Scale vs Conformal
in Non-Linear Sigma Model, Dipolar magnet and Parisi-Sourlas ~~supertranslation~~
Scalar supersymmetry

Yu Nakayama

Yukawa Institute for Theoretical Physics

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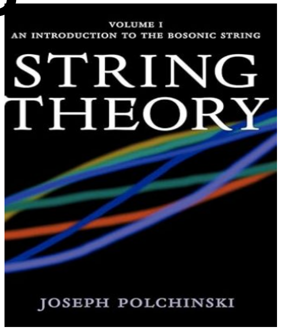
Scale vs Conformal in string theory

- Most of you studied Polchinski's book



- Let me see if you solved Exercise 15.12
- "(For the massless graviton vertex operator) Find the weaker condition for invariance under rigid Weyl transformations, and find solutions that have only this smaller invariance."
  - Rigid Weyl = Scale (but not conformal) invariance
- Since this is not solved in Matt Headrick's solution manual 0812.4408, let me show the short answer

Scale vs Conformal in string theory

"(For the massless closed string vertex operator) Find the weaker condition for invariance under rigid Weyl transformations, and find solutions that have only this smaller invariance."

- Solution: consider graviton vertex operator $\epsilon_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu e^{ikX}$
- Scale invariance: $k^2 = 0$
- Conformal invariance: $k^2 = 0, \quad k^\mu \epsilon_{\mu\nu} = 0$

- If we are working in (compact) Euclidean space, scale invariance does imply conformal: $k^2 = 0 \rightarrow k^\mu = 0$
- This is in accordance with the general theorem (Lüscher-Mack ubpublished, Zamolodchikov, Polchinski) scale implies conformal in d=2 (compact and unitary) QFTs

Scale vs Conformal in 2D NLSM

- General curved target space (Friedan)
- Conformal = Ricci flat: $R_{\mu\nu} = 0$
- Scale = quasi-Ricci flat (steady Ricci soliton) $R_{\mu\nu} = D_\mu V_\nu + D_\nu V_\mu$
- When vector is gradient $V_\mu = \partial_\mu \Phi$, one can introduce dilaton coupling to make it conformal (Ricci soliton of gradient type)

- Classical geometry theorem: compact Ricci solitons are all gradient type
- C-theorem like approach by Perelman

$$F[g_{\mu\nu}, \phi] = \int d^D x \sqrt{g} (4\partial_\mu \phi \partial^\mu \phi + R) e^{-2\phi}$$

- See also for more recent discussions Papadopoulos-Witten (with B-field, higher derivative corrections, SUSY)

Scale vs Conformal in $2+\epsilon$ D NLSM

- Much more non-trivial in $2+\epsilon$ dimensions

$$-\epsilon g_{\mu\nu} + R_{\mu\nu} = D_\mu V_\nu + D_\nu V_\mu$$

- (As easy as 2d when epsilon is negative: no Expanding non-gradient Ricci soliton, scale conformal : Hamilton, Perelman)

- Positive epsilon is significantly more difficult: The corresponding math problem was first solved around 2002 by Perelman in his seminal work on Poincare conjecture (c.f. strings 2006)
- Math theorem: (Compact) Shrinking Ricci solitons (quasi-Einstein manifold) are all gradient type!
- Perelman's entropy: (stationary when conformal)

$$W[t, g_{\mu\nu}, \phi] = -\int d^D x \sqrt{g}(t(4\partial_\mu\phi\partial^\mu\phi + R) + 2\phi - D)t^{-\frac{D}{2}} e^{-2\phi}$$

- There is no general theorem in 2+epsilon (or 3) dimensions, so this is much more non-trivial

General idea and puzzle

- The response to the Weyl transformation in QFTs can be "total derivative"

$$\delta S = \int d^d x \sqrt{g} \delta\sigma D^\mu V_\mu$$

In NLSM $V_\mu = V_M \partial_\mu X^M$

- Still a symmetry under constant dilatation, but not under special conformal
- This (non-conserved) current is called virial current (for historical reasons)

$$T^\mu_\mu = \partial^\mu V_\mu$$

- A puzzle: suppose we have (non-unitary) scale but nonconformal interacting field theories, can we have non-conserved operator with dimension exactly d-1?
- Is there any clever mechanism to protect dimensions of non-conserved vector operators?

Interlude

- Over the last decade or so, there has been significant progress to understand scale invariance vs conformal invariance in d=4 (with the assumption of unitarity)
  - Luty, Rattazi, Polchinski, Komargodski, Schwimmer, Theisen Dymarsky, Zhiboedov, Yonekura, Jack, Osborn, Stergiou, Fortin, Grinstein, Stone, Farnsworth, Prilepina, Bzowski, Skenderis, Naseh, Shore, Keren-Zur, Baume, Vitale ….
- Closely related to "a-theorem"
  - (If scale without conformal, "a" decreases forever)
- My focus today is d=3 and statistical models without reflection positivity (or unitarity)
- I'll show some examples of scale without conformal but they do not contradict the discussions there
- Physically realized (or realizable) in real material

Dipolar magnet with Gimenez-Grau and Rychkov

- Aharony and Fisher argued that dipolar interaction (photon exchange) in Heisenberg magnet is relevant in RG sense

$$H_{\text{dipolar}} = \int d^3k \phi^\mu \frac{k_\mu k_\nu}{k^2} \phi^\mu$$

- Leading to the transversality constraint on magnetization

$$\partial^\mu \phi_\mu = 0$$

- IR fixed point should be described by dipolar model

$$S = \int d^3x \left(\partial^\mu \phi_\nu \partial_\mu \phi^\nu + U \partial_\mu \phi^\mu + \lambda (\phi_\mu \phi^\mu)^2 \right)$$

- The fixed point is scale invariant but not conformal
    - Transversality constraint is not compatible with conformal invariance
    - Two-point function is not conformal form
- There exist real material such as EuO or EuS that showed scale without conformal invariance (Iron, Nickel??)

How is the virial current protected?

- The Lagrangian description includes the Lagrange multiplier field U (physically scalar magnetic potential)

$$S = \int d^3x \left(\partial^\mu \phi_\nu \partial_\mu \phi^\nu + U \partial_\mu \phi^\mu + \lambda(\phi_\mu \phi^\mu)^2\right)$$

- The (no-conserved) virial current

$$V_\mu = \Delta_U U \phi_\mu$$

- It has a shift symmetry (magnetic gauge symmetry)

$$U \to U + \text{const} \qquad s_\mu = \phi_\mu$$

- The dimension of virial current is protected due to the shift symmetry (non-perturbative argument)!

$$\langle \partial^\mu \phi_\mu(x) U \phi_\nu(y) \phi_\rho(z) \rangle = \delta^d(x-y) \langle \phi_\nu(y) \phi_\rho(z) \rangle$$

$$\Delta_{V_\nu} = d - 1$$

- Conjecture: interacting fixed points with shift symmetries are most likely scale but not conformal (e.g. higher derivative fixed point of scalar field theories)

Parisi-Sourlas ~~supertranslation~~ Scalar supersymmetry

- Consider an ensemble average of stochastic PDE

$$-\partial_\mu^2\varphi + V'(\varphi) = h$$

- Introduce Lagrange multiplier ω and FP-like trick to promote Jacobian into scalar supersymmetry

$$Z = \int \mathcal{D}\omega\mathcal{D}\varphi\mathcal{D}h e^{\int d^dx\omega(-\partial_\mu^2\varphi + V'(\varphi) - h) + R(h)} J[\varphi]$$

$$J[\varphi] = |\mathrm{Det}(-\partial_\mu^2 + V''(\varphi))|$$

- Fine tune parameters and do RG in $d = 4 - \epsilon$

$$S = \int d^dx(-\omega\partial_\mu^2\varphi + \Psi\partial_\mu^2\bar{\Psi} + \lambda_{-2}\omega^4 + \lambda_1(\omega\varphi^3 - 3\Psi\bar{\Psi}\varphi^2) + \lambda_0(\omega^2\varphi^2 - 2\Psi\bar{\Psi}\omega\varphi) + \lambda_{-1}(\omega^3\varphi - \Psi\bar{\Psi}\omega^2)).$$

- Look for fixed points

$$\beta_{\lambda_1} = -\epsilon\lambda_1 + 24\lambda_0\lambda_1$$
$$\beta_{\lambda_0} = -\epsilon\lambda_0 + 48\lambda_1\lambda_{-1} + 16\lambda_0^2$$
$$\beta_{\lambda_{-1}} = -\epsilon\lambda_{-1} + 72\lambda_{-2}\lambda_1 + 32\lambda_0\lambda_{-1}$$
$$\beta_{\lambda_{-2}} = -\epsilon\lambda_{-2} + 24\lambda_0\lambda_{-2} + 8\lambda_{-1}^2$$

- If we demand beta functions vanish (at one-loop) we obtain a conformal fixed point

$$\lambda_0 = \frac{\epsilon}{16}\ , \lambda_{-2} = \lambda_{-1} = \lambda_1 = 0$$

Scalar supersymmetry

Parisi-Sourlas ~~supertranslation~~

- But condition for scale invariance is weaker (because beta function can be a total derivative)

$$\beta_{\lambda_1} = -\epsilon\lambda_1 + 24\lambda_0\lambda_1 = 2\gamma\lambda_1,$$
$$\beta_{\lambda_0} = -\epsilon\lambda_0 + 48\lambda_1\lambda_{-1} + 16\lambda_0^2 = 0,$$
$$\beta_{\lambda_{-1}} = -\epsilon\lambda_{-1} + 72\lambda_{-2}\lambda_1 + 32\lambda_0\lambda_{-1} = -2\gamma\lambda_{-1},$$
$$\beta_{\lambda_{-2}} = -\epsilon\lambda_{-2} + 24\lambda_0\lambda_{-2} + 8\lambda_{-1}^2 = -4\gamma\lambda_{-2}.$$

- Eight additional scale invariant but not conformal fixed points!

Label	λ_1	λ_0	λ_{-1}	λ_{-2}	γ	instability
I-0	0	$\frac{\epsilon}{16}$	0	0	0	0
I-1	0	0	ϵ	$-\epsilon$	$\frac{\epsilon}{2}$	2
I-2	0	$\frac{\epsilon}{16}$	ϵ	$\frac{16\epsilon}{3}$	$-\frac{\epsilon}{2}$	1
I-3	0	0	0	ϵ	$\frac{\epsilon}{4}$	3
I-4	ϵ	0	0	0	$-\frac{\epsilon}{2}$	3
I-5	0	$\frac{\epsilon}{16}$	0	ϵ	$-\frac{\epsilon}{8}$	0
I-6	ϵ	$\frac{\epsilon}{16}$	0	0	$\frac{\epsilon}{4}$	0
I-7	$\frac{\epsilon}{3072}$	$\frac{\epsilon}{32}$	ϵ	$\frac{32\epsilon}{3}$	$-\frac{\epsilon}{8}$	2
I-8	$\frac{\epsilon}{3872}$	$\frac{\epsilon}{22}$	ϵ	$-\frac{88\epsilon}{3}$	$\frac{\epsilon}{22}$	1

- Show (hyperbolic) "RG-cylces" (observed in e.g. Fortin-Grinstein-Sterigou Jepsen-Klebanov-Popov in other models)

Scalar supersymmetry
PS ~~supertranslation~~ protects virial current

- (non-conserved) Virial current

$$V_\mu = \gamma(\omega\partial_\mu\varphi - \varphi\partial_\mu\omega)$$

- Supertranslation of the virial current is supercurrent

$$q_\mu = \Psi\partial_\mu\omega - \omega\partial_\mu\Psi$$

$$\langle\partial^\mu q_\mu(x_1)V_\nu(x_2)O(x_3)\rangle = \delta^d(x_1 - x_2)\langle q_\nu(x_2)O(x_3)\rangle,$$

$$\Delta_V = d - 1$$

- As in shift symmetry, scalar supersymmetry protects not itself but something else non-perturbatively

Summary

- It is generally believed scale invariance implies conformal invariance in sufficiently good (interacting) field theories (compact, unitary etc)
- Without unitarity, scale without conformal QFT may exist
- On the other hand, it is puzzling why there exists non-conserved non-renormalized vector operator in scale but not conformal field theories
- It is technically interesting how this happen

- Clever mechanism such as shift symmetry or Parisi-Sourlas scalar supersymmetry exist
- Scalar SUSY appears in (topological) twist, and…
- Don't be afraid of non-unitarity (c.f. Klebanov's comment)!