The Bunke–Naumann secondary invariant of a product

1 The secondary invariant of a product

Let $\varphi : \mathrm{TMF} \to \mathrm{KO}((q))$ be the standard map of ring spectra provided by the Tate curve. The aim of this section is to compute the Bunke–Naumann secondary invariant of a product $aa' \in \mathrm{TMF}_{8(k+k')+3}$, where $a \in \mathrm{TMF}_{8k+1}$ and $a' \in \mathrm{TMF}_{8k'+2}$, purely in terms of the primary invariants $\varphi(a)$ and $\varphi(a')$ of the two factors. Throughout, $k, k' \in \mathbb{Z}$ are arbitrary; we work with the periodic TMF. The final subsection, Sec. 1.5, records how the result generalizes to an arbitrary map of ring spectra.

1.1 Conventions, and the definition of the invariant

Recall that

$$\mathrm{KO}_* = \mathbb{Z}[\eta, \alpha, \beta^{\pm 1}]/(2\eta,\ \eta^3,\ \eta\alpha,\ \alpha^2 - 4\beta), \qquad |\eta| = 1, \quad |\alpha| = 4, \quad |\beta| = 8, \tag{1.1}$$

where the complexification map $c : \mathrm{KO}_* \to \mathrm{KU}_* = \mathbb{Z}[\beta_{\mathbb{C}}^{\pm 1}]$ sends $\alpha \mapsto 2\beta_{\mathbb{C}}^2$ and $\beta \mapsto \beta_{\mathbb{C}}^4$. One has $\mathrm{KO}_n((q)) = \mathrm{KO}_n \otimes \mathbb{Z}((q))$, and we fix the identifications

$$\mathrm{KO}_{8k+1}((q)) = \mathbb{Z}_2((q))\,\eta\beta^k, \qquad \mathrm{KO}_{8k+2}((q)) = \mathbb{Z}_2((q))\,\eta^2\beta^k, \qquad \mathrm{KO}_{8K+4}((q)) = \mathbb{Z}((q))\,\alpha\beta^K, \tag{1.2}$$

while $\mathrm{KO}_n((q)) = 0$ for $n \equiv 3, 5, 6, 7 \bmod 8$.

Rationally, $\mathrm{TMF}_{2w} \otimes \mathbb{Q} \simeq \mathrm{MF}_w^{\mathbb{Q}} := \mathrm{MF}_w^{\mathbb{C}} \cap \mathbb{Q}((q))$ and $\mathrm{TMF}_{\mathrm{odd}} \otimes \mathbb{Q} = 0$. In degree $8K + 4$, with respect to the basis (1.2), the rationalization of φ reads

$$\varphi(f) = \tfrac{1}{2}\, f(q)\, \alpha\beta^K, \qquad f \in \mathrm{MF}_{4K+2}^{\mathbb{Q}}, \tag{1.3}$$

the factor $\frac{1}{2}$ coming from $c(\alpha) = 2\beta_{\mathbb{C}}^2$. Since $\mathrm{MF}_{4K+2}^{\mathbb{Q}}$ is a $\mathbb{Q}$-vector space, the image of $\varphi \otimes \mathbb{Q}$ in degree $8K + 4$ is still the subspace $\mathrm{MF}_{4K+2}^{\mathbb{Q}} \subset \mathbb{Q}((q))$, so this factor of $\frac{1}{2}$ does not affect the value group introduced below.

Now fix $n = 8K + 3$ and let $C := \mathrm{KO}((q))/\mathrm{TMF}$ be the cofiber of φ, with its long exact sequence

$$\cdots \to \mathrm{TMF}_{n+1} \xrightarrow{\varphi} \mathrm{KO}_{n+1}((q)) \to C_{n+1} \xrightarrow{\partial} \mathrm{TMF}_n \xrightarrow{\varphi} \mathrm{KO}_n((q)) \to \cdots. \tag{1.4}$$

As $\mathrm{KO}_n((q)) = 0$, every $x \in \mathrm{TMF}_n$ admits a lift $\widetilde{x} \in C_{n+1}$ with $\partial\widetilde{x} = x$, unique up to the image of $\mathrm{KO}_{n+1}((q)) \simeq \mathbb{Z}((q))$. Rationalizing the sequence and using $\mathrm{TMF}_n \otimes \mathbb{Q} = 0$, we find $C_{n+1} \otimes \mathbb{Q} \simeq \mathbb{Q}((q))/\mathrm{MF}^{\mathbb{Q}}_{4K+2}$, so that the rationalization of $\widetilde{x}$ determines a well-defined class

$$\mathrm{BN}(x) \ \in \ \frac{\mathbb{Q}((q))}{\mathrm{MF}^{\mathbb{Q}}_{4K+2} + \mathbb{Z}((q))}. \tag{1.5}$$

This is the secondary invariant of Bunke and Naumann [BN09]; it is clearly additive in x.

1.2 Statement

Proposition 1.1 *Let $a \in \mathrm{TMF}_{8k+1}$ and $a' \in \mathrm{TMF}_{8k'+2}$, and set $K := k + k'$, so that $aa' \in \mathrm{TMF}_{8K+3}$. Write*

$$\varphi(a) = f\,\eta\beta^k, \qquad \varphi(a') = f'\,\eta^2\beta^{k'}, \qquad f, f' \in \mathbb{Z}_2((q)), \tag{1.6}$$

and let $\widetilde{ff'} \in \mathbb{Z}((q))$ be an arbitrary lift of $ff' \in \mathbb{Z}_2((q))$. Then

$$\mathrm{BN}(aa') = \frac{1}{2}\,\widetilde{ff'} \ \in \ \frac{\mathbb{Q}((q))}{\mathrm{MF}^{\mathbb{Q}}_{4K+2} + \mathbb{Z}((q))}. \tag{1.7}$$

Some comments are in order. First, the right hand side is well defined: two lifts of ff' differ by an element of $2\mathbb{Z}((q))$, which changes $\frac{1}{2}\widetilde{ff'}$ by an element of $\mathbb{Z}((q))$. Second, the class (1.7) is annihilated by 2, since $2 \cdot \frac{1}{2}\widetilde{ff'} = \widetilde{ff'} \in \mathbb{Z}((q))$. For this reason the overall sign of (1.7), which would depend on the orientation conventions entering the definition of BN, is immaterial. Third, $\mathrm{BN}(aa')$ depends on (a, a') only through the single mod-2 series $ff' \in \mathbb{Z}_2((q))$; in particular it depends only on $\varphi(a)$ and $\varphi(a')$, as promised.

The kernel of the resulting pairing is also easy to describe. Let $\mathrm{MF}^{\mathbb{Z}_2}_w$ denote the image of $\mathrm{MF}^{\mathbb{Z}}_w$ in $\mathbb{Z}_2((q))$ under the mod-2 reduction of q-expansions.

Proposition 1.2 *In the situation of Prop. 1.1, $\mathrm{BN}(aa') = 0$ if and only if $ff' \in \mathrm{MF}^{\mathbb{Z}_2}_{4K+2}$.*

Proof. Suppose $\frac{1}{2}\widetilde{ff'} = h + P$ with $h \in \mathrm{MF}^{\mathbb{Q}}_{4K+2}$ and $P \in \mathbb{Z}((q))$. Then $2h = \widetilde{ff'} - 2P \in \mathrm{MF}^{\mathbb{Q}}_{4K+2} \cap \mathbb{Z}((q)) = \mathrm{MF}^{\mathbb{Z}}_{4K+2}$, where the last equality holds by the very definition of $\mathrm{MF}^{\mathbb{Z}}$, and reducing $\widetilde{ff'} = 2h + 2P$ mod 2 gives $ff' = \overline{2h} \in \mathrm{MF}^{\mathbb{Z}_2}_{4K+2}$. Conversely, if ff' is the reduction of $m \in \mathrm{MF}^{\mathbb{Z}}_{4K+2}$, take $\widetilde{ff'} = m$; then $\frac{1}{2}m \in \mathrm{MF}^{\mathbb{Q}}_{4K+2}$ and the class vanishes. □

In other words, Prop. 1.1 says that $\mathrm{BN}(aa')$ is the image of $\varphi(a)\,\varphi(a')$, computed as a product of mod-2 q-series, under the injection

$$\frac{\mathbb{Z}_2((q))}{\mathrm{MF}^{\mathbb{Z}_2}_{4K+2}} \hookrightarrow \frac{\mathbb{Q}((q))}{\mathrm{MF}^{\mathbb{Q}}_{4K+2} + \mathbb{Z}((q))}, \qquad [g] \mapsto \left[\tfrac{1}{2}\,\widetilde{g}\right]. \tag{1.8}$$

1.3 Proof of Prop. 1.1

We will freely use the following standard notions. For a spectrum Z and a map $g : S^m \to Z$, a null-homotopy of g is an extension $J : D^{m+1} \to Z$ of g. Given two null-homotopies J, J' of the same g, gluing them along their common boundary yields a class $[J \cup J'] \in \pi_{m+1}Z$; for fixed g, the set of null-homotopies up to homotopy is a torsor over $\pi_{m+1}Z$ under this difference. In particular, if g is null-homotopic and $\pi_{m+1}Z = 0$, its null-homotopy is unique up to homotopy. When Z is a ring spectrum, an element $u \in \pi_m Z$ acts on maps and on null-homotopies by multiplication, denoted $u \cdot (-)$, compatibly with gluing: $[\, u \cdot J \cup u \cdot J' \,] = u \cdot [J \cup J']$. Finally, we will manipulate maps up to homotopy without further comment; each time a homotopy between two representatives is chosen, the resulting ambiguity lies in a homotopy group which will be either zero or already quotiented out in the value group (1.5).

Step 1: the invariant as a difference of null-homotopies. Let F denote the fiber of φ, so that $C \simeq \Sigma F$ and $C_{n+1} \simeq \pi_n F$, with ∂ becoming the natural map $\pi_n F \to \mathrm{TMF}_n$. An element of $\pi_n F$ is represented by a pair (x, H), where $x : S^n \to \mathrm{TMF}$ and H is a null-homotopy of $\varphi \circ x$; the map ∂ forgets H. Thus a lift $\widetilde{x}$ of $x \in \mathrm{TMF}_n$, $n = 8K + 3$, is exactly a choice of a null-homotopy H of $\varphi \circ x$ (which exists as $\mathrm{KO}_n((q)) = 0$), and changing H changes $\widetilde{x}$ by the image of $\mathrm{KO}_{n+1}((q)) = \mathbb{Z}((q))$, as it should.

Choose in addition a null-homotopy G of x in the rationalization $\mathrm{TMF}_{\mathbb{Q}}$; it exists because $\mathrm{TMF}_n \otimes \mathbb{Q} = 0$, and it is ambiguous by $\mathrm{TMF}_{n+1} \otimes \mathbb{Q} = \mathrm{MF}^{\mathbb{Q}}_{4K+2}$. Then $\varphi(G)$ and $H_{\mathbb{Q}}$ are two null-homotopies of $(\varphi \circ x)_{\mathbb{Q}}$, and we can form

$$\delta(H, G) := \big[\varphi(G) \cup H_{\mathbb{Q}}\big] \;\in\; \mathrm{KO}_{n+1}((q)) \otimes \mathbb{Q} = \mathbb{Q}((q))\, \alpha\beta^K. \tag{1.9}$$

We claim that

$$\mathrm{BN}(x) = \pm\, \delta(H, G) \quad \text{in} \quad \frac{\mathbb{Q}((q))}{\mathrm{MF}^{\mathbb{Q}}_{4K+2} + \mathbb{Z}((q))}. \tag{1.10}$$

Indeed, rationalize the fiber sequence and consider

$$\mathrm{TMF}_{n+1} \otimes \mathbb{Q} \xrightarrow{\varphi} \mathrm{KO}_{n+1}((q)) \otimes \mathbb{Q} \xrightarrow{\iota} \pi_n F \otimes \mathbb{Q} \to \mathrm{TMF}_n \otimes \mathbb{Q} = 0, \tag{1.11}$$

which identifies $\pi_n F \otimes \mathbb{Q} \simeq C_{n+1} \otimes \mathbb{Q}$ with $\big(\mathbb{Q}((q))\, \alpha\beta^K\big)/\mathrm{MF}^{\mathbb{Q}}_{4K+2}$. Deforming the first entry of the pair $(x_{\mathbb{Q}}, H_{\mathbb{Q}})$ to the constant map along G, and carrying $H_{\mathbb{Q}}$ along, turns the pair into $\big(0,\ \varphi(G) \cup H_{\mathbb{Q}}\big)$, which is precisely $\iota\big(\delta(H, G)\big)$ up to a sign coming from orientation conventions. This proves (1.10); the ambiguities of H and G move δ by $\mathbb{Z}((q))$ and by $\mathrm{MF}^{\mathbb{Q}}_{4K+2}$ respectively, matching the value group. As noted after Prop. 1.1, the class we are computing will be 2-torsion, so the sign in (1.10) can and will be ignored.

Step 2: choices for $x = aa'$. Fix lifts $\widetilde{f}, \widetilde{f}' \in \mathbb{Z}((q))$ of f, f', and set $u := \widetilde{f}\widetilde{f}'\beta^K \in \mathrm{KO}_{8K}((q))$. As homotopy classes,

$$\varphi(a) = (\widetilde{f}\beta^k) \cdot \eta, \qquad \varphi(a') = (\widetilde{f'}\beta^{k'}) \cdot \eta^2, \qquad \varphi(aa') = \varphi(a)\,\varphi(a') = u \cdot \eta^3, \tag{1.12}$$

where η^j stands for the composite $S^j \xrightarrow{\eta^j} S^0 \to \mathrm{KO}((q))$; here we used that $(\widetilde{f}\beta^k)\cdot\eta$ only depends on $\widetilde{f}$ mod 2, since $2\eta = 0$. Since $\eta^3 = 0 \in \mathrm{KO}_3$, we may choose a null-homotopy H_0 of η^3 within $\mathrm{KO} \subset \mathrm{KO}((q))$, and take as our integral null-homotopy of $\varphi(aa')$

$$H := u \cdot H_0. \tag{1.13}$$

On the rational side, choose a null-homotopy G_a of a in $\mathrm{TMF}_{\mathbb{Q}}$; it exists because $\mathrm{TMF}_{8k+1} \otimes \mathbb{Q} = 0$, and it is unique up to homotopy because $\mathrm{TMF}_{8k+2} \otimes \mathbb{Q} = \mathrm{MF}^{\mathbb{Q}}_{4k+1} \otimes \mathbb{Q} = 0$, there being no odd-weight modular forms at level one. Set $G := G_a \cdot a'$, a null-homotopy of $(aa')_{\mathbb{Q}}$; then $\varphi(G) = \varphi(G_a) \cdot \varphi(a')$.

Step 3: reduction to a universal constant. The map $\varphi(G_a)$ is a rational null-homotopy of $\varphi(a)_{\mathbb{Q}}$. Now

$$\pi_{8k+2}\big(\mathrm{KO}((q)) \otimes \mathbb{Q}\big) = \mathbb{Z}_2((q))\, \eta^2\beta^k \otimes \mathbb{Q} = 0, \tag{1.14}$$

so by the torsor property recalled above, *all* rational null-homotopies of $\varphi(a)_{\mathbb{Q}}$ agree up to homotopy.[1] In particular,

$$\varphi(G_a) \;\simeq\; (\widetilde{f}\beta^k) \cdot N, \tag{1.15}$$

where N is the essentially unique rational null-homotopy of $\eta : S^1 \to \mathrm{KO}_{\mathbb{Q}}$ (which exists and is unique up to homotopy since $\pi_1\mathrm{KO}\otimes\mathbb{Q} = \pi_2\mathrm{KO}\otimes\mathbb{Q} = 0$). Multiplying by $\varphi(a') = (\widetilde{f}'\beta^{k'})\cdot\eta^2$ we get

$$\varphi(G) \;\simeq\; u \cdot \big(N \cdot \eta^2\big), \tag{1.16}$$

and $N \cdot \eta^2$ is a rational null-homotopy of η^3 in $\mathrm{KO}_{\mathbb{Q}}$. Therefore

$$\delta(H, G) = u\cdot\big[(N\cdot\eta^2)\cup H_0\big] = \widetilde{f}\widetilde{f}'\; d_0\, \alpha\beta^K, \qquad d_0\, \alpha := \big[(N\cdot\eta^2)\cup H_0\big] \in \pi_4\mathrm{KO}\otimes\mathbb{Q} = \mathbb{Q}\,\alpha. \tag{1.17}$$

The number $d_0 \in \mathbb{Q}$ is universal: it is independent of a, a', k, k', and is well defined modulo $\mathbb{Z}$ (the residual ambiguity being the choice of H_0, which moves $d_0\alpha$ by $\pi_4\mathrm{KO} = \mathbb{Z}\alpha$). Note that the whole computation has now been reduced to one taking place inside KO.

Step 4: $d_0 \equiv \frac{1}{2}$ mod $\mathbb{Z}$. Applying the difference construction of Step 1 to the unit map $\iota : S \to \mathrm{KO}$ instead of φ yields the classical real e-invariant[2] $e_{\mathbb{R}} : \pi_3 S \to \mathbb{Q}/\mathbb{Z}$: for $y \in \pi_3 S$,

$$e_{\mathbb{R}}(y) = \pm\big[\iota(G_y) \cup H_y\big] \in \frac{\pi_4\mathrm{KO}\otimes\mathbb{Q}}{\pi_4\mathrm{KO}} = \mathbb{Q}/\mathbb{Z}\,, \tag{1.18}$$

with G_y a rational null-homotopy of y in $S_{\mathbb{Q}}$ and H_y an integral null-homotopy of $\iota(y)$ in KO (which exists as $\pi_3\mathrm{KO} = 0$). Take $y = \eta^3$. Since $\pi_2 S \otimes \mathbb{Q} = \pi_4 S \otimes \mathbb{Q} = 0$, rational null-homotopies of η and of η^3 in the sphere are unique up to homotopy, so that $G_{\eta^3} \simeq N_S \cdot \eta^2$

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[1] To compare null-homotopies of the two homotopic representatives $\varphi\circ a$ and $(\widetilde{f}\beta^k)\cdot\eta$ of the same class (1.12), one transports along a chosen homotopy between them; the ambiguity so introduced also lies in $\pi_{8k+2}(\mathrm{KO}((q))\otimes\mathbb{Q}) = 0$.

[2] J. F. Adams, *On the groups $J(X)$. IV*, Topology **5** (1966) 21–71. The formulation of the e-invariant as a difference of an integral and a rational null-homotopy, used here, is also the one adopted in [BN09].

with N_S the rational null-homotopy of η in $S_{\mathbb{Q}}$; applying ι and using the uniqueness of N once more,

$$\iota(G_{\eta^3}) \;\simeq\; N \cdot \eta^2. \tag{1.19}$$

Choosing $H_{\eta^3} = H_0$, we conclude from (1.17) that

$$d_0 \equiv \pm e_{\mathbb{R}}(\eta^3) \bmod \mathbb{Z}. \tag{1.20}$$

Classically, $\eta^3 = 12\nu$ in $\pi_3 S \simeq \mathbb{Z}/24$, and $e_{\mathbb{R}}(\nu) = \pm\frac{1}{24}$,[3] so that

$$d_0 \;\equiv\; 12 \cdot \left(\pm\frac{1}{24}\right) \;\equiv\; \frac{1}{2} \bmod \mathbb{Z}. \tag{1.21}$$

Combining (1.10) and (1.17), and noting that $\widetilde{f}\widetilde{f'}$ is one possible lift $\widetilde{ff'}$, we obtain

$$\mathrm{BN}(aa') = \pm\frac{1}{2}\,\widetilde{f}\widetilde{f'} = \frac{1}{2}\,\widetilde{ff'}\,, \tag{1.22}$$

where the sign was dropped as the class is 2-torsion. This proves Prop. 1.1. ☐

1.4 Example and remarks

The fundamental example. Take $k = k' = 0$ and let $a = \eta \in \mathrm{TMF}_1$, $a' = \eta^2 \in \mathrm{TMF}_2$ be the images of the Hopf classes under $S \to \mathrm{TMF}$, so that $f = f' = 1$. Prop. 1.1 gives

$$\mathrm{BN}(\eta^3) = \frac{1}{2} \in \frac{\mathbb{Q}((q))}{\mathrm{MF}_2^{\mathbb{Q}} + \mathbb{Z}((q))}\,, \tag{1.23}$$

and this class is *nonzero*: if $\frac{1}{2} = h + P$ with $h \in \mathrm{MF}_2^{\mathbb{Q}}$ and $P \in \mathbb{Z}((q))$, taking the coefficient of q^0 would give $\frac{1}{2} \in \mathbb{Z}$, since the q^0 coefficient of any weakly holomorphic weight-2 form h vanishes (the differential $h\,dq/q$ is holomorphic on the j-line $\simeq \mathbb{C}$, and the circle $|q| = \epsilon$ maps to a contractible loop there). Note that $\mathrm{MF}_2^{\mathbb{Z}_2} = \mathbb{Z}_2\langle \bar{\Delta}^{-1}, \bar{\Delta}^{-2}, \ldots\rangle$ indeed does not contain the constant $1 = ff'$, in agreement with Prop. 1.2.

This is consistent with the known value of the invariant on ν. By naturality of the construction along $S \to \mathrm{TMF}$, $\mathrm{BN}(\nu)$ is the constant series given by the real e-invariant, $\mathrm{BN}(\nu) = \pm\frac{1}{24} \bmod \mathrm{MF}_2^{\mathbb{Q}} + \mathbb{Z}((q))$, and $\eta^3 = 12\nu$ then gives $\mathrm{BN}(\eta^3) = 12 \cdot (\pm\frac{1}{24}) = \frac{1}{2}$, matching the formula. (In the literature the value of the invariant on ν is often written as $\pm E_2/24$; since $E_2 = 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n$, this agrees with $\pm\frac{1}{24}$ modulo $\mathbb{Z}((q))$.) In particular, the product formula detects $\eta \cdot \eta^2 \neq 0$ in TMF_3, even though $\varphi(\eta)\varphi(\eta^2) = \eta^3 = 0$ in $\mathrm{KO}_3((q))$.

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[3] For $\eta^3 = 12\nu$ see H. Toda, *Composition methods in homotopy groups of spheres*, Princeton Univ. Press, 1962. For $e_{\mathbb{R}}(\nu) = \pm\frac{1}{24}$ see Adams, *op. cit.*; this is the statement that $e_{\mathbb{R}}$ detects the full image of J in $\pi_3 S$, which is the whole $\mathbb{Z}/24$.

Which degrees can carry a nonzero invariant. Recall that the primary invariants of TMF classes are in fact mod-2 modular forms: for $a' \in \mathrm{TMF}_{8k'+2}$ one has $f' \in \mathrm{MF}^{\mathbb{Z}_2}_{4k'}$, and the same holds for f, since $\varphi(a\eta) = \varphi(a)\,\eta$ shows that $a \in \mathrm{TMF}_{8k+1}$ and $a\eta \in \mathrm{TMF}_{8k+2}$ have the same mod-2 series. Recall also that, since $E_4 \equiv E_6 \equiv 1 \bmod 2$, one has $\mathrm{MF}^{\mathbb{Z}_2}_w = \bigoplus_{c \le c_w} \mathbb{Z}_2\,\bar{\Delta}^c$ with

$$c_w = \begin{cases} \lfloor w/12 \rfloor & (w \equiv 0 \bmod 4), \\ \lfloor (w-6)/12 \rfloor & (w \equiv 2 \bmod 4). \end{cases} \tag{1.24}$$

Write $f = \sum_{c \le \lfloor k/3 \rfloor} f_c \bar{\Delta}^c$ and $f' = \sum_{c \le \lfloor k'/3 \rfloor} f'_c \bar{\Delta}^c$. Then $ff' \in \mathrm{MF}^{\mathbb{Z}_2}_{4K}$, and by Prop. 1.2 the invariant $\mathrm{BN}(aa')$ only sees the image of ff' in $\mathrm{MF}^{\mathbb{Z}_2}_{4K}/\mathrm{MF}^{\mathbb{Z}_2}_{4K+2}$, which is nonzero (namely $\mathbb{Z}_2\langle \bar{\Delta}^{K/3} \rangle$) only when $K \equiv 0 \bmod 3$. Moreover the coefficient of $\bar{\Delta}^{K/3}$ in ff' can be nonzero only if $\lfloor k/3 \rfloor + \lfloor k'/3 \rfloor = K/3$, i.e. only if $k \equiv k' \equiv 0 \bmod 3$. Summarizing:

$$\mathrm{BN}(aa') = \begin{cases} \dfrac{1}{2}\, f_{k/3}\, f'_{k'/3}\, \Delta^{K/3}(q) & (k \equiv k' \equiv 0 \bmod 3), \\ 0 & (\text{otherwise}). \end{cases} \tag{1.25}$$

In particular, such products can have a nonzero secondary invariant only in degrees $8K+3$ with $K \equiv 0 \bmod 3$, i.e. in degrees $n \equiv 3 \bmod 24$, and there the invariant is at most the single $\mathbb{Z}_2$ generated by $\frac{1}{2}\Delta^{K/3}$. This is consistent with the general structure of the possible values of the secondary invariant on TMF_n. (Whether a given pair (f, f') with $f_{k/3} = f'_{k'/3} = 1$ is actually realized by classes a, a' is a separate question about the precise image of φ, which we do not address here.)

1.5 Generalization: arbitrary maps of ring spectra

The proof given in Sec. 1.3 consumes remarkably little that is specific to TMF and $\mathrm{KO}((q))$: Steps 1–3 are completely general, and only Step 4, the evaluation of the universal constant d_0, used the actual homotopy groups involved. Let us record the general statement.

Let $\varphi : \mathsf{E} \to \mathsf{F}$ be a map of homotopy-commutative ring spectra, with homotopy groups $\mathsf{E}_n := \pi_n \mathsf{E}$, $\mathsf{F}_n := \pi_n \mathsf{F}$. Let $a \in \mathsf{E}_n$ and $a' \in \mathsf{E}_{n'}$, and set $N := n + n'$. We assume:

(i) a is a torsion element of E_n;

(ii) $\varphi(a')$ is a torsion element of $\mathsf{F}_{n'}$ (e.g. $\mathsf{F}_{n'}$ is entirely torsion);

(iii) $\mathsf{F}_N = 0$.

By (iii), aa' lifts to $(\mathsf{F}/\mathsf{E})_{N+1}$, uniquely up to the image of F_{N+1}. By (i), aa' is torsion, so the rationalization of the lift comes from $\mathsf{F}_{N+1} \otimes \mathbb{Q}$, well-defined up to $\varphi(\mathsf{E}_{N+1} \otimes \mathbb{Q})$. Exactly as in (1.5) we therefore obtain

$$\mathrm{BN}(aa') \;\in\; \frac{\mathsf{F}_{N+1} \otimes \mathbb{Q}}{\overline{\mathsf{F}_{N+1}} + \varphi(\mathsf{E}_{N+1} \otimes \mathbb{Q})}\,, \tag{1.26}$$

where $\overline{\mathsf{F}_{N+1}}$ denotes the image of F_{N+1} in $\mathsf{F}_{N+1}\otimes\mathbb{Q}$.

To state the answer, we use homotopy with $\mathbb{Q}/\mathbb{Z}$ coefficients, $\pi_*(\mathsf{F};\mathbb{Q}/\mathbb{Z}) := \pi_*(\mathsf{F}\wedge S(\mathbb{Q}/\mathbb{Z}))$ with $S(\mathbb{Q}/\mathbb{Z})$ the Moore spectrum, together with the Bockstein long exact sequence

$$\cdots\to\mathsf{F}_{n+1}\otimes\mathbb{Q}\to\pi_{n+1}(\mathsf{F};\mathbb{Q}/\mathbb{Z})\xrightarrow{\partial}\mathsf{F}_n\to\mathsf{F}_n\otimes\mathbb{Q}\to\cdots, \tag{1.27}$$

in which the image of ∂ is exactly the torsion of F_n. Note that (iii) and (1.27) identify $\pi_{N+1}(\mathsf{F};\mathbb{Q}/\mathbb{Z})\simeq(\mathsf{F}_{N+1}\otimes\mathbb{Q})/\overline{\mathsf{F}_{N+1}}$.

Proposition 1.3 *Assume (i)–(iii), and write $u:=\varphi(a)$, $v:=\varphi(a')$. Let $\widehat{u}\in\pi_{n+1}(\mathsf{F};\mathbb{Q}/\mathbb{Z})$ be any class with $\partial\widehat{u}=u$; it exists since u is torsion. Then the product*

$$\widehat{u}\cdot v\ \in\ \pi_{N+1}(\mathsf{F};\mathbb{Q}/\mathbb{Z})\simeq\frac{\mathsf{F}_{N+1}\otimes\mathbb{Q}}{\overline{\mathsf{F}_{N+1}}} \tag{1.28}$$

is independent of the choice of $\widehat{u}$, and

$$\mathrm{BN}(aa')=\pm\,\widehat{u}\cdot v \bmod \varphi(\mathsf{E}_{N+1}\otimes\mathbb{Q}). \tag{1.29}$$

In particular, $\mathrm{BN}(aa')$ *is determined by $\varphi(a)$ and $\varphi(a')$ alone, and is annihilated by the greatest common divisor of their orders.*

Proof. It is convenient to realize the $\mathbb{Q}/\mathbb{Z}$-coefficient homotopy by the fiber $W := \mathrm{fib}(\mathsf{F}\to\mathsf{F}_{\mathbb{Q}})\simeq\Sigma^{-1}\big(\mathsf{F}\wedge S(\mathbb{Q}/\mathbb{Z})\big)$, so that $\pi_n W=\pi_{n+1}(\mathsf{F};\mathbb{Q}/\mathbb{Z})$. As in Step 1, an element of $\pi_n W$ is a homotopy class of pairs (g,J), where $g:S^n\to\mathsf{F}$ and J is a null-homotopy of g in $\mathsf{F}_{\mathbb{Q}}$; the map ∂ forgets J; and W is an F-module compatibly with this description.

Existence and independence. Since u is torsion it is rationally null-homotopic, so a lift $\widehat{u}=(u,J)$ exists. Two lifts differ by the image of some $y\in\pi_{n+1}\mathsf{F}_{\mathbb{Q}}=\mathsf{F}_{n+1}\otimes\mathbb{Q}$, which changes $\widehat{u}\cdot v$ by the image of $y\cdot v_{\mathbb{Q}}\in\pi_{N+1}\mathsf{F}_{\mathbb{Q}}$; this vanishes because v is rationally zero by (ii). The same computation gives the annihilation statement: if $mu=0$ then $m\widehat{u}$ is a lift of 0, hence comes from $\mathsf{F}_{n+1}\otimes\mathbb{Q}$, so $m(\widehat{u}\cdot v)=(m\widehat{u})\cdot v=0$; and $\widehat{u}\cdot(m'v)=0$ if $m'v=0$.

The identity (1.29). Choose a rational null-homotopy G_a of a in $\mathsf{E}_{\mathbb{Q}}$, which exists by (i), and an integral null-homotopy H of $\varphi(a)\varphi(a')$, whose obstruction lies in $\mathsf{F}_N=0$. Exactly as in Steps 1 and 2, with $G:=G_a\cdot a'$ and $\varphi(G)=\varphi(G_a)\cdot\varphi(a')$,

$$\mathrm{BN}(aa')=\pm\big[\varphi(G_a)\cdot v\,\cup\,H\big] \bmod \varphi(\mathsf{E}_{N+1}\otimes\mathbb{Q}). \tag{1.30}$$

On the other hand, the pair $\widehat{u}_0:=(u,\varphi(G_a))$ is a lift of u to $\pi_n W$, and by the module structure

$$\widehat{u}_0\cdot v=\big(uv,\ \varphi(G_a)\cdot v\big)\in\pi_N W. \tag{1.31}$$

Applying the argument of Step 1 to $\mathsf{F}\to\mathsf{F}_{\mathbb{Q}}$ in place of φ: under the identification $\pi_N W\simeq(\mathsf{F}_{N+1}\otimes\mathbb{Q})/\overline{\mathsf{F}_{N+1}}$, a pair (g,J) whose first entry is null-homotopic equals $[J\cup H_g]$ for any integral null-homotopy H_g of g. Taking $(g,J)=(uv,\varphi(G_a)\cdot v)$ and $H_g=H$ gives $\widehat{u}_0\cdot v=[\varphi(G_a)\cdot v\cup H]$, and combining the two displays proves (1.29). □

Note that, in contrast to Step 3, no uniqueness of rational null-homotopies is needed: the relevant ambiguity has been absorbed into the well-definedness of $\widehat{u}\cdot v$. In the situation of Prop. 1.1, one has $\widehat{u}=(\widetilde{f\beta^k})\cdot\widehat{\eta}$ with $\widehat{\eta}\in\pi_2(\mathrm{KO};\mathbb{Q}/\mathbb{Z})$ the lift of η, and Step 4 amounts to the evaluation $\widehat{\eta}\cdot\eta^2=\frac{1}{2}\alpha$ in $\pi_4(\mathrm{KO};\mathbb{Q}/\mathbb{Z})=(\mathbb{Q}/\mathbb{Z})\,\alpha$.

On hypothesis (i). It cannot be weakened to "$\varphi(a)$ is torsion in F_n": if $s \in (\operatorname{Ker}\varphi)_n$ is a non-torsion element, then $b := a + s$ satisfies $\varphi(b) = \varphi(a)$, but $\mathrm{BN}(ba')$ (now valued in $(\mathsf{F}/\mathsf{E})_{N+1} \otimes \mathbb{Q}$ modulo the image of F_{N+1}) differs from $\mathrm{BN}(aa')$ by the rationalization of $\widetilde{s} \cdot a'$, where $\widetilde{s} \in (\mathsf{F}/\mathsf{E})_{n+1}$ is a lift of s; this is in general nonzero when a' is rationally nontrivial in E. Thus $\mathrm{BN}(aa')$ is a function of $(\varphi(a), \varphi(a'))$ on *torsion* classes only. In the situation of Prop. 1.1 this subtlety is invisible, as TMF_{8k+1} and $\mathrm{TMF}_{8k'+2}$ are entirely torsion.

Toda-bracket form. If $mu = 0$, the lift $\widehat{u}$ may be taken to come from $\mathbb{Z}/m$ coefficients via $\mathbb{Z}/m \simeq \frac{1}{m}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$, and unwinding the definitions one finds

$$\mathrm{BN}(aa') = \mp\frac{1}{m}\,\langle v,\, u,\, m\rangle_{\mathsf{F}} \bmod \overline{\mathsf{F}_{N+1}} + \varphi(\mathsf{E}_{N+1} \otimes \mathbb{Q}), \tag{1.32}$$

where the indeterminacy $v \cdot \mathsf{F}_{n+1} + m \cdot \mathsf{F}_{N+1}$ of the Toda bracket is absorbed by the quotient: elements of $v \cdot \mathsf{F}_{n+1}$ are torsion, hence die in $\mathsf{F}_{N+1} \otimes \mathbb{Q}$, and $\frac{1}{m}(m \cdot \mathsf{F}_{N+1}) = \overline{\mathsf{F}_{N+1}}$. In the situation of Prop. 1.1, this reads $\mathrm{BN}(aa') = \frac{1}{2}\langle \eta^2, \eta, 2\rangle_{\mathrm{KO}} \cdot \widetilde{f}\widetilde{f}'\beta^K$, and Step 4 amounts to the statement that $\langle \eta^2, \eta, 2\rangle_{\mathrm{KO}}$ is an *odd* multiple of α. Amusingly, this bracket does not exist in the sphere, where $\eta^2 \cdot \eta = 12\nu \neq 0$.

In what sense the sphere is the universal case. For $\mathsf{E} = S$ the sphere spectrum and $\varphi = \iota$ the unit of F, hypothesis (i) is automatic in positive degrees, BN is the F-based Adams e-invariant, and the value group (1.26) is the finest possible, since $\pi_{N+1}S \otimes \mathbb{Q} = 0$ for $N \geq 0$. As S is the initial ring spectrum, the construction is natural in E: whenever a, a' are pulled back from $\pi_* S$, the E-level invariant is the image of the sphere-level one under the further quotient by $\varphi(\mathsf{E}_{N+1} \otimes \mathbb{Q})$. Step 4 above is precisely this naturality, applied to $\eta, \eta^2 \in \pi_* S$ with $\mathsf{F} = \mathrm{KO}$.

However, the sphere does not tell the whole story, because interesting classes of E need not come from $\pi_* S$: in Prop. 1.1, any a whose series f is nonconstant is not in the image of $\pi_* S \to \mathrm{TMF}_*$. The formulation that deserves to be called universal is rather (1.29) itself, which is internal to F: the right hand side is a pairing defined on torsion classes $(u, v) \in \mathsf{F}_n \times \mathsf{F}_{n'}$ whenever $\mathsf{F}_N = 0$, and E enters only in two ways — by certifying the torsion trivialization of a on the E side, which is what makes the secondary invariant of aa' defined at all, and through the subgroup $\varphi(\mathsf{E}_{N+1} \otimes \mathbb{Q})$ that is quotiented out. Along $S \to \mathsf{E}$ there is thus a genuine trade-off: a larger E defines the invariant on more classes, but with values in a coarser group (in Prop. 1.1, modulo $\mathrm{MF}^{\mathbb{Q}}_{4K+2}$).

References

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