Toy Models of Fortuity

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Based on arXiv:2412.06902 with
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Motivation: Black Hole Microstates at finite N

Horizon is a ubiquitous feature of classical general relativity.

We've learned many of its signatures in holography, such as thermalization, scrambling, random matrix statistics, etc.

These universal features of black hole microstates become sharpest in the $N \to \infty$ limit.

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These universal features of black hole microstates become sharpest in the $N \to \infty$ limit.

A feature called **fortuity** [Chang, Lin, '24] reveals curious finite N properties of supersymmetric black hole microstates.

[Building on earlier work by Chang Lin, '22; Choi, Kim, Lee, Park, '22; Choi, Kim, Lee, Lee, Park, '23; Chang, Feng, Lin, Tao, '23; Budzik, Murali, Vieira, '23, Choi, Choi, Kim, Lee, Lee, '23; ...]

Review of Fortuity and Monotony

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A central role is played by **trace relations** - finite N identities relating multi-trace operators that were independent at $N = \infty$.

$$\text{e.g. when } N \leq 2, \quad 2\,\mathrm{Tr}Z^3 - 3\mathrm{Tr}Z\mathrm{Tr}Z^2 + (\mathrm{Tr}Z)^3 = 0.$$

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Heuristically, one can classify supersymmetric (BPS) states as

- **Monotonous**

$$QO = 0 \text{ without using relations}$$

- **Fortuitous**

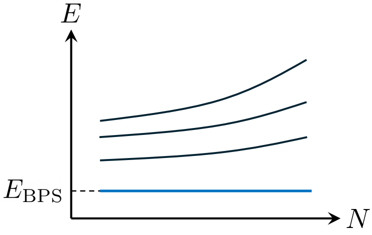
$$QO \in \text{trace relations at } N \leq N_*$$

Can be formulated precisely in terms of Q-cohomology and a long exact sequence [Chang, Lin].

A nice way to "see" fortuity is discussed in [Budzik, Murali, Vieira], where one follows the spectrum of the theory in N by treating N as a continuous variable.

- **Monotonous**

$$QO = 0 \text{ without using relations}$$

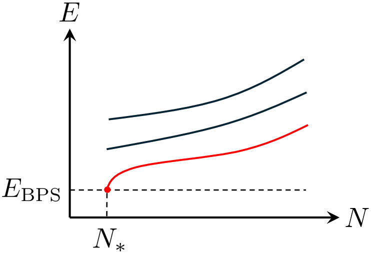


$N \to \infty$ limit (with fixed charges) being multi-graviton BPS states.

At small N, various evidences suggest that they correspond to microstates of horizonless geometries.

- **Fortuitous**

$$Q\mathcal{O} \in \text{trace relations at } N \leq N_*$$



They have non-BPS "origins".

The entropy of such states matches with the black hole entropy.

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The study of fortuitous states has so far been limited to small N.

- First fortuitous cohomology and the corresponding BPS state in the $\mathrm{SU}(2)$ theory.

  [Chang, Lin; Choi, Kim, Lee, Park; Budzik, Murali, Vieira]

- Infinite towers of fortuitous cohomologies in the $\mathrm{SU}(2)$ and $\mathrm{SU}(3)$ theory.

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Finding them involves laborious numerics and hard guesswork. The general emerging features at large N remain mysterious.

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Goal today: understand fortuity in toy models and extract universal lessons!

Toy Model 1: $\mathcal{N} = 2$ SUSY SYK Model

The $\mathcal{N} = 2$ SYK model is a quantum mechanical model with N complex fermions [Fu, Gaiotto, Maldacena, Sachdev]

$$\{\psi_i, \bar{\psi}_j\} = \delta_{ij}, \quad i, j = 1, ..., N$$

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$$\{\psi_i, \bar{\psi}_j\} = \delta_{ij}, \quad i, j = 1, ..., N$$

$$Q = \sum_{i_1,...,i_q} C_{i_1,...,i_q} \psi_{i_1} \psi_{i_2} ... \psi_{i_q}, \quad H = \{Q, Q^\dagger\}.$$

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$C_{i_1,\dots,i_q}$ is fully anti-symmetric, with components drawn randomly from a complex Gaussian ensemble - a **ramdom q-form** C_q.

- Solvable in the large N limit using collective fields.
- Contains an extensive amount of zero energy states ($S \sim N$).
- Low energy dynamics governed by a supersymmetric ($\mathcal{N}=2$) Schwarzian theory shared by near-BPS black holes.

  [$AdS_5 \times S^5$: Boruch, Heydeman, Iliesiu, Turiaci; $AdS_4 \times S^7$: Heydeman, Toldo]

The Hilbert space is graded by the R-charge = number of fermions.

An arbitrary state with $R = p$ is given by

$$|\alpha_p\rangle = \frac{1}{p!} \sum_{i_1,\ldots,i_p} \alpha_{i_1,\ldots,i_p} \psi_{i_1} \ldots \psi_{i_p} |\Omega\rangle \ ,$$

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where α_p is a p-form.

$$\boxed{\alpha_p \xrightarrow{Q} C_q \wedge \alpha_p}$$

Underlying cohomology problem (cohomologies $\leftrightarrow$ BPS states):

Q-closed: $C_q \wedge \alpha_p = 0$;
Q-exact: $\alpha_p = C_q \wedge \beta_{p-q}$.

Where are the BPS states?

Fix p and take $N \to \infty$, the map $\alpha_p \xrightarrow{Q} C_q \wedge \alpha_p$ looks like

$$\dim \mathcal{H}_{p+q} \sim N^{p+q}$$

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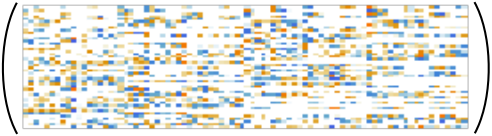

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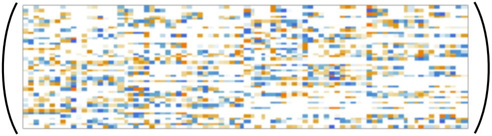
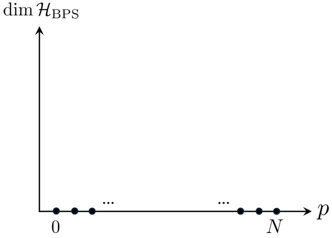


With generic couplings C_q, we expect such a matrix to easily saturate its possible rank - it doesn't have additional kernel other than the trivial ones it is constrained to have: $\alpha_p = C_q \wedge \beta_{p-q}$.

One expects no BPS states with fixed p and $N \to \infty$.

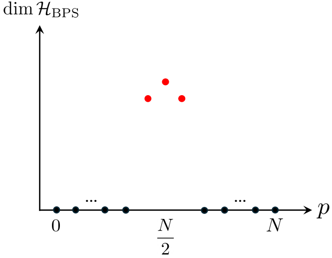
Similar argument for $Q^\dagger$ suggests no BPS states at $N - p \sim \mathcal{O}(1)$.



Strikingly, exact diagonalization reveals that all the BPS states are concentrated in an extremely narrow window.

[Fu, Gaiotto, Maldacena, Sachdev; Kanazawa, Wettig]

Example with $q = 3$ (N even), BPS only at $N/2, N/2 \pm 1$.



One can further refine the discussion with a $\mathbb{Z}_q$ charge $\equiv R \bmod q$.

Fixing $\mathbb{Z}_q$ charge, only a **single** charge sector contains BPS states.

R-Charge Concentration

The phenomenon of all BPS states concentrated in a single R-charge sector (a single degree along an irreducible cochain complex)

$$\dots \xrightarrow{Q} \mathcal{H}_{R_*-q} \xrightarrow{Q} {\color{red}\mathcal{H}_{R_*}} \xrightarrow{Q} \mathcal{H}_{R_*+q} \xrightarrow{Q} \dots$$

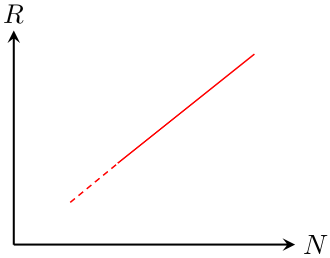
is quite sharp.

We call it "R-charge concentration".

It is highly robust - persisting even when we make C_q very sparse.

R-Charge Concentration and Fortuity

R-charge concentration implies fortuity in a natural way.



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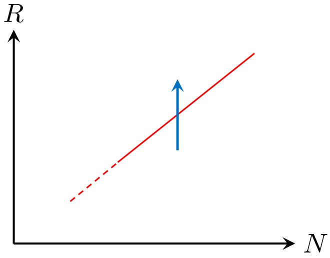

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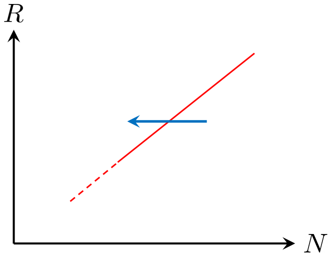

As N is decreased, one should suddenly encounter an exponential amount of BPS states - they are all fortuitous!

Further decreasing N, they cannot "survive" for too long, quickly turn into "trace relations" [Budzik, Murali, Vieira].

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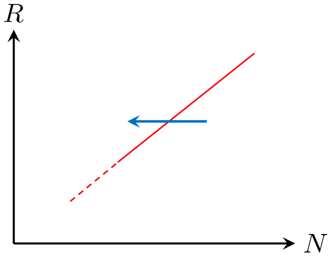

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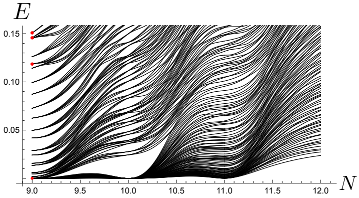
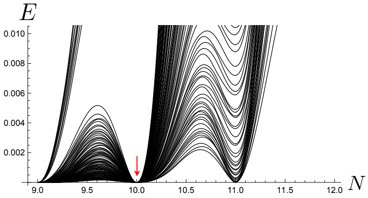
Further decreasing N, they cannot "survive" for too long, quickly turn into "trace relations" [Budzik, Murali, Vieira].

The analogy of trace relations are the forms $\alpha_{i_1,i_2,\ldots,i_p}$, $\forall\, i_k > N$.

The story of following N can be fleshed out quite explicitly in SYK models. Ask me later!





Connection to the Schwarzian Theory

R-charge concentration, from the microscopic point of view, reflects the genericity of the supercharge.

From the effective theory point of view, R-charge concentration is consistent with the prediction of the super-Schwarzian description.

[Stanford, Witten] showed that by quantizing the $\mathcal{N} = 2$ super-Schwarzian, all BPS states (in the large N sense) are concentrated within a narrow window

$$|R_{\rm IR}| < \frac{q}{2} \, .$$

Therefore, we see that fortuity of supersymmetric black hole microstates fits naturally with the Schwarzian description.

Speculations on “Non-linear Charge Constraints”

In many cases, it is observed that supersymmetric Lorentzian black hole solutions only exist when their charges satisfy a constraint. For instance, in $\mathrm{AdS}_5 \times S^5$, angular momenta (J_1, J_2) and electric charges (Q_1, Q_2, Q_3) are constrained as [Chong, Cvetic, Lu, Pope]

$$Q_1 Q_2 Q_3 + \frac{1}{2}N^2 J_1 J_2 = \left(\frac{1}{2}N^2 + Q_1 + Q_2 + Q_3\right)\left(Q_1 Q_2 + Q_2 Q_3 + Q_3 Q_1 - \frac{1}{2}N^2(J_1 + J_2)\right)$$

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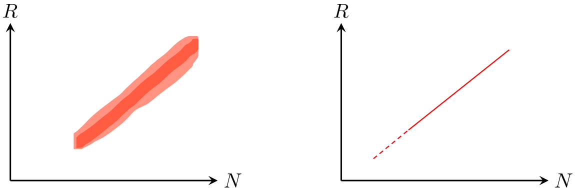

coarse-grained microscopically?

"Minimal" expectation from the near horizon AdS_2 region and its Schwarzian description [Boruch, Heydeman, Iliesiu, Turiaci] [Cabo-Bizet].

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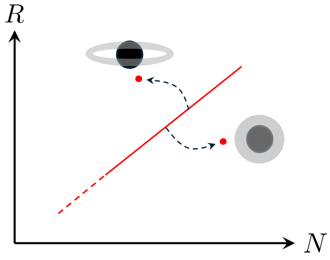

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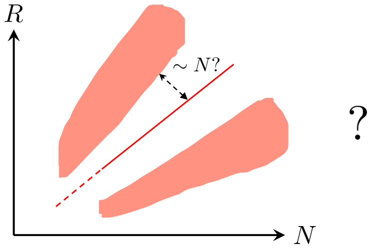
- Dressing might need to carry large enough charge.

  [Choi, Choi, Kim, Lee, Lee; de Mello Koch, Kim, Kim, Lee, Lee]

  [Dias, Mitra, Santos]



Toy Model 2: Models with Monotonous States

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We achieve this by generalizing a two-flavor model of [Heydeman, Turiaci, Zhao]

$$Q = \sum_{i,j,k} C_{ijk} \psi_i \psi_j \bar{\chi}_k, \quad H = \{Q, Q^\dagger\}.$$

The model also has a large number of fortuitous states ($S \sim N$), again exhibiting R-charge concentration.

It is a nice toy model that illustrates earlier comments about "non-linear charge constraint" [See paper for detail].

Monotonous States

We can arrange the couplings to have additional structures such that we also get monotonous states.

$$Q = \sum_{i,j,k} C_{ijk} \psi_i \psi_{(j} \bar{\chi}_{k)}$$

Existence of an opearator V (analogues to BPS gravitons)

$$[Q, V] = 0 \,, \quad V = \sum_{i=1}^{N} \psi_i \chi_i = \vec{\psi} \cdot \vec{\chi} \,,$$

A tower of monotonous states

$$V^n \left| \Omega \right\rangle , \quad n = 1, ..., N \,.$$

Only weak dependence on N hidden in the range of summation.

Further generalizations admit multiple such “graviton” operators

$$V_1, ..., V_k$$

One can form “larger” BPS states by taking product of them

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Some properties:

- Consider $N_\psi, N_\chi \sim \mathcal{O}(N)$, we have

$$\#(\text{monotonous}) \sim N^{k-1} \ll \#(\text{fortuitous}) \sim e^{\mathcal{O}(N)} .$$

- Interesting underlying algebraic structure. They form representations under $\mathrm{Sp}(2K)$ ($k = K(K+1)/2$).
- Model still solvable in the large N limit, in fact, the monotonous states do not seem to modify large N properties.

Can we provide further evidence for the connection of fortuity with the emergence of horizon?

In particular, how does the distinction of fortuitous versus monotonous relate to other universal features of black hole microstates, such as **chaos**?

BPS Chaos

In the BPS subspace, we do not have the usual notion of quantum chaos in terms of the level repulsion between energy levels.

However, we can study the notion of **BPS chaos** - whether the BPS subspace resemble a *random subspace* with respect to simple bases of the Hilbert space (e.g. eigenbasis of simple operators).

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In [YC, Henry Lin, Stephen Shenker], we explored BPS chaos of various monotonous (horizonless) subspaces in field theories:

- $\frac{1}{2}$, $\frac{1}{4}$-BPS sectors of the $\mathcal{N} = 4$ SYM theory
- $\frac{1}{2}$-BPS sector of the D1-D5 CFT
  (see [Chang, Lin, Zhang, to appear] for fortuity in D1-D5)

and verified that they *do not* display the extent of randomness for black hole microstates.

BPS Chaos

Our previous exploration was cut short in the fortuitous case.

We can do a better job in our toy models. We explored various probes of BPS chaos

- LMRS prescription [Lin, Maldacena, Rozenberg, Shan]
- Information entropy of typical states [Budzik, Murali, Vieira]
- Entanglement entropy of typical states

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The non-randomness of monotonous states like $(V_1)^{n_1}...(V_k)^{n_k}\,|\Omega\rangle$ in our model is intuitive.

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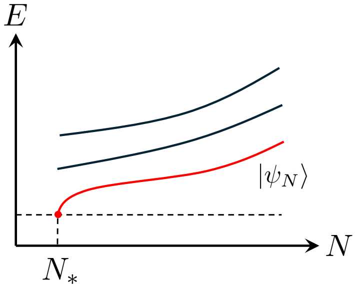
Nonetheless, it is natural to expect them to display similar randomness as the chaotic non-BPS states.



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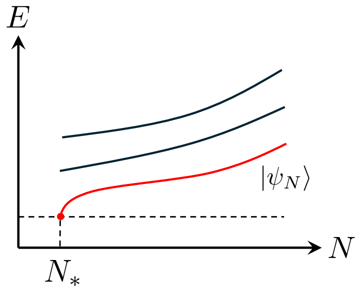

By following N, the BPS subspace gets invaded by the randomness from non-BPS states through fortuity.

In SYK models, this picture can be verified quantitatively.

Future Directions

- Is R-charge concentration a universal feature of generic q-local supercharges?
  Does it play a special role in the $\mathcal{N} = 2$ supersymmetric random matrix ensembles of [Turiaci, Witten]?
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Thank you!

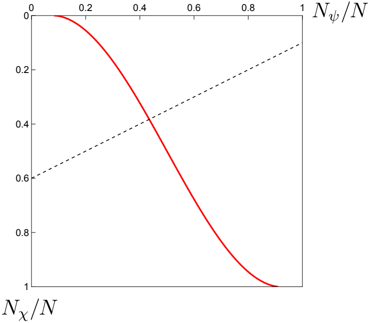
Back up slides

Phase diagram of the two flavor model

The model has two conserved quantities N_ψ, N_χ,

$$J = N_\psi + 2N_\chi, \quad [J, Q] = 0; \quad R = -N_\chi, \quad [R, Q] = Q.$$

With generic choices of couplings C_{ijk}, the model has a large number of fortuitous states.



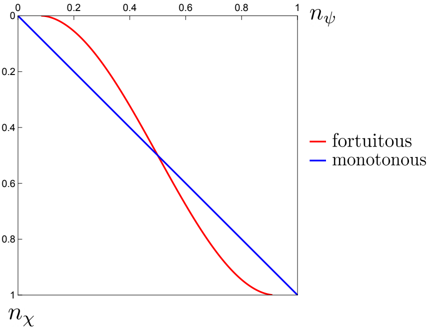
From large N equations

$\longrightarrow N_\psi$ (columns); $\downarrow N_\chi$ (rows)

1	9	27	3	0	0	0	0	0	0
0	0	162	306	0	0	0	0	0	0
0	0	162	1170	891	0	0	0	0	0
0	0	0	1709	3231	0	0	0	0	0
0	0	0	153	4692	3060	0	0	0	0
0	0	0	0	3060	4692	153	0	0	0
0	0	0	0	0	3231	1709	0	0	0
0	0	0	0	0	891	1170	162	0	0
0	0	0	0	0	0	306	162	0	0
0	0	0	0	0	0	3	27	9	1

Exact diagonalization of $N = 9$

Dashed line: fixed commuting charge J.



(a)

$\longrightarrow \mathbf{N}_\psi$

1	9	27	4	0	0	0	0	0	0
0	1	162	306	0	0	0	0	0	0
0	0	162	1170	891	0	0	0	0	0
0	0	0	1709	3231	0	0	0	0	0
0	0	0	153	4692	3060	0	0	0	0
0	0	0	0	3060	4692	153	0	0	0
0	0	0	0	0	3231	1709	0	0	0
0	0	0	0	0	891	1170	162	0	0
0	0	0	0	0	0	306	162	1	0
0	0	0	0	0	0	4	27	9	1

$\downarrow \mathbf{N}_\chi$

(b)