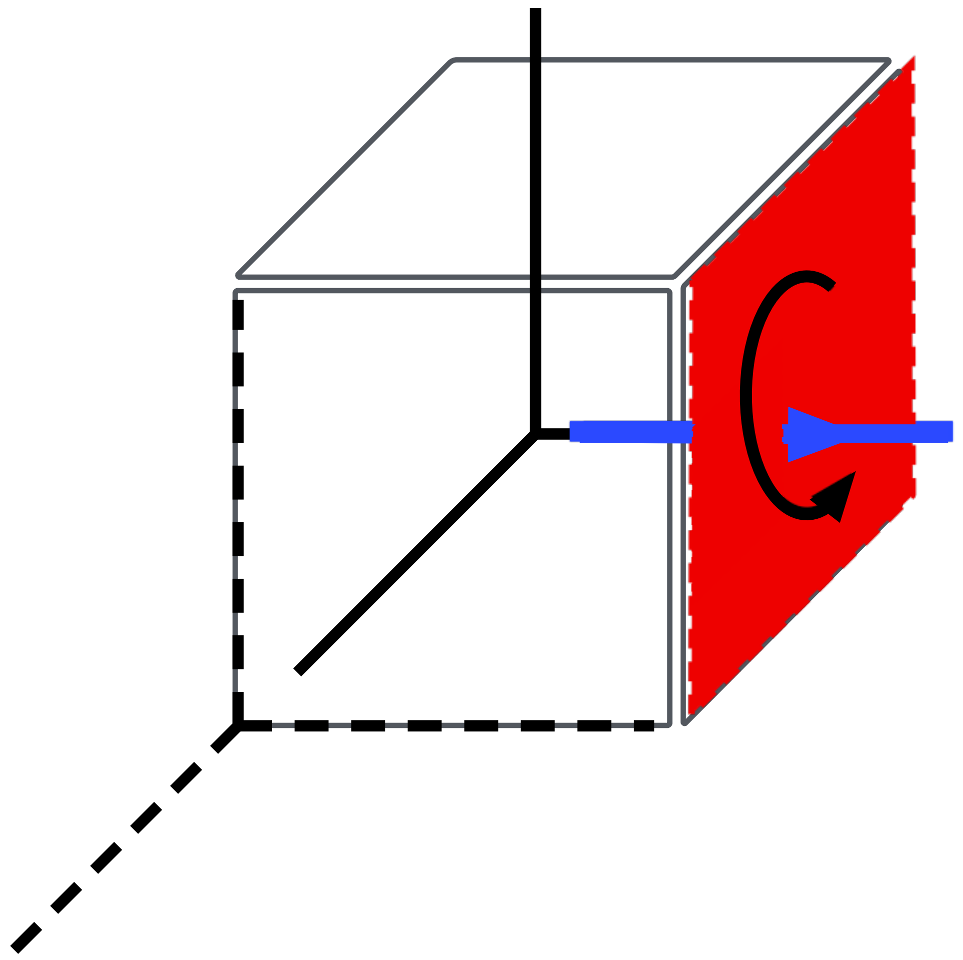

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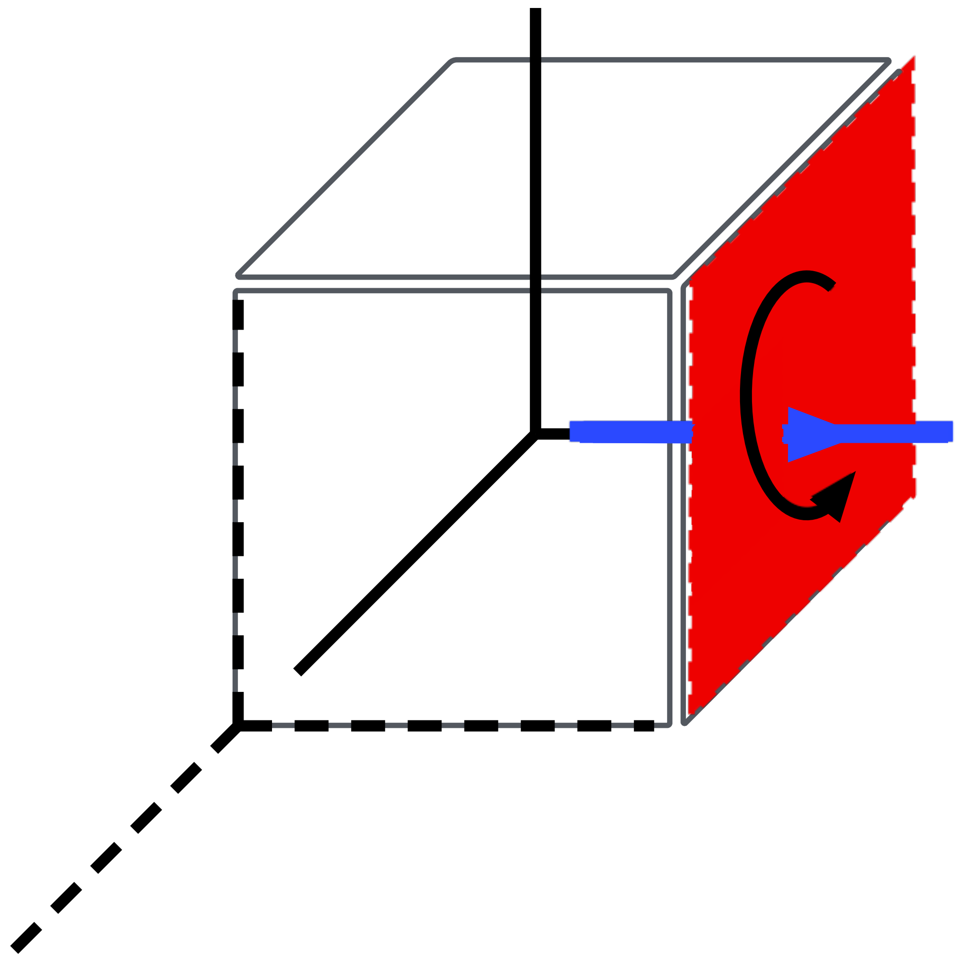

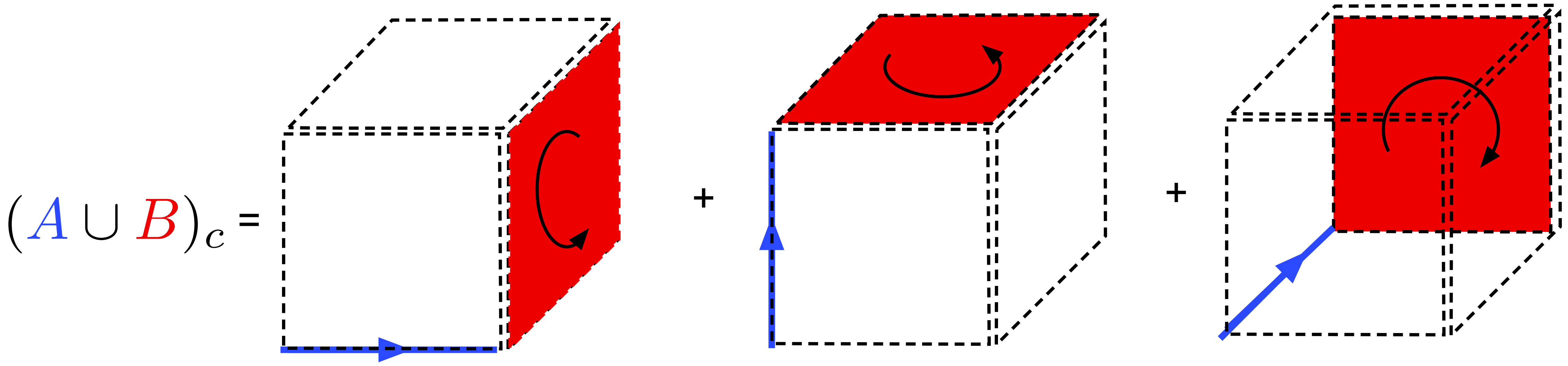

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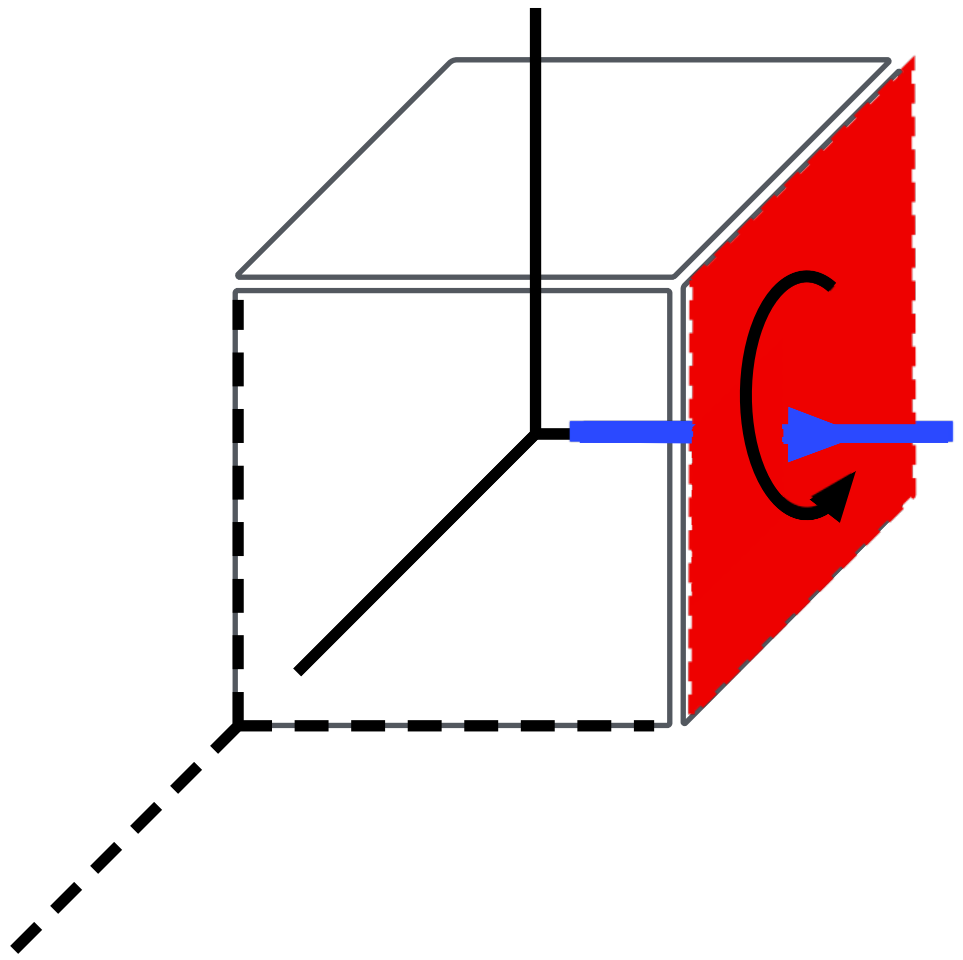

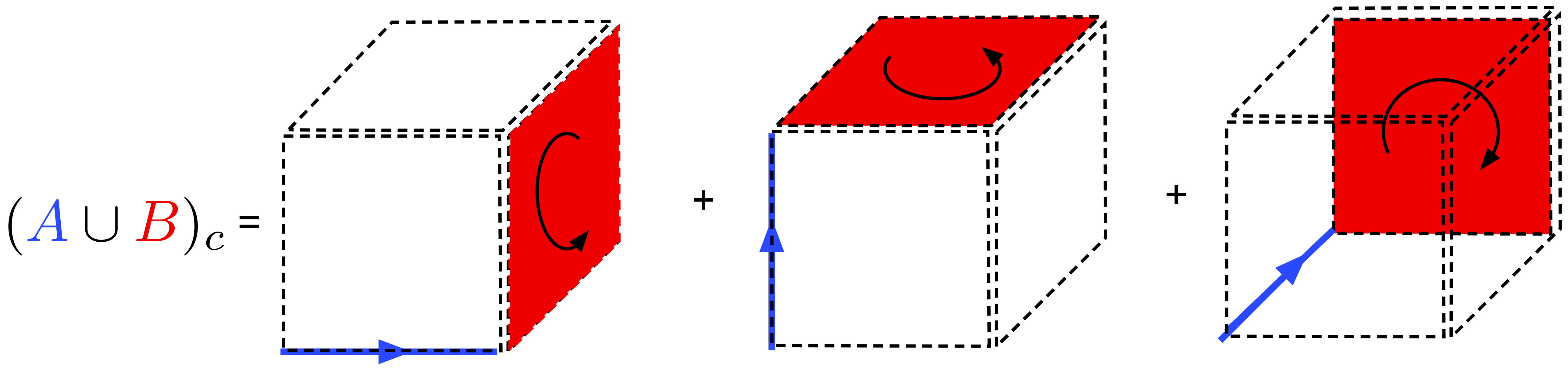

$$S = \frac{k}{4\pi} \sum_c (A \cup dA)_c$$

Invariant under $A \to A + d\lambda$

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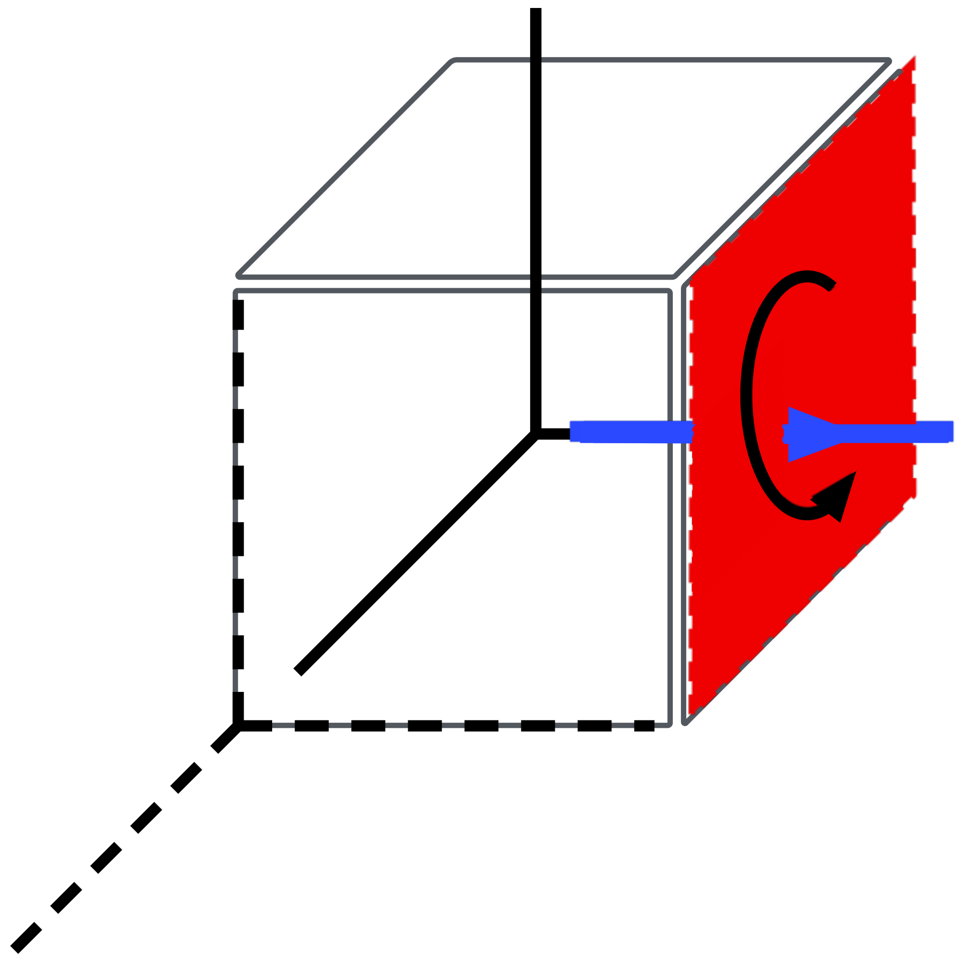

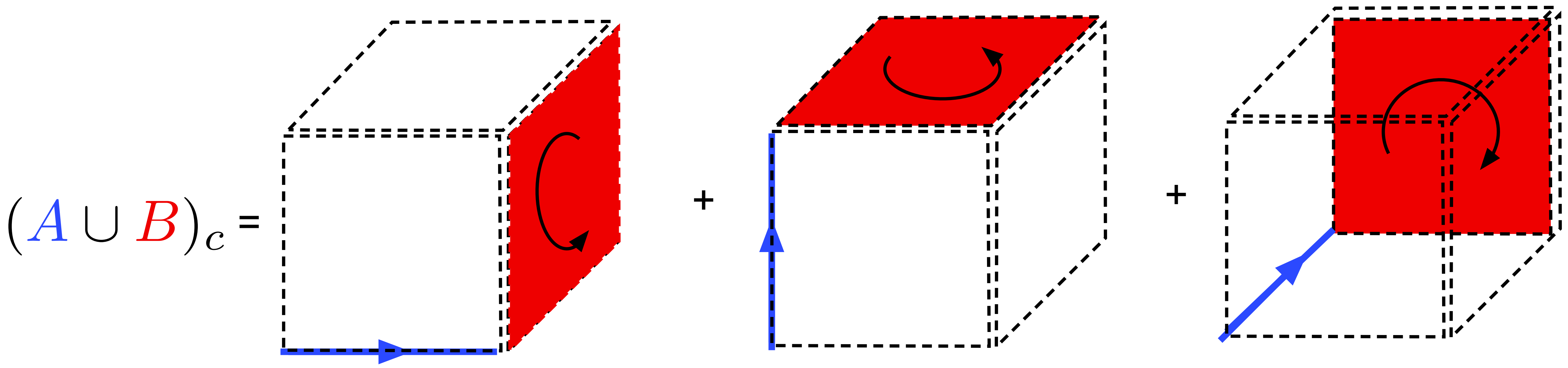

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We also must have invariance: $A_\ell \to A_\ell + 2\pi k_\ell,\ k_\ell \in \mathbb{Z}$

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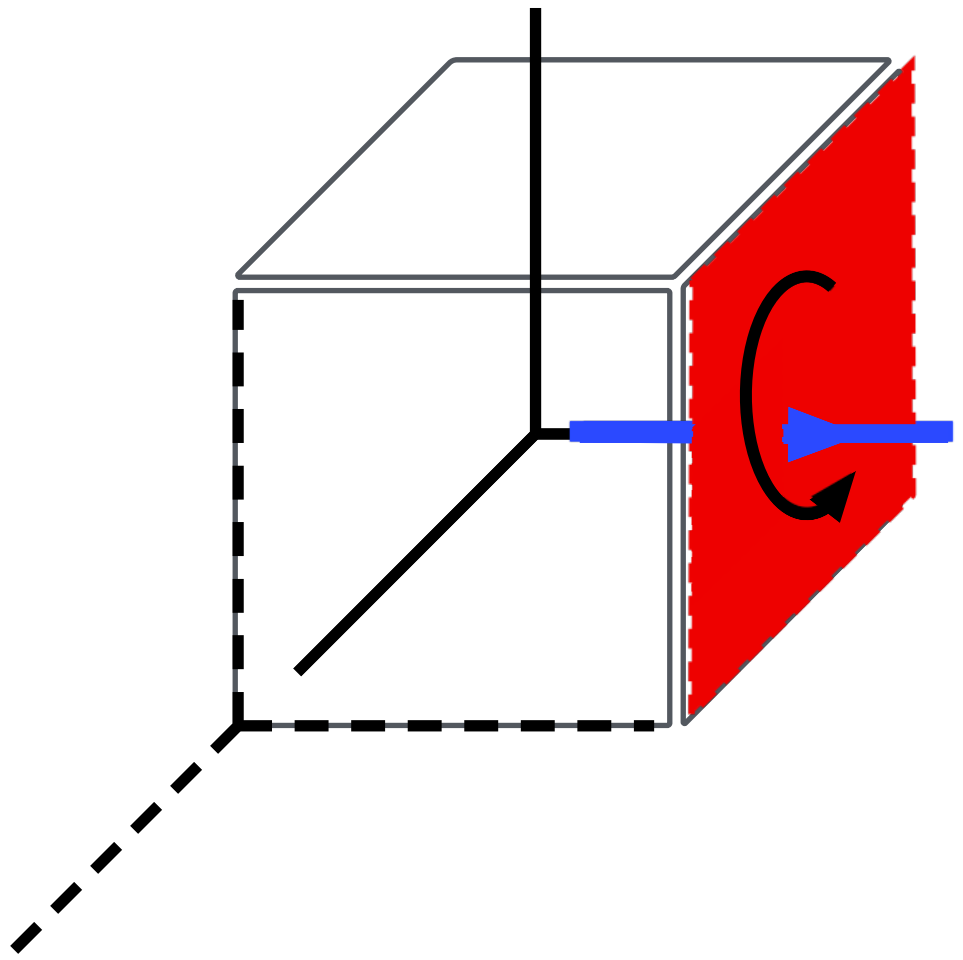

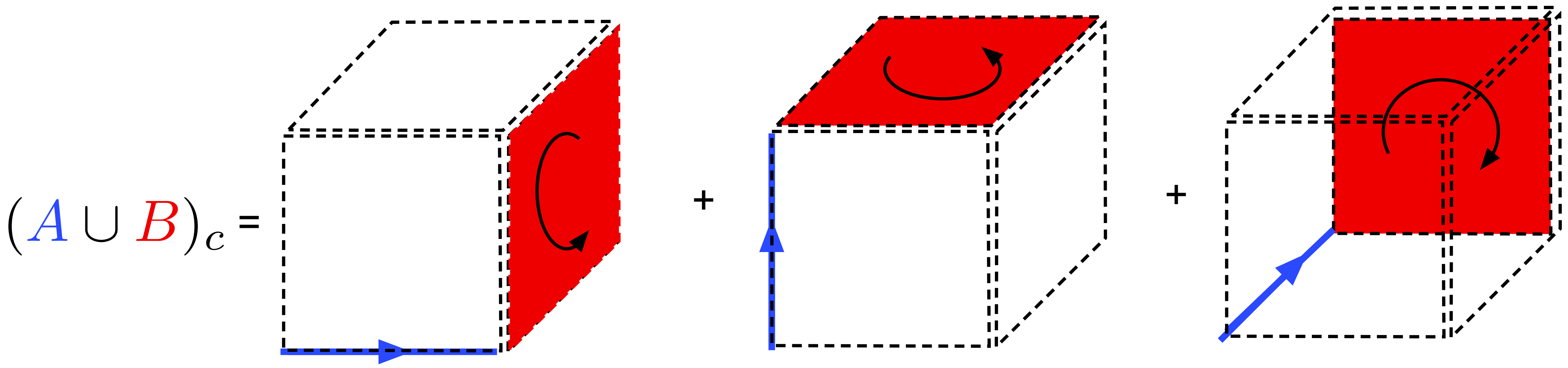

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But first... there is a problem

“The doubler” problem

$$L = \frac{k}{4\pi} \int d^3p\, A_\mu(-p) G_{\mu\nu}(p) A_\nu(p)$$

Spectrum of $G_{\mu\nu} : \lambda(p) = \pm \sqrt{3 - \cos p_1 - \cos p_2 - \cos p_3}\sqrt{1 + \cos(p_1 + p_2 + p_3)}$

Zeromode if $p_1 + p_2 + p_3 = \pi$ [Frolich and Marchetti, ’89]

These zero modes always exist if G is chosen such that the action is

[Berruto, Diamantini, Sodano, ’00]

1. gauge invariant
2. local
3. parity odd

These modes are a consequence of a symmetry

$$A_\ell \to A_\ell + \epsilon_\ell$$

$$\Delta S = \sum_c (\epsilon \cup X + X \cup \epsilon)$$

$$X = dA + \frac{1}{2} d\epsilon$$

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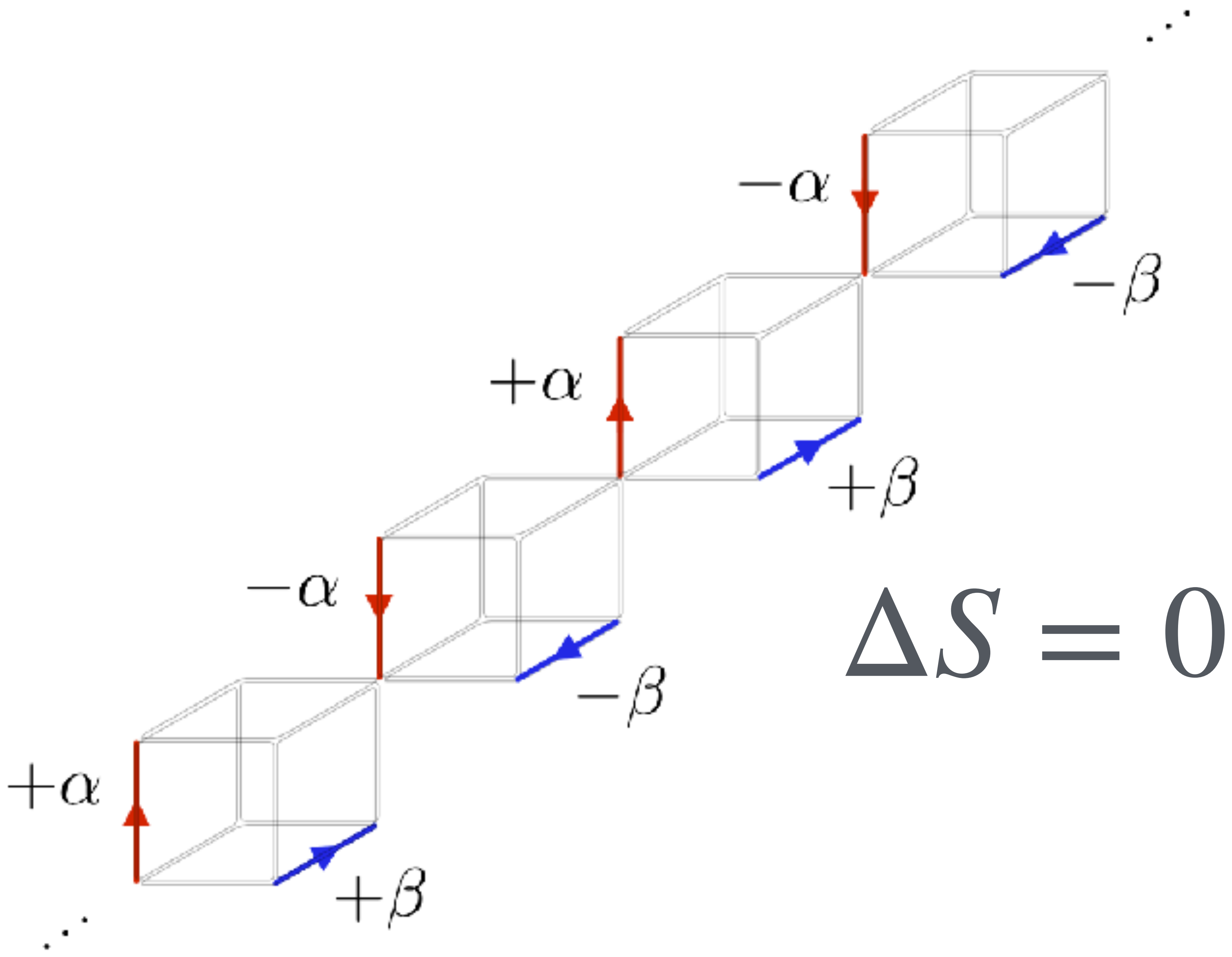

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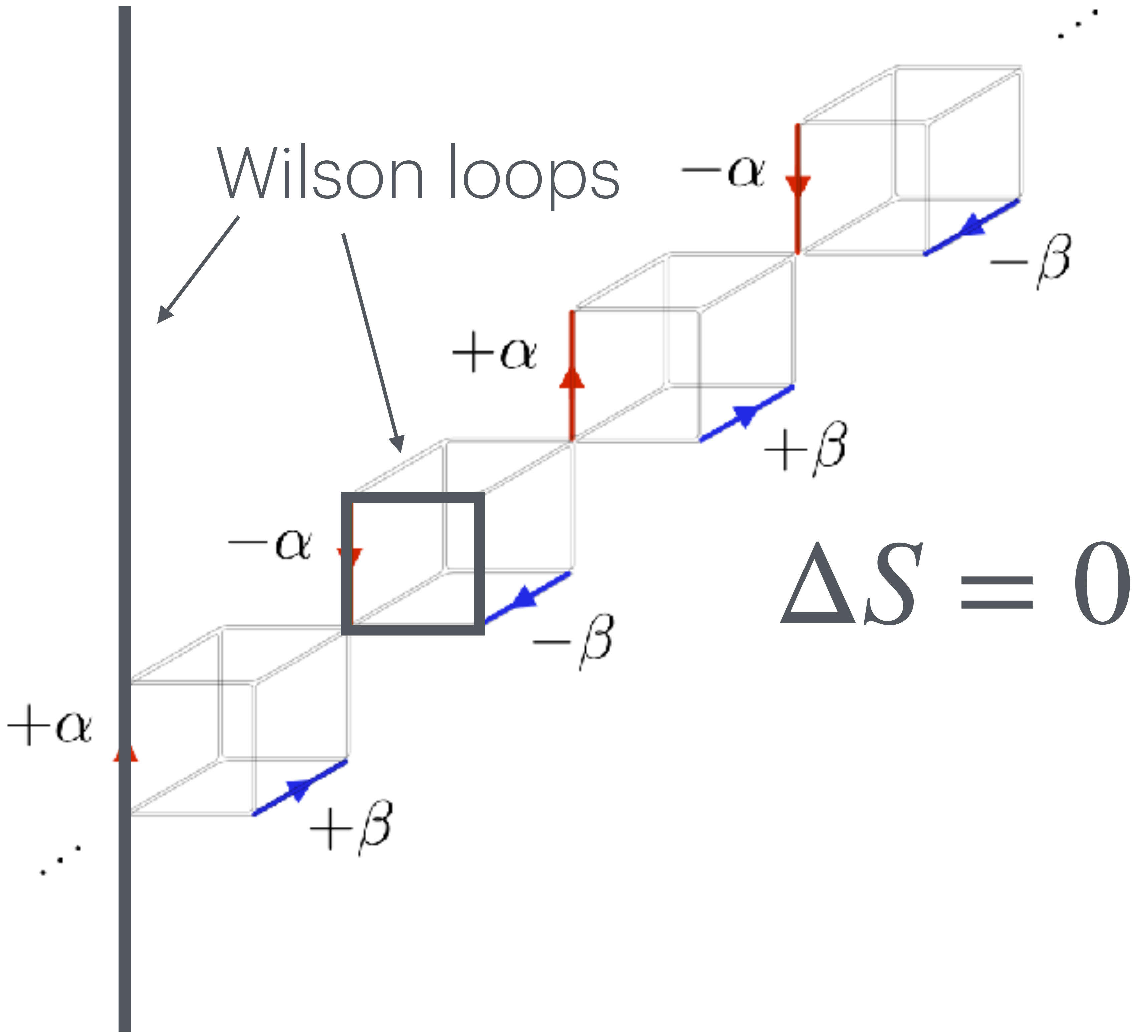

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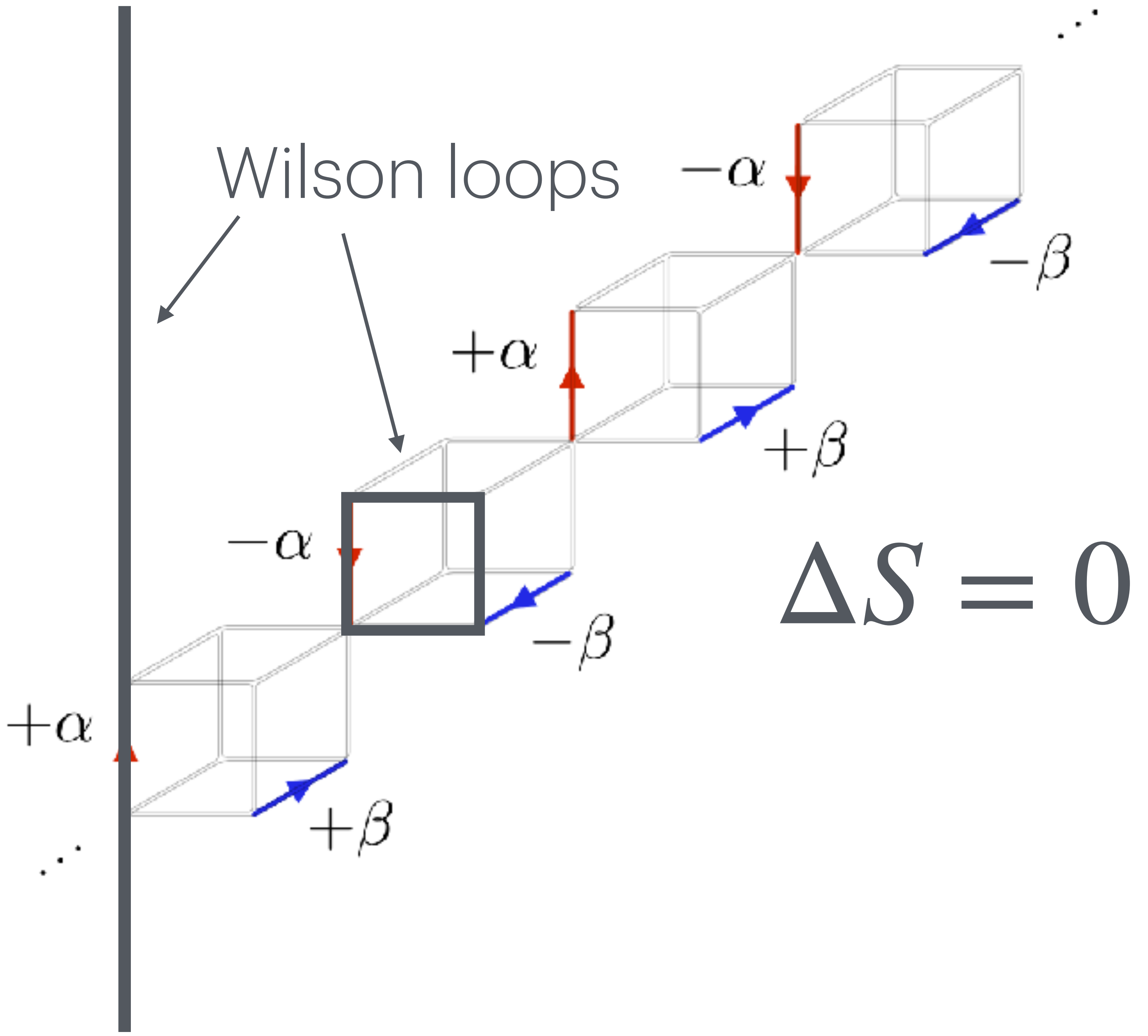

Wilson loops seem to be charged under this weird symmetry.
Since this symmetry cannot be spontanously broken, almost all Wilson loops vanish!
But... let's ignore this for now and make the theory compact

Now we make the theory compact

$$S = \frac{k}{4\pi} \sum_c \left(A \cup dA - 2\pi(a \cup n + n \cup a) \right)$$

with the constraint $dn = 0$

$$A_\ell \to A_\ell + (d\lambda)_\ell + 2\pi m_\ell$$
$$n_p \to n_p + (dm)_p$$

$$\Delta S \in 2\pi\mathbb{Z}, \text{ if } k \in 2\mathbb{Z}$$

Now we make the theory compact

$$S = \frac{k}{4\pi}\sum_c \big(A \cup dA - 2\pi(a \cup n + n \cup a)\big) - \frac{k}{2} A \cup_1 dn + \varphi \cup dn$$

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$$A_\ell \to A_\ell + (d\lambda)_\ell + 2\pi m_\ell$$

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$e^{i\varphi(x)}$- Monopole operator

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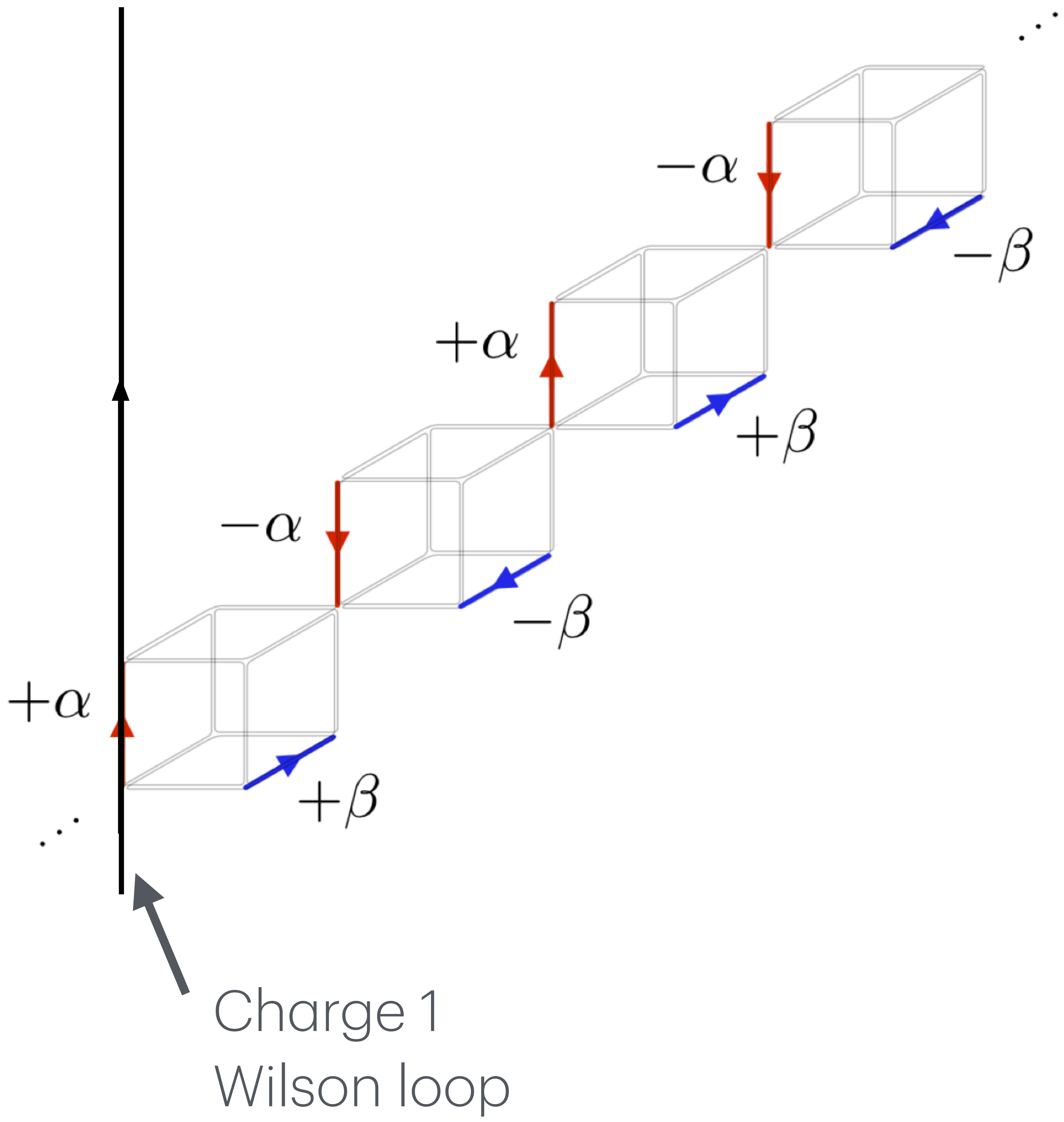
Properties of the theory

$$S = \frac{k}{4\pi} \sum_c \left(A \cup dA - 2\pi(a \cup n + n \cup a) \right)$$

with the constraint $dn = 0$

- It has a Z(N) 1-form symmetry $A_\ell \to A_\ell + \frac{2\pi\omega_\ell}{N}$, $d\omega = 0$ and $\omega_\ell \in \mathbb{Z}$
- It has an appropreate Z(N) 't Hooft anomaly
- Still has a weird staggared symmetry, which causes Wilson loops to vanish

Staggered symmetry revisited



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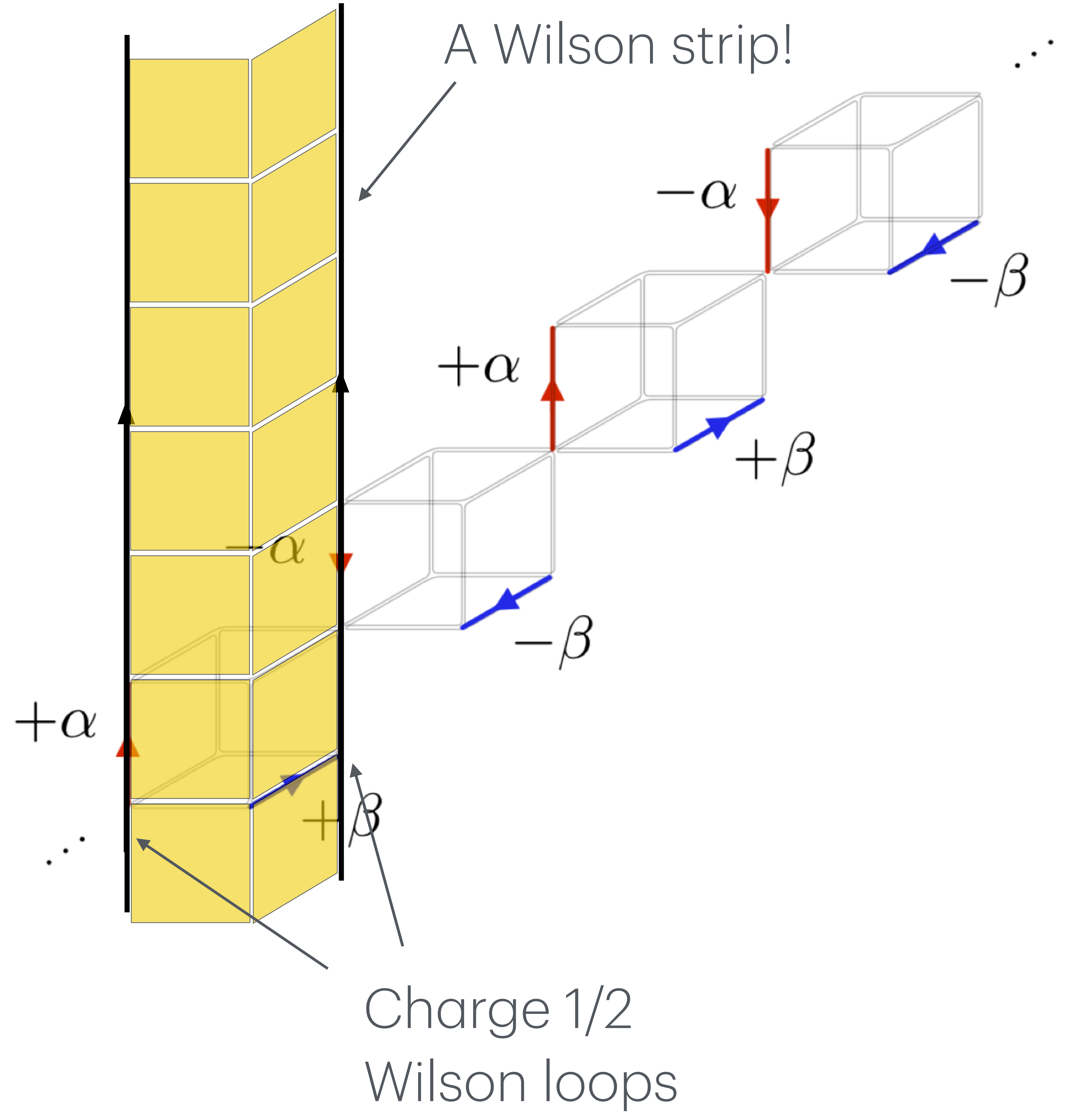

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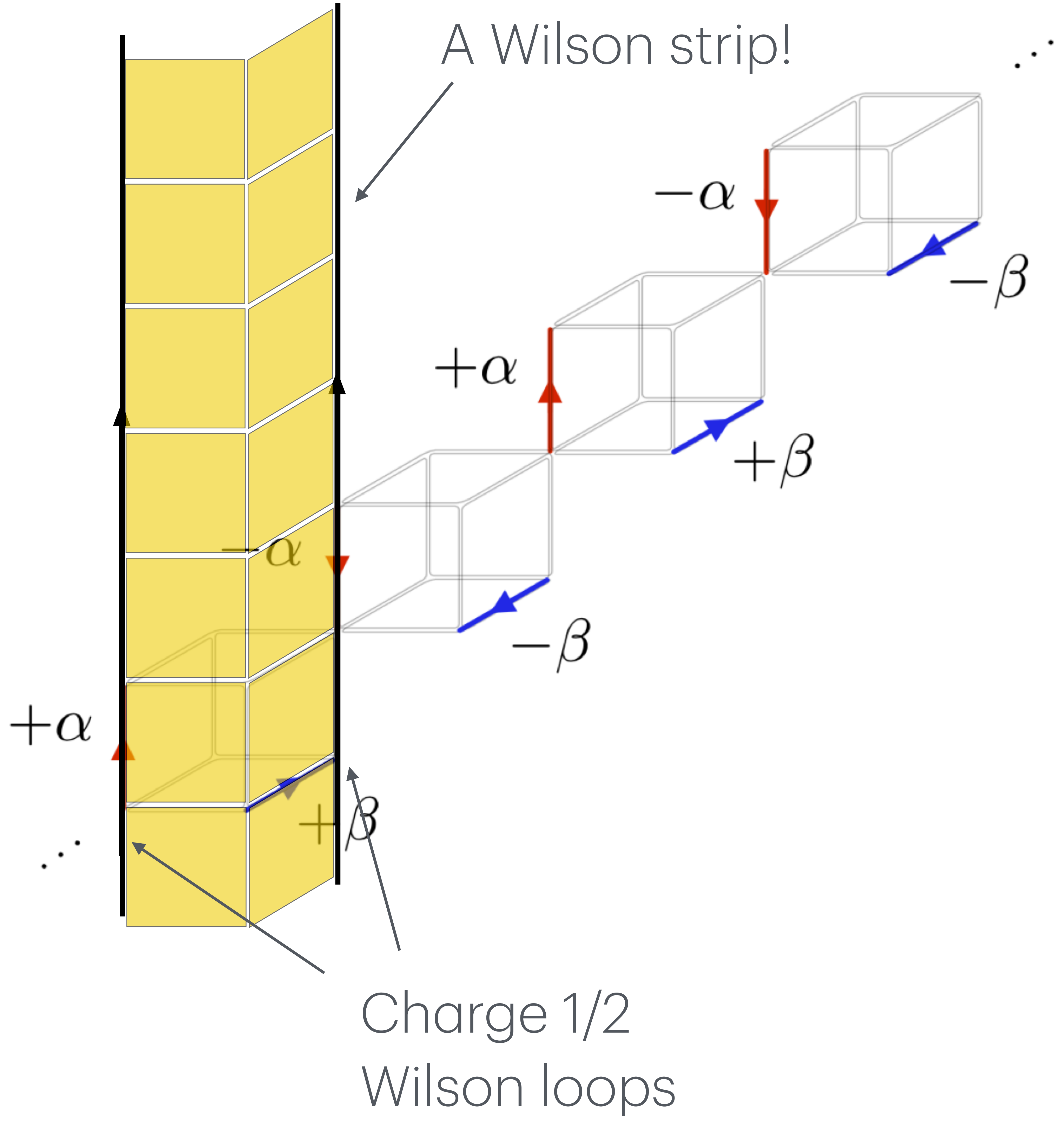

- These strips turn out to be the charges of the Z(N) symmetry, guaranteeing their topological nature

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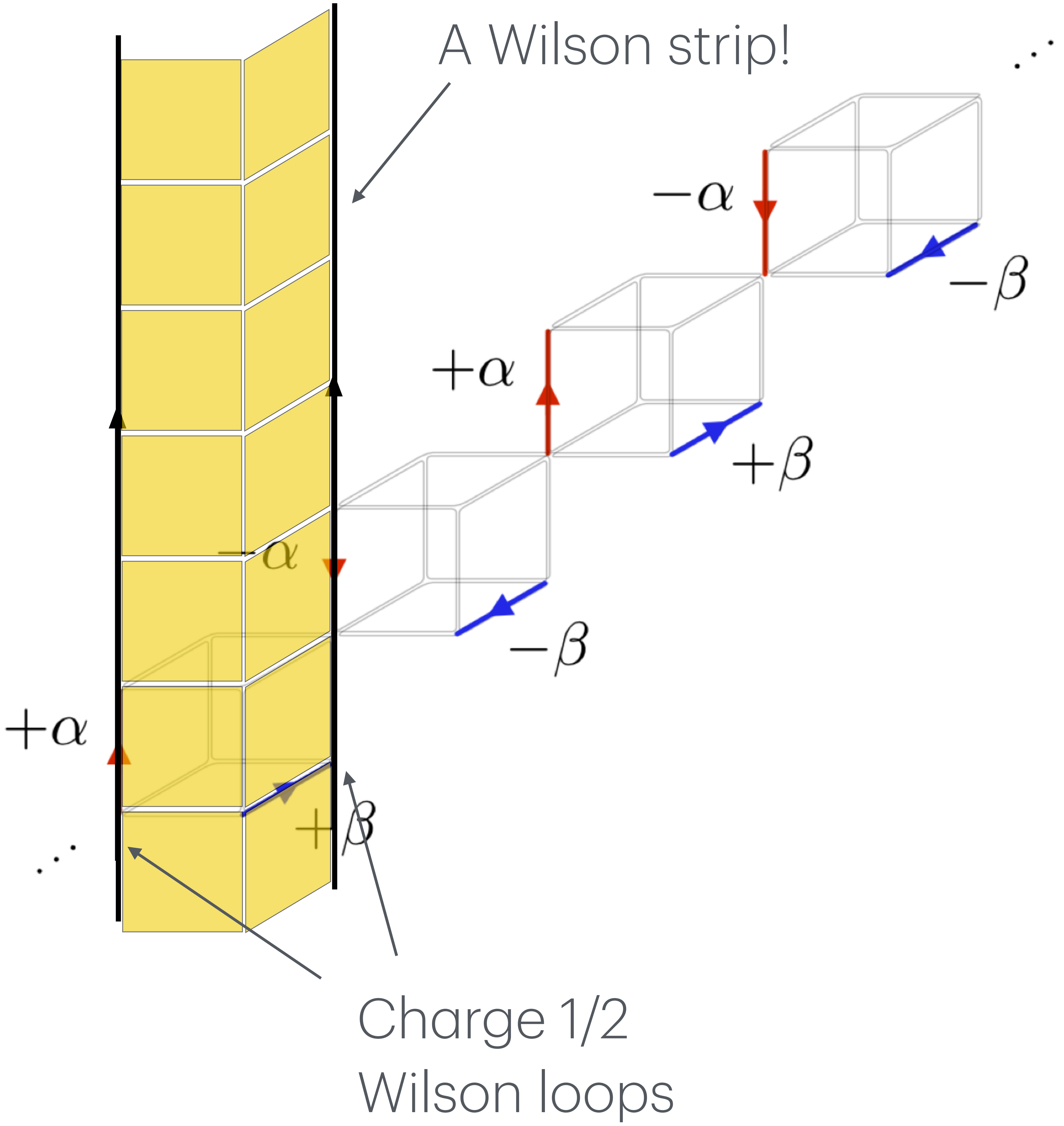

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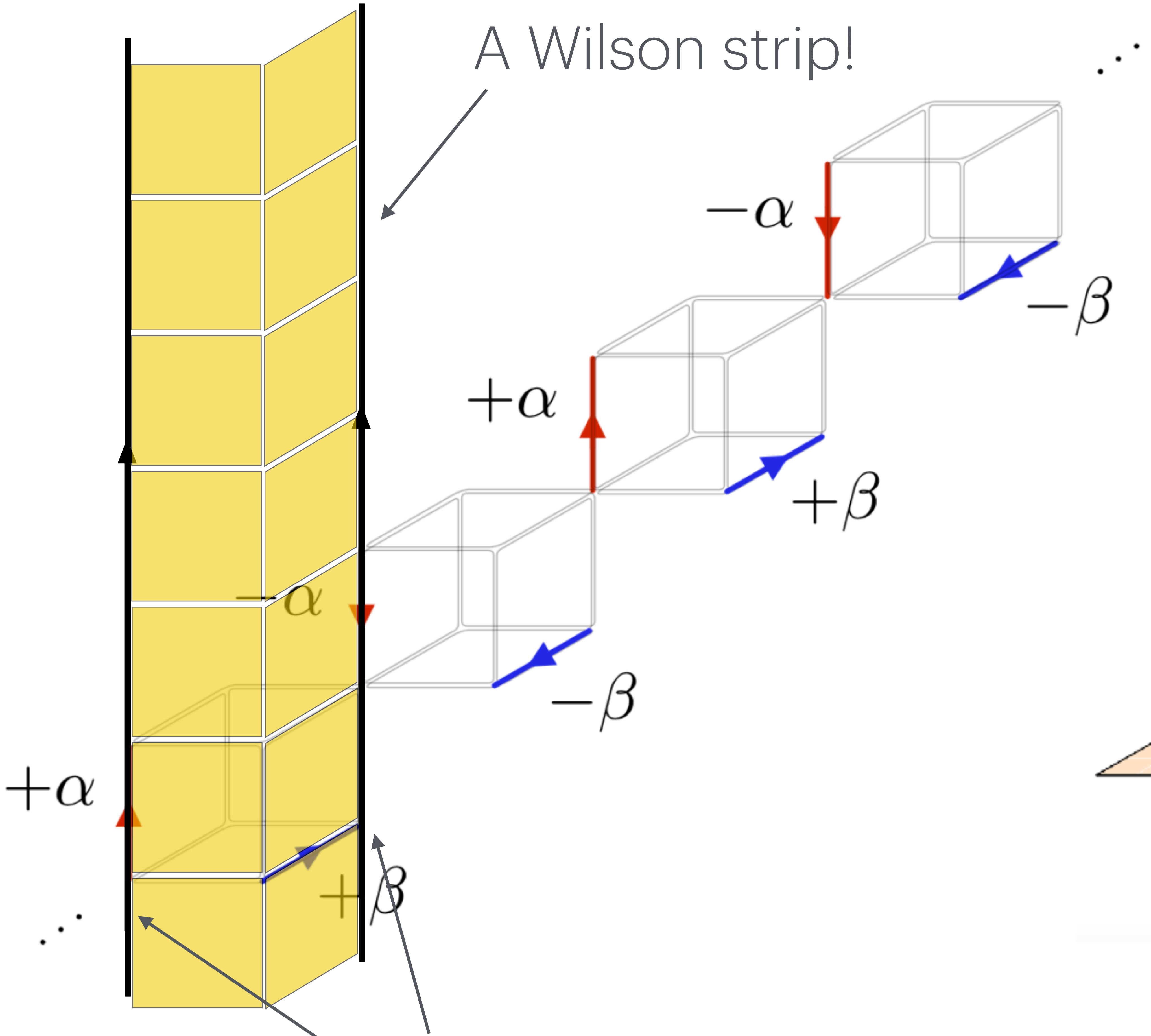

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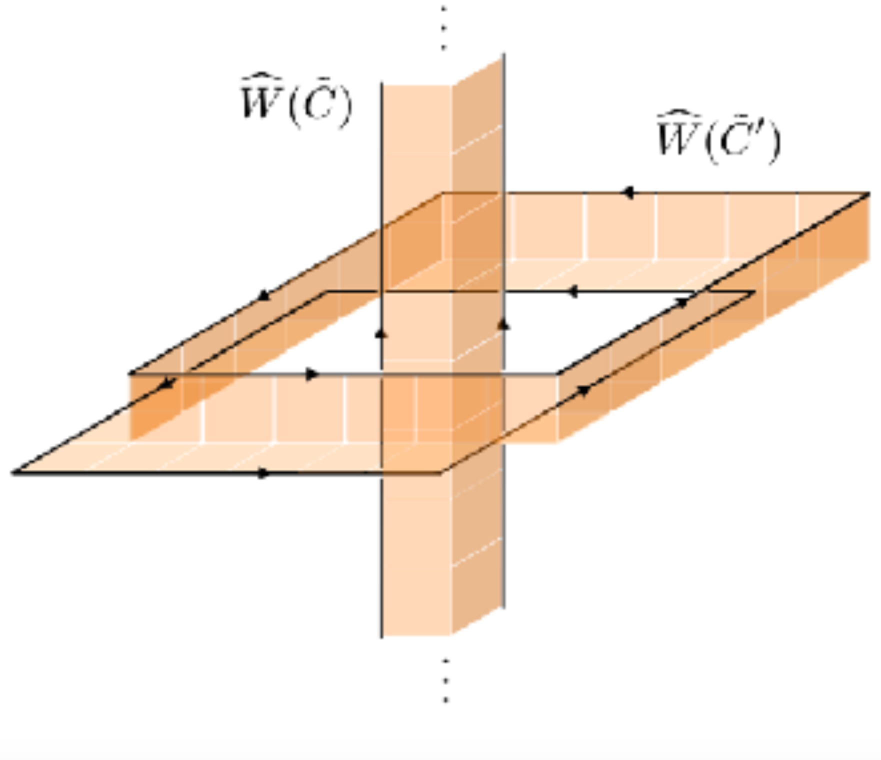

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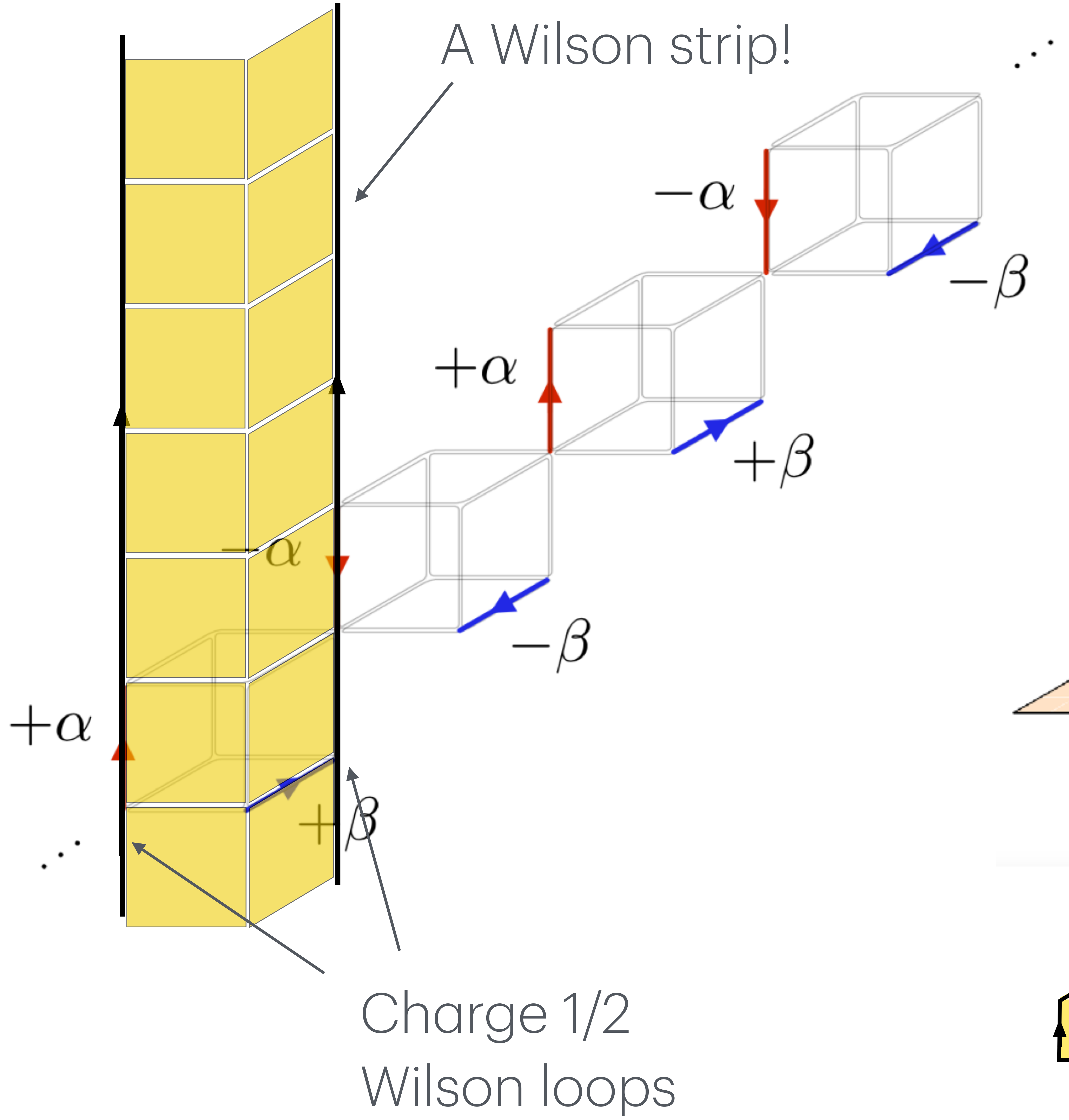

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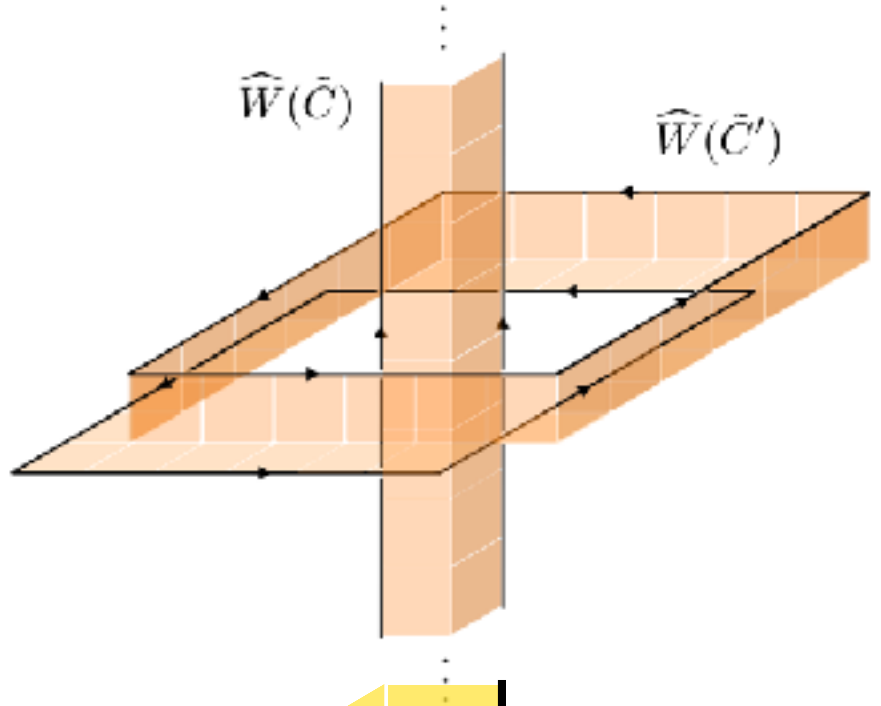

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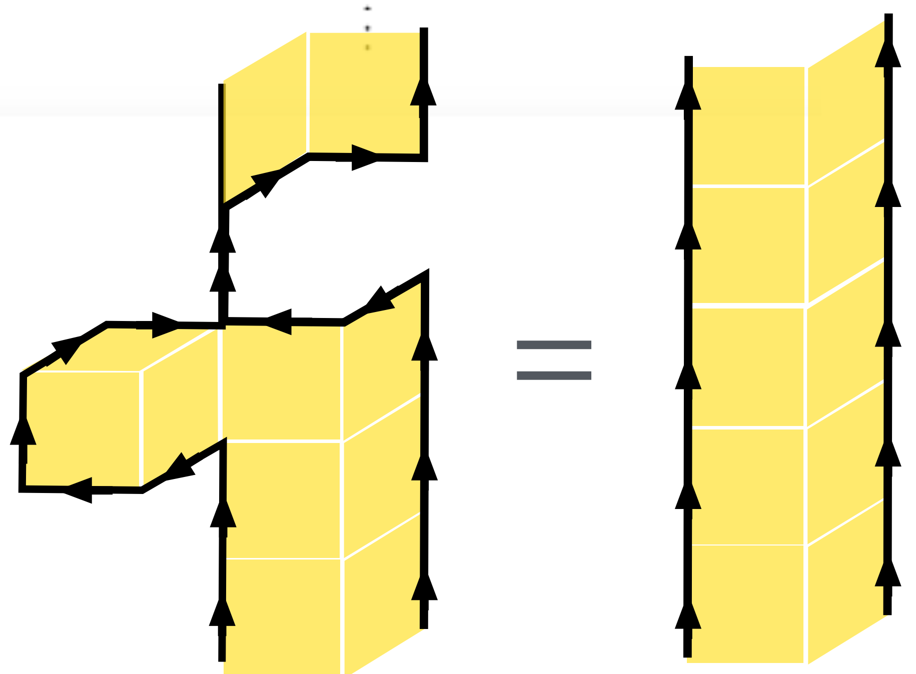
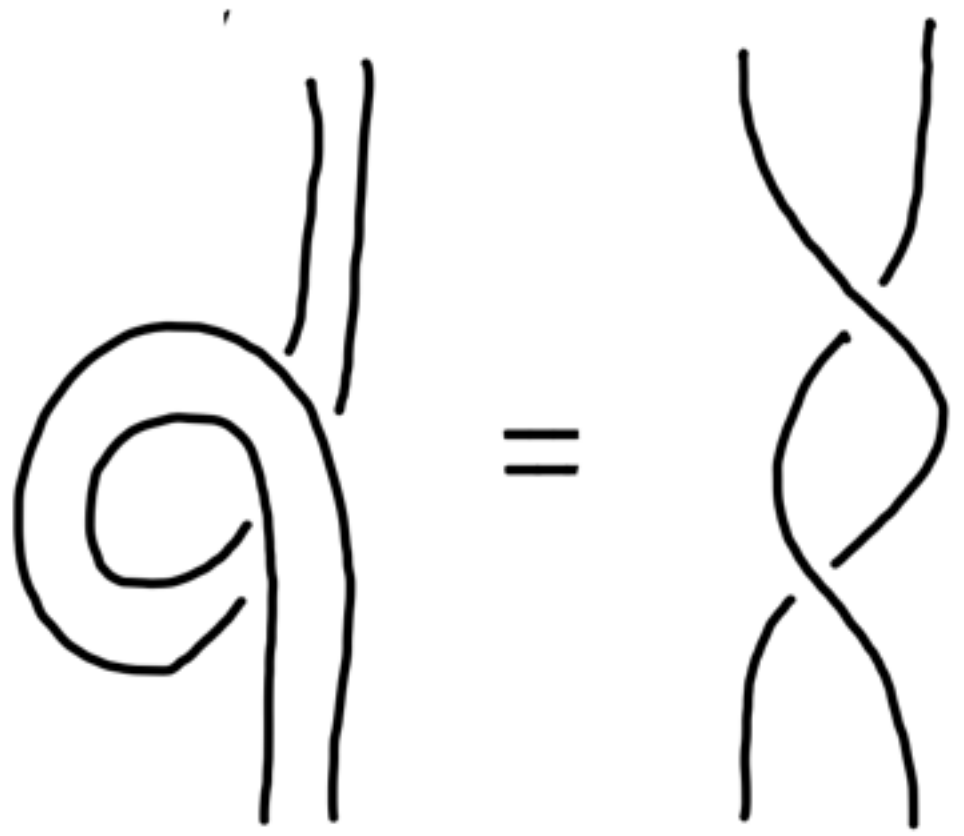


$$e^{\frac{2\pi i}{2k}}$$



Are weird zeromodes a problem? Do they imply criticality?

Introduce a staggared-symmetry invraint source H

$$e^{i\sum_c H\cup(da-2\pi n)+(da-2\pi n)\cup H}$$

can be removed, up to universal contact terms, by a shift $A \to A + \frac{4\pi}{k}H$

Conclusion: the theory has only a topological sector.

Some comments

- The system can be quantized canonically (continuous time) [Theo Jacobson, TS '24]

- Similar spatial staggered symmetry presist, but this time it can be explicitly shown that it is a consequence of the constraint on the Hilbert space, i.e. it is a gauge symmetry.

- The k odd case can be treated by introducing a fermion [Theo Jacobson, TS '24] [Ze-An Xu, Jing-Yuan Chen '24]

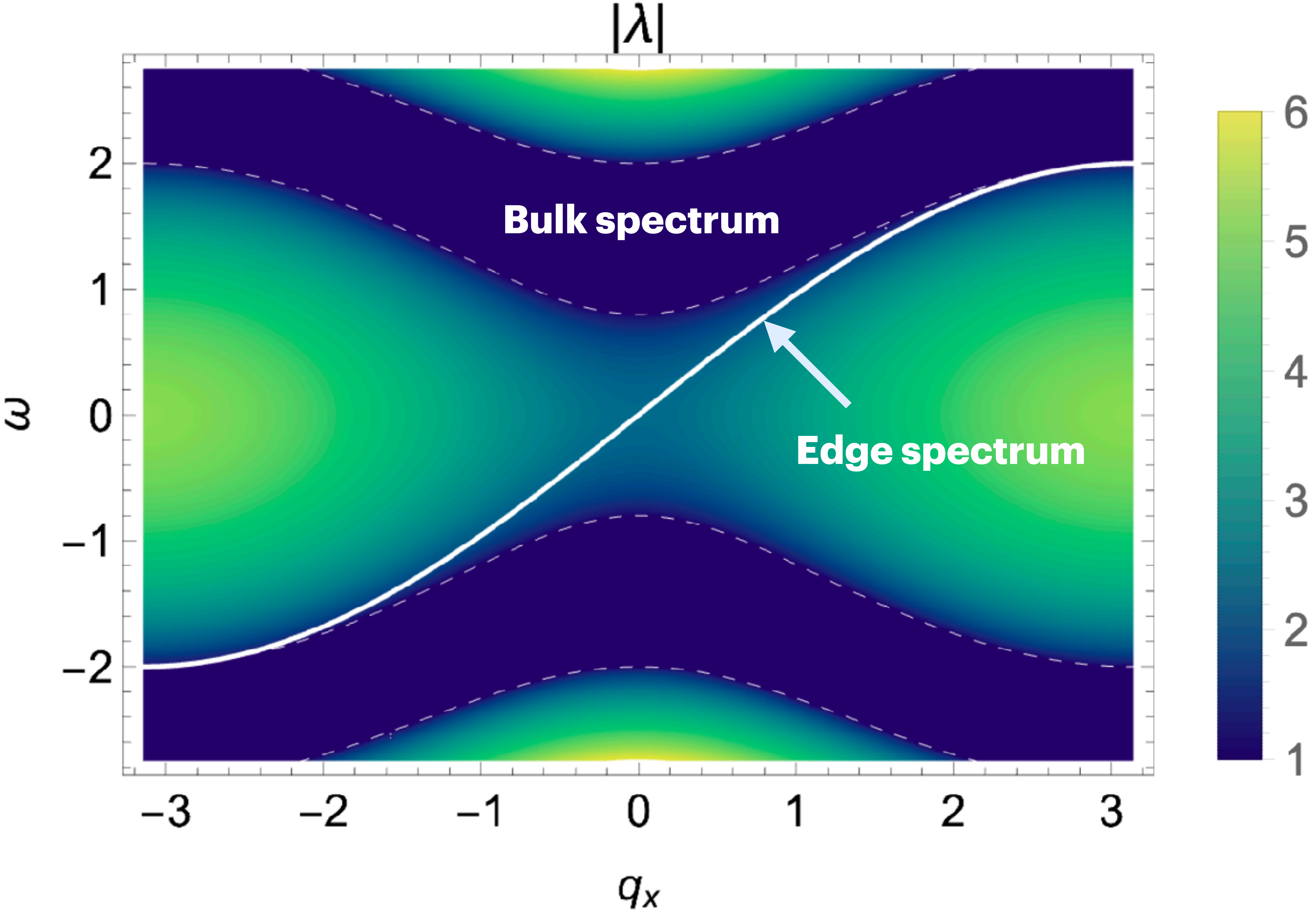
- The staggered symmetry is lifted by introducing the Maxwell term. Nevertheless the topological 1-form symmetry charges are still strips: Wilson loops attached to dangling ladder of canonical momentum of A_ℓ [Ze-An Xu, Jing-Yuan Chen '24], [C. Peng, M. C. Diamantini, L. Funcke, S.M.A. Hassan, K. Jensen '24]

- Introducing the "metric-dependent" Maxwell term may be necessary for understanding the gravitation anomaly [Ze-An Xu, Jing-Yuan Chen '24]

- On the space-time lattice with boundary Chern-Simons-Maxwell theory has an edge mode [Ze-An Xu, Jing-Yuan Chen '24]

- The same paper makes claims that the gravitational anomaly can be extracted from the Maxwell-Chern-Simons theory.

Work in progress: Edge mode in canonically quantized Maxwell-Chern-Simons



Conclusions

- The U(1) CS theory on the lattice has a natural formulation in terms of the Modified Villain formalism
- The construction has the correct $\mathbb{Z}_k$ 1-form symmetry and ’t Hooft anomaly
- The ominous-sounding “doubler” zero-modes serve to eliminate unframed Wilson loops
- The theory is void of any other content except the topological sector
- Maxwell-Chern-Simons theory has an appropriate edge mode