IPMU 13-0???
UT-13-??
preliminary version 2.0

A pseudo-mathematical pseudo-review on 4d $\mathcal{N}=2$ supersymmetric quantum field theories

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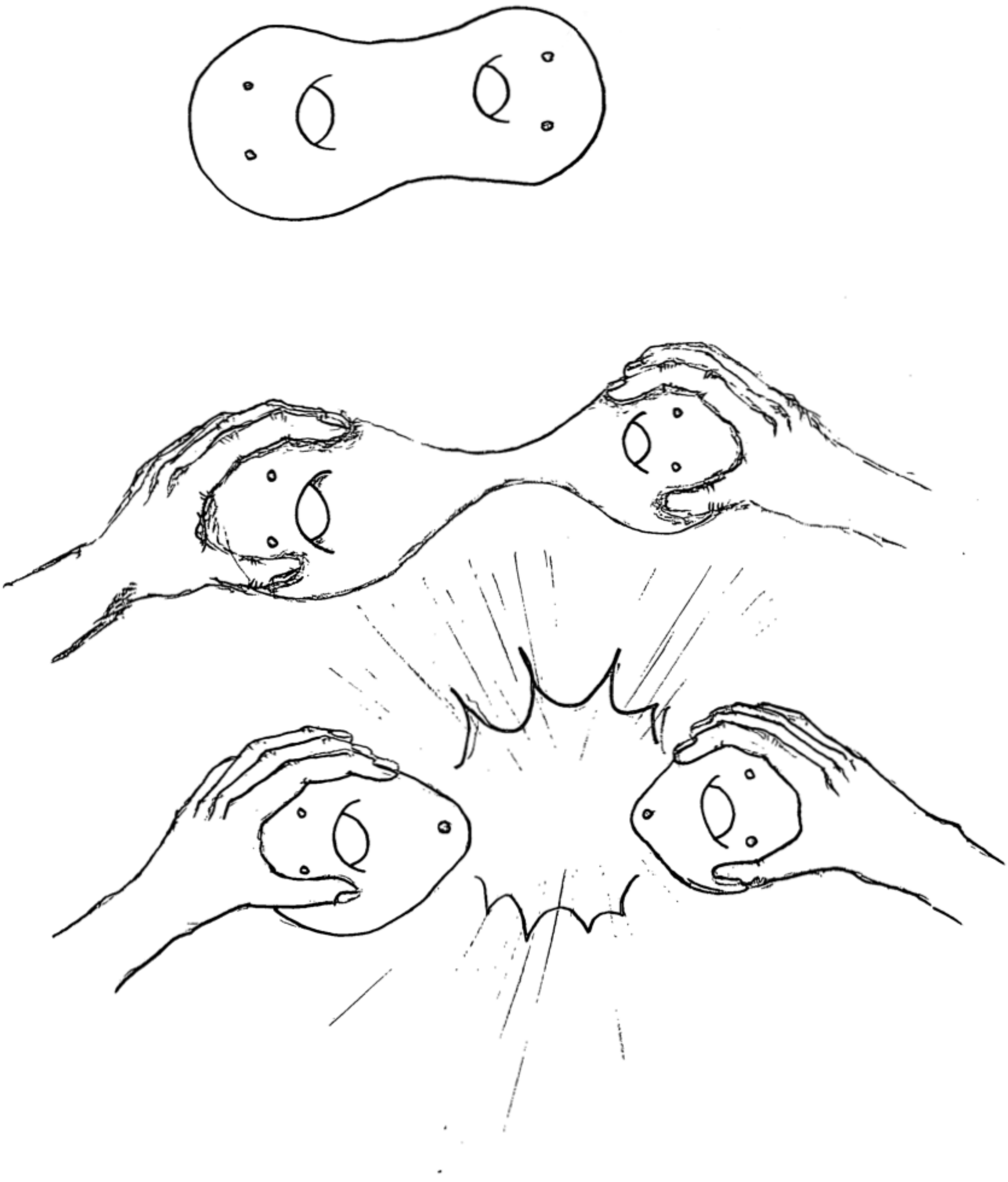
abstract

Supersymmetric quantum field theories in four spacetime dimensions with $\mathcal{N}=2$ supersymmetry will be introduced in a pseudo-mathematical language. Topics covered include the idea of categories of quantum field theories, general properties of $\mathcal{N}=2$ supersymmetric theories and their relation to W-algebras and to elliptic generalizations of Macdonald functions. This is a combined write-up of the lectures given by the author at IPMU and at RIMS in 2012 and at Komaba in 2013.



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0 Introduction

0.1 Useless forewords

The study of supersymmetric quantum field theory (QFT) by physicists has led to a few mathematical conjectures, such as mirror symmetry and the relation of instantons and vertex operator algebras. This clearly shows that the QFT itself should be a rich subject for mathematicians. Indeed, there have been many mathematical formulations of QFTs. But none of them really explains how physicists sometimes come up with new mathematical results, because the formulations so far available were based on QFTs as understood by physicists a few decades ago. It seems to the author, therefore, that it would not be completely useless if someone tries to formulate the concept of QFTs mathematically once again, so that it captures what physicists do with them in this 21st century. The author likes to compare mathematicians with civilized city-dwellers and physicists with barbaric tribes in the rain forests. Civilized city-dwellers are puzzled how those barbarians, speaking a strange tongue, can sometimes dig out precious stones from their soil. However, these should not stop civilized city-dwellers to try to make contact with them. Every language has a grammar, even the one spoken by unseemly barbarians. With the general method of linguistics at hand, civilized city-dwellers can start deciphering their language, and communicating with them. It might even happen that some of the barbarians have already learnt to speak English, albeit with a very strong accent, and that s/he can help explain barbarians' cultures to the city-dwellers. Once the city-dwellers are somewhat acquainted with the barbaric way of life, they can directly come to the rain forests, introduce the civilization to the barbarians, and effectively excavate all the precious materials from their land. As a barbarian who has a partial knowledge of English, the author thinks that he might be able to help the city-dwellers understand how barbarians speak to each other. This lecture note contains the author's first attempt in this direction. It does not contain a fully developed grammar of the barbarians' language, because it is clearly beyond the author's ability. The real grammar of the barbarians' language needs to be written by civilized city-dwellers themselves in the future. Hopefully that will not induce civilized city-dwellers coming to the rain forests en masse, burning down all the beautiful trees here without caring the rights of the barbaric inhabitants here.

Let us now turn to a more practical side. There are many mathematical papers where QFTs are analyzed using the language of (higher) categories. The prototypical example is the Atiyah-Segal formulation of the topological QFT, where the topological QFT is formulated as a functor between two categories.

The author's opinion is that we need to push this view point one step further, by regarding QFTs themselves as objects in something like a category. The important point is that a QFT (although not usually rigorously constructed mathematically) is a mathematical object, much like a group, a space or an algebra. Then, similarly to those more familiar mathematical objects, we can consider morphisms between two QFTs and various operations on QFTs. In this review a central role is played by the concept of a G-symmetric QFT, for a group G. This is not a QFT with a G-action in a naive sense. But it has almost all the familiar properties of "something with G-action". For example, given a G-symmetric Q and a subgroup $H \subset G$, one can be forgetful and think Q as an H-symmetric QFT. One can construct from G_i-symmetric QFTs Q_i a $G_1 \times G_2$-symmetric $Q_1 \times Q_2$, and from $G \times F$-symmetric QFT Q we can construct F-symmetric QFT $Q /\!\!\!/ G$, once the operations $\times$ and $/\!\!\!/$ are defined with care. One can also extract various invariants from Q. One is the vacuum manifold $\mathcal{M}_{\text{vac}}(Q)$, which gives a Riemanninan manifold with G-action from a G-symmetric QFT Q. Then $\mathcal{M}_{\text{vac}}$ can be thought of as a functor from the category of QFTs to the category of Riemanninan manifolds.

Another point is that the difficulty of QFTs is often associated to the difficulty of making sense of the concept of the path integrals, i.e. an infinite-dimensional integral over the space of maps. There is definitely a lot of truths in this statement, but physicists have learned a lot from experience when the path integrals make sense to which extent, and these properties can be stated quite precisely. Then mathematicians might be able to work on them as a kind of a set of axioms from which one can be inspired, rather as in the situation when Weil supposed the existence of a certain good cohomology theory yet to be constructed, but with a good properties, to deduce many interesting conjectures. Also, not all QFTs can be defined as a path integral, and there are many QFTs which can be at present only defined as something which satisfies the basic axioms of QFTs with a certain number of additional known properties. Therefore, there seem to be many parts of the QFTs which even mathematicians can learn, formalize and work on without completely ironing out the details of what a path integral is. Once this exercise is developed to a certain extent, mathematicians will hopefully be able to understand how physicists come up with mathematical conjectures in their own terms.

A final point the author wants to make is the following. There have been many attempts to axiomatize quantum field theories in the past. Every time, when one great mathematician and/or mathematical physicist axiomatizes the quantum field theories as practised by physicists in his/her days, a mathematical community forms around that work, deepening the understanding. This is not necessarily a bad thing. However, the quantum field theories as practised by physicists have been a moving target, and mathematicians who are already working on a formulation of quantum field theories should look, once in a while, at what physicists do in practice with regard to quantum field theories. They can then hopefully try to incorporate what physicists developed or found important in the meantime into their already great axiomatizations. The authors hope that this lecture note would serve a rough guide for mathematicians to have a glimpse of what theoretical physicists at the first third of 2010s are doing with quantum field theories.

0.2 Organization of the contents

In Sec. 1 we develop a pseudo-mathematical language describing quantum field theories (QFTs) in general. We basically follow the formulation of Atiyah and Segal, adopted to QFTs in the presence of the Riemannian metric. A d-dimensional G-symmetric QFT Q, very naively, gives a complex number $Z_Q(X)$ given a d-dimensional manifold X with Riemannian metric, together with a G-bundle with connection on it. We call $Z_Q(X)$ the partition function of Q on X. We introduce three central concepts:

- The product of two QFTs Q_1 and Q_2. It is simply given by $Z_{Q_1\times Q_2}(X) = Z_{Q_1}(X)Z_{Q_2}(X)$.
- The operation which we call gauging by a group G. Given a $G\times F$-symmetric QFT Q, this operation produces $Q/\!\!\!/G$, which is an F-symmetric QFT. The symbol $/\!\!\!/$ is chosen to suggest to the reader that its formal property is somewhat akin to the quotient operation of a space X with a group action G. Just as X/G does not have a G action, the result of the gauging $Q/\!\!\!/G$ no longer has the G symmetry.
- The functors called free bosons B_d and free fermions F_d. They map a finite-dimensional representation V of G to d-dimensional G-symmetric QFTs. Moreover, $B_d(V\oplus W) = B_d(V)\times B_d(W)$, and similarly for F_d.

In Sec. 1 we state the properties of QFTs matter-of-factly, and the reader is not expected to understand this section. The formalism which will be presented is a certain mixture of standard formalisms:

- As in the standard functorial formulations, for a manifold X with boundary $\partial X = Y_1 \sqcup -Y_2$, we have vector spaces $\mathcal{H}_Q(Y_{1,2})$ and a linear map
$$Z_Q(X) : \mathcal{H}_Q(Y_1) \to \mathcal{H}_Q(Y_2) \tag{0.2.1}$$
- As in the standard Osterwalder-Schroeder or Wightman axioms, for a manifold X without boundary with n points $p_1,\ldots,p_n \in X$ marked by labels $v_1,\ldots,v_n$, we have a complex number
$$\langle v_1(p_1)\cdots v_n(p_n)\rangle_X \equiv Z_Q(X;(p_1,v_1),\ldots,(p_n,v_n)) \in \mathbb{C}. \tag{0.2.2}$$
- In general, a QFT Q associates linear maps as in (0.2.1) to a manifold with boundary with points marked by labels. More generally, a manifold with boundary can have various submanifolds with various dimensions marked by various labels.

The section concludes with the discussion of the Standard Model of the particle physics phrased in the language of this review.

In Sec. 2, we develop the concept of four-dimensional $\mathcal{N}=2$ supersymmetric QFTs. Correspondingly to the three operations in Sec. 1, we will discuss

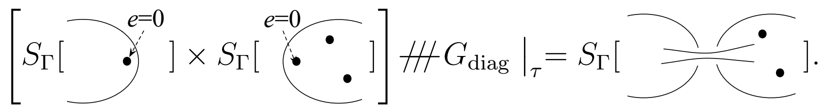
- The product of two $\mathcal{N}=2$ supersymmetric QFTs. This is just the same as the non-supersymmetric version.
- $\mathcal{N}=2$ supersymmetric version of the gauging. Given a $G\times F$-symmetric $\mathcal{N}=2$ supersymmetric QFT Q, this operation creates $Q/\!\!\!/\!\!\!/G$ which is an F-symmetric $\mathcal{N}=2$ supersymmetric QFT.

- The functor called Hyp. Given a pseudoreal representation V of G, $\mathrm{Hyp}(V)$ is a G-symmetric $\mathcal{N}=2$ supersymmetric QFT.

For each such QFT Q, we discuss the Donagi-Witten integrable system $DW(Q) \to \mathcal{M}_{\mathrm{Coulomb}}(Q)$ and the Higgs branch $\mathcal{M}_{\mathrm{Higgs}}(Q)$ which is a hyperkähler manifold, and various other invariants associated to Q.

An $\mathcal{N}=2$ supersymmetric QFT of the form $\mathrm{Hyp}(V)/\!\!/\!\!/G$ is called an $\mathcal{N}=2$ supersymmetric gauge theory. To determine its Donagi-Witten integrable system is what is usually referred to as the Seiberg-Witten theory in the physics literature. This is related but distinct from what is called the theory of the Seiberg-Witten invariants of four-dimensional manifolds, about which we do not have the space to discuss in this review. We discuss many examples of the Donagi-Witten integrable system for the theories of the form $\mathrm{Hyp}(V)/\!\!/\!\!/G$, and discuss the relation to the Hitchin system on an auxiliary Riemann surface with punctures.

In Sec. 3, we first introduce the concept of the dimensional reduction. Very roughly, the idea is the following. We start from a d-dimensional QFT Q and a d'-dimensional manifold Y. Then we define the $d-d'$-dimensional QFT $Q[Y]$ by declaring $Z_{Q[Y]}(X) = Z_Q(X \times Y)$. We introduce a class of six-dimensional theory S_Γ, where Γ is a simply-laced Dynkin diagram. Let G be a simple group of type Γ. Given a Riemann surface C with punctures p_i labeled by nilpotent elements e_i, the dimensional reduction $S_\Gamma[C, \{e_i\}]$ is a four-dimensional $\mathcal{N}=2$ supersymmetric $\prod_i G^{e_i}$-symmetric theory. These are the class S theories. In particular, when $e=0$ the symmetry is $G^e = G$ itself. One of the most important features is Gaiotto's gluing operation, which maps the gluing of two Riemann surfaces to the gauging of the product of $\mathcal{N}=2$ QFTs:



(0.2.3)

We also explain various cases when $S_\Gamma[C, \{e_i\}]$ is an $\mathcal{N}=2$ gauge theory. Together with the general fact that the Donagi-Witten system of $S_\Gamma[C, \{e_i\}]$ is the G-Hitchin system on C with singularities given by the dual orbit of e_i, it explains the form of many of the Donagi-Witten system of $\mathcal{N}=2$ gauge theories.

After these preparations, we discuss in Sec. 4 and in Sec. 5 two applications. In Sec. 4 we study Nekrasov's partition function of the class S theories. Nekrasov's partition function of an $\mathcal{N}=2$ gauge theory is a certain equivariant integral over the moduli space of instantons. When the $\mathcal{N}=2$ gauge theory is a class S theory, we will argue, based on the general properties developed in the preceding sections, that Nekrasov's partition function of it has another interpretation as the conformal block of the W-algebra. In Sec. 5, we consider the partition function of class S theories on $S^3 \times S^1$. We explain that this is governed by an elliptic generalization of Macdonald functions. In a certain limit, this provides an explicit formula of the Hilbert series of various hyperkähler cones, including instanton moduli spaces of exceptional groups.

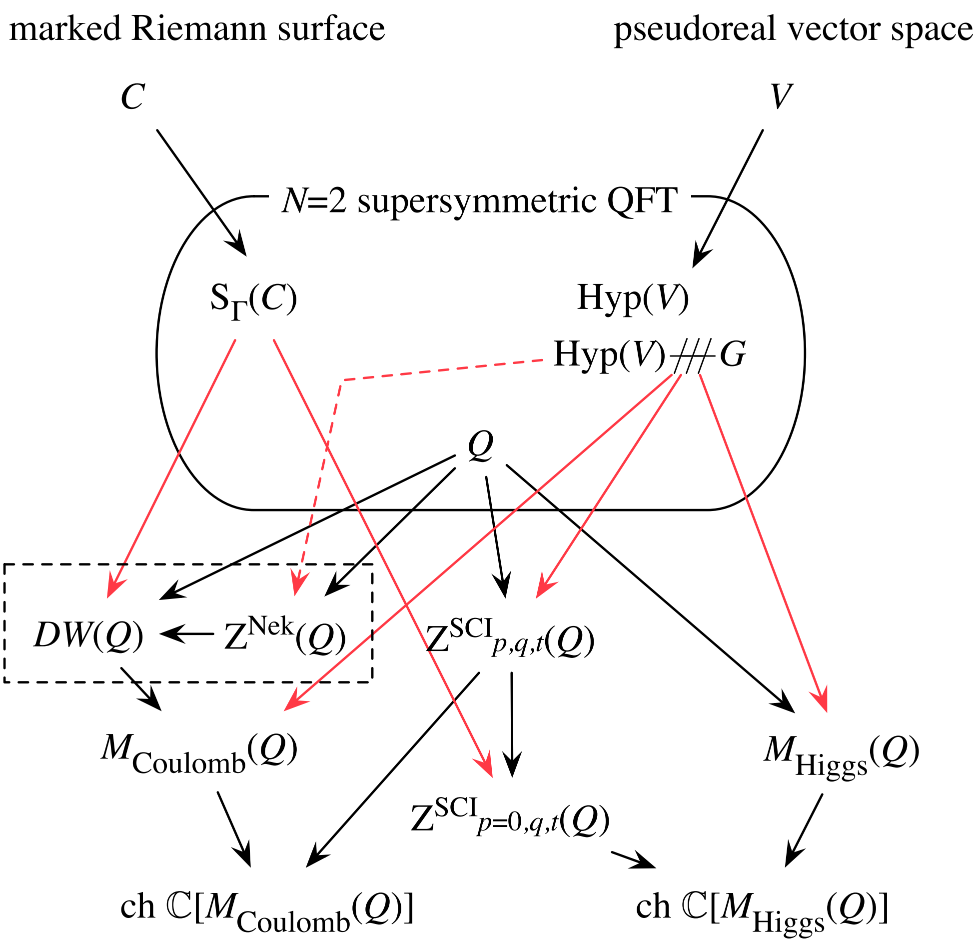


Figure 1: Interrelation of the objects we discuss concerning $\mathcal{N}=2$ supersymmetric theories. Black arrows show that the object at the head follows from the object at the tail. Red arrows show easily computable structures; $Z^{\text{Nek}}(Q)$ for $Q = \text{Hyp}(V)/\!\!/\!\!/G$ is practically computable only when V is zero dimensional or G is a product of type A groups, thus the dotted red arrow. The ones in the dotted box, $DW(Q)$ and $Z^{\text{Nek}}(Q)$, do depend on the continuous deformation of Q. But the other objects derived from Q are independent of the continous deformation of Q.

0.3 Properties of four-dimensional $\mathcal{N} = 2$ theories that we discuss

During the course of this lecture note, we visit various structures associated to $\mathcal{N} = 2$ supersymmetric theories, which are summarized in Fig. 1. We learn two methods to construct $\mathcal{N} = 2$ theories:

- Starting from a pseudoreal representation V of G, we have $\mathrm{Hyp}(V) /\!/\!/ G$, see Sec. 2.5 and 2.6.
- Starting from Γ a simply-laced Dynkin diagram, a Riemann surface C with points p_i marked by nilpotent orbits e_i of $\mathfrak{g}_{\mathbb{C}}$ where $\mathfrak{g}$ is a Lie algebra of type Γ , we have $S_\Gamma(C; (p_i, e_i))$. See Sec. 3.6.

Given an $\mathcal{N} = 2$ supersymmetric QFT Q, we discuss the following objects associated to it:

- a hyperkähler manifold $\mathcal{M}_{\text{Higgs}}(Q)$, which will be introduced in Sec. 2.4.
- a holomorphic integral system $DW(Q)$, whose base is $\mathcal{M}_{\text{Coulomb}}(Q)$. $\mathcal{M}_{\text{Coulomb}}(Q)$ is introduced in Sec. 2.4, and $DW(Q)$ is presented in Sec. 2.9.
- Nekrasov partition function $Z^{\text{Nek}}(Q)$, discussed in Sec. 4. This is essentially $Z_Q(\mathbb{R}^4)$ with an extra equivariant twist. By taking a limit, $DW(Q)$ can be reconstructed, as discussed in Sec. 4.1.
- the superconformal index $Z^{\text{SCI}}_{p,q,t}(Q)$, discussed in Sec. 5. This is essentially $Z_Q(S^3 \times S^1)$.

Most of these objects, except $DW(Q)$ and $Z^{\text{Nek}}(Q)$, do not change under a continuous deformation of Q.

When $Q = \mathrm{Hyp}(V) /\!/\!/ G$, $\mathcal{M}_{\text{Higgs}}(Q)$, $\mathcal{M}_{\text{Coulomb}}(Q)$ are both easily computable, as discussed in Sec. 2.6. $Z^{\text{SCI}}(Q)$ also has an explicit formula given in Sec. 5.2. When G is a product of SU gauge groups, we have an explicit formula for $Z^{\text{Nek}}(Q)$, since we can evaluate the definition given in Sec. 4.1 by localization. To determine the Donagi-Witten integrable system $DW(Q)$ is the main content of the Seiberg-Witten theory as known in the physics literature. But there is no known uniform way to do this.

When $Q = S_\Gamma(C)$, its Donagi-Witten integrable system $DW(Q)$ is essentially the G-Hitchin system on C, as will be discussed in detail in Sec. 3.8. We have a good control on its superconformal index when $p = 0$, as we review in Sec. 5.3.

Therefore, the objects which are easy to compute are complementary between the two cases when $Q = \mathrm{Hyp}(V) /\!/\!/ G$ and when $Q = S_\Gamma(C)$. If we somehow know that $S_\Gamma(C) = \mathrm{Hyp}(V) /\!/\!/ G'$, we can learn about $DW(\mathrm{Hyp}(V) /\!/\!/ G')$ which is in general hard to compute; conversely, we can learn about $\mathcal{M}_{\text{Higgs}}(S_\Gamma(C))$ which is in general hard to compute. The operation $/\!/\!/$ on the side of $S_\Gamma(C)$ can be performed via (0.2.3), so it is basic to understand the case when $S_\Gamma(C) = \mathrm{Hyp}(V)$. This is explored in Sec. 3.11.

We discuss the partial results known in the physics literature obtained in these indirect methods on $DW(\mathrm{Hyp}(V) /\!/\!/ G)$ in Sec. 2.11, and on $\mathcal{M}_{\text{Higgs}}(S_\Gamma(C))$ in Sec. 3.10.

Finally, the Hilbert series, denoted by ch in Fig. 1, of the function rings of $\mathcal{M}_{\text{Coulomb}}(Q)$ and $\mathcal{M}_{\text{Higgs}}(Q)$ are closely related to a specialization of $Z^{\text{SCI}}(Q)$. These relations will be discussed in Sec. 5.4 and 5.5.

0.4 Disclaimer

Admittedly the formulations presented in this review are not quite finished, but hopefully are not completely in the wrong direction and will be completed by a collaboration between mathematicians and physicists. The author would welcome constructive comments from readers.

One immediate problem would be that the notations which will be introduced in the review is not at all standard in the literature either on the physics side or on the mathematical side. We will cite various works in the later sections, but those works use the standard notations in the physics literature and will not be understandable unless the reader is more or less acquainted with them. Therefore, once a mathematician is sufficiently motivated, s/he is encouraged to pick up standard textbooks on non-supersymmetric QFTs and supersymmetric QFTs and to learn from those books.

Up until the latter part of Sec. 2, references to previous works will not be systematically given, because many of the statements in the physics terminology can be found scattered in physics textbooks, mathematical formulations are already given in related terms in various articles, and the original papers in which the particular points are discussed are hard to pin down. Again, the author would welcome comments from readers.

Another obvious defect of this review is that distinct compact groups with the same Lie algebra are not carefully distinguished. When one finds a compact group G in the review, it needs to be understood as a compact group whose Lie algebra is $\mathfrak{g}$.

Before proceeding, we list standard books and articles on mathematical formulations of QFTs. For the operator approaches, see [SW00, Haa96]. For the functorial approaches, see [Seg04, Ati88]. For a modern approach to perturbative renormalization, see [Cos11] and references therein. A collection of lectures for mathematicians can be found in [DEF$^+$99]. A very nice concise summary and insightful comments on various mathematical formulations of QFTs can be found in a review article [Dou12].

Acknowledgements

The formulation in this review grew out of many discussions of the author with his colleagues, and during contemplations while preparing various talks the author gave in various workshops, conferences and informal lecture series. In particular, the author thanks

- the organizers of the workshop *Langlands-type Dualities in QFT* at KITP in August 2010,

- the organizers of the conference *String-Math 2011* in July at Philadelphia,

- Akihiro Tsuchiya who organized the mini-workshop at the foot of Mt. Fuji in August 2011,
- Kyoji Saito who organized an introductory informal lecture series in spring 2012 at IPMU,
- Hiraku Nakajima who organized an intensive three-day informal lecture series at RIMS in October 2012,
- Tositake Kohno who organized a five-day lecture series at Dept. of Math. in U. Tokyo, May 2013,
- and Yosihisa Saito who organized the *13th workshop on Representations of Algebraic and Quantum groups* at Hakone, May 2013.

He thanks many valuable feedbacks from the participants of the lectures and the talks there. The author is particularly grateful for Hiroaki Kanno for his careful reading of the manuscript and many insightful comments. It is also a pleasure for the author to thank Yasuhiro Yamamoto for helping him choose the right symbol for the operation $Q{-\!\!\!\!/}G$. He also thanks Ryo Sato for providing a nice cover figure.

The manuscript was mostly written during a three-week stay of the author at the Institute for Advanced Study in March 2013. The author thanks the hospitality given there. This work is supported in part by World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan through the Institute for the Physics and Mathematics of the Universe, the University of Tokyo.

1 QFTs

Pick an integer d, and an additional structure S one can put on a compact manifold of dimension d. Here, S can be a Riemannian metric, or a G-bundle together with a connection, or just a smooth structure, etc. A d-dimensional S-structured QFT Q is a mathematical object, consisting of its partition function Z_Q, its space of states $\mathcal{H}_Q$, and its submanifold operators $\mathcal{V}_Q$, satisfying various axioms.

1.1 Partition function

First, we have the partition function

$$Z_Q \in \Gamma(\mathcal{M}, L_Q) \tag{1.1.1}$$

where $\mathcal{M}$ is the moduli space of the d-dimensional compact manifold with structure S without boundary and L is a line bundle with connection on $\mathcal{M}$. When L is a trivial line bundle with trivial connection, Q is called S-anomaly-free, and Z_Q is really a function

$$Z_Q : \mathcal{M} \to \mathbb{C}, \quad X \mapsto Z_Q(X). \tag{1.1.2}$$

We will consider extensions to noncompact X in Sec. 1.25.

1.2 Space of states

Second, choose another structure S' which we can put on a compact $(d-1)$-dimensional manifold. When S is the Riemannian structure, S' can also be the Riemannian structure. When S is the complex structure, S' will be the CR structure. In general, we need to specify a QFT with respect to both S and S'. Usually there is a conventional choice of S' given S, and we often just refer to a QFT to be S-structured.

Then $\mathcal{H}_Q$ assigns to a compact $(d-1)$-dimensional manifold Y with structure S' a vector space

$$Y \mapsto \mathcal{H}_Q(Y) \tag{1.2.1}$$

such that

$$\mathcal{H}_Q(Y_1 \sqcup Y_2) = \mathcal{H}_Q(Y_1) \otimes \mathcal{H}_Q(Y_2), \qquad \mathcal{H}_Q(\varnothing) = \mathbb{C}, \qquad \mathcal{H}_Q(-Y) = \mathcal{H}_Q(Y)^*. \tag{1.2.2}$$

Given S-structured Y_1 and Y_2, consider an S-structured manifold X such that $\partial X = Y_1 \sqcup -Y_2$. Here $-Y$ denotes Y with reversed orientation. We call components of Y_1, Y_2 the incoming and the outgoing boundaries, respectively. Let $\mathcal{M}_{Y_1,Y_2}$ be the moduli space of S-structured compact d-dimensional manifold with incoming boundaries Y_1 and outgoing boundaries Y_2. Then we have

$$Z_{Q,Y_1,Y_2} \in \Gamma(\mathcal{M}_{Y_1,Y_2}, V) \tag{1.2.3}$$

where V is a $\mathrm{Hom}(\mathcal{H}_Q(Y_1), \mathcal{H}_Q(Y_2)) = \mathcal{H}_Q(Y_1 \sqcup -Y_2)$ bundle with a connection.

This Z_{Q,Y_1,Y_2} should behave naturally with respect to reassignment of boundary components from incoming to outgoing, and the gluing of d-dimensional manifolds with boundary. In expressions, we require a natural identification

$$Z_{Q,Y_1,Y_2} \simeq Z_{Q,Y_1 \sqcup -Y_2,\varnothing} \tag{1.2.4}$$

and

$$Z_{Q,Y_1,Y_2} Z_{Q,Y_2,Y_3} \simeq \iota^* Z_{Q,Y_1,Y_3} \tag{1.2.5}$$

where $\iota : \mathcal{M}_{Y_1,Y_2} \times \mathcal{M}_{Y_2,Y_3} \to \mathcal{M}_{Y_1,Y_3}$ comes from the gluing of two d-dimensional manifolds at a common subset of boundary Y_2.

When Q is anomaly-free, for $\partial X = Y_1 \sqcup -Y_2$ we have

$$Z_Q(X) \in \mathrm{Hom}(\mathcal{H}_Q(Y_1), \mathcal{H}_Q(Y_2)) \tag{1.2.6}$$

satisfying the gluing axiom. This will make the QFT Q a functor from the category of cobordisms with structure S to the category of vector spaces. The non-triviality of the bundle L over $\mathcal{M}$ when Q is not anomaly-free will play a crucial role in our discussion in this review.

1.3 Trivial QFT

Let us introduce the trivial QFT which we denote by triv here. It has $\mathcal{H}_{\mathrm{triv}}(Y) = \mathbb{C}$ for all Y, $L \to \mathcal{M}$ is a trivial line bundle, and Z_{triv} is just a constant section.

1.4 Submanifold operators

Third, a QFT Q comes with a 'space' of labels which we can assign on submanifolds

$$\mathcal{V}_Q^0, \quad \mathcal{V}_Q^1, \quad \ldots, \quad \mathcal{V}_Q^{d-2}, \quad \mathcal{V}_{Q,Q'}^{d-1} \tag{1.4.1}$$

so that the whole structures described so far can be generalized to the moduli space of d-dimensional compact manifold X with a submanifold $W = \sqcup_i W_i$ with markings $v_i \in \mathcal{V}_Q^{\dim W_i}$ for each of the connected component W_i. As will be explained soon, the $\mathcal{V}^{d-1}$ is somewhat special in that it is defined with respect to two QFTs Q and Q'.

We allow W to intersect transversally with the boundary of X. Therefore, for Y of dimension $(d-1)$ with submanifolds $W = \sqcup_i W_i$, we have a vector space

$$\mathcal{H}_Q(Y, (W_i, v_i)) \tag{1.4.2}$$

where $v_i \in \mathcal{V}_Q^{1+\dim W_i}$, and we have the section

$$Z_{Q;Y,(W_i,v_i);Y',(W_i',v_i')} \in \Gamma(\mathcal{M}_{Y,(W_i,v_i);Y',(W_i',v_i')}, V) \tag{1.4.3}$$

where V is an $\mathrm{Hom}(\mathcal{H}_Q(Y, (W_i, v_i)), \mathcal{H}_Q(Y', (W_i', v_i')))$-bundle over the moduli space, etc.

The author does not understand yet how to precisely formulate the mathematical nature of $\mathcal{V}_Q^d$ in general. The axioms of $\mathcal{V}_Q^0$, when the structure S is the complex structure for real two-dimensional surfaces, are those of the vertex operator algebras. We discuss in Sec. 1.10 a possible formulation of $\mathcal{V}_Q^0$ when S is the Riemannian structure with metric. We will abbreviate $\mathcal{V}_Q^0$ by $\mathcal{V}_Q$. For $i \geq 1$, the space of labels $\mathcal{V}_Q^i$ is some version of (higher) categories. By abuse of terminology, we call elements of $\mathcal{V}^i$ for any i submanifold operators. It is not clear to the author how singular submanifolds with labels are allowed to be.

The $(d-1)$-dimensional submanifold operators in $\mathcal{V}^{d-1}$ is defined with respect to two QFTs, as a $(d-1)$-dimensional submanifold cuts the original manifold X into two: $X = X_1 \sqcup_Y X_2$ where $Y \subset \partial X_1$ and $-Y \subset \partial X_2$. Then we can consider putting the QFT Q_1 on X_1, and Q_2 on X_2. Then for $v \in \mathcal{V}_{Q_1,Q_2}^{d-1}$ we have

$$Z_{Q_1,v,Q_2} \in \Gamma(\mathcal{M}, L_{Q_1,v,Q_2}) \tag{1.4.4}$$

where $\mathcal{M}$ is now the moduli space of X with a splitting $X = X_1 \sqcup_Y X_2$. This $\mathcal{V}_{Q,Q'}^{d-1}$ associated to $(d-1)$-dimensional manifolds needs to be distinguished from $\mathcal{H}_Q$ which are associated to $(d-1)$-dimensional boundaries, as the $(d-1)$-dimensional submanifold of which $v \in \mathcal{V}_{Q,Q'}^{d-1}$ is a mark can intersect transversally with the boundary of X. So, for a $(d-1)$ dimensional manifold with a splitting, $Y = Y_1 \sqcup_Z Y_2$, we have a vector space

$$\mathcal{H}_{Q_1,v,Q_2}(Y_1 \sqcup_Z Y_2). \tag{1.4.5}$$

The point is that $\mathcal{V}_{Q_1,Q_2}^{d-1}$ is the space of morphisms between Q_1 and Q_2 in the category of QFTs, and the category of d-dimensional QFTs themselves is in some sense the space $\mathcal{V}^d$ of d-dimensional operators.

When $Q_2 = \text{triv}$, such a morphism v is called a *brane* of Q_1. In this case, $\mathcal{H}_{Q_1,v,\text{triv}}$ does not depend on Y_2, and we have a well-defined

$$\mathcal{H}_{Q_1,v_i}(Y) \quad \text{when} \quad \partial Y = \sqcup_i Z_i \tag{1.4.6}$$

where each component Z_i has a label v_i.

1.5 Generalized QFTs

We can also consider generalized QFTs with S structure, where we associate

$$Z_Q \in \Gamma(\mathcal{M}, E_Q) \tag{1.5.1}$$

where the vector bundle E has rank more than one even for the moduli space $\mathcal{M}$ of the d-dimensional compact space without boundary. Typical examples are

- the holomorphic part of a two dimensional conformal field theory, where E_Q is the bundle of the conformal blocks over the moduli space of Riemann surfaces, and
- six-dimensional $\mathcal{N} = (2,0)$ supersymmetric theories, which will be discussed in Sec. 3.2.

The formulation of the gluing law is beyond the author's comprehension.

1.6 Products of QFTs

Given two d-dimensional S-structured QFTs Q_1 and Q_2, its product $Q_1 \times Q_2$ is defined by an obvious formula

$$Z_{Q_1\times Q_2} = Z_{Q_1} Z_{Q_2}, \qquad \mathcal{H}_{Q_1\times Q_2} = \mathcal{H}_{Q_1} \otimes \mathcal{H}_{Q_2}. \tag{1.6.1}$$

The trivial QFT triv introduced in Sec 1.3 is a unit of the multiplication of the QFTs.

1.7 Topological QFTs

Consider a d-dimensional topological QFTs (TQFTs), in the sense that the structure S imposed on the d-dimensional space is just the smooth structure. An extremely nice exposition for mathematicians is [Fre93]. A TQFT Q, if we only talk about Z_Q and $\mathcal{H}_Q$, is then a functor assigning $Y \mapsto \mathcal{H}_Q(Y)$ to $(d-1)$-dimensional manifolds, and a linear map $Z_Q(X) : \mathcal{H}_Q(Y_1) \to \mathcal{H}_Q(Y_2)$ when a d-dimensional manifold X is a cobordism from Y_1 to Y_2. When $d = 2$, the information contained in $\mathcal{H}_Q$ and $\mathcal{V}_Q$ can be summarized as the structure of a commutative Frobenius algebra on $\mathcal{H}_Q(S^1)$, as detailed e.g. in [Koc04].

Consider two d-dimensional TQFTs Q_1 and Q_2. Then, a morphism between the two

$$v \in \text{Hom}(Q_1, Q_2) = \mathcal{V}^{d-1}_{Q_1,Q_2} \tag{1.7.1}$$

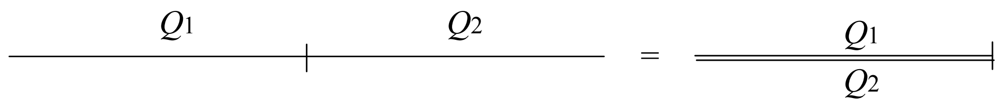


Figure 2: Q_1 and Q_2 sharing a boundary is equivalent to $Q_1 \times Q_2$ with a boundary.

gives an assignment as in (1.4.5). Note that this is *not* a natural transformation from Q_1 to Q_2 as functors from the cobordism category to the category of vector spaces. In other words, the category of TQFTs has the same objects as the category of functors from the category of cobordisms to the category of vector spaces, but the morphisms are different.

Consider a two-dimensional TQFT Q. Let Y be a one-dimensional segment with two boundary points. Then, for two branes $v_1, v_2 \in \mathrm{Hom}(Q, \mathrm{triv})$, we have a linear space

$$H(v_1, v_2) := \mathcal{H}_{v_1,Q,v_2}(Y). \tag{1.7.2}$$

One can define a composition of elements between $H(v_1, v_2)$ and $H(v_2, v_3)$ as is familiar. Then it makes $\mathrm{Hom}(Q, \mathrm{triv})$ itself a category.

This should be familiar to people who study mirror symmetry. Here, we have a 'functor' B which maps a complex variety M to a 2d TQFT $B(M)$, called the B-model on M, and another 'functor' A which maps a symplectic variety W to a 2d TQFT $A(W)$, called the A-model on W.

The category of branes of $B(M)$ is

$$\mathrm{Hom}(B(M), \mathrm{triv}) = D(M), \tag{1.7.3}$$

the derived category of coherent sheaves on M, and the category of branes of $A(W)$ is

$$\mathrm{Hom}(A(W), \mathrm{triv}) = \mathrm{Fuk}(W), \tag{1.7.4}$$

the Fukaya category of W. The homological mirror symmetry is then that there is a natural association between M and W such that the two categories of branes are equivalent

$$D(M) \simeq \mathrm{Fuk}(W). \tag{1.7.5}$$

In a two-dimensional case we have $\mathrm{Hom}(Q_1, Q_2) = \mathrm{Hom}(Q_1 \times Q_2, \mathrm{triv})$. This is called the folding trick, and can be roughly understood by referring to Fig. 2. This implies that

$$\mathrm{Hom}(B(M), B(M')) = D(M \times M') \tag{1.7.6}$$

and also

$$\mathrm{Hom}(A(W), A(W')) = \mathrm{Fuk}(W \times W'). \tag{1.7.7}$$

The general consideration so far means that an object in $D(M \times M')$ and another in $D(M' \times M'')$ can be composed to give an object in $D(M \times M'')$. This should be a derived version of the convolution product. Similarly, we should be able to compose an object in $\mathrm{Fuk}(W \times W')$ and another in $\mathrm{Fuk}(W' \times W'')$ to give an object in $\mathrm{Fuk}(W \times W'')$.

Therefore, homological mirror symmetry assigning W to M should not only be an equivalence between category $D(M)$ and $A(W)$, but should also be an equivalence of categories whose objects are $D(M)$ and $A(W)$, respectively.

1.8 2d Yang-Mills theory

1.8.1 2d Yang-Mills for finite group G

A nice example of 2d TQFT is the 2d Yang-Mills theory $Q = \mathrm{YM}_2(G)$ for a finite group G. This QFT is defined as follows. A more detailed exposition can be found in [Fre93].

First, we let

$$\mathcal{H} := \mathcal{H}_Q(S^1) = \{f : G \to \mathbb{C} \mid f(ghg^{-1}) = f(h)\}. \tag{1.8.1}$$

We define the inner product on $\mathcal{H}$ to be defined by

$$(f, f') = \sum_g f(g)f'(g^{-1}), \tag{1.8.2}$$

and identify $\mathcal{H} \simeq \mathcal{H}^*$. With this we can freely replace incoming boundaries and outgoing boundaries on a 2d surface. We then assume all boundaries to be outgoing unless otherwise specified.

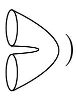
Let X be a genus γ surface with n boundaries. Then $Z_Q(X)$ is an element $f \in \mathcal{H}^{\otimes n}$, which we define as

$$f(g_1, \ldots, g_n) = |G|^{1-\gamma-n} \sum_P \frac{\prod_i |C(g_i)|}{|\operatorname{Aut} P|}. \tag{1.8.3}$$

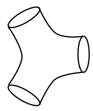
where the sum is over isomorphism classes of G-bundles P over X such that the restriction of P to the i-th boundary S^1 has a holonomy conjugate to g_i, $C(g)$ is the centralizer of g and $\operatorname{Aut} P$ is the bundle automorphism group of P. This is an easily mathematically well-defined case of path integrals of gauge theories, to which we come back at Sec. 1.22. It is straightforward to check that Z_Q defined via the formula above behaves correctly under the gluing of boundaries, and when X has no boundary, the definition (1.8.3) translates to

$$Z_Q(X) = |G|^{-\gamma} |\operatorname{Hom}(\pi_1(X), G)|. \tag{1.8.4}$$

Let us see some examples: the map

$Z_Q($  $) : \mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$ (1.8.5)

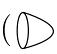
agrees with the inner product (1.8.2). A pair of pants defines a map

$Z_Q($  $) : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ (1.8.6)

given by

$$f \otimes f' \mapsto (f \circ f')(h) = \sum_h f(gh)f'(g^{-1}). \tag{1.8.7}$$

Similarly, we have

$Z_Q($  $) : \mathcal{H} \to \mathbb{C}$ (1.8.8) is given by

$$f \mapsto f(e). \tag{1.8.9}$$

Let us denote by $\operatorname{Irr} G$ the set of irreducible representations Then $\mathcal{H}$ has a natural basis given by the character χ_ρ for $\rho \in \operatorname{Irr} G$. The inner product (1.8.2), (1.8.5) is now given by

$$(\chi_\rho, \chi_{\rho'}) = \delta_{\rho\rho'} |G| \tag{1.8.10}$$

and a pair of pants (1.8.6) is

$$\chi_\rho \otimes \chi'_\rho \mapsto \delta_{\rho\rho'} \chi_\rho |G| / \dim \rho. \tag{1.8.11}$$

Then we have another formula for the Z_Q of a surface X of genus γ without boundary:

$$Z_Q(X) = |G|^{\gamma-1} \sum_{\rho \in \operatorname{Irr} G} \frac{1}{(\dim \rho)^{2\gamma-2}}. \tag{1.8.12}$$

The equality of this and (1.8.4) is a classic identify of finite group theory.

1.8.2 $\mathcal{V}_Q^1$ for 2d Yang-Mills

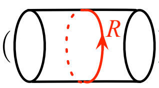
Now let us discuss the labels we can put on the submanifolds, for $Q = \mathrm{YM}_2(G)$. $\mathcal{V}_Q^0$ is trivial, and $\mathcal{V}_Q^0 \simeq \mathbb{C}$. $\mathcal{V}_Q^1 = \operatorname{Hom}(Q, Q)$ contains the category of representations of G. The trivial representation of G gives a trivial label for a one-dimensional submanifold, which is equivalent to having no one-dimensional submanifold to start with.

Let X be a genus γ surface with n boundaries. Pick k embedded S^1's, $L_{1,\ldots,k}$, of X, which we assume not to intersect with the boundaries, for simplicity. Put the labels $R_{1,\ldots,k}$ which are representations of G. Then $Z_Q(X, (L_1, R_1), \ldots, (L_k, R_k)$ is an element $f \in \mathcal{H}^{\otimes n}$, which we define as

$$f(g_1, \ldots, g_n) = |G|^{1-\gamma-n} \sum_P \frac{\prod_i |C(g_i)|}{|\operatorname{Aut} P|} \prod_{i=1}^k \operatorname{tr}_{R_i} \operatorname{Hol}(P, L_i) \tag{1.8.13}$$

where most of the symbols are as in (1.8.3), and $\operatorname{Hol}(P, L_i)$ is the holonomy of the G-bundle P around L_i.

For example, when we have a line labeled by a representation R around the cylinder, we have a map

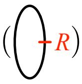
$Z_Q($ 

$) : \mathcal{H} \to \mathcal{H}$

$$\begin{aligned} f &\mapsto \chi_R f, \quad (\chi_R f)(g) = \chi_R(g) f(g) \\ \chi_\rho &\mapsto \sum_{\rho'} n_{R\rho}^{\rho'} \chi_{\rho'}. \end{aligned} \tag{1.8.14}$$

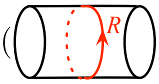
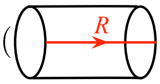
Here, ρ and ρ' are in $\operatorname{Irr} G$ and $R \otimes \rho = \rho'^{\oplus n_{R\rho}^{\rho'}}$. When R is trivial this operator is just the identity.

When we have a line labeled by R intersecting transversally with a boundary S^1, we have

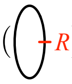
$\mathcal{H}_Q($  $) = \{f : G \to R \mid f(g^{-1}hg) = g(f(h))\}$ (1.8.15)

When R is trivial this reduces to $\mathcal{H}_Q(S^1)$, see (1.8.1).

Then we can compute Z_Q of a torus with a line labeled by R in two ways:

$\operatorname{tr} Z_Q($  $) = \operatorname{tr} Z_Q($  $)$ (1.8.16)

which is

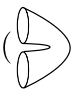
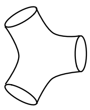
$\operatorname{tr}_{\mathcal{H}} \chi_R = \dim \mathcal{H}_Q($  $) = \sum_{\rho \in \operatorname{Irr} G} n_{R\rho}^{\rho}.$ (1.8.17)

Properties of $\mathcal{V}_Q^1$ for general 2d TQFTs have been formulated and explored in [DKR11, CR12].

1.8.3 2d Yang-Mills for compact continuous G

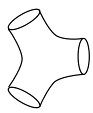
Now let us try to extend our discussions so far on $\mathrm{YM}_2(G)$ from just finite group G to general compact group G. Many formulas can be modified slightly to make sense. For example, we can keep (1.8.1) except we demand the smoothness of f. The inner product (1.8.2) can be replaced by

$$(f, f') = \int_G f(g) f'(g^{-1}) dg \tag{1.8.18}$$

where dg stands for the Harr measure with total volume 1. The *path integral* definition of Z_Q, (1.8.3) does not make sense as it is. So, let us try to directly define $Z_Q($  $)$, $Z_Q($  $)$ etc. This can be most easily done in the representation basis, as in (1.8.10), (1.8.11), (1.8.12). We pick a constant c to replace $|G|$ and we just demand

$$(\chi_\rho, \chi_{\rho'}) = c\delta_{\rho\rho'} \tag{1.8.19}$$

and for a pair of pants we have

$Z_Q($  $) : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ (1.8.20)

$$\chi_\rho \otimes \chi'_\rho \mapsto \delta_{\rho\rho'} \chi_\rho c / \dim \rho.$$

Then we have

$$Z_Q(X) = c^{\gamma-1} \sum_{\rho \in \operatorname{Irr} G} \frac{1}{(\dim \rho)^{2\gamma-2}} \tag{1.8.21}$$

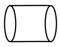
a surface X of genus γ without boundary, but the crucial point is that this converges only for large enough γ. For example, when $\gamma = 1$, we formally have

$$Z_Q(X) = \mathrm{tr}_{\mathcal{H}}\, 1 \tag{1.8.22}$$

which does not naively make sense.

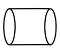
There are a few ways out. One way is to declare that we only allow X such that $Z_Q(X)$ converges. Another way is to consider not just TQFTs defined over $\mathbb{C}$ but also TQFTs defined over $\mathbb{C} \cup \{\infty\}$. What physicists usually do is to give up having a topological QFT. Instead, 2d Yang-Mills $Q = \mathrm{YM}_2(G)$ for a compact group G can be defined without any problem as an *area-ed* QFT, i.e. as the structure S in the definition of a QFT, we require that there is a real positive number A which we call the area assigned to the 2d surface X. On the boundary one-dimensional manifold, we do not put additional structure, so the structure S' we introduced in Sec. 1.2 is trivial. When we glue two area-ed surface, the areas are added together.

Then, for a surface X of genus γ without boundary, we define for example

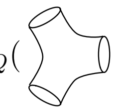
$Z_Q($  $) : \mathcal{H} \to \mathcal{H},$

$$\chi_\rho \mapsto e^{-Ac_2(\rho)} \chi_\rho. \tag{1.8.23}$$

Here, A is the area of the tube, and $c_2(\rho)$ is the quadratic Casimir of the irreducible representation ρ. In other words, we have

$Z_Q($  $) = e^{-A\triangle_G}$ (1.8.24)

where $\triangle_G$ is the standard Laplacian on the group manifold G. Similarly, we define

$Z_Q($  $) : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$

$$\chi_\rho \otimes \chi'_\rho \mapsto \delta_{\rho\rho'} \chi_\rho c e^{-Ac_2(\rho)} / \dim \rho. \tag{1.8.25}$$

Then, for a genus γ surface X without boundary, we have

$$Z_Q(X) = c^{\gamma-1} \sum_{\rho \in \mathrm{Irr}\, G} \frac{e^{-Ac_2(\rho)}}{(\dim \rho)^{2\gamma-2}}. \tag{1.8.26}$$

The path integral definition for the finite group, (1.8.3), can be generalized to the area-ed case, as an integral over the space of connections on G-bundles over a given 2d surface. This is a special case of what we discuss in Sec. 1.22. In the limit $A \to 0$, which corresponds to the not-quite-existent TQFT discussed above, the path integral becomes an integral over the moduli space of flat G-bundles over a given surface, which was discussed at length in [Wit91, Wit92]. A thorough discussion of 2d Yang-Mills for compact G can be found in the review article [CMR95].

1.9 Physical unitary QFTs

Mathematicians are already familiar with the topological QFTs where the structure S above is the smooth structure, or the two-dimensional conformal QFTs where the structure S on a two-dimensional manifold is the complex structure. In these cases, the axioms in the previous section, once precisely formulated, should reduce to the Atiyah's axioms of TQFT and the Segal's axioms of conformal field theory, respectively. We discussed TQFTs briefly above.

In the high energy physics theory community, people mostly care about the case when the structure S consists of a spin structure, a Riemannian structure with metric, and a G-bundle with a connection.[1] Let us call a d-dimensional QFT with this structure S a d-dimensional G-symmetric QFT. It is easy to see that if $H \subset G$ there is a forgetful map which makes a G-symmetric QFT a H-symmetric QFT. Also, the product of a G_1-symmetric Q_1 and G_2-symmetric Q_2 is $G_1 \times G_2$-symmetric. When $G_1 = G_2 = G$, we can take the diagonal subgroup $G_{\text{diag}} \subset G \times G$ and consider $Q_1 \times Q_2$ as G-symmetric.

Physicists also usually impose the unitarity condition, which says that

- $\mathcal{H}_Q(Y)$ has the Hilbert space structure (i.e. a positive definite sesquilinear form on it) and therefore there is a canonical conjugate-linear identification $\mathcal{H}_Q(Y) \simeq \mathcal{H}_Q(-Y)$.[2]
- This conjugate linear identification is compatible with the sections

$$Z_{Q,Y} \in \Gamma(\mathcal{M}_Y, V), \qquad Z_{Q,-Y} \in \Gamma(\mathcal{M}_{-Y}, \bar{V}). \tag{1.9.1}$$

  This is called the reflection positivity.

In the following, we only deal with unitary QFTs.

1.10 Point operators

Let us discuss the properties of the space of operators $\mathcal{V}_Q = \mathcal{V}_Q^0$ for a G-symmetric QFT Q. This is a $\mathbb{C}$-linear space with the following properties

- $\mathcal{V}$ is a representation of $G \times \mathrm{Spin}(d)$, and is filtered by $D \in \mathbb{R}_{\geq 0}$

$$\mathcal{V}_D \subset \mathcal{V}_{D'} \subset \mathcal{V}, \quad (D < D') \tag{1.10.1}$$

  such that $\mathcal{V}_D$ is a finite-dimensional representation of $G \times \mathrm{Spin}(d)$. When $v \in \mathcal{V}_D$ it is said that v has *mass dimension* less than or equal to D.

[1]Comparison against experiments require a QFT when S consists of a four-dimensional Lorentzian metric of signature $(- + ++)$, instead of a Euclidean Riemannian metric. As there is a one-to-one map between unitary Lorentizan QFTs and unitary Euclidean QFTs, we formulate everything in terms of Euclidean QFTs in this review.

[2]It is often the case in physics literature that the Hilbert space is defined as a cohomology, $\mathcal{H}_Q(Y) = H(\underline{\mathcal{H}_Q}(Y), \delta)$ where $\underline{\mathcal{H}_Q}(Y)$ does not necessarily have a Hilbert space structure. In this case δ is usually called the BRST operator.

- There is a linear map ∇

$$v \in \mathcal{V} \mapsto \nabla v \in \mathbb{R}^d \otimes \mathcal{V}. \tag{1.10.2}$$

This satisfies

$$\nabla \mathcal{V}_D \subset \mathbb{R}^d \otimes \mathcal{V}_{D+1}. \tag{1.10.3}$$

- $\mathcal{V}$ has a family of non-commutative products $\circ_x$ parameterized by $x \in \mathbb{R}^d \setminus \{0\}$:

$$(v, w, x) \in \mathcal{V} \times \mathcal{V} \times (\mathbb{R}^d \setminus \{0\}) \mapsto v \circ_x w \in \mathcal{V} \tag{1.10.4}$$

called the *operator product expansion*. This is continuous in x, compatible with the $\mathrm{Spin}(d)$ action on $\mathcal{V}$ and $\mathbb{R}^d$, and when $v \in V_D$ and $v' \in V_{D'}$ the limit

$$\lim_{x \to 0} |x|^{D+D'} v \circ_x v' \tag{1.10.5}$$

exists.

- The family of products $\circ_x$ are associative in the following sense:

$$(v \circ_x v') \circ_{x'} v'' = v \circ_{x+x'} (v' \circ_{x'} v''). \tag{1.10.6}$$

- The product $\circ_x$ and the derivative ∇ is compatible, in the sense that

$$\partial(v \circ_x w) = (\nabla v) \circ_x w \tag{1.10.7}$$

where ∂ on the left hand side is the partial derivative with respect to x.

We note that the concept of the algebra of point operators of a 2d conformal field theory is already axiomatized as vertex operator algebras, see e.g. [Bor86].

1.11 Multipoint functions

Let X be a d-dimensional compact spin manifold with a metric with distinct marked points $p_1, \ldots, p_n$, with a G-bundle P with connection. Let

$$F_{G \times \mathrm{Spin}(d)} X = P \times_X F_{\mathrm{Spin}(d)} X \to X \tag{1.11.1}$$

where $F_{\mathrm{Spin}(d)} X$ is the frame bundle of the spin structure, together with the connection determined by the metric. For a vector space V with an action of $G \times \mathrm{Spin}(d)$, we denote by $\underline{V}$ the associated line bundle over X:

$$\underline{V} = F_{G \times \mathrm{Spin}(d)} X \times_{G \times \mathrm{Spin}(d)} V. \tag{1.11.2}$$

Then the markings for the marked points p_i are given by $v_i^* \in \underline{\mathcal{V}^*}|_{p_i}$ for each i. We then have

$$Z_Q((p_1, v_1^*), (p_2, v_2^*), \ldots, (p_n, v_n^*)) \in \Gamma(\mathcal{M}, L_Q). \tag{1.11.3}$$

The left hand side determines a section of a bundle

$$\underbrace{\underline{\mathcal{V}} \boxtimes \underline{\mathcal{V}} \boxtimes \cdots \boxtimes \underline{\mathcal{V}}}_{n \text{ times}} \to X^n \tag{1.11.4}$$

which we denote by

$$\langle v_1(p_1) v_2(p_2) \cdots v_n(p_n) \rangle_X. \tag{1.11.5}$$

This is called the n-point function. Note that for vector bundles $E_i \to X_i$, $i = 1, 2$ and $p_i : X_1 \times X_2 \to X_i$, we define $E_1 \boxtimes E_2 = p_1^*(E_1) \otimes p_2^*(E_2)$.

The n-point function is compatible with the product structure on $\mathcal{V}$ in the following sense:

- The derivative ∇ satisfies

$$\langle (\nabla v)(p_1) \cdots v_n(p_n) \rangle_X = \nabla \langle v(p_1) \cdots v_n(p_n) \rangle_X \tag{1.11.6}$$

  where ∂ on the right hand side is the covariant derivative with respect to p_1.

- Pick $v \in \mathcal{V}_D$ and $v' \in \mathcal{V}_{D'}$. Pick a patch of X by taking $\{0\} \subset U \subset \mathbb{R}^d$ and $\iota : U \to X$. Then we have

$$|x|^{D+D'} \langle v(\iota(x)) v'(\iota(0)) \cdots v_n(p_n) \rangle_X \tag{1.11.7}$$

  and

$$|x|^{D+D'} \langle (v \circ_x v')(\iota(0)) \cdots v_n(p_n) \rangle_X \tag{1.11.8}$$

  become the same in the limit $x \to 0$.

We note that the concept of multipoint functions of 2d conformal field theories is already axiomatized in [GG00].

1.12 Energy-momentum tensor and currents

Given a d-dimensional QFT Q, let us consider the behavior of $Z_Q((X, g_X))$ under an infinitesimal change of the metric

$$g_X \to g_X + \epsilon \delta g \tag{1.12.1}$$

where δg is a section of $\mathrm{Sym}^2 TX$. The dependence of Z_Q with respect to δg is given by an element $T \in \mathcal{V}_{Q,d-2}$, transforming as $\mathrm{Sym}^2 \mathbb{R}^d$ under the $\mathrm{Spin}(d)$ action, as follows:

$$\begin{aligned} Z_Q((X, g_X + \epsilon \delta g)) = \langle 1 \rangle_{X, g_X} + \epsilon \int_{p \in X} (\langle T(p) \rangle_{X, g_X}, \delta g(p)) d\,\mathrm{vol}_X \\ + \frac{\epsilon^2}{2} \int_{(p,q) \in X \times X} (\langle T(p) T(q) \rangle_{X, g_X}, \delta g(p) \delta g(q)) d\,\mathrm{vol}_{X \times X} \\ + \frac{\epsilon^3}{6} \int_{(p,q,r) \in X \times X \times X} (\langle T(p) T(q) T(r) \rangle_{X, g_X}, \delta g(p) \delta g(q) \delta g(r)) d\,\mathrm{vol}_{X \times X} + \cdots . \end{aligned} \tag{1.12.2}$$

This point operator T is called the energy momentum tensor. The leading divergence of $T \circ_x T$ when $x \to 0$ has the form

$$\lim_{x \to 0} |x|^{2(d-2)} T \circ_x T \to c(Q)X \tag{1.12.3}$$

where $c(Q)$ is a positive real number called the c central charge of Q, and X is a certain $\mathrm{Spin}(d)$-invariant element in $\mathrm{Sym}^2(\mathrm{Sym}^2 \mathbb{R}^d)$ fixed by convention. This c is additive: $c(Q_1 \times Q_2) = c(Q_1) + c(Q_2)$.

For some choice of δg, (X, g_X) and $(X, g_X + \epsilon\delta g)$ can correspond to isometric manifolds related by a certain diffeomorphism on X. This implies that $f(\nabla T) = 0$, where f is given by the composition

$$f : \mathbb{R}^d \otimes \mathrm{Sym}^2 \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d \otimes \mathbb{R}^d \xrightarrow{(,)\otimes 1} \mathbb{R}^d. \tag{1.12.4}$$

Similarly, given a d-dimensional G-symmetric QFT Q and a manifold X with G-bundle $P \to X$ with connection D, we consider an infinitesimal change

$$D \to D + \epsilon A \tag{1.12.5}$$

where A is a $\mathfrak{g}$-valued one-form. We have an element $J \in \mathcal{V}_{Q,d-1}$, transforming as $\mathfrak{g} \otimes \mathbb{R}^d$ under the $G \times \mathrm{Spin}(d)$ action, such that

$$\begin{aligned} Z_Q((P, D + \epsilon A)) = \langle 1 \rangle_{P,D} + \epsilon \int_{p \in X} (\langle J(p) \rangle_{P,D}, A(p)) d \,\mathrm{vol}_X \\ + \frac{\epsilon^2}{2} \int_{(p,q) \in X \times X} (\langle J(p)J(q) \rangle_{P,D}, A(p)A(q)) d \,\mathrm{vol}_{X \times X} \\ + \frac{\epsilon^3}{6} \int_{(p,q,r) \in X \times X \times X} (\langle J(p)J(q)J(r) \rangle_{P,D}, A(p)A(q)A(r)) d \,\mathrm{vol}_{X \times X} + \cdots . \end{aligned} \tag{1.12.6}$$

This operator J is called the G-current. The leading divergence of $J \circ_x J$ when $x \to 0$ has the form

$$\lim_{x \to 0} |x|^{2d-2} J \circ_x J = \langle , \rangle \otimes \mathrm{id} \in (\mathrm{Sym}^2 \mathfrak{g}) \otimes (\mathrm{Sym}^2 \mathbb{R}^d) \tag{1.12.7}$$

where $\langle , \rangle$ is a positive bilinear form on $\mathfrak{g}$, and id is the standard bilinear form on $\mathbb{R}^d$. When $\mathfrak{g}$ is simple, the form $\langle , \rangle$ is determined by a positive number $k_G(Q)$ times the Killing form. This k_G is additive: $k_G(Q_1 \times Q_2) = k_G(Q_1) + k_G(Q_2)$.

For some choice of A, (P, D) and $(P, D + \delta A)$ corresponds to a G-connection equivalent to the original one D related by a gauge transformation on P. This implies that $f(\nabla J) = 0$, where f is given by the inner product $\mathbb{R}^d \otimes \mathbb{R}^d \to \mathbb{R}$.

1.13 1d QFTs

Now let us consider a rather simple case of 1d QFTs Q with Riemannian structure. A boundary of one-dimensional manifolds is just a disjoint union of points. Let $\mathcal{H}_Q(pt) = \mathcal{H}$. A segment of length s gives a linear map

$$Z(s) : \mathcal{H} \to \mathcal{H} \tag{1.13.1}$$

satisfying

$$Z(s+t) = Z(s)Z(t). \tag{1.13.2}$$

We write

$$Z(s) = e^{-sH} \tag{1.13.3}$$

and call H the Hamiltonian. This is the energy-momentum tensor introduced above. We can also identify the zero-dimensional operators $\mathcal{V}_Q^0$ as a subset of $\mathrm{Hom}(\mathcal{H},\mathcal{H})$. Then the multi-point function on S^1 with circumference s is given by

$$Z_Q(S^1,(p_1,v_1),\ldots,(p_n,v_n)) = \mathrm{tr}_{\mathcal{H}}\, v_1(p_1)v_2(p_2)\cdots v_n(p_n)e^{-sH} \tag{1.13.4}$$

where

$$v(p) = e^{-pH}\ v\ e^{pH}, \quad v \in \mathcal{V}_Q^0 \subset \mathrm{Hom}(\mathcal{H},\mathcal{H}). \tag{1.13.5}$$

When Q is unitary and $\mathcal{H}$ is a Hilbert space, then H is Hermitean.

1.14 CPT conjugation

When the theory is unitary, the $\mathrm{Spin}(d)$ $\mathbb{C}$-representation on $\mathcal{V}$ is extended to $\mathrm{Pin}(d)$ $\mathbb{R}$-representation such that elements in $\mathrm{Pin}(d)$ connected to the identity is represented $\mathbb{C}$-linearly and those not connected to the identity is represented conjugate-linearly, i.e. an element $g \in \mathrm{Pin}(d) \setminus \mathrm{Spin}(d)$ determines a conjugate linear map

$$\mathcal{V} \ni v \mapsto \bar{v} \in \mathcal{V}. \tag{1.14.1}$$

This map is called *the CPT conjugation.* This $\mathrm{Pin}(d)$ action is compatible with the filtration by the mass dimension, the derivative, and the product. Most importantly, this is compatible with the reflection positivity of the n-point function, i.e.

$$\overline{\langle v_1(p_1)v_2(p_2)\cdots v_n(p_n)\rangle_X} = \langle \bar{v}_1(p_1)\bar{v}_2(p_2)\cdots \bar{v}_n(p_n)\rangle_{-X} \tag{1.14.2}$$

where $-X$ is X with the reverse orientation, and the conjugate linear map $v_i \mapsto \bar{v}_i$ are chosen according to the orientation reversal at p_i. Note that the boundary of X can be non empty.

On $\mathrm{Spin}(d)$-invariant part of $\mathcal{V}$, the part of the $\mathrm{Pin}(d)$ action disconnected to the identity gives a unique real structure

$$\bar{\cdot} : \mathcal{V}^{\mathrm{Spin}(d)} \to \mathcal{V}^{\mathrm{Spin}(d)}. \tag{1.14.3}$$

The subspace $\mathrm{Re}\,\mathcal{V}^{\mathrm{Spin}(d)}$ fixed by $\bar{\cdot}$ plays an important role in Sec. 1.20.

1.15 Renormalization Group

We have an action of the multiplicative group $\mathbb{R}_{>0}$ on the space of Riemannian QFTs. Namely, given a QFT Q, we define $\mathcal{RG}_t\,Q$ via the formula

$$Z_{\mathcal{RG}_t\,Q}((X,g)) = Z_Q((X,tg)). \tag{1.15.1}$$

If $Q \simeq \mathcal{RG}_t Q$ the theory Q is called scale-invariant. In this case the space of operators become not just filtered but graded, and we have

$$\mathcal{V}_Q = \oplus_d \mathcal{V}_{Q,d}. \tag{1.15.2}$$

Then $\mathcal{RG}_t$ acts on $\mathcal{V}_{Q,d}$ by the multiplication by t^{-d}. When Q is believed to be unitary, a scale-invariant Q is automatically conformally invariant, in the sense that $Z_Q((X, e^{-f}g))$ for a function $f : X \to \mathbb{R}$ can be written in terms of $Z_Q((X, g))$. Furthermore $\mathcal{V}$ has an action of the conformal group $\mathrm{Spin}(d, 1)$. For more on this topic, consult [Nak13] and references therein.

1.16 Free Bosons

1.16.1 Massless and massive free bosons

After all these abstract discussions, it would be appropriate to discuss a few examples. First is the free boson theory. Let V be a real representation of a group G. For any $d > 2$, there is a d-dimensional G-symmetric QFT $B_d(V)$, called a real boson valued in V. For a compact Riemannian manifold X with a G-bundle with connection $P \to X$, we define the partition function of $B_d(V)$ there via

$$Z_{B_d(V)}(X) = \frac{1}{\det -\triangle_V}. \tag{1.16.1}$$

Here, $\triangle_V$ is the natural Laplacian on the real vector bundle $\underline{V}$ on X associated to V, recall the definition given in (1.11.2). det is a regularized determinant. We have

$$B_d(V \oplus W) = B_d(V) \times B_d(W). \tag{1.16.2}$$

More generally, given a positive real number ω^2, we have $Q = B_d(V, \omega^2)$ for any d; $B_d(V)$ above is the limit when $\omega^2 \to 0$. The definition (1.16.1) is modified to

$$Z_Q(X) = \frac{1}{\det \omega^2 - \triangle_V}. \tag{1.16.3}$$

1.16.2 Space of states and the vacuum energy

The space of states is given by

$$\mathcal{H}_Q(P \to Y) = \mathbb{C} \oplus \mathcal{A} \oplus \mathrm{Sym}^2 \mathcal{A} \oplus \mathrm{Sym}^3 \mathcal{A} \oplus \cdots \tag{1.16.4}$$

where

$$\mathcal{A} = \Gamma(Y, P \times_G V_{\mathbb{C}}). \tag{1.16.5}$$

In physics literature we call an element $|0\rangle = \mathbb{C} \subset \mathcal{H}_Q$ as the vacuum, and denote

$$\mathcal{A} = \bigoplus_i a_i^\dagger |0\rangle \tag{1.16.6}$$

where $a_i^\dagger$ corresponds to an eigenfunction of $\omega^2 - \triangle_V$ on Y with eigenvalue ω_i^2. Then the space of states (1.16.4) can be identified with the polynomial algebra of $a_i^\dagger$. We also introduce a_i so that $[a_i, a_j^\dagger] = \delta_{ij}$.

1.16.3 Examples: $d = 1$ and $d = 2$

$Z_Q(Y \times [0, \beta])$ then defines an operator $e^{-\beta H}$ on $\mathcal{H}_Q(Y)$, given by

$$H = E_Q(Y) + \sum |\omega_i| a_i^\dagger a_i. \tag{1.16.7}$$

$E_Q(Y)$ is a number called the vacuum energy or the Casimir energy, determined by demanding that $Q = B_d(V, \omega^2)$ satisfies the axioms of unitary QFTs. We demonstrate how this is done below when $d = 2$.

Let us first examine the case $d = 1$, G is a trivial group, and $V = \mathbb{R}$. Consider $Q = B_1(V, \omega^2)$. We have

$$\mathcal{H}_Q(pt) = \mathbb{C}[a^\dagger] \tag{1.16.8}$$

and

$$H = E_Q(pt) + \omega a^\dagger a. \tag{1.16.9}$$

We then have, for a circle S^1 of circumference β,

$$Z_Q(S^1_\beta) = \mathrm{tr}_{\mathcal{H}_Q(pt)}\, e^{-\beta H} = \frac{e^{-\beta E_Q(pt)}}{1 - e^{-\beta\omega}} = \frac{1}{e^{+\beta\omega/2} - e^{-\beta\omega/2}}. \tag{1.16.10}$$

Here, in the last inequality, we used the conventional choice $E_Q(pt) = \omega/2$. This is called the zero-point energy. This is the quantum harmonic oscillator.

With the direct definition of Z_Q (1.16.3), we instead have

$$Z_Q(S^1_\beta) = \text{“}\frac{1}{\omega} \prod_{n \geq 1} \frac{1}{\omega^2 + (\frac{2\pi n}{\beta})^2}\text{”} \tag{1.16.11}$$

by examining the spectrum of $\triangle_V$. We can make a further manipulation so that we have

$$= \text{“}\frac{1}{\omega} \prod_{n \geq 1} \frac{1}{1 + (\frac{\beta\omega}{2\pi n})^2}\text{”} = \frac{1}{\sinh \beta\omega/2}, \tag{1.16.12}$$

which equals with (1.16.10). These are made into rigorous mathematics, by carefully defining the regularized determinant without this formal manipulation.

Generalizing $E_Q(pt) = \omega/2$ for $d = 1$ free boson, it is often written in the physics literature that for $Q = B_d(V, \omega^2)$

$$E_Q(Y) = \sum_i \frac{1}{2} |\omega_i| \tag{1.16.13}$$

where ω_i^2 run over the eigenvalues of the operator $\omega^2 - \triangle_V$ over Y. However, the expression above does not make much sense without properly defining the divergent sum. It is often then said that we should use the zeta-function regularization, which is again not quite well-motivated. Rather, the principle to determine $E_Q(Y)$ is to make $B_d(V, \omega^2)$ to satisfy the axioms.

Let us take for simplicity $d = 2$, G is trivial, and $V = \mathbb{R}$. Consider $Q = B_d(V, \omega^2)$. We can then evaluate

$$Z_Q(S^1_{\beta_1} \times S^1_{\beta_2}) = \mathrm{tr}_{\mathcal{H}_Q(S^1_{\beta_1})}\, e^{-\beta_2 H} = e^{-\beta_2 E(\beta_1)} \frac{1}{e^{-\beta_2 \omega}} \left[\prod_{n \geq 1}^{\infty} \frac{1}{1 - e^{-\beta_2 \sqrt{\omega^2 (2\pi n/\beta_1)^2}}} \right]^2 \tag{1.16.14}$$

where $E(\beta_1) = E_Q(S^1_{\beta_1})$.

The right hand side of (1.16.14) is not manifestly symmetric under the exchange of β_1 and β_2; we need to choose $E(\beta)$ so that it becomes symmetric. It is not very obvious that there is such a function $E(\beta)$; its existence is guaranteed once the regularized determinants of Laplacians are defined with care.

Here, let us content ourselves by studying the $\omega \to 0$ limit. We see that

$$\lim_{\omega \to 0} \omega Z_Q(S^1_{\beta_1} \times S^1_{\beta_2}) = e^{-\beta_2 E(\beta_1)} \frac{1}{\beta_2} \left[\prod_{n \geq 1} \frac{1}{1 - q^n} \right]^2 \tag{1.16.15}$$

where $q = e^{2\pi i \tau}$, $\tau = i\beta_2/\beta_1$. Then, by the modular property of the Dedekind eta function $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty}(1 - q^n)$ which is

$$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau), \tag{1.16.16}$$

we see that (1.16.15) is symmetric under $\beta_1 \leftrightarrow \beta_2$ when

$$E(\beta) = -\frac{2\pi}{12} \frac{1}{\beta} + c\beta \tag{1.16.17}$$

for an undetermined constant c. Compared with (1.16.13), it is often written suggestively as

$$\frac{2\pi}{\beta}(1 + 2 + 3 + 4 + \cdots) = -\frac{2\pi}{12\beta}. \tag{1.16.18}$$

1.16.4 Point operators

The space of operators $\mathcal{V}_{B_d(V,\omega^2)}$ is, as a vector space, equal to

$$\mathcal{V}_{B_d(V,\omega^2)} = \mathbb{C} \otimes \mathrm{Sym}^\bullet[\mathrm{Sym}^\bullet[\mathbb{R}^d] \otimes_{\mathbb{R}} V], \tag{1.16.19}$$

i.e. a polynomial algebra on V together with an action of a formal differential operator ∇ in the vector representation of $\mathrm{SO}(d)$. Here V is in $\mathcal{V}_{d/2-1}$; recall the subscript refers to the filtration, (1.10.1). The CPT conjugation fixes V. For $v_i \in V^*$, we can consider a multi-point function

$$\begin{aligned} &Z_{B_d(V,\omega^2)}(P \to X; (x_1, v_1), (x_2, v_2), \cdots, (x_{2n}, v_{2n})) \\ &\qquad = \langle v_1(x_1) \cdots v_{2n}(x_{2n}) \rangle_X = \frac{1}{\det \triangle_V} \sum_S \prod_{(i,j) \subset S} \langle v_i, K(x_i, x_j) v_j \rangle \end{aligned} \tag{1.16.20}$$

where K is the Green function of $\omega^2 - \triangle_V$, and S runs over sets of n pairs (i,j) such that $\cup S = \{1,\ldots,2n\}$. For example, when $2n=4$, S is either $\{(1,2),(3,4)\}$, $\{(1,3),(2,4)\}$ or $\{(1,4),(2,3)\}$. This is called Wick's theorem in physics literature.

When V is a complex representation of a group G, we define $B_d(V)$ mostly similarly. This is called a complex boson. For a real representation V and its complexification $V_\mathbb{C}$ we have

$$B_d(V_\mathbb{C}) = B_d(V) \times B_d(V). \tag{1.16.21}$$

When G is simple, $k_G(V)$ for a complex representation V is given as follows. We decompose

$$V = \oplus_i R_i \tag{1.16.22}$$

into irreducible G representations R_i, and then

$$k_G(B(V)) = \frac{2}{3}\sum c_2(R_i) \tag{1.16.23}$$

where $c_2(R)$ is the eigenvalue of the quadratic Casimir operator normalized so that $c_2(\mathfrak{g}_\mathbb{C}) = h^\vee(G)$.

1.17 Free Fermions

Another fundamental example is the free-fermion theory. As its property is intrinsically linked to that of spinors, its precise definition depends on $d \mod 8$. Here we just discuss so-called Weyl fermions in even dimensions.

1.17.1 Dirac operator and the partition function

Recall that $\mathrm{Spin}(d)$ for even d has two spinor representations $S^\pm$ such that

$$\begin{cases} S^{+*} = S^+, & S^{-*} = S^- \quad \text{if } d = 0 \mod 4, \\ S^{+*} = S^-, & S^{-*} = S^+ \quad \text{if } d = 2 \mod 4. \end{cases} \tag{1.17.1}$$

Given a spin d-manifold X with G connection, let $F_{G\times\mathrm{Spin}(d)}X$ be its frame bundle. Given a complex representation V of G, we can consider the associated vector bundle $\underline{V\otimes S^\pm} = F_{G\times\mathrm{Spin}(d)}X \times_{G\times\mathrm{Spin}(d)} V\otimes S^\pm$. Consider the Dirac operator $\not{D}^\pm$ which is a linear operator

$$\begin{aligned} \not{D}^+ :&\Gamma(X, \underline{V\otimes S^+}) \to \Gamma(X, \underline{V\otimes S^-}), \\ \not{D}^- :&\Gamma(X, \underline{V\otimes S^-}) \to \Gamma(X, \underline{V\otimes S^+}). \end{aligned} \tag{1.17.2}$$

Using this we define the free fermion theory $F_d^\pm(V)$ by

$$Z_{F_d^\pm(V)} \in \Gamma(\mathcal{M}, \mathrm{Det}\,\not{D}^\pm) \tag{1.17.3}$$

where $\mathrm{Det}\,\not{D}^\pm$ is the determinant line bundle of the Dirac operator $\not{D}^\pm$ and $Z_{F_d^\pm(V)}$ is its natural section. We have the property

$$F_d^+(V\oplus W) = F_d^+(V)\times F_d^+(W), \qquad F_d^-(V\oplus W) = F_d^-(V)\times F_d^-(W). \tag{1.17.4}$$

The point operators are given by

$$\mathcal{V}_{F_d^+(V)} = \Lambda^\bullet[\mathrm{Sym}^\bullet[\mathbb{R}^d] \otimes_{\mathbb{C}} (V \otimes S^+ \oplus \bar{V} \otimes (S^-)^*)] \tag{1.17.5}$$

and similarly for $F_d^-(V)$. The CPT conjugation maps $V \otimes S^+$ to $\bar{V} \otimes (S^-)^*$.

The combination $V \otimes S^+ \oplus \bar{V} \otimes (S^-)^*$ is made because the the Green function $K^+(x,y)$ of the Dirac operator $\not{D}^+$ is a section of

$$\underline{(\bar{V} \otimes (S^-)^*)^*} \boxtimes \underline{(V \otimes S^+)^*} \tag{1.17.6}$$

on $X \times X$. We can then define

$$\begin{aligned} &Z_{F_d^+(V)}(X; (x_1, v_1), (y_1, w_1), \cdots, (x_n, v_n), (y_n, w_n)) \\ &\qquad = Z_{F_d^+(V)}(X) \sum_\sigma \prod (-1)^\sigma \langle w_i, K^+(y_i, x_{\sigma(i)}) v_{\sigma(i)} \rangle \end{aligned} \tag{1.17.7}$$

where $v_i \in V \otimes S^+$ and $w_i \in \bar{V} \otimes (S^-)^*$. The sum is taken over all permutations σ of $\{1, \ldots, n\}$, and $(-1)^\sigma$ denotes the sign of the permutation. We define $Z_{F_d^-(V)}$ in a similar manner.

Comparing with (1.17.1), we see that

$$\begin{cases} F_d^+(V) = F_d^-(\bar{V}), \qquad F_d^-(V) = F_d^+(\bar{V}) & \text{if } d = 0 \mod 4, \\ F_d^+(V) = F_d^+(\bar{V}), \qquad F_d^-(V) = F_d^-(\bar{V}) & \text{if } d = 2 \mod 4. \end{cases} \tag{1.17.8}$$

Because of this, we use a shorthand notation $F_d(V) = F_d^+(V)$ when $d = 0 \mod 4$. When G is simple, $k_G(F_4(V))$ is given as in the free boson case. We have

$$k_G(F_4(V)) = 2k_G(B_4(V)). \tag{1.17.9}$$

1.17.2 Space of states

Let $Q = F_d^+(V)$. Let Y be a spin $(d-1)$ dimensional manifold Y with G-bundle $P \to Y$ with connection, and let us discuss $\mathcal{H}_Q(Y)$. Consider

$$\mathcal{B} = \Gamma(Y, \underline{V \otimes S \oplus \bar{V} \otimes S}) \tag{1.17.10}$$

where S is the irreducible spinor representation of $\mathrm{Spin}(d-1)$

For simplicity we assume that there is no zero eigenvalue of the Dirac operator $\not{D}$ on $\mathcal{B}$. Then we can split

$$\mathcal{B} = \mathcal{B}^+ \oplus \mathcal{B}^- \tag{1.17.11}$$

where $\mathcal{B}^+$ is the subspace where the eigenvalue of $\not{D}$ is positive. Then we have

$$\mathcal{H}_Q(Y) = \mathbb{C} \oplus \mathcal{B}^+ \oplus \Lambda^2\mathcal{B}^+ \oplus \Lambda^3\mathcal{B}^+ \oplus \cdots. \tag{1.17.12}$$

As in the case of free bosons, we call an element $|0\rangle \in \mathbb{C} \subset \mathcal{H}_Q(Y)$ the vacuum, and write

$$\mathcal{B}^+ = \bigoplus_i b_i^+|0\rangle \qquad (1.17.13)$$

for each positive eigenvalue ω_i of the Dirac operator on $\mathcal{B}^+$. Then $\mathcal{H}_Q(Y)$ as a vector space can be identified with the exterior algebra generated by $b_i^\dagger$. We introduce operators b_i so that

$$[b_i, b_j^\dagger]_+ = b_i b_j^\dagger + b_j^\dagger b_i = \delta_{ij}. \qquad (1.17.14)$$

Then $Z_Q(Y \times [0,\beta])$ is an operator $e^{-\beta H}$ on $\mathcal{H}_Q(Y)$ given by

$$H = E_Q(Y) + \sum_i \omega_i b_i^\dagger b_i. \qquad (1.17.15)$$

We introduce an operator usually denoted by $(-1)^F$ which is $(-1)^n$ on $\Lambda^n\mathcal{B}^+ \subset \mathcal{H}_Q(Y)$. Then we have

$$\mathrm{tr}_{\mathcal{H}_Q(Y)}(-1)^F e^{-\beta H} = e^{-\beta E_Q(Y)} \prod_{i=1}^{\infty}(1 - e^{-\beta\omega_i}). \qquad (1.17.16)$$

Conventionally, $E_Q(Y)$ is written as

$$E_Q(Y) = \text{``} - \sum_i \frac{\omega_i}{2}\text{''} \qquad (1.17.17)$$

but is needed to be determined so that $F_d^+(Q)$ satisfies the axioms of the unitary quantum field theory.

1.18 Anomaly polynomial

For a G-symmetric d-dimensional QFT Q, recall

$$Z_Q(X) \in \Gamma(\mathcal{M}, L_Q) \qquad (1.18.1)$$

where $\mathcal{M}$ is the moduli space of compact spin d-manifolds with Riemannian metric and G bundle with connection, and L_Q is a line bundle determined by Q. The anomaly polynomial $A(Q)$ encodes $c_1(L_Q)$ in the following way. We have the universal G-bundle $\mathcal{P}$ over the universal family $\mathcal{X}$ of d-dimensional spin manifold over $\mathcal{M}$,

$$\mathcal{P} \to \mathcal{X} \to \mathcal{M}. \qquad (1.18.2)$$

Then $A(Q)$ is a degree $(d+2)$ characteristic class on $\mathcal{X}$ of $T\mathcal{X}$ and $\mathcal{P}$ such that $c_1(L)$ is given by its integral along the fiber of $\mathcal{X} \to \mathcal{M}$.

$B_d(V)$ is an anomaly-free theory, so $A_{B(V)} = 0$. $F_d^\pm(V)$ is not in general anomaly-free. The anomaly polynomial is given by the family index theorem,

$$A(F_d^\pm(V)) = \pm(\hat{A}(\mathcal{X})\,\mathrm{ch}(\mathcal{V}))_{d+2} \qquad (1.18.3)$$

where $\mathcal{V} = \mathcal{P} \times_G V$.

The topology of L_Q captured by the anomaly polynomial is called a local anomaly, as it can be written in terms of curvatures of the connections of $T\mathcal{X}$ and $\mathcal{P}$ via the Chern-Weil homomorphism. Other anomalies are called global. For example, take $d = 4$, $G = \mathrm{Sp}(n)$ and $V = \mathbb{C}^{2n}$, the defining vector representation. Then $A(F_4(V)) = 0$ but is anomalous in the following way. One can consider a family of G-connections on S^4 parameterized by S^1, corresponding to the nontrivial generator $KSp(S^5) \simeq \mathbb{Z}/\mathbb{Z}_2$. In this case the determinant line bundle $\mathrm{Det}\, \not{D} \to S^1$ has a nontrivial holonomy -1 around it. For a through discussion on these issues, see [Wit85].

1.19 Path integrals and QFTs

The free boson theory $B(V)$ has a path-integral definition. Namely, we consider the space of maps

$$\mathcal{B}(V) = \Gamma(X, \underline{V}) \tag{1.19.1}$$

and the action functional S on it

$$S(\phi) = \frac{1}{2} \int_X \langle D\phi, D\phi \rangle d\,\mathrm{vol}_X \tag{1.19.2}$$

Then we have

$$\langle v_1(x_1) \cdots v_{2n}(x_{2n}) \rangle_X = \int_{\mathcal{B}(V)} v_1(\phi(x_1)) \cdots v_{2n}(\phi(x_{2n})) e^{-S(\phi)} d\,\mathrm{vol}_{\mathcal{B}}\,. \tag{1.19.3}$$

The integration measure needs to be defined that a formal Gaussian integral can be then applied.

The free fermion theory $F_4(V)$ has a path integral definition too. Namely, we take

$$\mathcal{F}(V) = \Gamma(X, \underline{V \otimes S^+}) \tag{1.19.4}$$

and

$$\bar{\mathcal{F}}(\bar{V}) = \Gamma(X, \underline{\bar{V} \otimes S^-}). \tag{1.19.5}$$

Then for $\psi \oplus \bar{\psi} \in \mathcal{F}(V) \oplus \bar{\mathcal{F}}(\bar{V})$ we define the action functional

$$S(\psi, \bar{\psi}) = \int_X \langle \bar{\psi}, \not{D}\psi \rangle d\,\mathrm{vol}_X\,. \tag{1.19.6}$$

Then the Berezin integration over $\mathcal{F}(V)$ and $\bar{\mathcal{F}}(\bar{V})$ gives

$$\begin{aligned} &Z_{F_4(V)}(P \to X; x_1, v_1; y_1, w_1; \cdots; x_n, v_n; y_n, w_n) \\ &\quad = \int_{\mathcal{F}(V) \oplus \bar{\mathcal{F}}(\bar{V})} v_1(\psi(x_1)) w_1(\bar{\psi}(y_1)) \cdots v_n(\psi(x_n)) w_n(\bar{\psi}(y_n)) e^{-S(\psi, \bar{\psi})} d\,\mathrm{vol}_{\mathcal{F}}\, d\,\mathrm{vol}_{\bar{\mathcal{F}}}\,. \end{aligned} \tag{1.19.7}$$

In view of the path integral definitions of the free fields above, it is tempting to pick V, W, consider a more general functional $S(\phi, \psi, \bar\psi)$ on

$$\mathcal{B}(V) \oplus \mathcal{F}(W) \oplus \bar{\mathcal{F}}(W) \tag{1.19.8}$$

and try to define a QFT $Q(S)$ via

$$Z_{Q(S)}(X) = \int_{\mathcal{B}(V)\oplus\mathcal{F}(W)\oplus\bar{\mathcal{F}}(W)} e^{-S(\phi,\psi,\bar\psi)} d\,\mathrm{vol}_{\mathcal{B}(V)}\, d\,\mathrm{vol}_{\mathcal{F}(W)}\, d\,\mathrm{vol}_{\bar{\mathcal{F}}(W)}\,. \tag{1.19.9}$$

Physicists have accumulated knowledge when and to what degree and in which sense this is possible, for which class of functionals S. A rather literal pseudo-mathematical translation of what physicists usually say is the following. We pick an element $L(\phi, \psi, \bar\psi) \in \operatorname{Re} \mathcal{V}_{B(V)\times F(W)}{}^{\mathrm{Spin}(d)}$, and consider $S(\phi, \psi, \bar\psi) = \int_X L(\phi, \psi, \bar\psi) d\,\mathrm{vol}_X$. We also pick something called a renormalization-regularization scheme $\mathcal{RRS}$ which encapsulates various algorithmic procedure which removes infinities appearing in the intermediate computations. The famous ones are the "naive momentum cutoff", MS, $\overline{MS}$, DR, $\overline{DR}$, etc. Then we say

- (Perturbative renormalizability) $Q(L, \mathcal{RRS})$ can be defined as an effective QFT:

  $$Z_{Q(L,\mathcal{RRS})}(P \to X) = \int_{\mathcal{B}(V)\oplus\mathcal{F}(W)\oplus\bar{\mathcal{F}}(W)} \mathcal{RRS}[e^{-S(\phi,\psi,\bar\psi)} d\,\mathrm{vol}_{\mathcal{B}(V)}\, d\,\mathrm{vol}_{\mathcal{F}(W)}\, d\,\mathrm{vol}_{\bar{\mathcal{F}}(W)}]. \tag{1.19.10}$$

  Here the effectiveness is used in the technical sense that things make sense only as an asymptotic series of various parameters. QFTs, when emphasized against effective QFTs, are often called ultraviolet-complete QFTs.

- (Regularization independence) If $L \in \operatorname{Re} \mathcal{V}_{B(V)\times F(W)}{}^{\mathrm{Spin}(d)}{}_{,d}$, then for any other regularization scheme $\mathcal{RRS}'$ we have another $L' \in \mathcal{V}_{B(V)\times F(W)}{}^{\mathrm{Spin}(d)}{}_{,d}$ such that

  $$Q(L, \mathcal{RRS}) = Q(L', \mathcal{RRS}'). \tag{1.19.11}$$

  Recall that the subscript d is the degree in the filtration, introduced in (1.10.1).

These properties are well-established mathematically, in the sense that at least there should not be any serious obstacles to make the physics statements into a rigorous mathematics. Usually experimental results are reported by specifying L and $\mathcal{RRS}$.

1.20 Deformations of QFTs

An equivalent but more invariant statement, perhaps preferable to mathematicians, is as follows. Given a QFT Q (not necessary defined via path integrals as above), there is a family of effective QFTs $Q\,\big|_{u\in\mathcal{U}}$ such that $Q = Q_0$ at $0 \in \mathcal{U}$ and moreover

$$T\mathcal{U}|_{u=0} \simeq \operatorname{Re}(\mathcal{V}_{Q,d}/\operatorname{Image}\nabla)^{\mathrm{Spin}(d)}. \tag{1.20.1}$$

The statements in the previous sections are what we would get when Q is a free theory, $Q = B(V) \times F(W)$.

For a G-symmetric QFT Q, there is a natural action of G on $\mathcal{U}$ which is compatible with the identification (1.20.1), so that there is an equivalence

$$Q \big|_u \simeq Q \big|_{gu} \tag{1.20.2}$$

for $g \in G$. Also, there is a subfamily of effective G-symmetric QFTs $Q \big|_{u \in \mathcal{U}^G}$ where

$$T\mathcal{U}^G|_{u=0} \simeq \mathrm{Re}(\mathcal{V}_{Q,d}/\operatorname{Image}\nabla)^{\mathrm{Spin}(d)\times G}. \tag{1.20.3}$$

1.21 Non-linear sigma model

So far, the integration region used in the previous sections are the linear spaces $\mathcal{B}(V)$ and $\mathcal{F}(V)$. A natural generalization is to pick a Riemannian manifold Σ to consider the space of maps

$$\mathrm{Map}(X, \Sigma) = \{f : X \to \Sigma\} \tag{1.21.1}$$

from d-dimensional Riemannian manifold X. Then we consider the action functional on this space of maps given by

$$S(f) = \frac{1}{2}\int_X |df|^2 \operatorname{vol}_X . \tag{1.21.2}$$

Here $|df|^2$ is defined by using the metric of both X and Σ.

We can try to define a d-dimensional QFT $Q = \sigma_d(\Sigma)$ by

$$Z_Q(X) = \int_{\mathrm{Map}(X,\Sigma)} e^{-S(f)} d\operatorname{vol}_{\mathrm{Map}(X,\Sigma)} . \tag{1.21.3}$$

This is called a non-linear sigma model with the target space Σ. When Σ is flat, this is a UV complete QFT for any d. It is a UV complete QFT when $d = 2$. Otherwise, $\sigma_d(\Sigma)$ only exists as an effective QFT in general.

1.22 Gauging of QFTs

Another important operation we need to discuss is the coupling to the gauge field, or gauging in short. This is an operation which, given a $G \times H$-symmetric QFT Q, creates a family of H-symmetric effective QFT $Q /\!\!\!/ G$. H is called the flavor symmetry and G is called the gauge symmetry in the physics literature.

The symbol $/\!\!\!/$ is chosen to suggest that its formal property is similar to the quotient of a G-space X by the G-action: X/G no longer has the action by G. Similarly, Q is G-symmetric but $Q /\!\!\!/ G$ is not G-symmetric. The reader will surely find a slightly misguided but historical terminology in the physics literature, referring to G as 'the gauge symmetry of the theory $Q /\!\!\!/ G$'. In this review we avoid this terminology 'gauge symmetry' in the hope of reducing the confusion.

By $Q \times Q' \mathbin{/\!\!\!\!/} G$ we mean $(Q \times Q') \mathbin{/\!\!\!\!/} G$. In general the symbol $\mathbin{/\!\!\!\!/}$ is assumed to have the same precedence as the symbols $+$ or $-$ within equations.

This QFT $Q \mathbin{/\!\!\!\!/} G$ is defined via a path integral. Denote by F the curvature of a G-bundle with connection $P \to X$. For simplicity assume G is simple or U(1). Then we try to define a one-parameter family $Q \mathbin{/\!\!\!\!/} G\,\big|_{u \in \mathbb{R}_{>0}}$

$$Z_{Q \mathbin{/\!\!\!\!/} G\big|_u}(X) = \int_{\mathcal{M}_{G,X}} \mathcal{RRS}[Z_Q(P \to X) e^{-\frac{1}{g^2}\int_X \langle F, \wedge * F\rangle} d\,\mathrm{vol}_{\mathcal{M}_G}] \tag{1.22.1}$$

where $\mathcal{M}_{G,X}$ is the moduli space of G-bundles with connections on X, and u and $1/g^2$ are related by $\mathcal{RRS}$. For this to make sense, first of all we need to require that Q is G-anomaly-free so that $Z_Q(P \to X)$ is really a function.[3] We then have

- (Perturbative renormalizability) The left hand side exists as an effective theory when $d \leq 4$. This is proved.
- (Existence as UV complete theory, $d < 4$) The left hand side exists as a UV-complete theory when $d < 4$. It should not be hard to prove this.
- (Existence as UV complete theory, $d = 4$) The left hand side exists as a UV-complete theory when $d = 4$ and
$$k_G(Q) \leq \frac{22}{3} h^\vee(G). \tag{1.22.2}$$

The last item implies that $\mathrm{triv}_4 \mathbin{/\!\!\!\!/} G$ for any simple G should exist since $k_G(\mathrm{triv}_4) = 0$. Any reader is encouraged to prove this statement and receive the Clay prize. The RG acts within the family $\mathrm{triv} \mathbin{/\!\!\!\!/} G\,\big|_{u \in \mathbb{R}_{>0}}$ by changing u.

The space of operators is given by

$$\mathcal{V}_{Q \mathbin{/\!\!\!\!/} G} = (\mathrm{Sym}^\bullet[\mathfrak{g} \otimes \wedge^2 \mathbb{R}^d] \otimes \mathcal{V}_Q)^G. \tag{1.22.3}$$

The elements in $\mathfrak{g} \otimes \wedge^2 \mathbb{R}^d$ correspond to the curvature of the G-connection.

When $d = 4$, we can slightly generalize the construction so that we consider the family $Q \mathbin{/\!\!\!\!/} G\,\big|_{u,\theta}$ where

$$Z_{Q \mathbin{/\!\!\!\!/} G\big|_{u,\theta}}(X) = \int_{\mathcal{M}_{G,X}} \mathcal{RRS}[Z_Q(P \to X) e^{-\int_X u\langle F, \wedge * F\rangle + i\theta \int_X \langle F \wedge F\rangle} d\,\mathrm{vol}_{\mathcal{M}_G}]. \tag{1.22.4}$$

Here θ takes values in $\mathbb{R}/\mathbb{Z}$, by appropriately normalizing the invariant inner product on $\mathfrak{g}$. When $d = 3$ we can instead consider the family $Q \mathbin{/\!\!\!\!/} G\,\big|_{u,k}$

$$Z_{Q \mathbin{/\!\!\!\!/} G\big|_{u,k}}(X) = \int_{\mathcal{M}_{G,X}} \mathcal{RRS}[Z_Q(P \to X) e^{-\int_X u\langle F, \wedge * F\rangle + ikCS(P)} d\,\mathrm{vol}_{\mathcal{M}_G}] \tag{1.22.5}$$

[3] It is often suggested by the audience that one might be able to choose $d\,\mathrm{vol}_{\mathcal{M}_G}$ to be a section of a compensating bundle to allow for non-anomaly-free Q. We consider such nontrivial $d\,\mathrm{vol}_{\mathcal{M}_G}$ to be another QFT Q' by definition. Then it is a gauging of $Q \times Q'$ which is anomaly free.

where $CS(P)$ is the Chern-Simons invariant of P, and k takes values in $\mathbb{Z}$. The usual Chern-Simons theory with group G of level k is in this notation $\mathrm{triv}_3 \not\!\!/ G \,\Big|_{0,k}$.

The discussions above can be generalized to the case when G is reductive and Q itself comes in a G-symmetric family $Q \,\Big|_{u \in U}$. Then there is a family $Q \not\!\!/ G \,\Big|_{x \in \mathcal{X}}$ where there is a non-canonical identification

$$\mathcal{X} \simeq \mathcal{U} \times (\text{space of invariant positive bilinear form on } \mathfrak{g}). \tag{1.22.6}$$

The 2d Yang-Mills theory $\mathrm{YM}_2(G)$ discussed in Sec. 1.8.3 is, in the notation in this section,

$$\mathrm{YM}_2(G) \simeq (\mathrm{triv}_2 \not\!\!/ G)_{u_*}. \tag{1.22.7}$$

For general u, we need to replace the factors $e^{-Ac_2(\rho)}$ there by $e^{-A(u/u_*)c_2(\rho)}$.

1.23 Gauging and submanifold operators

In a d-dimensional gauge theory $Q = Q' \not\!\!/ G$, we have natural elements in $\mathcal{V}^1_Q$ labeled by representations of G. Namely, given X and a closed one-dimensional curve $C \subset X$, we define $Z_Q(X, C, R)$ by inserting $\mathrm{tr}_R \, \mathrm{Hol}(C)$ in the path integral (1.22.1). Here $\mathrm{Hol}(C)$ is the holonomy of the G-connection and tr_R is its trace in the representation R. These are called Wilson lines in physics literature. We can consider the same thing in (1.22.5), which is used in giving a path integral expression to Jones' polynomial by Witten [Wit89].

We also naturally have elements in $\mathcal{V}^{d-3}_Q$, labeled by elements φ of the coroot lattice of G, modulo the action of the Weyl group. Equivalently, we have a homomorphism

$$\varphi : U(1) \to G \tag{1.23.1}$$

up to conjugation. Given X and a $d-3$-dimensional submanifold D, we define $Z_Q(X, D, \varphi)$ as follows. We let $X' = X \setminus D$. Very close to D, the manifold can be approximated by an S^2 bundle over D times $\mathbb{R}_{>0}$. By regarding the $U(1)$ bundle with $c_1 = 1$ over this S^2 as a G-bundle via φ, we have a natural G-connection over this S^2 bundle over D. Then, we perform the path integral (1.22.1) over G-connections which approach this particular G-connection close to D. When $d = 4$, this construction also determines elements in $\mathcal{V}^1_Q$. These are called 't Hooft loops in physics literature.

Behaviors of Wilson loops and 't Hooft loops have played an essential role in the physical study of gauge theories in the last few decades. Mathematicians who wish to axiomatize quantum field theories absolutely need to incorporate them in their formulations.

1.24 The Standard Model

After all these preparations, we can state what is the Standard Model, which describes all of the real world, including you who is reading this review, and the activity in the neurons in your brain trying to make out the meaning of this sentence.

Take $G_0 = \mathrm{Spin}(10)$ and its irreducible spinor representation $\mathcal{S}$ of dimension 16. Take a standard subgroup $\mathrm{U}(5) \subset \mathrm{Spin}(10)$, whose embedding is induced from $\mathbb{C}^5 \simeq \mathbb{R}^{10}$ as $\mathbb{R}$ vector spaces. Let G be the Levi subgroup

$$G = \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) \subset \mathrm{U}(5) \tag{1.24.1}$$

which is the stabilizer of $\mathrm{U}(1) \subset G \subset \mathrm{U}(5)$, where we embed $e^{\sqrt{-1}t} \in \mathrm{U}(1)$ to

$$e^{\sqrt{-1}\,\mathrm{diag}(2,2,2,-3,-3)t} \in \mathrm{U}(5). \tag{1.24.2}$$

Under G, the representation $\mathcal{S}$ decomposes as

$$\mathcal{S} = (\bar{W} \otimes V \otimes T) \oplus (W \otimes T^{\otimes -4}) \oplus (W \otimes T^{\otimes 2}) \oplus (V \otimes T^{\otimes -3}) \oplus T \oplus \mathbb{C} \tag{1.24.3}$$

where $T \simeq \mathbb{C}$, $V \simeq \mathbb{C}^2$ and $W \simeq \mathbb{C}^3$ are the defining representations of $\mathrm{U}(1)$, $\mathrm{SU}(2)$, $\mathrm{SU}(3)$ respectively.

We consider a G-symmetric four-dimensional QFT

$$F_4(\mathcal{S} \oplus \mathcal{S} \oplus \mathcal{S}) \times B_4(V \otimes T^{\otimes 3}). \tag{1.24.4}$$

This is anomaly-free, because $F_4(\mathcal{S} \oplus \mathcal{S} \oplus \mathcal{S})$ is anomaly-free as a G_0-symmetric theory, since $[\hat{A}(T\mathcal{X})\,\mathrm{ch}(\mathcal{S} \times_{G_0} \mathcal{P})]_6 = 0$ due to a simple reason that there is no characteristic class of $\mathrm{Spin}(10)$ of degree 2 or 6.

Then we can form the family

$$SM\ \Big|_{u_1,u_2,u_3} = \left[F_4(\mathcal{S} \oplus \mathcal{S} \oplus \mathcal{S}) \times B_4(V \otimes T^{\otimes 3})\right] \not{}G\ \Big|_{u_1,u_2,u_3}. \tag{1.24.5}$$

over $(u_1, u_2, u_3) \in \mathbb{R}^3_{>0}$. This family is a subfamily of a bigger family $SM\ \Big|_{u \in U}$ where U is of real dimension 38. The real world is a fiber of this family $SM\ \Big|_{u_0}$ at a particular point $u_0 \in U$.

The deformations of this family can be found by studying

$$\mathrm{Re}(\mathcal{V}_{SM_{u_1,u_2,u_3},4}/\,\mathrm{Image}\,\nabla)^{\mathrm{SO}(4)}. \tag{1.24.6}$$

Recall $(\mathcal{S} \oplus \mathcal{S} \oplus \mathcal{S}) \otimes S^+ \in \mathcal{V}_{F_4(\mathcal{S}\oplus\mathcal{S}\oplus\mathcal{S}),3/2}$. We denote an element of it by $\psi_1 \oplus \psi_2 \oplus \psi_3$. We further decompose ψ_i according to (1.24.3) and denote

$$\psi_i = Q_i \oplus \bar{u}_i \oplus \bar{d}_i \oplus E_i \oplus \bar{e}_i \oplus \bar{\nu}_i. \tag{1.24.7}$$

Note that $F_4(\mathcal{S} \oplus \mathcal{S} \oplus \mathcal{S})$ is a $G \times \mathrm{U}(3)_Q \times \mathrm{U}(3)_{\bar{u}} \times \mathrm{U}(3)_{\bar{d}} \times \mathrm{U}(3)_E \times \mathrm{U}(3)_{\bar{e}} \times \mathrm{U}(3)_{\bar{\nu}}$-symmetric theory, where $\mathrm{U}(3)_X$ acts on $X_{i=1,2,3}$.

Recall also $V \otimes T^{\otimes 3} \in \mathcal{V}_{B(V\otimes T^{\otimes 3}),1}$. We denote an element of it by ϕ. Next, recall $(\mathfrak{g} \otimes \Lambda^2(\mathbb{R}^d))^G \in \mathcal{V}_{Q\not{}G,2}$, corresponding to invariant polynomials of curvatures of the G-connection. We denote an element of $\mathfrak{g} \otimes \Lambda^2(\mathbb{R}^d)$ by $F_1 \oplus F_2 \oplus F_3$, according to the direct product structure $G = \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$. Terms in $\mathcal{V}_4$ involving ∇, such as $\nabla\phi\nabla\phi$, are all easily seen to be in the image of ∇. Then, possible deformations in (1.24.6) are given by polynomials of ψ_i, ϕ and F_i which are invariant under $G \times \mathrm{Spin}(4)$, with mass dimension less than or equal to 4.

The basis of such polynomials are given by the following: First,

$$m^2|\phi^2|, \quad \lambda|\phi^2|^2 \tag{1.24.8}$$

are called the Higgs mass and the Higgs quartic coupling,

$$\mathrm{Re}\sum_{ij} y^u_{ij}\phi Q_i\bar{u}_j, \quad \mathrm{Re}\sum_{ij} y^d_{ij}\bar{\phi}Q_i\bar{d}_j, \quad \mathrm{Re}\sum_{ij} y^e_{ij}\phi E_i\bar{e}_j, \quad \mathrm{Re}\sum_{ij} y^\nu_{ij}\bar{\phi}E_i\bar{\nu}_j \tag{1.24.9}$$

are called the up-type Yukawa couplings, the down-type Yukawa couplings, the lepton Yukawa couplings, and the Dirac neutrino mass terms, and

$$\mathrm{Re}\sum_{ij}\mu_{ij}\bar{\nu}_i\bar{\nu}_j \tag{1.24.10}$$

are called the Majorana neutrino mass terms, and

$$u_i\langle F_i, \wedge * F_i\rangle, \quad \theta_i\langle F_i, \wedge F_i\rangle \tag{1.24.11}$$

are called the gauge coupling constants, and the theta angles.

The parameters m^2, λ, α_i and θ_i are real, and the rest $y^{u,d,e,\nu}_{ij}$ and μ_{ij} are complex. The Majorana mass term μ_{ij} is symmetric in its two subscripts. $\mathrm{U}(3)_Q \times \mathrm{U}(3)_{\bar{u}} \times \mathrm{U}(3)_{\bar{d}}$ acts on on the space of y^u_{ij} and y^d_{ij}. The stabilizer of a typical point is $\mathrm{U}(1)_B$, which is called the baryon number symmetry. $\mathrm{U}(1)_B$ acts on θ_2 by shifting it, due to 't Hooft anomalies. This effect is not explained in this review. $\mathrm{U}(3)_E \times \mathrm{U}(3)_{\bar{e}} \times \mathrm{U}(3)_{\bar{\nu}}$ acts on on the space of y^e_{ij} and y^d_ν. The stabilizer of a typical point is again $\mathrm{U}(1)_L$, which is called the lepton number symmetry. This $\mathrm{U}(1)_L$ acts on the space of μ_{ij}. So in total we have

$$2 + 6 + 72 + 12 - 54 = 38 \tag{1.24.12}$$

parameters in the Standard Model.

Before going further, we should emphasize that the Standard Model is not UV-complete. It exists only as an effective theory, and various quantities such as $Z(X)$ only exists as an asymptotic series. This signifies physically that there are phenomena in the real world not described by the Standard Model. Presumably there is a four-dimensional UV-complete quantum gravity theory which describes the whole physical phenomenon, whose approximation is the Standard Model.

1.25 Vacua of QFT

So far in this review, given a QFT Q, $Z_Q(X)$ is defined only for compact X. When Q is unitary, by studying the behavior of $Z_Q(X)$ when X is large, one can extract a finite-dimensional Riemannian manifold

$$\mathcal{M}_{\mathrm{vac}}(Q) \tag{1.25.1}$$

called the moduli space of vacua of Q. When Q is G-symmetric, there is a natural G action on $\mathcal{M}_{\text{vac}}(Q)$. Essentially, we find that $\mathcal{RG}_t\,Q$ with t very large, can be approximated by the effective QFT $\sigma_d(\mathcal{M}_{\text{vac}}(Q))$, introduced in Sec. 1.21.

A point $u \in \mathcal{M}_{\text{vac}}(Q)$ is called a vacuum of Q. Then, for $(d-1)$-dimensional noncompact Y with infinite volume,

$$\mathcal{H}_Q(Y, u) \tag{1.25.2}$$

can be defined. For d-dimensional noncompact X with infinite volume, we can also define

$$Z_{Q,u} \in \Gamma(\mathcal{M}_X, L) \tag{1.25.3}$$

where $\mathcal{M}$ is the moduli space of d-dimensional noncompact spin manifolds X' such that

$$X \setminus K = X' \setminus K' \tag{1.25.4}$$

for compact submanifolds K and K', respectively. L is a line bundle with connection on $\mathcal{M}_X$.

The vacua and the OPE algebra $\mathcal{V}$ are related as follows:

- The continuous functions on $\mathcal{M}_{\text{vac}}(Q)$ is a subspace of $\mathcal{V}$:

$$C^\infty(\mathcal{M}_{\text{vac}}(Q)) \subset \mathcal{V}. \tag{1.25.5}$$

  The action of $\mathrm{Spin}(d)$ on $C^\infty(\mathcal{M}_{\text{vac}}(Q))$ is trivial. The algebra structure does not necessarily match.

- For $f \in C^\infty(\mathcal{M}_{\text{vac}}(Q)) \subset \mathcal{V}$, we have

$$\langle f(p) \rangle_{X,u} = f(u). \tag{1.25.6}$$

  The left hand side is the one-point function $Z_{Q,u}(X; p, f)$, and the right hand side is the evaluation of a function at u.

The theorem by Coleman, Mermin and Wagner states that when $d \leq 2$, $\mathcal{M}_{\text{vac}}(Q)$ is discrete.

From the axioms it follows that $\mathcal{H}_Q(\mathbb{R}^{d-1}, u)$ with a standard flat metric on $\mathbb{R}^{d-1}$ carries an action of its isometry $\mathrm{Spin}(d-1)$. This is known to enhance to an action of $\mathrm{Spin}(d-1, 1)$. Once the contents of this section are fully formally developed, it should be straightforward to restrict the axioms to the case where $X = \mathbb{R}^d$, which should reproduce the standard Osterwalder-Schroeder axioms.

2 Supersymmetric QFTs

The rest of the lecture note is mainly devoted to the discussion of $\mathcal{N} = 2$ supersymmetric QFTs in four dimensions. We discuss various structures associated to them. The readers are advised to refer to Fig. 1 at the beginning of the lecture note as a summary. Below, we start from generalities and gradually restrict our attention to four-dimensional $\mathcal{N} = 2$ QFTs.

2.1 Generalities

A supersymmetric d-dimensional QFT is, morally speaking, a QFT for a d-dimensional manifold with super-Riemannian structure. Here, a super-Riemannian structure is a 'super' version which adds additional structure on top of a standard Riemannian structure with a Riemannian metric. In each spacetime dimension d, there are a few kinds of super-Riemannian structure, first of all labeled by $\mathcal{N}$, the so-called the number of the supersymmetry. Even with d and $\mathcal{N}$ fixed, there are usually several different super-Riemannian structures known in the physics literature, usually called the off-shell supergravity multiplets. The author does not know a concise definition of what a super-Riemannian structure on a manifold is, encompassing various known versions.

Most of the time, physicists considers supersymmetric theories only with $d \leq 11$.[4] The structure of the supersymmetry also depends strongly on $d \mod 8$, as it uses the structure of spin representations of $\mathfrak{so}(d)$. Therefore, the discussions of the supersymmetry requires each of $d = 1, 2, \ldots, 11$ almost separately, one by one.

These limitations force the author to phrase the following discussions in a rather ad-hoc manner. In this lecture note, we mainly discuss the case $d = 4$. At the end of this section we will briefly discuss the $d = 2$ case in relation to the mirror symmetry. In the next section we will also have a little to say about the $d = 6$ case. In the following, QFTs are assumed to be four-dimensional unless otherwise specified.

A supersymmetric QFT which is conformally invariant as introduced in Sec. 1.15 is called a superconformal field theory (SCFT). Many of the supersymmetric QFTs we deal with below are superconformal.

2.2 Generalities in $d = 4$

A four-dimensional $\mathcal{N}$-extended supersymmetric QFT Q is a QFT with a lot of additional properties. First, Q is $\mathrm{SU}(\mathcal{N})$-symmetric[5]. We write by $\mathcal{R} \simeq \mathbb{C}^{\mathcal{N}}$ the defining representation of this $\mathrm{SU}(\mathcal{N})$. Second, the space of point operators $\mathcal{V}_Q$ has an action of the super Lie algebra

$$(\mathfrak{su}(\mathcal{N}) \times \mathfrak{so}(4)) \ltimes (\mathbb{R}^4 \oplus S^+ \otimes \mathcal{R} \oplus S^- \otimes \bar{\mathcal{R}}) \tag{2.2.1}$$

where the even part $\mathbb{R}^4$ corresponds to the action of ∇, the part $S^+ \otimes \mathcal{R} \oplus S^- \otimes \bar{\mathcal{R}}$ is the odd part. The commutator between an element in $S^+ \otimes \mathcal{R}$ and $S^- \otimes \bar{\mathcal{R}}$ is given by the tensor product of the natural maps $S^+ \otimes S^- \simeq \mathbb{R}^4$ and $\mathcal{R} \otimes \bar{\mathcal{R}} \to \mathbb{C}$. The elements in $S^+ \otimes \mathcal{R} \oplus S^- \otimes \bar{\mathcal{R}}$ are called supersymmetry generators. They map an element of $\mathcal{V}_D$ to $\mathcal{V}_{D+1/2}$.

[4]This is due to the following reason. For a representation R of $\mathfrak{so}(d)$, let us define its spin j by requiring that the largest irreducible representation of $\mathfrak{so}(3) \subset \mathfrak{so}(3) \times \mathfrak{so}(d-3) \subset \mathfrak{so}(d)$ appearing in the irreducible decomposition of R has dimension $2j + 1$. Physicists know very little about how to deal with theories involving $\mathfrak{so}(d)$ representations of spin greater than 2, and any nontrivial representation of supersymmetry algebra for $d \geq 12$ necessarily contains such representations. This forces d to be less than or equal to 11.

[5]Strictly speaking, there are supersymmetric QFTs with no $\mathrm{SU}(\mathcal{N})$ R-symmetry known in the physics literature, but as they do not play role in this review, we require the existence of $\mathrm{SU}(\mathcal{N})$ R-symmetry in the definition.

An $\mathcal{N}$-extended supersymmetric QFT is automatically $\mathcal{N}'$-extended supersymmetric QFT for any $\mathcal{N}' < \mathcal{N}$. A 1-extended, 2-extended or 4-extended QFT is usually called an $\mathcal{N} = 1$, $\mathcal{N} = 2$, $\mathcal{N} = 4$ supersymmetric QFT, respectively.

An $\mathcal{N}$-extended super-Riemannian structure on a 4-manifold X includes at least an $\mathrm{SU}(\mathcal{N})$-bundle with connection. Then we have its frame bundle as $F_{\mathrm{SU}(\mathcal{N})\times\mathrm{Spin}(4)}X \to X$. Now, consider the vector bundle

$$TX \oplus \mathcal{S}^+X \oplus \mathcal{S}^-X = \underline{\mathbb{R}^4 \oplus S^+ \otimes \mathcal{R} \oplus S^- \otimes \bar{\mathcal{R}}} \tag{2.2.2}$$

over X associated to (2.2.1). This determines three vector bundles TX, $\mathcal{S}^+X$ and $\mathcal{S}^-X$ over X. The first is the standard tangent bundle; the second and the third are what can be called the super-tangent bundles. A certain nice section of TX is an infinitesimal isometry, and is called a Killing vector. similarly, a certain nice section of $\mathcal{S}^+X$ or $\mathcal{S}^-X$ is an infinitesimal super-isometry, and is called a Killing spinor. A subcase is when the section is in fact covariantly constant with respect to the spin connection and the $\mathrm{SU}(\mathcal{N})$ connection. In this review we only explicitly use this case. The partition function $Z_Q(X)$ and the n-point functions of a supersymmetric QFT Q is invariant under the action of a super-isometry, just as those of a Riemannian-structured QFT are invariant under the action of an isometry.

A G-symmetric $\mathcal{N}$-extended supersymmetric QFT Q is an $\mathcal{N}$-extended supersymmetric QFT where G-action commutes with the action of the supersymmetry generators. The $\mathrm{SU}(\mathcal{N})$ symmetry acting on $\mathcal{R}$ is called the $\mathrm{SU}(\mathcal{N})$ R-symmetry to distinguish it from the non-R symmetry G just introduced above. A U(1) R-symmetric $\mathcal{N}$-extended supersymmetric QFT Q is one where Q is U(1)-symmetric such that it acts on $\mathcal{R}$ by a scalar multiplication.

2.3 $\mathcal{N} = 1$ supersymmetric QFTs

There are many interesting topics with $\mathcal{N} = 1$ supersymmetry, but we state only the bare basics to study $\mathcal{N} = 2$ supersymmetric QFTs. Let us consider $\mathcal{N} = 1$ susy QFT. Take a supersymmetry generator $\delta \in S^+ \otimes \mathcal{R}$ and fix it. This acts on $\mathcal{V}_Q$. We have $\delta^2 = 0$ from the super-Lie-algebra structure mentioned above, and thus we can define its cohomology $H(\mathcal{V}_Q, \delta)$.

Furthermore, δ has the following properties with respect to the OPE product, namely

- If $v, w \in \mathcal{V}_Q$ are δ-closed, $v \circ_x w$ is finite when $x \to 0$.
- If furthermore w is δ-exact, $v \circ_x w$ is 0 when $x \to 0$.

This means that the OPE product $\circ_x$ with $x \to 0$ induces a standard super algebra structure on $H(\mathcal{V}_Q, \delta)$. This is called the chiral ring of the theory.

These properties follow by considering n-point functions on a flat $\mathbb{R}^4$, where δ generates a superisometry. As the OPE product is determined by the short-distance behavior of n-point functions on arbitrary manifold, we can extract the statements above from the properties on $\mathbb{R}^4$.

The vacuum $\mathcal{M}_{\text{vacuum}}(Q)$ should be thought of as the bosonic part of a supermanifold $\mathcal{M}'(Q)$, on which there is a natural action of the supersymmetry. The fixed loci of the supersymmetry action, $\mathcal{M}_{\text{susyvac}}(Q) \subset \mathcal{M}'(Q)$, is then a non-super manifold which is a submanifold of $\mathcal{M}_{\text{vacuum}}(Q)$. This submanifold is naturally Kähler. It satisfies the important relation

$$H(\mathcal{V}_Q, \delta)^{\mathfrak{so}(d)} = \mathbb{C}[\mathcal{M}_{\text{susyvac}}(Q)] \tag{2.3.1}$$

and this is compatible with the property

$$\mathcal{V}_Q \supset C^\infty(\mathcal{M}_{\text{vac}}(Q)) \tag{2.3.2}$$

as vector spaces.

2.4 $\mathcal{N} = 2$ supersymmetric QFTs

Given a G-symmetric $\mathcal{N} = 2$ QFT Q, its vacuum moduli space $\mathcal{M}_{\text{susyvac}}(Q)$ has two projections

$$\mathcal{M}_{\text{susyvac}}(Q) \to \mathcal{M}_{\text{Coulomb}}(Q), \qquad \mathcal{M}_{\text{susyvac}}(Q) \to \mathcal{M}_{\text{Higgs}}(Q) \tag{2.4.1}$$

such that

$$\mathcal{M}_{\text{susyvac}}(Q) \to \mathcal{M}_{\text{Coulomb}}(Q) \times \mathcal{M}_{\text{Higgs}}(Q) \tag{2.4.2}$$

is an embedding.

The Coulomb branch $\mathcal{M}_{\text{Coulomb}}(Q)$ is a base space of a holomorphic integrable system as discussed below. As a complex variety it is an affine space $\simeq \mathbb{C}^r$, although there is no canonical vector space structure on it. The number r is called the rank of Q. The G action on it is trivial. The Higgs branch $\mathcal{M}_{\text{Higgs}}(Q)$ is a hyperkähler manifold with a triholomorphic G action with moment maps. $\text{SU}(2) \simeq \text{SO}(3)$ R-symmetry acts on $\mathcal{M}_{\text{Higgs}}(Q)$ by rotating three complex structures.

When Q has U(1) R-symmetry, we can define more invariants. First, we have numbers

$$n_v(Q), \qquad n_h(Q). \tag{2.4.3}$$

If Q is G-symmetric, we have numbers

$$k_{G_0}(Q) \tag{2.4.4}$$

for each simple factor $G_0 \subset G$. They are coefficients of the anomaly polynomial of Q as a linear combination of a conventionally-chosen characteristic classes. Namely, $A(Q)$ is a degree-6 characteristic class in terms of $T\mathcal{X}$, $\mathcal{P}_{\text{U}(1)}$, $\mathcal{P}_{\text{SU}(2)}$, $\mathcal{P}_G$:

$$\begin{aligned} A(Q) = \sum_G \frac{k_G}{2} c_1(\mathcal{P}_{\text{U}(1)}) c_2(\mathcal{P}_G) + \\ (n_v - n_h)[-\frac{1}{12} c_1(\mathcal{P}_{\text{U}(1)}) p_1(T\mathcal{X}) + \frac{1}{3} c_1(\mathcal{P}_{\text{U}(1)})^3] + n_v c_1(\mathcal{P}_{\text{U}(1)}) c_2(\mathcal{P}_{\text{SU}(2)}). \end{aligned} \tag{2.4.5}$$

k_G is also given by the short-distance behavior of two G-currents, and similarly $c = n_v/6 + n_h/12$ is given by the short-distance behavior of two energy-momentum tensor. They are the same quantities discussed in Sec. 1.12.

In this case $\mathcal{M}_{\text{Coulomb}} \simeq \mathbb{C}^r$ has an action of U(1) R-symmetry. In other words there is a natural $\mathbb{C}^\times$ action giving a degree on its function ring. Let us write, then,

$$\mathbb{C}[\mathcal{M}_{\text{Coulomb}}] = \mathbb{C}[u_1, \ldots, u_r] \tag{2.4.6}$$

where u_i has well-defined degrees. Then

$$n_v(Q) = \sum_i (2\deg(u_i) - 1) \tag{2.4.7}$$

in a standard convention where $\mathcal{R}$ in (2.2.1) has degree 1/2 as always.

For $Q_1 \times Q_2$, n_v, n_h, k_G are additive

$$n_v(Q_1 \times Q_2) = n_v(Q_1) + n_v(Q_2), \qquad n_h(Q_1 \times Q_2) = n_h(Q_1) + n_h(Q_2), \tag{2.4.8}$$
$$k_G(Q_1 \times Q_2) = k_G(Q_1) + k_G(Q_2), \tag{2.4.9}$$

whereas $\mathcal{M}_{\text{Higgs}}$ and $\mathcal{M}_{\text{Coulomb}}$ are multiplicative

$$\mathcal{M}_{\text{Coulomb}}(Q_1 \times Q_2) = \mathcal{M}_{\text{Coulomb}}(Q_1) \times \mathcal{M}_{\text{Coulomb}}(Q_2), \tag{2.4.10}$$
$$\mathcal{M}_{\text{Higgs}}(Q_1 \times Q_2) = \mathcal{M}_{\text{Higgs}}(Q_1) \times \mathcal{M}_{\text{Higgs}}(Q_2). \tag{2.4.11}$$

2.5 Hypermultiplets

Let us take a pseudoreal representation V of G, or equivalently, assume that V has a quaternionic structure and we have a homomorphism $G \to \mathrm{Sp}(V)$. Then there is a natural complex action of $G \times \mathrm{SU}(2)$ on V. We denote this $G \times \mathrm{SU}(2)$ representation by V'; the underlying vector space is the same as V. Then there is a free G-symmetric $\mathcal{N} = 2$ QFT which we denote by $\mathrm{Hyp}(V)$:

$$\mathrm{Hyp}(V) = B_4(V') \oplus F_4(V). \tag{2.5.1}$$

This is called a half-hypermultiplet based on V. When $V = W \oplus \bar{W}$ for a complex representation W of G, $\mathrm{Hyp}(W \oplus \bar{W})$ is called a hypermultiplet based on W.

We have

$$\mathcal{M}_{\text{Coulomb}}(\mathrm{Hyp}(V)) = \{pt\}, \tag{2.5.2}$$
$$\mathcal{M}_{\text{Higgs}}(\mathrm{Hyp}(V)) = V, \tag{2.5.3}$$
$$n_v(\mathrm{Hyp}(V)) = 0, \tag{2.5.4}$$
$$n_h(\mathrm{Hyp}(V)) = \dim_{\mathbb{H}} V. \tag{2.5.5}$$

For a simple component $G_0 \subset G$, $k_{G_0}(\mathrm{Hyp}(V))$ is given as follows. We decompose

$$V = \oplus_i R_i \tag{2.5.6}$$

into irreducible G_0 representations R_i, and then

$$k_{G_0}(\mathrm{Hyp}(V)) = 2\sum c_2(R_i) \tag{2.5.7}$$

where $c_2(R)$ is the eigenvalue of the quadratic Casimir operator normalized so that $c_2(\mathfrak{g}_{0,\mathbb{C}}) = h^\vee(G_0)$. This also follows from $k_{G_0}(B(V))$ and $k_{G_0}(F(V))$ given in Sec. 1.16 and Sec. 1.17.

A hypermultiplet $\mathrm{Hyp}(V)$ is G-anomaly-free, unless G has a simple component $G_0 = \mathrm{Sp}(n)$ and $k_{G_0}(\mathrm{Hyp}(V))$ is odd. This is related to Witten's global anomaly discussed previously in Sec. 1.18.

2.6 Quotients

Given a $G \times F$-symmetric $\mathcal{N} = 2$ QFT Q with no G-anomaly, we consider

$$[Q \times F_4(\mathfrak{g}_\mathbb{C} \otimes \mathcal{R}) \times B_4(\mathfrak{g}_\mathbb{C})] /\!\!\!\!/ G \,\big|_{u\in\mathbb{R}_{>0},\theta\in\mathbb{R}/2\pi\mathbb{Z}} \,. \tag{2.6.1}$$

For simplicity we assume G is simple. This family of effective QFT is embedded in a bigger family of QFT, whose complex-dimension-1 subfamily is again $\mathcal{N} = 2$ supersymmetric. Among others, one needs to add a deformation to (2.6.1) given by $|\vec{\mu}_G|^2 \subset \mathcal{V}_Q$, where

$$\vec{\mu}_G : \mathcal{M}_{\mathrm{Higgs}}(Q) \to \mathfrak{g} \otimes \mathbb{R}^3 \tag{2.6.2}$$

is the hyperkähler moment map of the G action. This is an F-symmetric effective $\mathcal{N} = 2$ supersymmetric QFT which we denote by

$$Q /\!\!/\!\!/ G \,\big|_\tau, \tag{2.6.3}$$

where $\tau = 4\pi\sqrt{-1}u + \theta/2\pi$. The notation $/\!\!/\!\!/$ is chosen to suggest its relation to the hyperkähler quotient below (2.6.7). The group F is called the flavor symmetry of this theory. In (2.6.1), the part $\times F_4(\mathfrak{g}_\mathbb{C} \otimes \mathcal{R}) \times B_4(\mathfrak{g}_\mathbb{C}) /\!\!\!\!/ G$ is called the $\mathcal{N} = 2$ vector multiplet, and the operation (2.6.3) is called the coupling of the vector multiplet of group G to the theory Q.

The theory $Q /\!\!/\!\!/ G \,\big|_\tau$ is a UV complete QFT if

$$k_G(Q) \le 4h^\vee(G). \tag{2.6.4}$$

Suppose Q is U(1) R-symmetric. Then $Q /\!\!/\!\!/ G \,\big|_\tau$ is U(1) R-symmetric if and only if $k_G(Q) = 4h^\vee(G)$. Otherwise the U(1) R-symmetry acts nontrivially on τ. The action is given as follows: define q and Λ via

$$q = e^{2\pi i\tau} = \Lambda^{2h^\vee(G) - k_G(Q)/2} \tag{2.6.5}$$

and say that Λ has degree 1. The data of simply-laced groups are given in Table 1. Note that $\dim G = \operatorname{rank} G(h^\vee(G) + 1)$.

Let $Q' = Q /\!\!/\!\!/ G$. Then

$$n_v(Q') = n_v(Q) + \dim G, \quad n_h(Q') = n_h(Q) \tag{2.6.6}$$

G	$\operatorname{rank} G$	$\dim G$	$h^\vee(G)$	$\{d_a\}$
A_{N-1}	$N-1$	N^2-1	N	$2,3,\ldots,N$
D_N	N	$N(2N-1)$	$2N-2$	$2,4,\ldots,2N-2;N$
E_6	6	78	12	$2,5,6,8,9,12$
E_7	7	133	18	$2,6,8,10,12,14,18$
E_8	8	248	30	$2,8,12,14,18,20,24,30$

Table 1: Data of the simply laced groups.

and

$$\mathcal{M}_{\text{Higgs}}(Q') = \mathcal{M}_{\text{Higgs}}(Q /\!\!/\!\!/ G) = \mathcal{M}_{\text{Higgs}}(Q)/\!/\!/G. \tag{2.6.7}$$

Here on the right hand side the symbol $/\!/\!/$ stands for the hyperkähler quotient[6]. As complex varieties

$$\mathcal{M}_{\text{Coulomb}}(Q') = \mathcal{M}_{\text{Coulomb}}(Q) \times \operatorname{Spec} \mathbb{C}[\mathfrak{g}_{\mathbb{C}}]^{G_{\mathbb{C}}} \tag{2.6.8}$$

where $\mathfrak{g}_{\mathbb{C}}$ has degree one. This is compatible with (2.4.7) because

$$\mathbb{C}[\mathfrak{g}_{\mathbb{C}}]^{G_{\mathbb{C}}} = \mathbb{C}[u_1, \ldots, u_r] \tag{2.6.9}$$

where $\deg u_i = e_i + 1$ and e_i is the i-th exponent of G, and

$$\dim G = \sum_i [2(e_i+1) - 1]. \tag{2.6.10}$$

2.7 Examples of $\mathcal{N}=2$ gauge theories

A straightforward subclass of effective $\mathcal{N}=2$ supersymmetric QFTs are the set of

$$\operatorname{Hyp}(V) /\!\!/\!\!/ G \tag{2.7.1}$$

for all possible V and G. These are called $\mathcal{N}=2$ gauge theories. We are mostly interested in UV complete ones, i.e. those with $k_{G_0}(\operatorname{Hyp}(V)) \leq 4h^\vee(G_0)$ for all simple component G_0 of G. Let us see some examples.

2.7.1 Pure theory

Take a simple gauge group G. The pure theory is

$$\operatorname{triv}_4 /\!\!/\!\!/ G \,\big|_\tau \,. \tag{2.7.2}$$

This is a special case of (2.7.1) where V is zero dimensional, so that $\operatorname{Hyp}(V) = \operatorname{triv}_4$. This is never U(1) R-symmetric.

[6]When Q is an $\mathcal{N}=1$ supersymmetric G-symmetric QFT, we can similarly define a gauging operation $Q /\!\!/ G\,\big|_\tau$, so that it is an $\mathcal{N}=1$ supersymmetric QFT. This quotient is a deformation of $[Q \times F_4(\mathfrak{g}_{\mathbb{C}})] / G\,\big|_{\alpha,\theta}$. Then $\mathcal{M}_{\text{susyvac}}(Q /\!\!/ G)$ is a submanifold of $\mathcal{M}_{\text{susyvac}}(Q)/\!/G$, where the symbol $/\!/$ stands for the Kähler quotient.

2.7.2 $\mathcal{N}=4$ theory

Take a simple gauge group G, and consider

$$\mathrm{SYM}_{\mathcal{N}=4}(G)\ \Big|_\tau := \mathrm{Hyp}(\mathfrak{g}_\mathbb{C}\oplus\mathfrak{g}_\mathbb{C}) /\!\!/\!\!/ G\ \Big|_\tau . \tag{2.7.3}$$

As $k_G(\mathfrak{g}_\mathbb{C}\oplus\mathfrak{g}_\mathbb{C}) = 4h^\vee(G)$, this gauge theory is conformal. By decomposing we see that

$$\mathrm{SYM}_{\mathcal{N}=4}(G)\ \Big|_\tau = B_4(\mathfrak{g}_\mathbb{R}\otimes_\mathbb{R}\mathbb{R}^6)\times F_4(\mathfrak{g}_\mathbb{C}\otimes_\mathbb{C}\mathbb{C}^4) / G\ \Big|_{\tau,\text{properly deformed}} \tag{2.7.4}$$

and there is in fact an action of $\mathcal{N}=4$ supersymmetry; the SU(4) R-symmetry acts naturally on $\mathbb{C}^4$ and on $\mathbb{R}^6$ via the isomorphism $\mathrm{SU}(4)\simeq\mathrm{Spin}(6)$.

It is believed

$$\mathrm{SYM}_{\mathcal{N}=4}(G)\ \Big|_\tau = \mathrm{SYM}_{\mathcal{N}=4}(G^\vee)\ \Big|_{-1/(n\tau)} \tag{2.7.5}$$

where $G^\vee$ is the group Langlands-dual to G and n is the ratio of the length squared of long roots and short roots. This is called the S-duality of the $\mathcal{N}=4$ super Yang-Mills theory, and underlies the proposed relation between geometric Langlands program and the gauge theory.

2.7.3 SQCD

Let $V\simeq\mathbb{C}^{N_c}$ and $W\simeq\mathbb{C}^{N_f}$. Let $G=\mathrm{SU}(V)$ and $F=\mathrm{SU}(W)$. We have

$$k_G(V\otimes\bar{W}\oplus W\otimes\bar{V}) = 2N_f. \tag{2.7.6}$$

Then we can consider the theory

$$\mathrm{Hyp}(V\otimes\bar{W}\oplus W\otimes\bar{V}) /\!\!/\!\!/ G \tag{2.7.7}$$

when $2N_f\le 4N_c$, i.e. $N_f\le 2N_c$. These are called $\mathcal{N}=2$ supersymmetric quantum chromodynamics (SQCD). N_c and N_f are called the number of colors and of flavors, respectively.

Similarly, let $V\simeq\mathbb{R}^N$ and $W\simeq\mathbb{H}^M$. Then $\mathrm{Hyp}(V\otimes_\mathbb{R} W)$ is $\mathrm{SO}(V)\times\mathrm{Sp}(W)$-symmetric. We find

$$k_{\mathrm{SO}(V)}(\mathrm{Hyp}(V\otimes_\mathbb{R} W)) = 4M,\qquad k_{\mathrm{Sp}(W)}(\mathrm{Hyp}(V\otimes_\mathbb{R} W)) = N. \tag{2.7.8}$$

Since $h^\vee(\mathrm{SO}(V)) = N-2$ and $h^\vee(\mathrm{Sp}(W)) = M+1$, we find that

$$\mathrm{Hyp}(V\otimes_\mathbb{R} W) /\!\!/\!\!/ \mathrm{SO}(V)\ \Big|_\tau \tag{2.7.9}$$

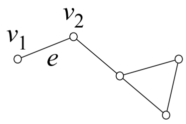
for $M\le N-2$ and

$$\mathrm{Hyp}(V\otimes_\mathbb{R} W) /\!\!/\!\!/ \mathrm{Sp}(W)\ \Big|_\tau \tag{2.7.10}$$

for $N\le 4(M+1)$, N even, are UV complete. Note that in the latter case odd N is not allowed due to the anomaly.

2.7.4 Quiver gauge theory

Let Γ be an unoriented graph



$$. \tag{2.7.11}$$

For each vertex v, introduce complex vector spaces V_v and W_v. Let

$$V_\Gamma := \bigoplus_e (V_{h(e)} \otimes \bar{V}_{t(e)} \oplus V_{t(e)} \otimes \bar{V}_{h(e)}) \oplus \bigoplus_v (V_v \otimes \bar{W}_v \oplus W_v \otimes \bar{V}_v), \tag{2.7.12}$$

$$G_\Gamma := \prod_v \mathrm{SU}(V_v). \tag{2.7.13}$$

We want to consider

$$\mathrm{Hyp}(V_\Gamma) /\!\!/\!\!/ G_\Gamma \,\Big|_{(\tau_v)\in(\text{upper half plane})^{\#\text{vertices}}} . \tag{2.7.14}$$

This is UV complete when

$$2 \dim V_v \geq \dim W_{v'} + \sum_{v'} \dim V_{v'} \tag{2.7.15}$$

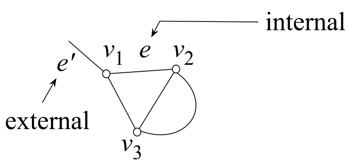
for all v, where the summation on the right hand side is over the vertices v' connected to v via an edge. This means that Γ is either a Dynkin graph or an affine Dynkin graph. In the latter case we also see that W_v is all zero dimensional.

2.7.5 An enumeration problem

As shown, the classification of UV-complete $\mathcal{N} = 2$ gauge theory $\mathrm{Hyp}(V) /\!\!/\!\!/ G$, if we restrict V and G to be associated to a quiver as above, is equivalent to the classification of the affine and non-affine Dynkin diagram. Therefore the classification of all UV-complete $\mathrm{Hyp}(V) /\!\!/\!\!/ G$ is a natural enumerative problem generalizing that question. It should not be too difficult a problem but this classification has not been done to the author's knowledge. Let us see below a few additional typical examples of a UV-complete $\mathcal{N} = 2$ gauge theory.

2.7.6 Trivalent gauge theory

Here we consider a different way to associate V and G given a combinatorial object. Let Γ be a trivalent graph



$$, \tag{2.7.16}$$

i.e. we only allow univalent or trivalent vertices. An edge connected to two trivalent vertices is called internal, and an edge connected to a univalent vertex and a trivalent vertex is called external. For each edge e, introduce $V_e \simeq \mathbb{C}^2$, and let

$$V_\Gamma := \bigoplus_{v:\text{trivalent}} V_{e_1(v)} \otimes_{\mathbb{C}} V_{e_2(v)} \otimes_{\mathbb{C}} V_{e_3(v)}, \tag{2.7.17}$$

$$G_\Gamma := \prod_{e:\text{internal}} \mathrm{SU}(V_e). \tag{2.7.18}$$

where $e_{1,2,3}(v)$ are the three edges connected to a trivalent vertex v. Then we consider

$$\mathrm{Hyp}(V_\Gamma) /\!\!/\!\!/ G_\Gamma \big|_{(\tau_e)\in(\text{upper half plane})^{\#\text{int. edges}}} . \tag{2.7.19}$$

This is a F_Γ symmetric theory, where

$$F_\Gamma := \prod_{e:\text{external}} \mathrm{SU}(V_e). \tag{2.7.20}$$

As we have

$$k_{\mathrm{SU}(V_e)}(V_\Gamma) = 8 = 4h^\vee(\mathrm{SU}(2)), \tag{2.7.21}$$

this theory is always conformal with respect to all $\mathrm{SU}(V_e)$. This construction does *not* generalize to any simple group G other than $\mathrm{SU}(2)$ if we only consider $\mathrm{Hyp}(V) /\!\!/\!\!/ (G)^n$. It is because there is no analogue of the pseudoreal representation $V \otimes V' \otimes V''$ where $V \simeq V' \simeq V''$, which can be used in an analogue of (2.7.17), that satisfies the constraint (2.6.4).

2.7.7 Exceptional gauge theories

Let $G = E_6$, $V \simeq \mathbb{C}^{27}$ its minuscule representation. This is a complex representation, with $k_{E_6}(\mathrm{Hyp}(V \oplus \bar{V})) = 12$. As $h^\vee(E_6) = 12$, we can consider

$$\mathrm{Hyp}(V \otimes \mathbb{C}^{N_f} \oplus \bar{V} \otimes \bar{\mathbb{C}}^{N_f}) /\!\!/\!\!/ E_6 \big|_\tau \tag{2.7.22}$$

for $0 \le N_f \le 4$. This is an $\mathrm{U}(N_f)$-symmetric theory.

Let $G = E_7$, $V \simeq \mathbb{H}^{28} \simeq \mathbb{C}^{56}$ its minuscule representation. This is a pseudoreal representation, with $k_{E_7}(\mathrm{Hyp}(V)) = 12$. As $h^\vee(E_7) = 18$, we can consider

$$\mathrm{Hyp}(V \otimes_{\mathbb{R}} \mathbb{R}^{N_f}) /\!\!/\!\!/ E_7 \big|_\tau \tag{2.7.23}$$

for $0 \le N_f \le 6$. This is an $\mathrm{SO}(N_f)$-symmetric theory.

Let $G = F_4$, $V \simeq \mathbb{R}^{26}$ its nontrivial real 26-dimensional representation. We find $k_{F_4}(\mathrm{Hyp}(V \otimes_{\mathbb{R}} \mathbb{H})) = 12$. As $h^\vee(F_4) = 9$, we can consider

$$\mathrm{Hyp}(V \otimes_{\mathbb{R}} \mathbb{H}^{N_f}) /\!\!/\!\!/ F_4 \big|_\tau \tag{2.7.24}$$

for $0 \le N_f \le 3$. This is an $\mathrm{Sp}(N_f)$-symmetric theory.

Let $G = G_2$, $V \simeq \mathbb{R}^7$ its nontrivial real 7-dimensional representation. We find $k_{G_2}(\mathrm{Hyp}(V \otimes_{\mathbb{R}} \mathbb{H})) = 4$. As $h^\vee(G_2) = 4$, we can consider

$$\mathrm{Hyp}(V \otimes_{\mathbb{R}} \mathbb{H}^{N_f}) /\!\!/\!\!/ G_2 \big|_\tau \tag{2.7.25}$$

for $0 \le N_f \le 4$. This is an $\mathrm{Sp}(N_f)$-symmetric theory.

2.8 Mass deformations

When Q is F-symmetric, there is a standard deformation Q_m where m is a semisimple element of $\mathfrak{g}_{\mathbb{C}}$. The parameter m is called the mass. Q_m and $Q_{m'}$ are equivalent if m and m' are conjugate. Q_m is F^m-symmetric. When Q is U(1) R-symmetric, the mass m has degree 1 under the U(1) R-symmetry.

As a complex manifold we have

$$\mathcal{M}_{\text{Coulomb}}(Q_m) \simeq \mathcal{M}_{\text{Coulomb}}(Q) \tag{2.8.1}$$

but other structures on them are different. The most important one is the following.

2.9 Donagi-Witten integrable system

Let Q be an F-symmetric $\mathcal{N} = 2$ supersymmetric QFT. We have the Donagi-Witten integrable system

$$DW(Q_m) \to \mathcal{M}_{\text{Coulomb}}(Q_m). \tag{2.9.1}$$

The standard review on this topic is [Don97]. The basic requirements are that

- $\dim DW(Q_m) = 2 \dim \mathcal{M}_{\text{Coulomb}}(Q_m) = 2r$.
- The generic fiber is an r-dimensional principally polarized Abelian variety.
- There is a holomorphic symplectic form Ω on DW such that its restriction to a generic fiber T is trivial: $\Omega|_T = 0$. These are why it is called an integrable system.
- There is a meromorphic one-form λ_{SW}, called the Seiberg-Witten differential, such that $\Omega = d\lambda_{SW}$.
- The polar divisor D of λ_{SW} has the structure

  $$D = \bigcup_{w \in P_F} D_w. \tag{2.9.2}$$

  Here and in the following, P_F and Q_F stands for the weight and the root lattice of F. Some of D_w can be empty.

Let $\mathsf{L} = H_1(T \setminus (T \cap D), \mathbb{Z})$. This has a skew-symmetric form $\langle , \rangle$ on it given by the polarization. There is a sequence

$$\mathsf{P}_F \to \mathsf{L} \to H_1(T, \mathbb{Z}), \tag{2.9.3}$$

and L has a skew-symmetric form with signature $(+^r, -^r, 0^{\operatorname{rank} F})$. Denote by $\mathrm{Sp}(\mathsf{L})$ the group of automorphism of L preserving this skew symmetric form. The differential λ_{SW} determines a homomorphism $a : \mathsf{L} \to \mathbb{C}$. Its restriction on P_F is constant on $\mathcal{M}_{\text{Coulomb}}$, as

$d\lambda_{SW}$ is holomorphic. This constant homomorphism $\mathsf{P}_F \to \mathbb{C}$ is identified with $m \in \mathfrak{f}_{\mathbb{C}}$ up to conjugation.

Let $\mathrm{Disc}(Q_m)$ be the discriminant of the fibration. We have an $\mathrm{Sp}(\mathsf{L})$ local system over

$$\mathcal{M}_{\mathrm{Coulomb}}(Q_m) \setminus \mathrm{Disc}(Q_m). \tag{2.9.4}$$

Locally we can take a basis of L

$$\alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_r; \gamma_1, \ldots, \gamma_{\mathrm{rank}\, F} \tag{2.9.5}$$

such that $\langle \alpha_i, \beta_j \rangle = \delta_{ij}$, otherwise $= 0$. We let

$$a_i = a(\alpha_i), \quad a_i^D = a(\beta_i), \quad m_i = a(\gamma_i). \tag{2.9.6}$$

We identify the sublattice generated by γ_i with P_F. Then $(m_1, \ldots, m_{\mathrm{rank}\, F})$ is identified with m of Q_m. We denote by L_E the maximally isotropic sublattice generated by $\{\alpha_i\}$ and $\{\gamma_j\}$.

Locally the tuple $(a_1, \ldots, a_r)$ gives a coordinate system on $\mathcal{M}_{\mathrm{Coulomb}}(Q_m)$. As the fiber is a polarized Abelian variety, we find that there is a holomorphic function

$$\mathcal{F}(a_1, \ldots, a_r; m_1, \ldots, m_{\mathrm{rank}\, F}) \tag{2.9.7}$$

such that

$$a_i^D = \frac{\partial \mathcal{F}}{\partial a_i} \tag{2.9.8}$$

and furthermore

$$\tau_{ij} = \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j} \tag{2.9.9}$$

is the period matrix of T, and in particular $\mathrm{Im}\, \tau_{ij}$ is symmetric positive definite.

The prepotential $\mathcal{F}$ is defined with respect to the choice of the maximally isotropic sublattice $\mathsf{L}_E \subset \mathsf{L}$. The relation (2.9.8) means that when we change the choice of L_E the prepotential is transformed by a Legendre transformation.

2.10 Donagi-Witten integrable system and gauging

It would be useful to consider a further fibration

$$\widetilde{DW}_F(Q) \to \mathfrak{f}_{\mathbb{C}}/F_{\mathbb{C}} \tag{2.10.1}$$

where the fiber at $m \in \mathfrak{f}_{\mathbb{C}}$ is $DW(Q_m)$. When Q is $G \times F$-symmetric, it should be possible to characterize $\widetilde{DW}_F(Q /\!\!/\!\!/ G \mid_\tau)$ in terms of $\widetilde{DW}_{F \times G}(Q)$, but the author does not currently know how to do it. Instead let us just state the condition when $Q = \mathrm{Hyp}(V)$ where V is a pseudoreal representation of $G \times F$.

Let us then consider $\mathrm{Hyp}(V) /\!\!/\!\!/ G \mid_{\tau,m}$. Here $\mathrm{Im}\, \tau$ can be non-canonically identified with an invariant positive bilinear form $(,)$ on $\mathfrak{g}$. Let us write $G = \prod_x G_x$ where G_x is simple.

Define τ_x by $(\,,\,)|_{\mathfrak{g}_x} = \tau_x(\,,\,)_0$ where $(\,,\,)_0$ is the invariant product normalized so that the length squared of the long root is 2. Note that $\operatorname{Im}\tau_x$ is positive. We then set $q_x = e^{2\pi\sqrt{-1}\tau_x}$. As stated in (2.6.5), q_x has degree $2h^\vee(G_x) - k_{G_x}(\mathrm{Hyp}(V))/2$ under U(1) R-symmetry, whereas a and m has degree one.

Then our aim is to find the fibration is

$$DW(\mathrm{Hyp}(V)/\!\!/\!\!/G) \to \mathcal{M}_{\mathrm{Coulomb}}(\mathrm{Hyp}(V)/\!\!/\!\!/G) = \mathfrak{h}_{\mathbb{C}}/W \tag{2.10.2}$$

where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$ and W is the Weyl group. This fibration depends furthermore on q_x and m. We pull back this family to

$$DW \to \mathfrak{h}_{\mathbb{C}}. \tag{2.10.3}$$

Let

$$\mathfrak{h}_{\mathbb{C}} \supset U_K = \{\underline{a} \in \mathfrak{h}_{\mathbb{C}} \mid |\alpha(\underline{a})| > K \text{ and } |w(\underline{a} \oplus m)| > K\} \tag{2.10.4}$$

where α runs over all roots of G and w is over all weights of V. Note that $\underline{a} \oplus m$ is in the Cartan subalgebra of $\mathfrak{g} \times \mathfrak{f}$ and therefore there is a natural pairing with a weight w of V.

A standard perturbative computation shows that the family restricted to U_K for a sufficiently large K,

$$DW \to U_K \tag{2.10.5}$$

satisfies the following properties.

- The monodromy of the local system on U_K preserves an isotropic sublattice $\mathsf{P}_G \times \mathsf{P}_F \subset \mathsf{L}$, where we identify P_G with the weight lattice of $\mathfrak{g}$.

- Let us then take a basis $\alpha_1, \ldots, \alpha_r$ of P_G. Locally on U_K, we can choose $\beta_1, \ldots, \beta_r$ generating the complementary sublattice Q_G such that $\mathsf{L} = \mathsf{P}_G \oplus \mathsf{Q}_G \oplus \mathsf{P}_F$. We identify Q_G with the root lattice of $\mathfrak{g}$.

- We let $\underline{a}_i = \alpha_i(\underline{a})$ be the coordinate functions of $\underline{a} \in \mathfrak{h}_{\mathbb{C}}$. We also introduce $a \in \mathfrak{h}_{\mathbb{C}}$ via $\alpha_i(a) = a_i = \int_{\alpha_i} \lambda_{SW}$. Both the set $\{\underline{a}_i\}$ and the set $\{a_i\}$ give a coordinate system in U_K.

- The most crucial condition is that the prepotential $F(a)$ has the power series expansion in terms of $\{q_x\}$

  $$F(a, m) = \sum_{d_x \geq 0} F_{\{d_x\}}(a, m) \prod_x q_x^{d_x} \tag{2.10.6}$$

  such that the leading term is

  $$F_{\{d_x=0\}}(a, m) = (a, a) - \sum_{v:\text{roots of } \mathfrak{g}} f(v(a)) + \frac{1}{2} \sum_{w:\text{weights of } V} f(w(a \oplus m)) \tag{2.10.7}$$

  where

  $$f(x) = \frac{1}{2\pi\sqrt{-1}} \left[\frac{x^2}{2} \log x - \frac{3}{4} x^2 \right] \tag{2.10.8}$$

is a function such that $f'''(x) = 1/x$, and other $F_{\{d_x\}}(a,m)$ are rational functions of $\{a_j\}$ and $\{m_k\}$. The degree of $F(a,m)$ under U(1) R-symmetry should be two. Recall that a and m has degree 1 and q_x has degree $2h^\vee(G_x) - k_{G_x}(\mathrm{Hyp}(V))/2$.

Note that the branch cut of $f(x)$ together with (2.9.8) determines the $\mathrm{Sp}(\mathsf{L})$ local system on U_K uniquely.

- The next condition is not so crucial as the previous one. It is on the property of $\underline{a}_i$ as a function of a_j, q_x and m. Namely, $\underline{a}_i$ has a power series expansion in terms of q_x

$$\underline{a}_i = \sum_{d_x \geq 0} f_{i,\{d_x\}}(a,m) \prod_x q_x^{d_x} \tag{2.10.9}$$

such that the leading term is

$$f_{i,\{d=0\}}(a,m) = a_i \tag{2.10.10}$$

and other $f_{i,d}(a,m)$ are rational functions of $\{a_j\}$ and $\{m_k\}$. $\underline{a}_i$ should furthermore have degree 1 under the U(1) R-symmetry. This just says that the coordinate a_i defined by λ_{SW} and $\underline{a}_i$ defined by the underlying $\mathfrak{h}$ are not very different.

The physics intuition says that such fibration should exist and is furthermore essentially unique, in the sense that if we have two solutions

$$F(a,m,\{q_x\}), \qquad \tilde{F}(a,m,\{q_x\}) \tag{2.10.11}$$

then there are power series with a definite degree under U(1) R-symmetry,

$$\tilde{q}_x = \sum_{d_y \geq 0} \tilde{q}_{x,\{d_y\}}(m) \prod_y q_y^{d_y} \tag{2.10.12}$$

where

$$\tilde{q}_{x,\{d_y=0\}} = q_x \tag{2.10.13}$$

and other $\tilde{q}_{x,\{d_y\}}(m)$ are rational in m, so that

$$F(a,m,\{q_x\}) = \tilde{F}(a,m,\{\tilde{q}_x\}). \tag{2.10.14}$$

Before proceeding we mention that there is a one-parameter family of hyperkähler structure on $DW(Q_m)$ which is compatible with the holomorphic symplectic structure discussed above. On this topic, see e.g. [GMN08].

2.11 Examples of Donagi-Witten integrable systems

It is not known how to construct the DW integrable system given $\mathrm{Hyp}(V) /\!/\!/\!/ G$ in complete generality. Even describing them is tricky. The methods often employed are the following.

1. One can start from a family of curves

$$\Sigma_{SW} \to \mathcal{M}_{\text{Coulomb}} \tag{2.11.1}$$

and take the Jacobian (or a nice subspace of it such as Prym) at each point on the base. In this case one needs to check that the resulting family is integrable. Σ_{SW} is called the Seiberg-Witten curve.

2. One can start from a Riemann surface C and a G'-Hitchin system on it, where G' is a group related to G. Then $DW \to \mathcal{M}_{\text{Coulomb}}$ is identified with a small modification of the Hitchin fibration. Given a representation R of G, one can construct an associated spectral curve $\Sigma_R \to \mathcal{M}_{\text{Coulomb}}$ which can then be regarded as the Seiberg-Witten curve.

3. One can also start from a family of compact Calabi-Yau 3-fold over the moduli space of its complex structure. In this case the fibration of its intermediate Jacobian is an integrable system but it is not principally polarized and $\operatorname{Im}\tau_{ij}$ is not positive definite. One needs to take a certain limit to extract a positive-definite subsystem. We usually end up with a family of non-compact 3-fold which is a fibration of deformed simple singularities over a Riemann surface C, $X \to \mathcal{M}_{\text{Coulomb}}$. This family can also arise as a spectral geometry of a Hitchin system on C.

4. Finally there are also cases where $DW(Q)$ is given by the moduli space of anti-self-dual G'-connections on a certain open four-manifold.

We review below some of the typical Donagi-Witten integrable system of $\text{Hyp}(V)/\!\!/\!\!/G$. We do not explain how to check that the conditions explained in Sec. 2.10 are satisfied. In the literature some of them were checked. There are some cases where the conditions have not been checked, although $DW(Q)$ is believed to be correct from various other considerations.

2.11.1 G-Hitchin system

We begin by a quick review of the Hitchin system. Let C be a Riemann surface with punctures $p_1, \ldots p_k$ with labels which we describe later. Let $P \to C$ be a $G_{\mathbb{C}}$-bundle with a reference connection d''. We take

$$\phi \in \Omega^{1,0}(C, \underline{\mathfrak{g}_{\mathbb{C}}}), \quad A'' \in \Omega^{0,1}(C, \underline{\mathfrak{g}_{\mathbb{C}}}). \tag{2.11.2}$$

Recall that we use $\underline{V}$ to denote a vector bundle associated to a representation V, (1.11.2). $D'' = d'' + A''$ is a connection. The labels determine the singularities allowed for ϕ and A. Suppose a singularity p is at the origin of a local coordinate $z = 0$. A tame (or regular) singularity is labeled by a $\mathfrak{g}_{\mathbb{C}}$-orbit O, and ϕ is of the form

$$\phi \sim X\frac{dz}{z} + \text{less singular terms}, \qquad X \in O. \tag{2.11.3}$$

A wild (or irregular) singularity is one where ϕ has a pole of order more than one.

We let

$$\mathcal{G} = \{f : C \to G_{\mathbb{C}}\}. \tag{2.11.4}$$

Then

$$\{D''\phi = 0\}/\mathcal{G} =: \mathcal{M}_{G\text{-Hitchin}}(C) \tag{2.11.5}$$

is a holomorphic symplectic manifold and there is the Hitchin map

$$h : \mathcal{M}_{G\text{-Hitchin}} \to \bigoplus_{a=1}^{r} H^0(K_C^{\otimes d_a} + p_{(a)}) \tag{2.11.6}$$

where K_C is the canonical divisor and $p_{(a)}$ is a linear combination of $p_1, \ldots, p_k$ determined by the labels. The Hitchin map h is given by

$$h : \phi \mapsto u_1(\phi) \oplus \cdots \oplus u_r(\phi) \tag{2.11.7}$$

where we fixed the isomorphism

$$\mathbb{C}[\mathfrak{g}_{\mathbb{C}}^*]^{G_{\mathbb{C}}} \simeq \mathbb{C}[u_1, \ldots, u_r] \tag{2.11.8}$$

so that u_a has degree d_a. Given a representation R of G we can consider the spectral curve of the Hitchin system. For example, when $G = A_{N-1}$, we take the vector representation as R and consider

$$\det_R(\lambda - \phi) = \lambda^N + u_2(\phi)\lambda^{N-2} + \cdots + u_N(\phi) = 0 \tag{2.11.9}$$

as an equation giving a curve within T^*C, where λ is the tautological one-form on T^*C. $\mathcal{M}_{G\text{-Hitchin}}(C)$ is recovered as its Jacobian.

The spectral curve has a spurious dependence on R. When G is simply-laced, a more invariant object is its spectral geometry [DDP06]. Let us illustrate the construction by considering two cases. First consider the case $G = E_6$. The deformation of the simple singularity of type E_6 is given by

$$W_{E_6} = x_1^4 + x_2^3 + x_3^2 + u_2 x_1^2 x_2 + u_5 x_1 x_2 + u_6 x_1^2 + u_8 x_2 + u_9 x_1 + u_{12} \tag{2.11.10}$$

where x_1, x_2 and x_3 have degree 3, 4, 6 respectively and u_k are the generators as in (2.11.8) where the subscripts are renamed to correspond to the degree. The whole expression has the degree $h^\vee(E_6) = 12$.

Then, given ϕ as in (2.11.2), we consider a three-fold X in the total space of the vector bundle

$$K_C^{\otimes 3} \oplus K_C^{\otimes 4} \oplus K_C^{\otimes 6} \to C \tag{2.11.11}$$

given by

$$0 = x_1^4 + x_2^3 + x_3^2 + u_2(\phi)x_1^2 x_2 + u_5(\phi)x_1 x_2 + u_6(\phi)x_1^2 + u_8(\phi)x_2 + u_9(\phi)x_1 + u_{12}(\phi) \tag{2.11.12}$$

where x_1, x_2, x_3 are now sections of $K_C^{\otimes 3}$, $K_C^{\otimes 4}$, $K_C^{\otimes 6}$, respectively. Then the fiber of the Hitchin system is given by the intermediate Jacobian of X.

Next, let us consider the case $G = A_{N-1}$. In this case the spectral geometry is given by

$$0 = x_2 x_3 + x_1^N + u_2(\phi) x_1^{N-2} + \cdots + u_N(\phi) \tag{2.11.13}$$

where x_1, x_2, x_3 are sections of K_C, $K_C^{\otimes 2}$, $K_C^{\otimes (N-2)}$, respectively. Note that this is essentially equivalent to the spectral curve (2.11.9).

2.11.2 Pure theory

For a simple gauge group G, consider the pure theory $Q \,\big|_\tau = \mathrm{triv}_4 /\!\!/\!\!/ G \,\big|_\tau$. We use the parameter $q = \Lambda^{2h^\vee} = e^{2\pi\sqrt{-1}\tau}$ introduced in (2.6.5). This has degree $2h^\vee$ under the U(1) R-symmetry.

Its Donagi-Witten integrable system $DW(Q)$ is the Toda integrable system of type G when G is simply-laced. For non-simply-laced G, it is the twisted Toda system associated to the Langlands dual of the affine Lie algebra $\hat{G}$ associated to G.

For type A_{N-1},

$$\mathcal{M}_{\mathrm{Coulomb}}(Q) = \mathrm{Spec}\, \mathbb{C}[\mathfrak{g}_{\mathbb{C}}]^{G_{\mathbb{C}}} = \mathrm{Spec}\, \mathbb{C}[u_2, \ldots, u_N] \tag{2.11.14}$$

and the Seiberg-Witten curve is the spectral curve of the Toda system of type A_{N-1} given by

$$\Lambda^N z + \frac{\Lambda^N}{z} = x^N + u_2 x^{N-2} + \cdots u_N. \tag{2.11.15}$$

By defining the one-form $\lambda = x dz/z$ we have

$$\lambda^N + u_2 (\frac{dz}{z})^2 \lambda^{N-2} + \cdots + (u_N + \Lambda^N z + \frac{\Lambda^N}{z})(\frac{dz}{z})^N = 0. \tag{2.11.16}$$

This is of the form of a spectral curve of SU(N)-Hitchin system on a sphere, with two marked points at $z = 0$ and $z = \infty$. $u_i(\phi)$ for $i < N$ has degree $\leq i$ poles at 0 and ∞, but $u_N(\phi)$ has order $N + 1$ poles there. The points 0 and ∞ are therefore irregular (wild) singularities.

For type E_6, say, the Seiberg-Witten geometry is given by

$$\Lambda^{12} z + \frac{\Lambda^{12}}{z} = x_1^4 + x_2^3 + x_3^2 + u_2 x_1^2 x_2 + u_5 x_1 x_2 + u_6 x_1^2 + u_8 x_2 + u_9 x_1 + u_{12} \tag{2.11.17}$$

and this is of the form of the spectral geometry of the E_6-Hitchin system on a sphere with two marked points at $z = 0$ and $z = \infty$, with

$$u_i(\phi) = u_i \frac{dz^i}{z^i}, \quad (i \neq 12), \qquad u_{12}(\phi) = (u_{12} + \Lambda^{12} z + \frac{\Lambda^{12}}{z}) \frac{dz^{12}}{z^{12}}. \tag{2.11.18}$$

The points 0 and ∞ are again irregular (wild) singularities.

2.11.3 $\mathcal{N} = 4$ theory and $\mathcal{N} = 2^*$ theory

Pick a simple $\mathfrak{g}$. Consider the $\mathcal{N} = 4$ system introduced in Sec. 2.7.2:

$$Q \big|_\tau = \mathrm{Hyp}(\mathfrak{g}_\mathbb{R} \otimes_\mathbb{R} \mathbb{H}) /\!\!/\!\!/ G \big|_\tau . \tag{2.11.19}$$

This is a Sp(1)-symmetric $\mathcal{N} = 2$ supersymmetric theory, and therefore one can consider the mass deformation $Q_{\tau,m}$ where m is in the Cartan subalgebra of $\mathfrak{su}(2)_\mathbb{C}$, i.e. a complex number up to sign. The theory when $m \neq 0$ is called the $\mathcal{N} = 2^*$ theory.

Here $q = e^{2\pi\sqrt{-1}\tau}$ is degree zero. The Donagi-Witten integrable system for a simply-laced G when $m = 0$ is the G-Hitchin system on the elliptic curve with modulus q without any puncture. When G is not simply-laced, it is given by the twisted Hitchin system associated to the Langlands dual of the affine Lie algebra $\hat{G}$ associated to G. In either case, the prepotential is just given by

$$F(a) = \tau(a, a)_0, \tag{2.11.20}$$

where $(\cdot, \cdot)_0$ is the positive-definite invariant form on $\mathfrak{g}$ introduced around (2.10.1).

When $m \neq 0$, $DW(Q_{\tau,m})$ is given by the elliptic Calogero-Moser system of type G when G is simply-laced, and by the twisted version associated to the Langlands dual of $\hat{G}$ when G is non-simply-laced [DP99]. When $G = A_{N-1}$ it is given by an SU(N)-Hitchin system on an elliptic curve with one puncture at $z = 0$, such that the $\mathfrak{g}_\mathbb{C}$-valued one-form ϕ at $z = 0$ has a residue conjugate to

$$\operatorname*{Res}_{z=0} \phi \sim m \,\mathrm{diag}(1, 1, \ldots, 1, 1 - N). \tag{2.11.21}$$

There is no known way to construct (twisted) elliptic Calogero-Moser systems of other types as a Hitchin system.

2.11.4 SQCD

Consider the SQCD introduced in Sec. 2.7.3:

$$Q \big|_\tau = \mathrm{Hyp}(V \otimes \bar{W} \oplus W \otimes \bar{V}) /\!\!/\!\!/ \mathrm{SU}(V) \big|_\tau \tag{2.11.22}$$

where $V \simeq \mathbb{C}^N$ and $W \simeq \mathbb{C}^{N_f}$. This is a U($W$)-symmetric theory, and therefore we can introduce mass deformations by $m = (m_1, \ldots, m_{N_f})$. The maximum N_f allowed is $2N$.

The Seiberg-Witten curve is given by the family

$$\Sigma_{SW} \ni (z, x): \quad z + \frac{q \prod_{i=1}^{N_f} (x - m_i)}{z} + x^N + u_2 x^{N-2} + \cdots + u_N = 0 \tag{2.11.23}$$

and $\lambda = x dz/z$. Here z has degree N and $q =$ has degree $2N - N_f$. From this one can construct

$$DW(Q) := \mathrm{Jac}(\Sigma_{SW}) \to M_{\mathrm{Coulomb}}(Q). \tag{2.11.24}$$

The one-form on DW is induced from the one-form $\lambda = xdz/z$ on Σ_{SW}. It is a good exercise to check that indeed this fibration satisfies the defining conditions stated in Sec. 2.10. We note that for $N_f \le 2N-2$ we can identify $m_i = \underline{m}_i$, but for $N_f > 2N-2$, $\underline{m}_i = m_i + O(q)$.

Consider the upper limit case $N_f = 2N$. Let us rewrite (2.11.23) as the spectral curve of the Hitchin system. We first redefine z to have

$$z\prod_{i=1}^{N}(x-\underline{m}_i) + \frac{q}{z}\prod_{i=N+1}^{2N}(x-\underline{m}_i) + x^N + u_2 x^{N-2} + \ldots + u_N = 0. \tag{2.11.25}$$

We make a few rewrites: first, we gather the same powers of x to have

$$(z + \frac{q}{z} + 1)x^N + \hat{u}_1(z)x^{N-1} + \cdots + \hat{u}_N(z) = 0. \tag{2.11.26}$$

By dividing by $z + q/z + 1$ and redefining $x_{\text{new}} = x_{\text{old}} - u_1(z)/(z+q/z+1)/N$, we have

$$x^N + \tilde{u}_2(z)x^{N-2} + \cdots + \tilde{u}_N(z) = 0. \tag{2.11.27}$$

Now $\tilde{u}_k(z)$ has degree k poles at $z_\pm$, where $z_\pm$ are two zeros of $z + q/z + 1 = 0$.

This last expression is of the form of the spectral curve of a Hitchin system,

$$\lambda^N + u_2(\phi)\lambda^{N-2} + \cdots + u_N(\phi) = 0. \tag{2.11.28}$$

where $\lambda = xdz/z$ and $u_k(\phi) = \tilde{u}_k(z)dz^k/z^k$. The field ϕ has four singularities on a sphere parameterized by z, all of which are regular. The cross ratio of four points is a function of q. When all $\underline{m}_i$ are generic, we find the following:

- At $z = 0, \infty$, we have a pole of the form

$$\phi \sim \operatorname{diag}(\tilde{m}_1, \ldots, \tilde{m}_N)dz/z, \quad \phi \sim \operatorname{diag}(\hat{m}_1, \ldots, \hat{m}_N)dz/z, \tag{2.11.29}$$

  so that $\sum \tilde{m}_i = \sum \hat{m}_i = 0$.

- At $z = z_\pm$, we have a pole of the form

$$\phi \sim \tilde{m}\operatorname{diag}(1,1,\ldots,1,1-N)\frac{dz}{z-z_+}, \quad \phi \sim \hat{m}\operatorname{diag}(1,1,\ldots,1,1-N)\frac{dz}{z-z_-}. \tag{2.11.30}$$

We thus see that there are two types of residues with distinct Levi types.

When some of the parameter, say $\tilde{m}$, is taken to zero, the residue of ϕ is no longer semisimple. Instead, we have

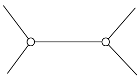
$$\phi \sim (J_2 \oplus \underbrace{J_1 \oplus \cdots \oplus J_1}_{N-2})\frac{dz}{z-z_+}, \tag{2.11.31}$$

where J_k is a $k\times k$ Jordan block. We will have more to say about it in the next section.

Recall that we saw in Sec. 2.11.2 that we found wild singularities for the pure theories. An experimental fact is that when we write the Seiberg-Witten curve in terms of a Hitchin system we usually have

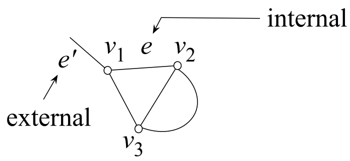
- some wild singularities if $2h^\vee(G) > k_G$ and
- all singularities are tame when $2h^\vee(G) = k_G$.

Let us consider a particularly simple case where $N = 2$ and $N_f = 4$. Then the residue $(1, 1-N)$ in (2.11.30) is equal to the residue $(1, -1)$ in (2.11.29), and the four singularities at $z = 0, \infty, z_+, z_-$ are all of the same type. This is in fact the simplest case of the trivalent theory, with

$\Gamma =$  . (2.11.32)

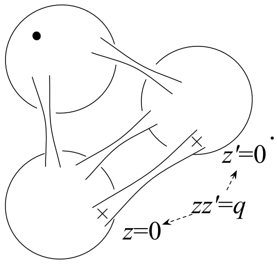
2.11.5 Trivalent theory

Let us then consider a general trivalent theory $Q_\Gamma \big|_\tau$ introduced in Sec. 2.7.6. Given

$\Gamma =$ 

, (2.11.33)

recall we have the theory $Q = \mathrm{Hyp}(V_\Gamma) /\!\!/\!\!/ G_\Gamma$ which is F_Γ-symmetric, see (2.7.17), (2.7.18) and (2.7.20). Note that mass deformation is given by $m = \{m_e\}_{e\text{:external}}$. We associate to Γ a Riemann surface by picking a three-punctured sphere P^1 for each vertex v, and for each edge with τ_e associated, we make the identification $zz' = q_e = e^{2\pi\sqrt{-1}\tau_e}$:

$C =$ 

. (2.11.34)

Note that each external edge e becomes a puncture p_e on C. Let us say p_e is at the origin of the local coordinate $z_e = 0$. Then we consider an SU(2)-Hitchin system on this Riemann surface with the boundary condition

$$\phi \sim \frac{dz_e}{z_e} \operatorname{diag}(m_e, -m_e) \tag{2.11.35}$$

at each puncture. This gives the Donagi-Witten integrable system of $Q_{\Gamma,\tau,m}$.

2.11.6 An exceptional gauge theory

Consider the theory

$$\mathrm{Hyp}(V \otimes \mathbb{C}^{N_f} \oplus \bar{V} \otimes \bar{\mathbb{C}}^{N_f}) /\!\!/\!\!/ E_6 \big|_\tau \tag{2.11.36}$$

as introduced in Sec. 2.7.7, where $V \simeq \mathbb{C}^{27}$ is the minuscule representation of E_6, and $0 \leq N_f \leq 4$. As this is $\mathrm{U}(N_f)$-symmetric, introduce the mass deformation $\vec{m} = (m_1, \ldots, m_{N_f})$. $q = e^{2\pi\sqrt{-1}\tau}$ has degree $24 - 6N_f$.

The Seiberg-Witten geometry is given by

$$z + \frac{q \prod_i^{N_f} X(\{x_1, x_2, x_3\}, \{u_d\}, \underline{m_i})}{z} = W_{E_6}(\{x_1, x_2, x_3\}, \{u_d\}) \tag{2.11.37}$$

where W_{E_6} was given in (2.11.10) and

$$\begin{aligned} X(\{x_1, x_2, x_3\}, \{u_d\}, m) = &-8(x_1^2 - \sqrt{-1}x_3 + \frac{1}{2}u_6) - 4u_2 x_2 \\ &+ 4mu_5 + m^2(u_2^2 - 12x_2) - 8m^3 x_1 + 2m^4 w_2 + m^6. \end{aligned} \tag{2.11.38}$$

Note first that when $N_f = 0$ it reduces to the geometry of the pure theory, (2.11.17). In particular it is the spectral geometry of a E_6-Hitchin system with two wild singularities.

The polynomial X above has the following important property. Consider

$$zX(\{x_1, x_2, x_3\}, \{u_d\}, m) = W_{E_6}(\{x_1, x_2, x_3\}, \{u_d\}) \tag{2.11.39}$$

as defining a family $\mathcal{X}$ of three-dimensional hypersurface in $(z, x_1, x_2, x_3) \in \mathbb{C}^4$ parameterized by m and $\{u_d\}$. By the identification $\mathbb{C}[\mathfrak{h}_\mathbb{C}]^W = \mathbb{C}[u_d]$ where $\mathfrak{h}_\mathbb{C}$ is the Cartan subalgebra of E_6 and W the Weyl group, we can think of $\mathcal{X}$ as a family

$$\mathcal{X} \to \mathbb{C} \oplus \mathfrak{h} \ni m \oplus a. \tag{2.11.40}$$

Then the fiber develops a singularity of the form $x^2 + y^2 + z^2 + w^2 = 0$ if and only if there is a weight of V such that $m = w(a)$. Many of the Donagi-Witten system of $\mathrm{Hyp}(\oplus_i V_i) /\!/\!/ G$ for a simple G can be found using a polynomial X_i satisfying this condition for V_i [TT11].

Let us next consider the case $N_f = 4$ so that the theory is conformal. Here it is more convenient to rewrite (2.11.37) to

$$zX(\underline{m_1})X(\underline{m_2}) + \frac{qX(\underline{m_3})X(\underline{m_4})}{z} = W_{E_6}. \tag{2.11.41}$$

As in the rewriting in the conformal $\mathrm{SU}(N)$ case starting at (2.11.25), we can transform it into the spectral geometry of a Hitchin system on a sphere with four tame singularities:

- At $z = 0$ and $z = \infty$, the Hitchin field behaves as

$$\phi \sim [3(m_1 + m_2)(v_2 - v_4) + (m_1 - m_2)(v_2 + v_4)]\frac{dz}{z}, \tag{2.11.42}$$

$$\phi \sim [3(m_3 + m_4)(v_2 - v_4) + (m_3 - m_4)(v_2 + v_4)]\frac{dz}{z} \tag{2.11.43}$$

  respectively, where v_i is the i-th fundamental weight where the ordering of the nodes is given by $\overset{\scriptstyle 6}{12345}$. When $m \to 0$ the residue is nilpotent, whose Bala-Carter label is $A_4 + A_1$.

- At $z = z_+$ and $z = z_-$ at the zeroes of $z + q/z + 1 = 0$, we have

$$\phi \sim E_\alpha \frac{dz}{z - z_+}, \qquad \phi \sim E_\alpha \frac{dz}{z - z_-}. \tag{2.11.44}$$

  where E_α is an element in the SL(2) triple $(E_\alpha, H_\alpha, F_\alpha)$ associated to a simple root. The Bala-Carter label is A_1.

2.11.7 Affine quiver theory

As a final example, consider the quiver gauge theory Q_Γ introduced in Sec. 2.7.4, in a particular case when the underlying graph Γ is an affine Dynkin diagram of type A_r, D_r or E_r. The gauge group is

$$G_\Gamma = \prod_{i=0}^{r} \mathrm{SU}(Na_i) \tag{2.11.45}$$

where d_i are the marks of the Dynkin diagram so that $\sum d_i = h^\vee(\Gamma_r)$. The flavor symmetry F_Γ is $\tilde{G}_\Gamma / G_\Gamma$ where

$$\tilde{G}_\Gamma = \prod_{i=0}^{r} \mathrm{U}(Na_i). \tag{2.11.46}$$

The gauge couplings are given by $q_i = e^{2\pi\sqrt{-1}\tau_i}$ for $i = 0, \ldots, r$. Then the Donagi-Witten integrable system $DW(Q)$ is given by the moduli space of anti-self-dual Γ_r-connections of instanton number N on $\mathcal{E}_q \times \mathbb{C}$ where $\mathcal{E}_q$ is an elliptic curve with the complex structure $q = q_0 \cdots q_r$ [NP12].

The fibration $DW(Q_\Gamma) \to \mathcal{M}_{\mathrm{Coulomb}}(Q_\Gamma)$ is given by using Looijenga's theorem, which states that the moduli of holomorphic Γ_r-bundle on $\mathcal{E}$ is isomorphic to the weighted projective space $\mathbb{WP}_{a_0,\ldots,a_r}$. Let us denote by x the coordinate on $\mathbb{C}$. Then, restricting the Γ_r-bundle on the fiber $\mathcal{E}_q$ at x, one has a holomorphic degree-N quasimap from $\mathbb{C}$ to $\mathbb{WP}_{a_0,\ldots,a_r}$. More explicitly, we have $r+1$ polynomials χ_i of degree Na_i of x:

$$\chi_i(x) = q_i x^{Na_i} + m_i x^{Na_i - 1} + u_{i,2} x^{Na_i - 2} + \cdots + u_{i,Na_i} \tag{2.11.47}$$

so that $[\chi_0(x) : \chi_1(x) : \cdots : \chi_r(x)] \in \mathbb{WP}_{a_0,\ldots,a_r}$. The coefficients are naturally associated to the coupling constants q_i, masses m_i of F_Γ, and the coordinates $u_{i,2}, \ldots, u_{i,Na_i}$ of $\mathcal{M}_{\mathrm{Coulomb}}(Q_\Gamma)$ which comes from $\mathbb{C}[\mathrm{SU}(Na_i)]^{\mathfrak{su}(Na_i)}$.

When $\Gamma_r = A_r$ or D_r, one can also describe the same integrable system as an SU(N)-Hitchin system or a twisted SU($2N$)-Hitchin system, respectively. For $\Gamma_r = A_r$, we have an SU(N)-Hitchin system on T^2 with complex structure q as above, with $r+1$ punctures with residue of the form (2.11.30). For $\Gamma_r = D_r$, we have a twisted SU($2N$)-Hitchin system on a sphere in the following sense. In addition to r singularities where the residue of ϕ is of the form (2.11.30), there are four singularities around which there is a monodromy by the outer automorphism of SU($2N$). These descriptions when the Dynkin diagram is of type A or D are obtained by applying the Nahm transformation to the descriptions given above in terms of instantons on $T^2 \times \mathbb{R}^2$.

2.12 BPS states and Wall crossing

Given an F-symmetric $\mathcal{N} = 2$ supersymmetric theory Q, consider $\mathcal{H}_{Q_m}(\mathbb{R}^3, p)$ for $p \in \mathcal{M}_{\text{Coulomb}}(Q_m) \setminus \text{Disc}(Q_m)$. This is an infinite dimensional Hilbert space, graded by L

$$\mathcal{H}_{Q_m}(\mathbb{R}^3, p) = \oplus_{l \in \mathsf{L}} \mathcal{H}_l(p). \tag{2.12.1}$$

There is an action of the supersymmetry $S^+ \otimes \mathcal{R} \oplus S^- \otimes \bar{\mathcal{R}}$ on $\mathcal{H}_{Q_m}(\mathbb{R}^3, p)$ compatible with the grading by L. Recall that we introduced a map $a : \mathsf{L} \to \mathbb{C}$. Pick $\delta^+ \in S^+ \otimes \mathcal{R}$ and $\delta^- \in S^- \otimes \bar{\mathcal{R}}$ and let

$$\delta_\varphi = \delta^+ + e^{i\varphi}\delta^- \tag{2.12.2}$$

for $\varphi \in \mathbb{R}$. It is known that

$$[\delta_\varphi, \delta_\varphi^\dagger]_+ = \delta_\varphi \delta_\varphi^\dagger + \delta_\varphi^\dagger \delta_\varphi = t - \text{Re}(e^{-i\varphi} a) \tag{2.12.3}$$

where t is an Hermitean operator on $\mathcal{H}_{Q_m}(\mathbb{R}^3, p)$ called the Hamiltonian, defined by the map

$$e^{-\beta t} = Z_{Q_m}([0, \beta] \times \mathbb{R}^3, p) : \mathcal{H}_{Q_m}(\mathbb{R}^3, p) \to \mathcal{H}_{Q_m}(\mathbb{R}^3, p). \tag{2.12.4}$$

Therefore, the eigenvalue of t on $\mathcal{H}_l$ is bounded below by $|a(l)|$. Let $\varphi = \operatorname{Arg} a(l)$. Then $\delta_\varphi v = 0$ for $v \in \mathcal{H}_l(p)$ if and only if $tv = |a(l)|v$. The subspace of $\mathcal{H}_l(p)$ satisfying this condition is called the space of BPS states and we denote it by $\text{BPS}_l(p)$. $\text{BPS}_l(p)$ is a $\mathbb{Z}/\mathbb{Z}_2$-graded finite-dimensional vector space. $\text{BPS}_l(p)$ is locally constant but it can jump at real-codimension-1 walls. Its wall-crossing behavior is intensively studied. See e.g. [GMN09].

2.13 Topological twisting

Let Q be any $\mathcal{N} = 2$ supersymmetric QFT. We define a new QFT Q_{top}, which is not a supersymmetric QFT, as follows. First, recall an $\mathcal{N} = 2$ supersymmetric QFT is SU(2) R-symmetric. Given a spin 4-manifold X, we decompose the frame bundle $F_{\text{Spin}(4)}X \to X$ to $P_{\text{SU}(2)} \times_X P'_{\text{SU}(2)} \to X$, and then we feed it to Z_Q to define $Z_{Q_{\text{top}}}$:

$$Z_{Q_{\text{top}}}(X) = Z_Q(P_{\text{SU}(2)} \to X). \tag{2.13.1}$$

In other words we choose a homomorphism

$$\varphi : \text{SU}(2)_R \to \text{Spin}(4). \tag{2.13.2}$$

The supertangent to X as defined in (2.2.2) is now, due to the identification of $\text{SU}(2) \subset \text{Spin}(4)$ and the SU(2) R-symmetry, given by

$$\mathcal{S}^+ X \oplus \mathcal{S}^- X = (\mathbb{C} \oplus \Lambda^{2+} TX) \oplus TX \tag{2.13.3}$$

where $\Lambda^{2+}TX$ is the bundle of self-dual two-forms. Therefore there is a trivial subbundle of the supertangent bundle, which then has a covariantly constant section. This gives a superisometry δ.

This can be identified with a fixed element in $S^+ \otimes \mathcal{R}$ acting on $\mathcal{V}_Q$, and we define the space of point operators of Q_{top} by

$$\mathcal{V}_{Q_{\mathrm{top}}} = H(\mathcal{V}_Q, \delta). \tag{2.13.4}$$

Using this superisometry δ, we can show the following properties of Q_{top}:

- $Z_{Q_{\mathrm{top}}}(X)$ depends only on smooth structure on X. To show this, consider changing the metric of X from g to $g + \epsilon\delta g$. Then, from the analysis in Sec. 1.12, we have

$$\frac{\partial}{\partial \epsilon} Z_{Q_{\mathrm{top}}}(X) = \int_X (\langle T(p) + \varphi(\nabla J(p))\rangle_X, \delta g(p)) d\,\mathrm{vol}_X \tag{2.13.5}$$

  where T is the energy-momentum tensor, J is the SU(2) R-current, and φ is the map $\mathbb{R}^4 \times \mathfrak{su}(2)_R \to \mathrm{Sym}^2\,\mathbb{R}^4$ induced from (2.13.2). Now it turns out the point operator $T + \varphi(\nabla J)$ is δ-exact, and therefore its one-point function on the right hand side of (2.13.5) vanishes. Therefore $Z_{Q_{\mathrm{top}}}(X)$ does not depend on the continuous deformation of the metric.

- For the quotient $Q /\!\!/\!\!/ G\,\big|_\tau$ we have

$$Z_{Q /\!\!/\!\!/ G\big|_\tau,\mathrm{top}} = \sum_n q^n \int_{\mathcal{M}_n} Z_{Q_{\mathrm{top}}}(P_G \to X) \tag{2.13.6}$$

  where $q = e^{2\pi\sqrt{-1}\tau}$ and $\mathcal{M}_n$ is the moduli space of ASD G-connections on X with $c_2 = n$. Morally speaking, this happens as there is an action of δ on the integration domain of the path integral which is a supermanifold based on the moduli space $\mathcal{M}$ of G-bundles with connections. Then the integral localizes to the integral over the δ-fixed points, which happen to be given by the ASD G-connections. As a corollary, we see $(\mathrm{triv} /\!\!/\!\!/ \mathrm{SU}(2))^{\mathrm{top}}$ is the Donaldson invariant.

2.14 Topological twisting and the mirror symmetry

Before continuing, let us have a look at a classic application of topological twisting. We start from a 2d supersymmetric theory. A 2d supersymmetry algebra is of the form

$$(\mathfrak{so}(\mathcal{N}^+) \times \mathfrak{so}(\mathcal{N}^-) \times \mathfrak{so}(2)) \ltimes (\mathbb{R}^2 \oplus S^+ \otimes \mathcal{R}^+ \oplus S^- \otimes \mathcal{R}^-) \tag{2.14.1}$$

where $\mathcal{R}^\pm \simeq \mathbb{C}^{\mathcal{N}^\pm}$. The R-symmetry group acting on $\mathcal{R}^\pm$ is only $\mathfrak{so}(\mathcal{N}^\pm)$, not $\mathfrak{u}(\mathcal{N}^\pm)$, in order for the action to be compatible with the CPT conjugation action on the superalgebra, which as introduced in Sec. 1.14 is an action of Pin(2) where the element disconnected from the identity acts by a conjugate-linear map.

Here we only consider the case when $(\mathcal{N}^+, \mathcal{N}^-) = (2, 2)$. In this case we write $\mathfrak{so}(\mathcal{N}^+) = \mathfrak{u}(1)_+$, $\mathfrak{so}(\mathcal{N}^-) = \mathfrak{u}(1)_-$:

$$(\mathfrak{u}(1)_+ \times \mathfrak{u}(1)_- \times \mathfrak{so}(2)) \ltimes (\mathbb{R}^2 \oplus S^+ \otimes \mathcal{R}^+ \oplus S^- \otimes \mathcal{R}^-) \tag{2.14.2}$$

We consider two subalgebras of $\mathfrak{u}(1)_+ \oplus \mathfrak{u}(1)_-$ given by

$$\mathfrak{u}(1)_A = \{x \oplus x \in \mathfrak{u}(1)_+ \oplus \mathfrak{u}(1)_-\}, \qquad \mathfrak{u}(1)_B = \{x \oplus (-x) \in \mathfrak{u}(1)_+ \oplus \mathfrak{u}(1)_-\}. \tag{2.14.3}$$

We then consider subalgebras

$$(\mathfrak{u}(1)_A \times \mathfrak{so}(2)) \ltimes (\mathbb{R}^2 \oplus S^+ \otimes \mathcal{R}^+ \oplus S^- \otimes \mathcal{R}^-), \tag{2.14.4}$$
$$(\mathfrak{u}(1)_B \times \mathfrak{so}(2)) \ltimes (\mathbb{R}^2 \oplus S^+ \otimes \mathcal{R}^+ \oplus S^- \otimes \mathcal{R}^-). \tag{2.14.5}$$

There is an outer automorphism of the supersymmetry algebra (2.14.2) which exchanges (2.14.4) and (2.14.5).[7]

Various 2d supersymmetric QFT is known. Some have the symmetry of (2.14.4), some have the symmetry of (2.14.5), and some have both actions, leading to the symmetry of the full algebra (2.14.2).

For example, there are supersymmetric versions of non-linear sigma models introduced in Sec. 1.21, which can be defined for a Kähler manifold M. Let us denote it by $\Sigma(M)$. This is always $\mathfrak{u}(1)_A$ symmetric, i.e. has the symmetry of (2.14.4). When M is Calabi-Yau, it is $\mathfrak{u}(1)_A \times \mathfrak{u}(1)_B$ symmetric.

As another set of examples, there are so-called Landau-Ginzburg models given a Calabi-Yau manifold M and a holomorphic function f on it; here M can in general be non-compact as long as the locus $df = 0$ is compact. Denote it by $LG(M, f)$. This is always $\mathfrak{u}(1)_B$ symmetric. When f is quasi-homogeneous, it is $\mathfrak{u}(1)_A \times \mathfrak{u}(1)_B$ symmetric. When $f = 0$, $LG(M, 0) = \Sigma(M)$.

Now, a $\mathfrak{u}(1)_A$ symmetric 2d supersymmetric theory Q can be topologically twisted by using the homomorphism

$$\varphi_A : \mathfrak{u}(1)_A \simeq \mathfrak{so}(2) \tag{2.14.6}$$

as in (2.13.2), where $\mathfrak{so}(2)$ is the Lie algebra of the structure group of the two-dimensional spacetime. Let us denote it $Q_{\mathrm{top},A}$. This is a 2d topological theory. For a Kähler manifold M, $\Sigma(M)_{\mathrm{top},A}$ is the topological A-model on M, denoted by $A(M)$ in Sec. 1.7.

Similarly, a $\mathfrak{u}(1)_B$ symmetric 2d supersymmetric theory Q can be topologically twisted by using the homomorphism

$$\varphi_B : \mathfrak{u}(1)_B \simeq \mathfrak{so}(2) \tag{2.14.7}$$

as in (2.13.2). Let us denote it $Q_{\mathrm{top},B}$. This is again a 2d topological theory. For a Calabi-Yau manifold M and the superpotential f, $LG(M, f)_{\mathrm{top},B}$ is the topological B-model on M. When M is compact and $f = 0$, this is the B-model on M, denoted by $B(M)$ in Sec. 1.7. Note that when M is a compact Calabi-Yau manifold, we can define both the A-twist and the B-twist, denoted by $A(M)$ and $B(M)$.

Mirror symmetry is a manifestation of the outer automorphism exchanging (2.14.4) and (2.14.5). This gives rise to an equivalence between the category of $\mathfrak{u}(1)_A$ symmetric 2d supersymmetric QFTs and the category of $\mathfrak{u}(1)_B$ symmetric 2d supersymmetric QFTs.

[7] In physics literature $\mathfrak{u}(1)_A$ here is denoted $\mathfrak{u}(1)_V$, and $\mathfrak{u}(1)_B$ here is denoted $\mathfrak{u}(1)_A$, where V is for vector and A is for axial vector.

Assuming that a large part of the category of the $\mathfrak{u}(1)_A$ symmetric theories is generated in some sense by $\Sigma(M)$, and similarly that a large part of the category of $\mathfrak{u}(1)_B$ symmetric theories is generated by $LG(M, f)$, there should be a correspondence

$$\Sigma(M) \simeq LG(W, f) \tag{2.14.8}$$

where M is determined by (W, f) and vice versa. In this relation, when M is Calabi-Yau, the left hand side is $\mathfrak{u}(1)_A \times \mathfrak{u}(1)_B$ symmetric. The right hand side should also be symmetric under $\mathfrak{u}(1)_A \times \mathfrak{u}(1)_B$. Therefore $f = 0$, and we have

$$\Sigma(M) \simeq LG(W, 0) = \Sigma(W). \tag{2.14.9}$$

Note that this equivalence involves the outer automorphism.

By taking the A-twist of the left hand side and the B-twist of the right hand side, this gives rise to the equivalence $A(M) \simeq B(W)$. Further taking the category of branes, we see that there is the equivalence of the Fukaya category $\mathrm{Fuk}(M)$ of M and the derived category of the coherent sheaves $D(W)$ on W. Similarly, by taking the B-twist of the left hand side and the A-twist of the right hand side, we obtain $B(M) \simeq A(W)$.

3 6d theory and 4d theories of class S

In this section we study four-dimensional $\mathcal{N} = 2$ supersymmetric QFTs arising from the so-called dimensional reduction of a class of six-dimensional $\mathcal{N} = (2, 0)$ supersymmetric QFTs.

3.1 Dimensional reduction

In general, given a d-dimensional QFT Q and a $d' < d$ dimensional Riemannian manifold K, we can define a $d - d'$ dimensional QFT $Q[K]$ via the relation

$$Z_{Q[K]}(X) = Z_Q(K \times X). \tag{3.1.1}$$

The definition of the space of operators $\mathcal{V}_{Q[K]}$ requires more care. This operation is called the dimensional reduction. The resulting theory $Q[K]$ depends on the Riemannian metric on K, and is too detailed. We want an operation which depends only on rougher structures on K so that it is more tractable. This can often be done if the original d-dimensional QFT is supersymmetric.

For definiteness, we start from a six-dimensional supersymmetric theory. A six-dimensional supersymmetry algebra is of the form

$$(\mathfrak{sp}(\mathcal{N}^+) \times \mathfrak{sp}(\mathcal{N}^-) \times \mathfrak{so}(6)) \ltimes (\mathbb{R}^6 \oplus S^+ \otimes \mathcal{R}^+ \oplus S^- \otimes \mathcal{R}^-) \tag{3.1.2}$$

where $\mathcal{R}^\pm \simeq \mathbb{H}^{\mathcal{N}^\pm}$. The R-symmetry group acting on $\mathcal{R}^\pm$ is only $\mathfrak{sp}(\mathcal{N}^\pm)$, not $\mathfrak{u}(2\mathcal{N}^\pm)$, in order for the action to be compatible with the CPT conjugation action on the superalgebra, which as introduced in Sec. 1.14 is an action of Pin(6) where the element disconnected from the identity acts by a conjugate-linear map.

This is called the $(\mathcal{N}^+, \mathcal{N}^-)$-extended six-dimensional supersymmetry, and the $\mathfrak{sp}(\mathcal{N}^+) \times \mathfrak{sp}(\mathcal{N}^-)$ part is the R-symmetry. An $(\mathcal{N}^+, \mathcal{N}^-)$-extended theory and an $(\mathcal{N}^-, \mathcal{N}^+)$-extended theory are essentially the same by a change of convention. We only deal with the case $(\mathcal{N}^+, \mathcal{N}^-) = (2, 0)$, for which the supersymmetry algebra is

$$(\mathfrak{sp}(2) \times \mathfrak{so}(6)) \ltimes (\mathbb{R}^6 \oplus S^+ \otimes \mathcal{R}^+). \tag{3.1.3}$$

Note that $\mathfrak{sp}(2) \simeq \mathfrak{so}(5)$ in our convention. This is usually called the six-dimensional $\mathcal{N} = (2, 0)$ supersymmetric theory.

Given a six-dimensional spin manifold Y with a Riemannian metric, together with an Sp(2) R-symmetry bundle with connection, we have the frame bundle $F_{\mathrm{Sp}(2)\times\mathrm{Spin}(6)} Y \to Y$. Then the algebra bundle (3.1.3) gives rise to the supertangent space

$$TY \oplus \mathcal{S}^+ Y = \underline{\mathbb{R}^6 \oplus S^+ \otimes \mathcal{R}^+}. \tag{3.1.4}$$

Now, given a d'-dimensional manifold K, we pick a homomorphism $\varphi : \mathfrak{so}(d') \to \mathfrak{sp}(2)$. Then we have an $\mathfrak{sp}(2)$ bundle $\varphi(F_{\mathfrak{so}(d')} K \times X)$ over $K \times X$ constructed from the frame bundle of TK, which we use to define $Q[K_\varphi]$

$$Z_{Q[K_\varphi]}(X) := Z_Q(\varphi(F_{\mathfrak{so}(d')} K) \times X). \tag{3.1.5}$$

When $\varphi(\mathfrak{so}(d'))$ has a nontrivial stabilizer G in $\mathfrak{sp}(2)$ R-symmetry group, $Q[K_\varphi]$ becomes a $d - d'$ dimensional supersymmetric theory with G R-symmetry. This procedure is called the partial twisting.

In this review we only consider the case when $d' = 2$ and the homomorphism φ is given by the diagonal embedding

$$\varphi : \mathfrak{so}(2) \subset \mathfrak{so}(2) \times \mathfrak{so}(3) \subset \mathfrak{so}(5) \simeq \mathfrak{sp}(2). \tag{3.1.6}$$

Its stabilizer is $\mathfrak{so}(2) \times \mathfrak{so}(3) \simeq \mathfrak{u}(1) \times \mathfrak{su}(2)$. Then the theory $Q[K_\varphi]$ is a four-dimensional $\mathcal{N} = 2$ supersymmetric theory with SU(2) × U(1) R-symmetry. Indeed, one can check that the supertangent bundle (3.1.4) over $K \times X$ contains a subbundle pulled back from the supertangent bundle (2.2.2) over X of an $\mathcal{N} = 2$ theory with SU(2) × U(1) R-symmetry. The properties of $Q[K_\varphi]$ we study only depends on the complex structure and the total area of K. This can be shown as in the derivation of the independence of Q_{top} from the metric given in Sec 2.13.

3.2 6d $\mathcal{N} = (2, 0)$ theory

Now we need 6d $\mathcal{N} = (2, 0)$ supersymmetric theory to be used in the dimensional reduction just introduced above. They are known to have an ADE classification, namely, for each

Dynkin diagram $\Gamma = A_n, D_n, E_n$, we have a 6d $\mathcal{N} = (2,0)$ supersymmetric theory S_Γ.[8] This is $\mathrm{Out}(\Gamma)$-symmetric, where $\mathrm{Out}(\Gamma)$ is the graph automorphism of Γ. The theory S_Γ itself is constructed by a dimensional reduction starting from 10d quantum gravity system called string theories. A description of this theory for mathematicians can be found e.g. in [Wit09b, Wit09a, Moo12].

A d-dimensional gauge theory of the form $Q/\!\!\!/G$ involves a path integral over the moduli space of the G-bundles with connections on a d-dimensional manifold X. A d-dimensional quantum gravity theory should involve a path integral over the moduli space of the Riemannian manifolds of dimension d. But physicists learned that it is almost impossible to start from a QFT Q and form $Q/\!\!\!/(\text{diffeo. on metric})$. A quantum gravity theory is constructed in a rather indirect way, and only a few of them are known to exist. Also, as we need to perform an integral over the Riemannian manifolds, we do not expect that a quantum gravity theory gives a number given a compact d-dimensional Riemannian manifold. Rather, given a $d-1$ dimensional Riemannian manifold Y, we expect that the path integral over the moduli space of d-dimensional Riemannian manifolds with metric whose boundary is Y would give rise to a number.

A well-established supersymmetric quantum gravity theory is the Type IIB string theory St_{IIB} in 10 dimensions. This means that it can produce a number given a 9-dimensional Riemannian manifold. We can then perform a dimensional reduction to define

$$S_\Gamma = St_{\mathrm{IIB}}[S^3/\Gamma] \tag{3.2.1}$$

where Γ is identified with the corresponding finite subgroup of $\mathrm{SU}(2)$. This is a 6d $\mathcal{N} = (2,0)$ supersymmetric QFT.

Its space of point operators is not completely known, but it at least satisfies

$$\mathcal{V}_{S_\Gamma} \supset \mathbb{C}[\mathfrak{h} \otimes_{\mathbb{R}} (\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{R})]^W. \tag{3.2.2}$$

Here, $\mathfrak{h}$ is the Cartan subalgebra of the Lie algebra of type Γ, $\mathrm{Spin}(5)$ acts naturally on $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{R} \simeq \mathbb{R}^5$, and $\mathbb{C}[\mathfrak{h} \otimes \mathbb{R}^3]^W$ comes from the deformation parameters of a hyperkähler asymptotically-flat metric filling S^3/Γ.

S_Γ is a $\mathrm{Spin}(5)$-symmetric QFT. Then its anomaly polynomial $A(S_\Gamma)$ is a degree-8 characteristic class in $T\mathcal{X}$ and $\mathcal{P}_{\mathrm{Spin}(5)}$, known to be of the form

$$A(S_\Gamma) = (\operatorname{rank} G) I_8 + \dim G h^\vee(G) \frac{p_2(\mathcal{P}_{\mathrm{Spin}(5)})}{24} \tag{3.2.3}$$

where G is a Lie group of type Γ and

$$I_8 = \frac{1}{48}\left[p_2(\mathcal{P}_{\mathrm{Spin}(5)}) - p_2(T\mathcal{X}) + \frac{1}{4}(p_1(\mathcal{P}_{\mathrm{Spin}(5)}) - p_1(T\mathcal{X}))^2\right]. \tag{3.2.4}$$

[8] This is a generalized QFT in the sense of Sec. 1.5, but in this review this important subtlety is grossed over. More on this point, see e.g. [FT12].

3.3 Dimensional reduction on S^1

Before studying S_Γ compactified on a Riemann surface, let us study $S_\Gamma^{6d}[S^1_\ell]$ where the subscript ℓ denotes the circumference of the circle. This turns out to be a 5d gauge theory as an effective theory. Let G be the simply-laced group of type Γ. Then

$$S_\Gamma[S^1_\ell] = B_{d=5}(\mathfrak{g}_\mathbb{R}\otimes\mathbb{R}^5)\times F_{d=5}(\mathfrak{g}_\mathbb{C}\otimes\mathbb{H}^2)/\!\!\!-G\,\big|_{\ell\in\mathbb{R}_{>0},\ \text{properly deformed}} \tag{3.3.1}$$

This is the $\mathcal{N}=2$ supersymmetric 5d gauge theory with $\mathrm{Spin}(5)_R\simeq\mathrm{Sp}(2)_R$ R-symmetry, which acts on $\mathbb{R}^5$ and $\mathbb{H}^2$ in a natural way. As we have the Spin(5) action on the tangent space of the five-manifolds and an additional Spin(5)-bundle associated to the Spin(5) R-symmetry, we distinguish the latter by putting R in the subscript in this section.

This effective 5d theory has a path integral expression:

$$Z_{S_\Gamma[S^1_\ell]}(X) = \int_\mathcal{M} e^{-I}d\,\mathrm{vol}_\mathcal{M} \tag{3.3.2}$$

where

$$I = \int_X \frac{1}{\ell}\left[\langle\bar\phi,\triangle\phi\rangle + \langle F,\star F\rangle + \langle\bar\psi \not{D}\psi\rangle + \cdots\right] d\,\mathrm{vol}_X, \tag{3.3.3}$$

$\mathcal{M}$ is the moduli space of principal G-bundles $P\to X$ with connection, and sections ϕ of $(\mathfrak{g}_\mathbb{R}\otimes\mathbb{R}^5)\times_G P\to X$, and sections ψ of $(\mathfrak{g}_\mathbb{C}\otimes\mathcal{S})\times_{G\times\mathrm{Spin}(5)} F_{G\times\mathrm{Spin}(5)}X\to X$ where $\mathcal{S}$ is the spinor representation of Spin(5).

When $X=S^1_{\ell'}\times Y$, and take the limit $\ell'\ell\to 0$ keeping ℓ'/ℓ fixed, we have

$$Z_{S_\Gamma[S^1_\ell]}(X) \to \int_\mathcal{M} e^{-I}dvol_\mathcal{M} \tag{3.3.4}$$

where $\mathcal{M}$ is now the moduli space of P, ϕ, ψ over Y, and

$$I = \int_Y \frac{\ell'}{\ell}\left[\langle\bar\phi,\triangle\phi\rangle + \langle F,\star F\rangle + \langle\bar\psi \not{D}\psi\rangle + \cdots\right] d\,\mathrm{vol}_Y\,. \tag{3.3.5}$$

The holonomy of G-connection around $S^1_{\ell'}$ gives another $\mathfrak{g}_\mathbb{R}$-valued function on Y, and so ϕ is now a section of $(\mathfrak{g}_\mathbb{R}\otimes\mathbb{R}^6)\times_G P\to Y$. In total we have

$$\begin{aligned}&S_\Gamma[S^1_\ell\times S^1_{\ell'}]\to\\ &B(\mathfrak{g}_\mathbb{R}\otimes\mathbb{R}^6)\times F(\mathfrak{g}_\mathbb{C}\otimes\mathbb{C}^4)/\!\!\!-G\,\big|_{\tau=i\ell'/\ell,\text{properly deformed}} = \mathrm{Hyp}(\mathfrak{g}_\mathbb{C}\oplus\mathfrak{g}_\mathbb{C})/\!\!/\!\!/G\,\big|_{\tau=i\ell'/\ell}\,.\end{aligned} \tag{3.3.6}$$

This is the four-dimensional $\mathcal{N}=4$ supersymmetric Yang-Mills with simply-laced gauge group G, introduced in Sec. 2.7.2. From the 6d construction we have the symmetry $\ell\leftrightarrow\ell'$, which is a nontrivial symmetry $\tau\leftrightarrow -1/\tau$ from the 4d point of view.

3.4 Properties of nilpotent orbits

Before continuing it is necessary to gather here the properties of nilpotent orbits and other conjugacy classes of $\mathfrak{g}_{\mathbb{C}}$. For more details, refer to [CM93]. In this section all Lie algebra is over $\mathbb{C}$ and drop the subscript $\mathbb{C}$. Given an element $x \in \mathfrak{g}$, it can be uniquely decomposed to $x = e + m$ where e is nilpotent and m is semisimple and is in $\mathfrak{g}^e$. A subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ of the form $\mathfrak{l} = \mathfrak{g}^m$ for a semisimple m is called a Levi subalgebra.

We denote the $\mathfrak{g}$-orbit containing x by O_x. This has a natural holomorphic symplectic structure on it. There is only a finite number of nilpotent orbits. Given two nilpotent orbits O_e and $O_{e'}$, we define a partial ordering $O_e \le O_{e'}$ if and only if $O_e \subset \bar{O}_{e'}$. There is a maximal object in this partial order called the principal orbit. The minimal object in the partial order is of course the zero orbit, and the next-to-minimal object is called the minimal nilpotent orbit.

Below, we often use the generators of $\mathfrak{su}(2)$ given by (e, h, f) with the commutation relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h. \tag{3.4.1}$$

A triple (e, h, f) in $\mathfrak{g}$ satisfying the relations above is called an SL(2) triple. The theorem of Jacobson and Morozov says that any nilpotent element e in $\mathfrak{g}$ can be completed to an SL(2) triple unique up to conjugation, and that classifying an SL(2) subalgebra in $\mathfrak{g}$ up to conjugation is equivalent to classifying e up to conjugation. For the principal nilpotent element $e_{\text{principal}}$, $h = 2\rho$ where ρ is the Weyl vector.

Given e, the subspace

$$e + S_e = \{e + x \mid [f, x] = 0,\ x \in \mathfrak{g}\} \tag{3.4.2}$$

is called the Slodowy slice at e.

A nilpotent element of $\mathfrak{g} = A_{N-1}$ is classified by its Jordan normal form, i.e. by a partition of N which we denote by $[n_1, n_2, \ldots]$ where $N = \sum n_i$ and $n_1 \le n_2 \le \cdots$. Nilpotent elements in classical algebras are similarly labeled by partitions with certain constraints. In general, a nilpotent orbit is specified by picking a nilpotent element e in it and specifying the smallest Levi subalgebra which contains e. This Levi subalgebra does not always uniquely specify a nilpotent orbit, in which case we add a discrete label. This pair of a Levi subalgebra and a discrete label if needed is the Bala-Carter label of a nilpotent orbit. The weighted Dynkin diagram is just the element h as specified as the set of $\alpha_i(h)$, where α_i is the i-th simple root and h is conjugated to the positive Weyl chamber.

Given a Levi subalgebra $\mathfrak{l}$ and an element $x \in \mathfrak{l}$, it is known that $x + e$ where e is a generic nilpotent element outside of $\mathfrak{l}$ is in a fixed conjugacy class. This conjugacy class is denoted by $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}\, x$ and called the induced orbit. There is an order-reversing map d_{LS} on the set of nilpotent orbits of $\mathfrak{g}$ called Lusztig-Spaltenstein map. This satisfies

$$d_{LS}^2 = \mathrm{id} \tag{3.4.3}$$

when $\mathfrak{g}$ is type A but it only satisfies

$$d_{LS}^3 = d_{LS} \tag{3.4.4}$$

if not. When $\mathfrak{g}$ is type A, d_{LS} is given by the transpose of the partition specifying the nilpotent orbit. One important property of d_{LS} is its compatibility with the induction,

$$\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}\, d_{LS}^{\mathfrak{l}}(O_e) = d_{LS}^{\mathfrak{g}}(\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}\, O_e). \tag{3.4.5}$$

A nilpotent orbit which is in the image of d_{LS} is called special. Given a special orbit O_e, the set of nilpotent orbits $O_{e'}$ such that $d_{LS}^2(O_{e'}) = O_e$ is the special piece of O_e. Within the special piece of O_e, O_e itself is the maximal element. The partial order among the special piece is encoded in a subgroup $\mathcal{C}(O'_e) \subset \bar{A}(O_e)$ [Som01, AS02, Ach03]. Here, $\bar{A}(O_e)$ is a reflection group defined as a certain quotient of the component group $A(O_e) = G^e/(G^e)^\circ$ introduced by Lusztig. Then when two orbits in the special piece of O_e then

$$O_{e'} \le O_{e''} \leftrightarrow \mathcal{C}(O_{e'}) \supset \mathcal{C}(O_{e''}). \tag{3.4.6}$$

In particular $\mathcal{C}(O_e) = \{\mathrm{id}\}$.

3.5 4d operator of 6d theory

From now on we fix a simply-laced Dynkin diagram Γ and a corresponding group G. We know that the theory S_Γ has various 4d operators, and therefore we have

$$Z_{S_\Gamma}(X^6 \supset D_1^4 \sqcup D_2^4 \sqcup \cdots) \tag{3.5.1}$$

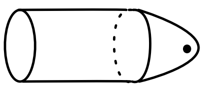
where each four-dimensional submanifold D_i^4 carries a certain label. In the following we sometimes indicate the dimension of a manifold by putting the dimension as a superscript as a way of clarification.

So far two classes of labels are known:

- Tame or regular operators. The label is a pair (e, m) up to conjugacy, where e is a nilpotent element of $\mathfrak{g}_\mathbb{C}$ and m a semisimple element of $\mathfrak{g}_\mathbb{C}^e$.
- Wild or irregular operators. The author does not quite know what are the available labels.

In this review we mainly talk about the regular operators.

To study a 4d operator, we consider the following setup:

$X^6 = Y^4 \times$  $\supset Y^4 \times \bullet = D^4$ (3.5.2)

We can dimensionally reduce around S^1 of the cigar. Then we can study $S_\Gamma[S^1]$ on

$$X^5 = Y^4 \times \relbar\joinrel\relbar\joinrel\relbar\joinrel\dashv \;\supset Y^4 \times \bullet = D^4 \tag{3.5.3}$$

using its description as a gauge theory we discussed in Sec. 3.3. Now we have a four-dimensional operator at a boundary of five-dimensional spacetime. We have a boson $B(\mathfrak{g}_\mathbb{R} \otimes \mathbb{R}^5)$ on X^5 and a G-bundle $P \to X$ with the connection. We decompose

$$\mathfrak{g}_\mathbb{R} \otimes \mathbb{R}^5 = \mathfrak{g}_\mathbb{C} \oplus \mathfrak{g}_\mathbb{R} \otimes \mathbb{R}^3, \tag{3.5.4}$$

and denote a section of $\mathfrak{g}_{\mathbb{C}} \times_G P$ by Φ and a section of $(\mathfrak{g}_{\mathbb{R}} \otimes \mathbb{R}^3) \times_G P$ by (ϕ_1, ϕ_2, ϕ_3). The $\mathfrak{so}(2) \simeq \mathfrak{u}(1)$ R-symmetry acts on Φ and the $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$ R-symmetry acts on (ϕ_1, ϕ_2, ϕ_3).

Let us introduce a coordinate s perpendicular to the boundary so that the boundary is at $s = 0$. A regular four-dimensional boundary operator is defined by the requirement that the fields $\phi_{1,2,3}$ to approach a singular solution of the Nahm equation

$$\frac{d}{ds}\phi_1 = [\phi_2, \phi_3], \quad \frac{d}{ds}\phi_2 = [\phi_3, \phi_1], \quad \frac{d}{ds}\phi_3 = [\phi_1, \phi_2] \tag{3.5.5}$$

given by

$$\phi_i = \rho(\sigma_i)/s \tag{3.5.6}$$

where $\sigma_{1,2,3}$ are the standard generators of $\mathfrak{su}(2)$ and

$$\rho : \mathfrak{su}(2) \to \mathfrak{g} \tag{3.5.7}$$

is a homomorphism. We then require

$$\lim_{s \to 0} \Phi = m \in \mathfrak{g}_{\mathbb{C}}^{\rho}. \tag{3.5.8}$$

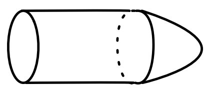
By the Jacobson-Morozov theorem, we can use the nilpotent element $\rho(e)$ instead of ρ to label a regular 4d operator. We often just write e instead of $\rho(e)$.

Note that with nonzero m we do not have U(1) R-symmetry any more, as nonzero m is not fixed by U(1) action. In contrast, even with nonzero ρ, the SU(2) R-symmetry action can be absorbed by a gauge transformation of the G-bundle P thanks to the form (3.5.6). Also note that when $m = 0$, one can introduce G^{ρ}-bundle with connection on the boundary D^4 without ruining the boundary condition above. This means that the 4d operator $(\rho, 0)$ is a G^{ρ}-symmetric 4d operator. We note that $(G^{\rho})_{\mathbb{C}}$ is the reductive part of $G_{\mathbb{C}}^{e}$.

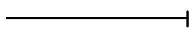
Two extreme types of regular 4d operators are:

- $e = 0$, i.e. $\rho : \mathfrak{sl}_2 \to \mathfrak{g}_{\mathbb{C}}$ is the zero map. Then $G^e = G$. So, if we insert a 4d operator with the label $(e = 0, m = 0)$, there is an additional G-symmetry. Under an S^1 reduction, this corresponds to the Neumann boundary condition for $\phi_{1,2,3}$ and the Dirichlet boundary condition for Φ at $s = 0$.

- $e = e_{\text{prin}}$, a principal nilpotent element, and $\rho_{\text{prin}} : \mathfrak{sl}_2 \to \mathfrak{g}_{\mathbb{C}}$ is a principal embedding. $G^{\text{prin}} = \{1\}$. This 4d operator corresponds to the absence of a 4d operator in 6d:

  $X^6 = Y^4 \times$  (3.5.9)

  and its S^1 reduction is

  $X^5 = Y^4 \times$  (3.5.10)

  with the boundary condition $\phi_i \to \rho_{\text{prin}}(\sigma_i)/s$. We have Neumann boundary condition for Φ.

It might be slightly counter-intuitive that nothing in 6d corresponds to a principal embedding, and that a G-symmetry in 6d corresponds to a zero embedding. The point is that zero does not always mean nothing.

A 4d operator with a label $(e,0)$ has its own anomaly polynomial of degree 6, in terms of characteristic polynomial of $\mathcal{P}_{\mathrm{Spin}(3)}$, $\mathcal{P}_{\mathrm{Spin}(2)}$, $\mathcal{P}_{G^\rho}$, TD and ND where ND is the normal bundle of D within X. The coefficients are known to be given by formulas involving h.

Below, the symbol ρ almost always refers to the Weyl vector. Instead of the representation (3.5.7) we use e to label the nilpotent orbit.

3.6 4d theory of class S

Given a Riemann surface C with points $x_1,\ldots,x_k$ and labels $(e_1,m_1),\ldots,(e_k,m_k)$, let us define a 4d QFT $Q = S_\Gamma[C;x_1,(e_1,m_1),\ldots,x_k,(e_k,m_k)]$ via

$$Z_Q(Y^4) = Z_{S_\Gamma}(Y^4\times C \supset \sqcup_i Y^4\times\{x_i\}) \tag{3.6.1}$$

with the given labels. We implicitly perform the topological twisting by φ given in (3.1.6), but for simplicity we do not explicitly denote them in the expressions. A 4d theory of class S is an $\mathcal{N}=2$ supersymmetric QFT Q obtained this way. When $m_i=0$ for all i, this is a $\prod_i G^{e_i}$-symmetric $\mathcal{N}=2$ supersymmetric QFT with U(1) R-symmetry. Apart from the labels, the theory depends only on the complex structure of the Riemann surface C with punctures and the total area.

The anomaly polynomial of $Q = S_\Gamma[C;x_i,(e_i,0)]$ is obtained from the anomaly polynomial of S_Γ integrating over C summed to the contributions of 4d operators. We have

$$n_v(Q) = \sum_i n_v(e_i) + (g-1)(\tfrac{4}{3}h^\vee(G)\dim G + \operatorname{rank} G), \tag{3.6.2}$$

$$n_h(Q) = \sum_i n_h(e_i) + (g-1)(\tfrac{4}{3}h^\vee(G)\dim G), \tag{3.6.3}$$

where

$$n_h(e) = 8\rho\cdot(\rho-\frac{h}{2}) + \frac{1}{2}\dim\mathfrak{g}_{1/2},\quad n_v(e) = 8\rho\cdot(\rho-\frac{h}{2}) + \frac{1}{2}(\operatorname{rank}G - \dim\mathfrak{g}_0). \tag{3.6.4}$$

Here ρ is the Weyl vector and h is an element in $\mathfrak{h}$ so that (e,h,f) is an SL(2) triple. The terms proportional to $g-1$ in (3.6.2) and (3.6.3) can be easily obtained by integrating $A(S_\Gamma)$, (3.2.3), over C, taking into account the homomorphism (3.1.6), and reading off n_v and n_h from the resulting anomaly polynomial by (2.4.5).

When e is principal, $h=2\rho$, and therefore $n_v(e)=n_h(e)=0$. This is consistent with the fact that a 4d operator with the label $e=\rho_{\mathrm{prin}}$ corresponds to the absence of any puncture. Therefore it should not add anything to $n_v(Q)$ or $n_h(Q)$. When $e=0$, we instead find

$$n_v(e=0) = 8\rho\cdot\rho + \frac{1}{2}(\operatorname{rank}G - \dim G),\quad n_h(e=0) = 8\rho\cdot\rho \tag{3.6.5}$$

where we used the relation $\rho \cdot \rho = h^\vee(G) \dim G/12$.

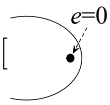
As for the flavor symmetry, $k_F(Q)$ for a simple component $F \subset G^{e_i}$ associated to the puncture at x_i is given by $k_F(Q) = k_F(e)$ where

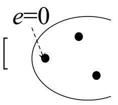
$$k_F(e) = 2 \sum_j c_2(R_j), \qquad \mathfrak{g}_{\mathbb{C}} = \oplus_d V_d \otimes R_d \tag{3.6.6}$$

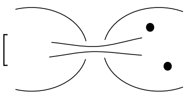
where the direct sum decomposition on the right hand side is with respect to $\rho(\mathrm{SU}(2)) \times F \subset G$ such that V_d is the irreducible representation of SU(2) of dimension d and R_d is a representation of F. As always we normalize the quadratic Casimir c_2 by $c_2(\mathfrak{f}_{\mathbb{C}}) = h^\vee(F)$. For example, $F = G$ when $e = 0$, and $k_G(e) = 2h^\vee(G)$.

3.7 Gaiotto construction

The most important observation by Gaiotto [Gai09] is pictorially given by

$S_\Gamma[$ 

$] \times S_\Gamma[$ 

$] /\!\!/\!\!/ G_{\text{diag}} \big|_\tau = S_\Gamma[$  $]$ (3.7.1)

where on the right hand side two Riemann surfaces are connected via the identification of the local coordinates z, z' around the punctures. The area of the surface on the right hand side is the sum of the area of the two surfaces on the left hand side. This procedure is only possible when both two punctures have the label $(e = 0, m = 0)$.

Let us describe the operation more carefully. Let us take two class S theories

$$Q_L = S_\Gamma[C_L; x_0, (e = 0, m = 0), x_i, (e_i, m_i)], \tag{3.7.2}$$
$$Q_R = S_\Gamma[C_R; x'_0, (e = 0, m = 0), x'_i, (e'_i, m'_i)]. \tag{3.7.3}$$

Both Q_L and Q_R is G-symmetric, associated to the puncture x_0 and x'_0 respectively. Then we can form a family

$$Q \big|_\tau = (Q_L \times Q_R) /\!\!/\!\!/ G_{\text{diag}} \big|_\tau . \tag{3.7.4}$$

When all m_i and m'_i are zero, both Q_L and Q_R are U(1) R-symmetric. As $k_{G_{\text{diag}}}(Q_L \times Q_R) = k_G(Q_L) + k_G(Q_R) = 4h^\vee(G)$, this family is also U(1) R-symmetric. Let us introduce $q_{\text{gauge}} = e^{2\pi\sqrt{-1}\tau}$.

Let us consider a family of Riemann surfaces C_q formed from C_L and C_R by gluing them at x_0, y_0 with the identification $zz' = q_{\text{geometric}}$, where x_0 is at $z = 0$ and x_0 is at $z' = 0$. The area of C_q is the sum of the area of C_L and C_R. We take another family of class S theory

$$\tilde{Q} \big|_{q_{\text{geometric}}} = S_\Gamma[C; x_i, (e_i, m_i), x'_i, (e'_i, m'_i)]. \tag{3.7.5}$$

When all m_i and m'_i are zero, this family is U(1) R-symmetric.

Then these two families are equivalent

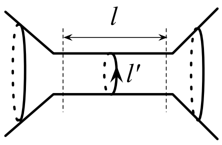
$$Q \big|_\tau \simeq \tilde{Q} \big|_{q_{\text{geometric}}} \tag{3.7.6}$$

under the identification

$$q_{\text{gauge}} = q_{\text{geometric}} + \sum_{n>1} c_n q_{\text{geometric}}^n \tag{3.7.7}$$

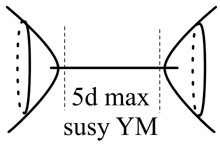
where c_n is a complicated function of the complex structure moduli of C_L and C_R, etc. There is not much use in specifying c_n precisely, because neither of q_{gauge} and q_{geometry} are canonically defined.

The reasoning behind this important relation (3.7.1) is as follows. Start from the right hand side:



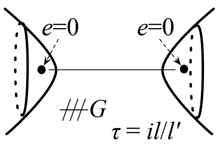
(3.7.8)

and perform the S^1 reduction around the neck:



(3.7.9)

we have a 5d super-Yang-Mills on the neck. Let us cut at two points slightly within the neck. Then the boundary condition for ϕ_i there is regular finite. Then this is further equal to



. (3.7.10)

Let us check that

$$n_{v,h}(Q_L \times Q_R /\!\!/\!\!/ G) = n_{v,h}(Q). \tag{3.7.11}$$

The left hand side can be computed using (3.6.2), (3.6.3) and (2.6.6). The right hand side can be computed using (3.6.2) and (3.6.3). Noting that the genus of C is the sum of the genus of C_L and C_R, the equality (3.7.11) boils down to the statement (3.6.5).

3.8 Donagi-Witten integrable system

For $Q = S_\Gamma[C; x_i, (e_i, m_i)]$ its Donagi-Witten integrable system $DW(Q) \to \mathcal{M}_{\text{Coulomb}}(Q)$ is given as follows [CDT12]. Consider G-Hitchin system on C, with the following singularities at x_i:

$$\Phi \simeq \alpha_i \frac{dz_i}{z_i} + \text{regular} + \cdots \tag{3.8.1}$$

where z_i is a local coordinate such that x_i is at $z_i = 0$ and

$$\alpha_i \in \text{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(m_i + d_{LS}^{\mathfrak{l}}(e_i)). \tag{3.8.2}$$

where $\mathfrak{l}$ is the smallest Levi subalgebra containing e_i. Two common cases are

- When $e_i = 0$, we just have $\alpha_i = m_i$, and
- When $m_i = 0$, we just have $\alpha_i \in d_{LS}(e_i)$.

The Coulomb branch has the dimension

$$\dim \mathcal{M}_{\text{Coulomb}}(S_\Gamma[C; x_i, (e_i, m_i)]) = (g-1)\dim G + \sum_i \frac{1}{2} \dim_{\mathbb{C}} \mathcal{O}_{\alpha_i}. \tag{3.8.3}$$

In the following we concentrate on the case $m_i = 0$. Not all of the group of gauge transformation

$$\mathcal{G} = \{f : C \to G_{\mathbb{C}}\} \tag{3.8.4}$$

preserves the boundary condition. We let

$$\mathcal{G}_0 = \{f : C \to G_{\mathbb{C}} \mid f(x_i) \in G_{\mathbb{C}}^{\alpha_i}\}. \tag{3.8.5}$$

Then we can consider the Hitchin map

$$h : \{D''\Phi = 0\}/\mathcal{G}_0 \to \bigoplus_a H^0(K_C^{\otimes d_a} + (d_a - 1)\sum x_i). \tag{3.8.6}$$

but this is *not quite* the Donagi-Witten integrable system.

First, let us describe the situation for type A_{N-1}. A label e is given by a nilpotent orbit, or equivalently a partition $[n_i]$ of N. The dual α is given by the transpose partition $[a_i]$. From this we define integers $p_d(\alpha) = d - \nu_d(\alpha)$ where

$$(\nu_1(\alpha), \nu_2(\alpha), \ldots, \nu_N(\alpha)) = (\underbrace{1, \ldots, 1}_{a_1}, \underbrace{2, \ldots, 2}_{a_2}, \ldots,). \tag{3.8.7}$$

Then we find that the image of the Hitchin map π is in fact onto

$$h : \{D''\Phi = 0\}/\mathcal{G}_0 \to \bigoplus_{d=2}^{N} H^0(K_C^{\otimes d} + \sum_i p_d(\alpha_i) x_i). \tag{3.8.8}$$

The right hand side is an affine space whose dimension is given by (3.8.3), and we identify it with $\mathcal{M}_{\text{Coulomb}}(S_\Gamma[C; x_i, (e_i, m_i)])$.

When G is not of type A and with general choice of labels e_i, the image of the Hitchin map h is not in itself affine. Instead we have the following structure. There is a natural projection

$$h : \mathcal{G}_0 \to \prod_i A(\alpha_i) \to \prod_i \bar{A}(\alpha_i) \tag{3.8.9}$$

where $A(\alpha) = G^\alpha / G^{\alpha\circ}$ is the component group of the stabilizer of α, and $\bar{A}(\alpha)$ is the Lusztig's component group. We introduced $\mathcal{C}(e) \subset \bar{A}(\alpha)$ in Sec. 3.4. Then we take

$$\mathcal{G}_0' = \pi^{-1} \prod_i \mathcal{C}(e_i). \tag{3.8.10}$$

Then we finally have

$$DW(Q) = \{D''\Phi = 0\}/\mathcal{G}'_0 \to \mathcal{M}_{\text{Coulomb}}(Q) \tag{3.8.11}$$

where $\mathcal{M}_{\text{Coulomb}}(Q)$ is affine and is of dimension (3.8.3), such that the Hitchin map

$$h : DW(Q) \to \bigoplus_a H^0(K_C^{\otimes d_a} + (d_a - 1)\sum x_i) \tag{3.8.12}$$

factors through $\mathcal{M}_{\text{Coulomb}}$ via a finite map:

$$h : DW(Q) \to \mathcal{M}_{\text{Coulomb}}(Q) \xrightarrow{\text{finite}} h(DW(Q)). \tag{3.8.13}$$

3.9 On degrees of generators

Let $Q = S_\Gamma[C; x_i, (e_i, m_i = 0)]$ be a class S theory. The number $n_v(Q)$ is given by the formula (3.6.2) as a class S theory. From the general property of $\mathcal{N} = 2$ theory it is given also by (2.4.7) applied to $\mathcal{M}_{\text{Coulomb}}(Q)$. For C with genus g without any punctures, the Donagi-Witten integrable system $DW(S_\Gamma[C])$ is the standard G-Hitchin system on C, and

$$\mathcal{M}_{\text{Coulomb}}(S_\Gamma[C]) = \bigoplus_a H^0(K_C^{\otimes d_a}). \tag{3.9.1}$$

Then it has $(2d_a - 1)(g - 1)$ generators of degree d_a, and so

$$n_v(S_\Gamma[C]) = \sum_a (2d_a - 1)^2 (g - 1) = (g - 1)(\tfrac{4}{3} h^\vee(G) \dim G + \operatorname{rank} G). \tag{3.9.2}$$

showing the agreement between (3.6.2) and (2.4.7).

In general, we conjecture there is a non canonical way to write

$$\mathcal{M}_{\text{Coulomb}}(S_\Gamma[C; x_i, e_i]) = \left[\bigoplus_a H^0(K_C^{\otimes d_a})\right] \oplus \bigoplus_i V(e_i) \tag{3.9.3}$$

where $V(e)$ is a $\mathbb{Z}$-graded affine space. Here the gradation is by the U(1) R-symmetry, and the equality is considered as elements in the Grothendieck group of the vector spaces with U(1) action. Furthermore, to be compatible with the structure (3.8.10) and (3.8.11), we demand that for a special orbit e, there is a linear action of the reflection group $\bar{A}(d_{LS}(e))$ on $V(e)$ compatible with the grading such that

$$V(e') = V(e)/\mathcal{C}(e') \tag{3.9.4}$$

when $d_{LS}(e) = d_{LS}(e')$. Note that $\mathcal{C}(e')$ is a reflection group, and therefore both $V(e)$ and $V'(e)$ can be affine spaces.

We deduce the following properties from (3.8.3) and (3.6.2). Its dimension is

$$\dim V(e) = \frac{1}{2} \dim_{\mathbb{C}} \mathcal{O}_\alpha \tag{3.9.5}$$

where $\mathcal{O}_\alpha$ is the Lusztig-Spaltenstein dual orbit of e. Let us call the basis of $V(e)$ with definite degrees as u_i, $i = 1, \ldots, \dim V(e)$. Then

$$\sum_i (2 \deg u_i - 1) = n_v(e) = 8\rho \cdot (\rho - \frac{h}{2}) + \frac{1}{2}(\operatorname{rank} G - \dim \mathfrak{g}_0). \tag{3.9.6}$$

This is interesting because the structure of $V(e)$ is governed both by e and its Lusztig-Spaltenstein dual α.

For type A we know what $V(e)$ is thanks to the explicit description of the base of the Hitchin fibration (3.8.8). The degree-d piece has the dimension

$$V(e)_d = p_d(\alpha) \tag{3.9.7}$$

where α is the dual orbit of e. Then the properties (3.9.5) and (3.9.6) are straightforward to check.

As a very nontrivial example, consider $G = E_8$ and a puncture with a label e in a special piece of $e_0 = E_8(a_7)$. Basic properties of each e are displayed in Table 2. The Spaltenstein dual is e_0 for all e in the table. $\bar{A}(e_0)$ is S_5, and the subgroup of S_5 assigned to each of the 7 nilpotent orbits by Sommers is also shown in the table, in terms of the generating reflections $(i, i+1)$, which act on the set $\{1, 2, 3, 4, 5\}$. Using (3.6.2) one can compute $n_v(e)$ for each nilpotent orbit, as h for each e is known. Since $\dim_{\mathbb{C}} O_{e_0} = 208$, $\dim V(e) = 104$ for all e. The degrees of four of the bases can be determined as follows.

Since $\bar{A}(E_8(a_7))$ is S_5, for the special nilpotent orbit e_0 we expect

$$V(e_0) = V \oplus V' \tag{3.9.8}$$

with $\dim V = 4$, $\dim V' = 100$ so that S_5 acts as the Weyl group of A_4 on V and acts trivially on V'. Let us say the degree of the bases of V is d. For Then, for $e = A_4 + A_3$ degrees of V are replaced by $\{2d, 3d, 4d, 5d\}$. These four numbers should be degrees of Casimir invariants of E_8, $\{2, 8, 12, 14, 18, 20, 24, 30\}$. The only possibility is $d = 6$. Then, for each of the 7 choices in the table, $\mathcal{C}(e)$ determines the degrees of these four generators , which are listed in the fourth column of Table 2, while the contribution to n_v from just these four generators is listed in the fifth column. The contribution from V' is not known but they should be completely the same for the 7 nilpotent elements. As a consistency check, the difference between $n_v(e)$ and the contribution to n_v from just the known 4 bases should be a constant. This is indeed so. The difference between entries on the same row in the third and fifth columns of Table 2 is always 4020.

3.10 Higgs branches

Let us study the Higgs branch of the class S theories

$$\mathcal{M}_{\text{Higgs}}(S_\Gamma[C; x_i, (e_i, m_i = 0)]). \tag{3.10.1}$$

$$
\begin{tikzcd}[column sep=small]
 & E_8(a_7) \arrow[d, dash, dotted] & \\
 & E_7(a_5) \arrow[dl, dash, dotted] \arrow[dr, dash, dotted] & \\
E_6(a_3)+A_1 \arrow[d, dash, dotted] \arrow[drr, dash, dotted] & & D_6(a_2) \arrow[d, dash, dotted] \arrow[dll, dash, dotted] \\
A_5+A_1 \arrow[dr, dash, dotted] & & D_5(a_1)+A_2 \arrow[dl, dash, dotted] \\
 & A_4+A_3 &
\end{tikzcd}
$$

e	h	$\mathcal{C}(e)$	$n_v(e)$	known ops	known n_v
$E_8(a_7)$	$\begin{matrix}0\\0002000\end{matrix}$	$\emptyset$	4064	$6,\ 6,\ 6,\ 6$	44
$E_7(a_5)$	$\begin{matrix}0\\0010100\end{matrix}$	(12)	4076	$6,\ 6,\ 6, 12$	56
$D_6(a_2)$	$\begin{matrix}1\\0100010\end{matrix}$	$(12), (34)$	4088	$6,\ 6, 12, 12$	68
$E_6(a_3)+A_1$	$\begin{matrix}0\\0101001\end{matrix}$	$(12), (23)$	4100	$6,\ 6, 12, 18$	80
A_5+A_1	$\begin{matrix}0\\1000101\end{matrix}$	$(12), (23), (45)$	4112	$6, 12, 12, 18$	92
$D_5(a_1)+A_2$	$\begin{matrix}0\\1010010\end{matrix}$	$(12), (23), (34)$	4136	$6, 12, 18, 24$	116
A_4+A_3	$\begin{matrix}0\\0100100\end{matrix}$	$(12), (23), (34), (45)$	4184	$12, 18, 24, 30$	164

Table 2: A special piece in the set of nilpotent orbits of E_8, h given as the inner products of h with simple roots, the corresponding subgroups of $S_5 = \bar{A}(E_8(a_7))$, n_v and the degrees of bases governed by subgroups of S_5. The sixth column shows the contribution to n_v just from the known 4 bases.

This object is a hyperkähler manifold, which depends on the area $\mathcal{A}$ of C but is independent of the complex structure of the punctured surface C. We denote this space by just

$$\eta_G(C,\{e_i\},\mathcal{A}). \tag{3.10.2}$$

The dependence on $\mathcal{A}$ is also known to be simple, as the underlying space of $\eta_G(C,e_i,\mathcal{A})$ is independent of $\mathcal{A}$ and the metric $g_\mathcal{A}$ on it satisfies

$$g_\mathcal{A} = \mathcal{A}^{-1} g_{\mathcal{A}=1}. \tag{3.10.3}$$

The holomorphic symplectic structure does not depend on $\mathcal{A}$. For now let us only consider the holomorphic symplectic structure; we come back to the $\mathcal{A}$ dependence at the end of this subsection.

Using the gluing property (3.7.1) of the class S theories and the behavior of the Higgs branch under the gauging (2.6.7), we have

$$[\eta_G(C_L,e=0,e_i)\times\eta_G(C_R,e'=0,e'_i)]///G = \eta_G(C,e_i,e'_i) \tag{3.10.4}$$

where C is obtained by gluing C_L and C_R at the two punctures with labels $e=0$ and $e'=0$. If we think of a point marked by $e=0$ as a boundary S^1, this means that η_G defines a functor from the category of cobordisms to the category $\mathcal{HS}$ of holomorphic symplectic spaces. Here, an object of $\mathcal{HS}$ is a compact group G, and an element in $\mathrm{Hom}_{\mathcal{HS}}(G,G')$ is a holomorphic symplectic manifold X with a Hamiltonian action of $G\times G'$. The composition of

$$X\in\mathrm{Hom}_{\mathcal{HS}}(G,G'),\quad X'\in\mathrm{Hom}_{\mathcal{HS}}(G',G'') \tag{3.10.5}$$

is given by the holomorphic symplectic quotient

$$(X\times X')///G'_{\mathrm{diag}}\in\mathrm{Hom}_{\mathcal{HS}}(G,G''). \tag{3.10.6}$$

Let us describe $\eta_G(S^2,e,e',\mathcal{A})$ explicitly. We put e and e' at the two poles of S^2, and perform the dimensional reduction around S^1. We have the $\mathcal{N}=2$ supersymmetric Yang-Mills theory on a segment of length proportional to $\mathcal{A}$, with the boundary conditions given by (3.5.6) at both ends. The Higgs branch of this system is known to be given by the moduli space of the Nahm equation with this boundary condition. When $e=e'=0$ it is particularly simple, the result as a holomorphic symplectic manifold is just

$$T^*G_\mathbb{C}\simeq G_\mathbb{C}\times\mathfrak{g}_C\ni(g,x) \tag{3.10.7}$$

which has an action of $G\times G$. The holomorphic moment maps are given by x and gxg^{-1}. The property (3.10.3) can be checked easily. This is indeed the identify homomorphism in $\mathrm{Hom}_{\mathcal{HS}}(G,G)$.

A more general case is given by

$$\eta_G(S^2,e=0,e')=\{(g,x)\subset G_\mathbb{C}\times\mathfrak{g}_\mathbb{C}\mid x\in e'+S_{e'}\}\subset T^*G_\mathbb{C} \tag{3.10.8}$$

where $e' + S'_e$ is the Slodowy slice at e'. The most general case is then

$$\eta_G(S^2, e, e') = \{(g, x) \subset G_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \mid gxg^{-1} \in e + S_e,\ x \in e' + S_{e'}\} \subset T^* G_{\mathbb{C}}. \tag{3.10.9}$$

As $\eta_G(S^2, e = 0, e')$ is already known (3.10.8), it suffices to know

$$W_{G,g,n} := \eta_G(C_g, n \text{ points with } e = 0) \tag{3.10.10}$$

where C_g is a genus-g surface. This is a hypekähler space with a triholomorphic action of

$$S_n \wr G = S_n \ltimes [\underbrace{G \times G \times \cdots \times G}_{n \text{ times}}] \tag{3.10.11}$$

where the permutation group S_n acts on G^n by permuting them.

These properties, together with the known case (3.10.7), uniquely fixes the dimension of η_G. We have

$$\dim_{\mathbb{H}} \eta_G(C; e_i) = \operatorname{rank} G + \sum_i \frac{1}{2}(\dim G - \operatorname{rank} G - \dim_{\mathbb{C}} \mathcal{O}_{e_i}). \tag{3.10.12}$$

By the pants decomposition, the determination of $X_{G,g,n}$ boils down to the determination of

$$W_G := W_{G,g=0,n=3}. \tag{3.10.13}$$

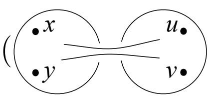
In an unpublished work Ginzburg and Kazhdan constructed $W_{G,g=0,n}$ in general and showed that they satisfy (3.10.4). Therefore in principle we know arbitrary $\eta_G(C, \{e_i\})$.

For $G = A_1$, it is known that

$$W_{A_1} = V_1 \otimes_{\mathbb{C}} V_2 \otimes_{\mathbb{C}} V_3 \tag{3.10.14}$$

where $V_i \simeq \mathbb{C}^2$ so that V_i is acted naturally by SU(2). It is instructive to check that this action of $S_3 \wr \mathrm{SU}(2)$ preserves the holomorphic symplectic structure. By the gluing property, we have

$$W_{A_1,g=0,n=4} = \eta_{A_1}(\quad) = [V_x \otimes V_y \otimes V \oplus V \otimes V_u \otimes V_v] /\!/\!/ \mathrm{SU}(V). \tag{3.10.15}$$



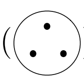
The right hand side should be invariant under the exchange $V_y \leftrightarrow V_u$ but this is not obvious in this notation. The right hand side, when written as

$$V \otimes_{\mathbb{R}} \mathbb{R}^8 /\!/\!/ \mathrm{SU}(V), \tag{3.10.16}$$

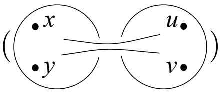
is the ADHM construction of the minimal nilpotent orbit of $\mathrm{SO}(8) \supset \mathrm{SU}(V_x) \times \mathrm{SU}(V_y) \times \mathrm{SU}(V_u) \times \mathrm{SU}(V_v)$, and the exchange $V_y \leftrightarrow V_u$ is given by an outer automorphism of SO(8).

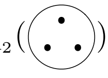
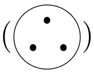
For $G = A_2$, it is conjectured that

$$W_{A_2} = \eta_{A_2}(\quad) = \text{minimal nilpotent orbit of } E_6. \tag{3.10.17}$$



This has $S_3 \wr \mathrm{SU}(3) \subset E_6$ triholomorphic action. Then

$\eta_{A_2}($

$) = \eta_{A_2}($$) \times \eta_{A_2}($$)/\!/\!/\, \mathrm{SU}(3).$ (3.10.18)

The action of $S_4 \wr \mathrm{SU}(3)$ is not manifest.

As a natural generalization of (3.10.15) and (3.10.17), it is known that

$$\eta_{A_{2n-1}}(S^2; [n^2], [n^2], [n^2], [n^2]) = \tilde{\mathcal{M}}_{D_4,n}, \tag{3.10.19}$$

$$\eta_{A_{3n-1}}(S^2; [n^3], [n^3], [n^3]) = \tilde{\mathcal{M}}_{E_6,n}, \tag{3.10.20}$$

$$\eta_{A_{4n-1}}(S^2; [2n^2], [n^4], [n^4]) = \tilde{\mathcal{M}}_{E_7,n}, \tag{3.10.21}$$

$$\eta_{A_{6n-1}}(S^2; [3n^2], [2n^3], [n^6]) = \tilde{\mathcal{M}}_{E_8,n} \tag{3.10.22}$$

where $\tilde{\mathcal{M}}_{G,n}$ is the centered framed moduli space of G-instantons on $\mathbb{R}^4$ with instanton number n, with real dimension $4h^\vee(G)(n-1)$; note that the minimal nilpotent orbit of G is the centered framed one-instanton moduli space of G. More details on this functor η_Γ can be found in [MT11].

Let us consider the $\mathcal{A}$ dependence [GMT11]. One problem is that $T^*G_\mathbb{C}$ is no longer an identity under the composition; instead, we have

$$(T^*G_\mathbb{C})_\mathcal{A} \times (T^*G_\mathbb{C})_{\mathcal{A}'}/\!/\!/G = (T^*G_\mathbb{C})_{\mathcal{A}+\mathcal{A}'} \tag{3.10.23}$$

where $X_\mathcal{A}$ is introduced in (3.10.3). This translates to the following slight problem to define the category $\mathcal{HK}$ of hyperkähler spaces in a way similar to the category $\mathcal{HS}$. An object of $\mathcal{HK}$ is a compact group G, and $\mathrm{Hom}_{\mathcal{HK}}(G, G')$ consists of hyperkähler spaces with triholomorphic $G \times G'$ action. But there is no identity element in $\mathrm{Hom}_{\mathcal{HK}}(G, G')$.

Correspondingly, the source category of η_G is not just the cobordism category, but the category wit h cobordisms with an area assigned, just as in the case of 2d Yang-Mills with continuous gauge group G Sec. 1.8.3.

3.11 When $S_\Gamma[C]$ is $\mathrm{Hyp}(V)$

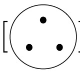
Let us consider when $Q = S_\Gamma[C] = \mathrm{Hyp}(V)$. If this is the case, we should have

- $n_v(Q) = 0$,
- $\mathrm{rank}\, Q = \dim_\mathbb{C} \mathcal{M}_{\mathrm{Coulomb}}(Q) = 0$,
- and $n_h(Q) = \dim_\mathbb{H} \mathcal{M}_{\mathrm{Higgs}}(Q)$.

It is believed that any one of these conditions implies all the others. Let us enumerate a few known cases. Enumerating all possible cases would be an interesting exercise.

3.11.1 Trifundamental of A_1

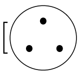
For $G = A_1$, the basic case is

$S_{A_1}[$  $] = \mathrm{Hyp}(V_1 \otimes V_2 \otimes V_3)$ (3.11.1)

where $V_i \simeq \mathbb{C}^2$. From this we can construct $\mathcal{N} = 2$ gauge theories associated to trivalent graphs introduced in Sec. 2.7.6 by Gaiotto's gluing (3.7.1). Then the Donagi-Witten integrable system of the trivalent theories, discussed in Sec. 2.11.5, naturally follows from the property of the class S theory, discussed in Sec. 3.8. The residue of the Hitchin field ϕ at the punctures are given by the formula (3.8.2), but it just becomes a semisimple element in $\mathfrak{su}(2)$, giving (2.11.35).

3.11.2 Bifundamental of A_{N-1}

One natural generalization of the trifundamental for A_1 in Sec. 3.11.1 is the bifundamental for or $G = A_{N-1}$. We have

$S_{A_{N-1}}[$  $, e = [N-1, 1], e = 0, e = 0] = \mathrm{Hyp}(V_1 \otimes \bar{V}_2 \otimes W \oplus \bar{V}_1 \otimes V_2 \otimes \bar{W}).$ (3.11.2)

Here $V_i \simeq \mathbb{C}^N$ on which $\mathrm{SU}(V_i)$ acts, and $W \simeq \mathbb{C}$ has an action of $G^{[N-1,1]} = \mathrm{U}(1)$. A Cartan element m of this $\mathrm{U}(1)$ is given by

$$m = \mu \, \mathrm{diag}(1, 1, \ldots, 1, 1-N). \tag{3.11.3}$$

Let us compute $n_v(Q)$ and $n_h(Q)$ in two ways. As $\mathrm{Hyp}(V)$, it is determined as in Sec. 2.5, then we should have $n_v(Q) = 0$ and $n_h(Q) = N^2$. As a class S theory, we start from

$$n_v(e = 0) = \frac{1}{6}N(N-1)(4N+1), \quad n_h(e = 0) = \frac{2}{3}N(N-1)(N+1) \tag{3.11.4}$$

and

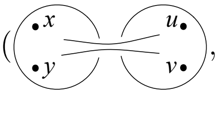
$$n_v(e = [N-1, 1]) = N^2 - 1, \quad n_h(e = [N-1, 1]) = N^2. \tag{3.11.5}$$

Plugging them to the formulas (3.6.2) and (3.6.3), we again find $n_v(Q) = 0$ and $n_h = N^2$.

As for the symmetry $\mathrm{SU}(N) \times \mathrm{SU}(N)$, we find $k_{\mathrm{SU}(N)}(Q) = k_{\mathrm{SU}(N)}(e = 0) = 2h^\vee(\mathrm{SU}(N)) = 2N$ as a class S theory. As $\mathrm{Hyp}(V)$, we already studied it in Sec. 2.7.3 and found it is $2N$.

Let us take two copies and apply Gaiotto's gluing construction. We find

$\mathrm{Hyp}(V \otimes \bar{W} \oplus \bar{V} \otimes W) /\!\!/\!\!/ \mathrm{SU}(V) \Big|_\tau =$

$S_{A_{N-1}}($ 

$, x, e = [N-1, 1], y, e = 0, u, e = [N-1, 1], v, e = 0)$ (3.11.6)

where

$$W = V_x \otimes W_y \oplus V_u \otimes W_v \simeq \mathbb{C}^{2N}. \tag{3.11.7}$$

This is the SQCD introduced in Sec. 2.7.3, with $N_f = 2N$. Its Donagi-Witten integrable system was discussed in Sec. 2.11.4. This now follows from the property of the Donagi-Witten integrable system of a class S theory, discussed in Sec. 3.8. For example, at the puncture $e = [N-1, 1]$, the residue α of the Hitchin field should be in its Lusztig-Spaltenstein orbit. The dual partition to $[N-1,1]$ is $[2, 1^{N-2}]$, which describes the Jordan block decomposition of α, and indeed it agrees with what we saw in (2.11.31). With the mass deformation of the form (3.11.3) at this puncture, the residue α of the Hitchin field is given by the formula (3.8.2), which just gives $\alpha = m$. This again reproduces what we saw in (2.11.30).

We can also construct a gauge theory of the form

$$\mathrm{Hyp}(\oplus_{i=1}^{n} V_i \otimes \bar{V}_{i+1} \oplus \bar{V}_i \otimes V_{i+1}) /\!\!/\!\!/ \prod_{i=1}^{n} \mathrm{SU}(V_i) \,\Big|_{\{\tau_i\}} \tag{3.11.8}$$

where we set $V_{n+1} = V_n$, via Gaiotto's gluing (3.7.1). This theory is therefore

$$= S_{A_{N-1}}[T^2, x_1, [N-1,1], x_2, [N-1,1], \ldots, x_n[N-1,1]] \tag{3.11.9}$$

where τ_i is encoded in the complex structure of the elliptic curve with n punctures.

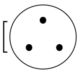
This is a case of the quiver gauge theory introduced in Sec. 2.7.4, where the underlying graph is of type $\hat{A}_{n-1}$. Its Donagi-Witten integrable system discussed in Sec. 2.11.7, in the Hitchin system formulation, immediately follows from this construction. It is known how to represent other quiver gauge theories as a class S theory if the underlying graph is of type A, D or $\hat{D}$, but we will not detail the construction here.

The Higgs branch of the theory above is

$$[\oplus_{i=1}^{n} V_i \otimes \bar{V}_{i+1} \oplus \bar{V}_i \otimes V_{i+1}] /\!\!/\!\!/ \prod_{i=1}^{n} \mathrm{SU}(V_i). \tag{3.11.10}$$

This is an SU version of a quiver variety.

3.11.3 E_6

As an example of enumeration of all class S theories which are $\mathrm{Hyp}(V)$, let us consider $Q = S_{E_6}[$  $; e_1, e_2, e_3 = 0]$. From the formula above,

$$n_v(Q) = -(\frac{4}{3} h^\vee(G) \dim G + \operatorname{rank} G) + n_v(e_1) + n_v(e_2) + n_v(e_3 = 0). \tag{3.11.11}$$

Scanning through the list of nilpotent orbits of E_6, one finds that there is only one solution to $n_v(Q) = 0$, namely with

$$e_1 = E_6(a_1), \quad e_2 = A_2 + 2A_1. \tag{3.11.12}$$

Here the notation $E_6(a_1)$ and $A_2 + 2A_1$ are the standard Bala-Carter labels. We then have

$$\dim \mathcal{M}_{\mathrm{Coulomb}}(Q) = 0, \quad \dim \mathcal{M}_{\mathrm{Higgs}} = n_h(Q) = 54 = 27 \times 2. \tag{3.11.13}$$

Recall that the minuscule representation of E_6 is $V_{\text{min}} \simeq \mathbb{C}^{27}$. It is likely, from the numerical data above, that

$$Q = \text{Hyp}(V_{\text{min}} \otimes F \oplus \bar{V}_{\text{min}} \otimes \bar{F}) \tag{3.11.14}$$

with $F \simeq \mathbb{C}^2$. This has a natural pseudoreal action of $E_6 \times \mathrm{U}(2)$. And indeed, $G^{E_6(a_1)} = 1$ and $E_6^{A_2+2A_1} = \mathrm{SU}(2) \times \mathrm{U}(1)$.

Let us first compute $k_{E_6}(Q)$ in two ways. As a class S theory, this is $k_{E_6}(e = 0) = 2h^\vee(E_6) = 24$. As $\text{Hyp}(V)$, we saw in Sec. 2.7.7

$$k_{E_6}(\text{Hyp}(V_{\text{min}} \otimes F \oplus \bar{V}_{\text{min}} \otimes \bar{F})) = 24 \tag{3.11.15}$$

and they nicely match.

We can also compute $k_{\mathrm{SU}(2)}(Q)$ in two ways, using the formula as class S theory and using the formula for $\text{Hyp}(V)$. In the former, we need to decompose $\mathfrak{e}_6$ by

$$G^{e_2} \otimes \rho_{e_2}(\mathrm{SU}(2)) \simeq \mathrm{SU}(2) \otimes \rho_e(\mathrm{SU}(2)). \tag{3.11.16}$$

We find

$$\mathfrak{e}_6 = V_5 \otimes V_3 \oplus V_3 \otimes V_5 \oplus V_4 \otimes V_2 \oplus V_2 \otimes V_4 \oplus V_3 \otimes V_3 \oplus V_1 \otimes V_3 \oplus V_3 \otimes V_1. \tag{3.11.17}$$

It turns out that $\mathrm{SU}(2) \subset G^{e_2} \subset G$ is also of type $A_2 + 2A_1$, explaining the symmetry. We find

$$k_{\mathrm{SU}(2)}(A_1 + 2A_2) = 54. \tag{3.11.18}$$

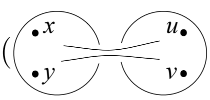
In the other way of computation,

$$k_{\mathrm{SU}(2)}(\text{Hyp}(V_{\text{min}} \otimes F \oplus \bar{V}_{\text{min}} \otimes \bar{F})) = 27 \times 2 = 54. \tag{3.11.19}$$

We can use this to determine the Donagi-Witten integrable system of some E_6 gauge theory. Namely, we have

$$DW[\text{Hyp}(V_{\text{min}} \otimes F \oplus \bar{V}_{\text{min}} \otimes \bar{F}) /\!/\!/ E_6]$$

$$= \mathcal{M}_{\text{Hitchin}}(\quad, E_6(a_1), E_6(a_1), A_2 + 2A_1, A_2 + 2A_1). \tag{3.11.20}$$



According to the property of the Hitchin system associated to the class S theories discussed in Sec. 3.8, the Hitchin system should have two regular singularities with residues in

$$d_{LS}(E_6(a_1)) = A_1 \tag{3.11.21}$$

and two more regular singularities with residues in

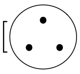
$$d_{LS}(A_2 + 2A_1) = A_4 + A_1 \tag{3.11.22}$$

when there is no mass deformation. For either puncture of type $e = A_2 + 2A_1$, we can add a mass deformation m in $\mathfrak{g}^e$. They can be conjugated to $av_2 + bv_4$ where v_i is the i-th fundamental weights where we labeled the nodes as $12\overset{6}{3}45$. Then the residue should be given by the formula (3.8.2):

$$\text{Ind}^{\mathfrak{e}_6}_{A_2+2A_1}[m + d_{LS}^{A_2+2A_1}(A_2 + 2A_1)] = m. \tag{3.11.23}$$

This is exactly what we saw in Sec. 2.11.6 previously.

3.11.4 E_7

Let $Q = S_{E_7}[$$; e_1, e_2, e_3 = 0]$. As in the E_6 case, we find only one combination where $n_v(Q) = 0$, namely with

$$e_1 = E_7(a_1), \quad e_2 = A_3 + A_2 + A_1. \tag{3.11.24}$$

One can check that automatically we have

$$\dim \mathcal{M}_{\text{Coulomb}}(Q) = 0, \quad \dim \mathcal{M}_{\text{Higgs}} = n_h(Q) = 84 = 28 \times 3. \tag{3.11.25}$$

The minuscule representation of E_7 is $V_{\min} \simeq \mathbb{H}^{28} \simeq \mathbb{C}^{56}$ and is pseudoreal. It is likely, from the numerical data above, that

$$Q = \text{Hyp}(V_{\min} \otimes_{\mathbb{R}} \mathbb{R}^3). \tag{3.11.26}$$

This has a natural pseudoreal action of $E_7 \times \text{SO}(3)$. And indeed, $G^{E_7(a_1)} = 1$ and $E_7^{A_3+A_2+A_1} = \text{SO}(3)$. $k_{E_7}(Q)$ can be computed both as a class S theory and as $\text{Hyp}(V)$ and they agree; it is 36.

We can compute $k_{\text{SO}(3)}(Q)$ in two ways, using the formula as class S theory and using the formula for $\text{Hyp}(V)$. In the former, we need to decompose $\mathfrak{e}_7$ by

$$G^{e_2} \otimes \rho_{e_2}(\text{SU}(2)) \simeq \text{SO}(3) \otimes \text{SU}(2). \tag{3.11.27}$$

We find

$$\mathfrak{e}_7 = V_5 \otimes V_7 \oplus V_7 \otimes V_5 \oplus V_5 \otimes V_3 \oplus V_3 \otimes V_5 \oplus V_1 \otimes V_3 \oplus V_3 \otimes V_1 \oplus V_9 \otimes V_3, \tag{3.11.28}$$

It happens that $\text{SO}(3) \simeq G^{e_2}$ has the type $A_4 + A_2$. We find

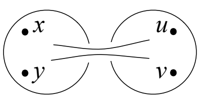
$$k_{\text{SO}(3)}(A_3 + A_2 + A_1) = 224. \tag{3.11.29}$$

In the latter, we have

$$k_{\text{SO}(3)}(\text{Hyp}(V \otimes_{\mathbb{R}} \mathbb{R}^3)) = 28 \times 8 = 224. \tag{3.11.30}$$

The Donagi-Witten integrable system of E_7 gauge theory is then

$$DW[\text{Hyp}(V_{\min} \otimes_{\mathbb{R}} \mathbb{R}^6) /\!\!/\!\!/ E_7]$$

$= \mathcal{M}_{\text{Hitchin}}($

$, E_7(a_1), E_7(a_1), A_3 + A_2 + A_1, A_3 + A_2 + A_1)$. (3.11.31)

The spectral geometry of this Hitchin system agrees with what was found before using string duality.

4 Nekrasov partition functions and the W-algebras

In the last section we obtained a 4d QFT $S_\Gamma[C^2]$ by dimensionally reducing a 6d theory S_Γ on a two-dimensional surface C^2. The partition function was given schematically by

$$Z_{S_\Gamma[C^2]}(X^4) = Z_{S_\Gamma}(X^4 \times C^2). \tag{4.0.32}$$

We can switch the role of X^4 and C^2, and consider the 2d theory $S_\Gamma[X^4]$, whose partition function is again given by

$$Z_{S_\Gamma[X^4]}(C^2) = Z_{S_\Gamma}(X^4 \times C^2). \tag{4.0.33}$$

Therefore we see the equality

$$Z_{S_\Gamma[C^2]}(X^4) = Z_{S_\Gamma[X^4]}(C^2) \tag{4.0.34}$$

which relates two-dimensional QFTs and four-dimensional QFTs. This is not surprising from the six-dimensional point of view, but for a person who only knows the theories $S_\Gamma[C^2]$ and $S_\Gamma[X^4]$ as defined intrinsically in respective dimensions, this is a rather mysterious relation.

As seen in the last section, the behavior of $S_\Gamma[C^2]$ under the cutting and the pasting of the two-dimensional surface is relatively well understood. It would be nice to have a way to understand $S_\Gamma[X^4]$ in a similar manner. Currently we have not come to this point. Instead, what has been done is to guess $S_\Gamma[X^4]$ by studying $Z_{S_\Gamma[C^2]}(X^4)$ using the knowledge of $S_\Gamma[C^2]$.

So far we have the understanding of $S_\Gamma[X^4]$ for basically two classes:

1. $\mathbb{R}^4$ with equivariance, S^4, and their variants
2. $S^1 \times S^3$ and its variants

In this section we discuss the former, and in the next section we discuss the latter.

4.1 Nekrasov's partition function

4.1.1 Definition

We first introduce the concept of Nekrasov's partition function of an $\mathcal{N} = 2$ supersymmetric F-symmetric QFT Q, which is basically $Z_Q(\mathbb{R}^4)$ with a few qualifications.

- We consider a general mass deformation Q_m for $m \in \mathfrak{f}$.
- We consider $\mathbb{R}^4 \simeq \mathbb{C}^2$ with equivariance under a natural $\mathrm{U}(1)^2$ action. We have an equality

  $$H^*_{\mathrm{U}(1)^2}(pt) = \mathbb{C}[\epsilon_1, \epsilon_2]. \tag{4.1.1}$$

  We call ϵ_1 and ϵ_2 the equivariant parameters.

- As $\mathbb{R}^4$ is noncompact, we need to specify a vacuum $p \in \mathcal{M}_{\text{susyvac}}(Q)$.
- We perform the topological twists to the theory as in Sec. 2.13. Then the partition function only depends on the projection of p to $\mathcal{M}_{\text{Coulomb}}(Q_m)$.
- We pick a maximally isotropic sublattice $\mathsf{L}_E \subset \mathsf{L}$ and introduce the coordinates $a_i = \int_{\alpha_i} \lambda$ of $\mathcal{M}_{\text{Coulomb}}$ and parameterize the mass deformation by $m_j = \int_{\gamma_j} \lambda$ as explained in Sec. 2.9.

Then we define

$$Z_Q^{\text{Nek}}(\epsilon_1, \epsilon_2; a_1, \ldots, a_r; \{m_j\}) := Z_{Q_{m,\text{top}}}(\mathbb{R}^4_{\epsilon_1,\epsilon_2}, p). \tag{4.1.2}$$

It is known that the prepotential as introduced in Sec. 2.9 is obtained from Nekrasov's partition function:

$$\lim_{\epsilon_1,\epsilon_2 \to 0} \epsilon_1 \epsilon_2 Z_Q^{\text{Nek}}(\epsilon_1, \epsilon_2; a_1, \ldots, a_r; \{m\}) = \mathcal{F}(a_1, \ldots, a_r; \{m\}). \tag{4.1.3}$$

The transformation of $\mathcal{F}(a_1, \ldots, a_r; \{m\})$ under the change of $\mathsf{L}_E \subset \mathsf{L}$ was via the Legendre transformation. To reproduce it in the limit $\epsilon_{1,2} \to 0$, $Z^{\text{Nek}}(a_1, \ldots, a_r; \{m\})$ should transform under the change of $\mathsf{L}_E \subset \mathsf{L}$ via the Fourier transformation, but the contour to be used in this Fourier transformation is not well understood. As the properties of Z^{Nek} globally over $\mathcal{M}_{\text{Coulomb}}(Q)$ is not understood, we fix a patch of $\mathcal{M}_{\text{Coulomb}}(Q)$ on which the monodromy of the $\text{Sp}(\mathsf{L})$ local system preserves the sublattice L_E.

This is the formalization of Nekrasov's partition function as used in physics literature. This concept was first introduced in [Nek04]. It is convenient for our purposes to extend the concept slightly. Namely, for an F-symmetric QFT Q, we can consider

$$Z_{Q_{\text{top}}}(P_F \to \mathbb{R}^4, p) \tag{4.1.4}$$

where P_F is an F-bundle with connection over $\mathbb{R}^4$. The object (4.1.4) determines a section of a bundle over the moduli space of F-bundles. When $P_F \to \mathbb{R}^4$ is further assumed to be anti-self-dual, this section descends to a closed equivariant differential form on $\mathcal{M}_F$, the moduli space of framed anti-self-dual F-connections on $\mathbb{R}^4$. We denote it by

$$Z^{\text{Nek}}(Q) \in H^*_{F \times \text{U}(1)^2}(\mathcal{M}_F) \otimes \text{Frac}(H^*_{F \times \text{U}(1)^2}(pt)) \otimes \mathbb{C}(a_1, \ldots, a_r) \tag{4.1.5}$$

where we identify

$$H^*_{F \times \text{U}(1)^2}(pt) = \mathbb{C}[m_1, \ldots, m_F][\epsilon_1, \epsilon_2]. \tag{4.1.6}$$

Note that

$$H^*_{F \times \text{U}(1)^2}(\mathcal{M}_F) = \oplus_{n \geq 0} H^*_{F \times \text{U}(1)^2}(\mathcal{M}_{F,n}) \tag{4.1.7}$$

where n is the instanton number and

$$\dim_{\mathbb{R}} \mathcal{M}_{F,n} = 4h^\vee(F)n. \tag{4.1.8}$$

We can obtain the standard Nekrasov function (4.1.2) by projecting the object (4.1.5) to the $n=0$ component in the decomposition (4.1.7), and evaluating the formal variables $\epsilon_{1,2}$ and m_i in (4.1.6) by assigning numbers. The integer $k_F(Q)$ determines the degree of Z_Q^{Nek}:

$$\deg Z^{\mathrm{Nek}}(Q)|_{H^*_{F\times \mathrm{U}(1)^2}(\mathcal{M}_{F,n}(\mathbb{R}^4))} = k_F(Q)n. \tag{4.1.9}$$

Therefore when $k_F(Q)=2h^\vee(F)$, Z_Q^{Nek} determines a middle-dimensional class on $\mathcal{M}_F$, and when $k_F(Q)=4h^\vee(F)$, Z_Q^{Nek} is a top form on $\mathcal{M}_F$.

4.1.2 For $\mathrm{Hyp}(V\oplus\bar{V})$

Let $Q=\mathrm{Hyp}(V\oplus\bar{V})$ for a complex F-representation V. $\mathcal{M}_{\mathrm{Coulomb}}(Q)$ is a point. Then

$$Z^{\mathrm{Nek}}(\mathrm{Hyp}(V\oplus\bar{V})) \in H^*_{F\times\mathrm{U}(1)^2}(\mathcal{M}_F)\otimes \mathrm{Frac}(H^*_{F\times\mathrm{U}(1)^2}(pt)) \tag{4.1.10}$$

is given by

$$Z^{\mathrm{Nek}}(\mathrm{Hyp}(V\oplus\bar{V})) = \prod_{w:\text{weights of }V} \Gamma_B(w(m)|\epsilon_1,\epsilon_2)\times e(\mathrm{Ind}\,\not{D}_V) \tag{4.1.11}$$

where $\not{D}_V$ is the Dirac operator associated to the F-bundle

$$V\times_F P_F \to \mathbb{R}^4, \tag{4.1.12}$$

$\mathrm{Ind}\,\not{D}_V$ is the index bundle determined by $\not{D}_V$ over $\mathcal{M}_{F,n}$, e is the equivariant Euler class, and

$$\Gamma_B(x|\epsilon_1,\epsilon_2) = \text{regularized version of } \prod_{m,n\geq 0}\frac{1}{x+n\epsilon_1+m\epsilon_2} \tag{4.1.13}$$

is the Barnes double gamma function.

4.1.3 For the products

For $Q=Q_1\times Q_2$, Nekrasov's partition function behaves multiplicatively:

$$Z^{\mathrm{Nek}}(Q) = Z^{\mathrm{Nek}}(Q_1)\times Z^{\mathrm{Nek}}(Q_2). \tag{4.1.14}$$

4.1.4 For the quotients

Let Q be $G\times F$-symmetric, and suppose we know

$$\begin{aligned} Z^{\mathrm{Nek}}(Q) &\in H^*_{G\times F\times\mathrm{U}(1)^2}(\mathcal{M}_G\times\mathcal{M}_F)\otimes \mathrm{Frac}\,H^*_{G\times F}(pt)\otimes\mathbb{C}(a_1,\ldots,a_{\mathrm{rank}\,Q}) \\ &= H^*_{G\times F\times\mathrm{U}(1)^2}(\mathcal{M}_G\times\mathcal{M}_F)\otimes \mathrm{Frac}\,H^*_{F}(pt)\otimes\mathbb{C}(a_1,\ldots,a_{\mathrm{rank}\,Q};a'_1,\ldots,a'_{\mathrm{rank}\,G}) \end{aligned} \tag{4.1.15}$$

where we introduced the variables $a'_1,\ldots,a'_{\mathrm{rank}\,G}$ via

$$H^*_G(pt)\simeq \mathbb{C}[a'_1,\ldots,a'_{\mathrm{rank}\,G}]^{W_G} \tag{4.1.16}$$

where W_G is the Weyl group of G.

Recall that the Coulomb branches of Q and $Q/\!/\!/G$ satisfy the relation (2.6.8)

$$\mathcal{M}_{\text{Coulomb}}(Q/\!/\!/G) = \mathcal{M}_{\text{Coulomb}}(Q) \times \operatorname{Spec} \mathbb{C}[\mathfrak{g}_{\mathbb{C}}]^{G_{\mathbb{C}}}. \tag{4.1.17}$$

Then Nekrasov's partition function for $Q/\!/\!/G \,\big|_\tau$ for us is defined over the patch

$$\mathcal{M}_{\text{Coulomb}}(Q/\!/\!/G) \supset \mathcal{M}_{\text{Coulomb}}(Q) \times U_K \tag{4.1.18}$$

for large K, where U_K was defined in (2.10.4). Then the algebras of functions on U_K we are interested in is contained in

$$\mathbb{C}(a'_1, \ldots, a'_{\operatorname{rank} G}) = \operatorname{Frac} H^*_G(pt), \tag{4.1.19}$$

and Nekrasov's partition function for $Q/\!/\!/G$ takes values in

$$H^*_{F\times \mathrm{U}(1)^2}(\mathcal{M}_F) \otimes \mathbb{C}(a_1, \ldots, a_{\operatorname{rank} Q}; a'_1, \ldots, a'_{\operatorname{rank} G}) \tag{4.1.20}$$

Then $Z^{\text{Nek}}(Q/\!/\!/G \,\big|_\tau)$ is obtained by a natural operation which sends an element in (4.1.15) to (4.1.20). Such a map is defined by using the fundamental class

$$[\mathcal{M}_G] = \oplus_{n\geq 0}[\mathcal{M}_{G,n}] \tag{4.1.21}$$

and we have

$$\begin{aligned} Z^{\text{Nek}}(Q/\!/\!/G \,\big|_\tau) &= q^{\frac{1}{\epsilon_1\epsilon_2}\langle a,a\rangle_0} \\ &\times \prod_{\alpha:\text{pos. roots}} \frac{1}{\Gamma_B(\alpha(a)|\epsilon_1,\epsilon_2)\Gamma_B(\epsilon_1+\epsilon_2-\alpha(a)|\epsilon_1,\epsilon_2)} \times \langle q^{\mathsf{N}}[\mathcal{M}_G], Z_G^{\text{Nek}}\rangle \end{aligned} \tag{4.1.22}$$

where N is an operator which is a multiplication by n on $H^*_G(\mathcal{M}_{G,n})$ and as always $q = e^{2\pi\sqrt{-1}\tau}$.

Combining (4.1.11) and (4.1.22) we can define and compute Nekrasov's partition function for $\mathcal{N}=2$ gauge theory $\operatorname{Hyp}(V)/\!/\!/G$, assuming that there is a good control of the moduli space $\mathcal{M}_G$ of antiselfdual G connections and the determinant line bundle $\operatorname{Ind} \not{D}_V$ on it. the Donagi-Witten integrable system of $\operatorname{Hyp}(V)/\!/\!/G$ can then be recovered by studying its small $\epsilon_1\epsilon_2$ behavior, (4.1.3). This is best developed when G is of type A, and there are a few scattered works for other classical G's. For the case when G is a product of type A groups, the most recent comprehensive discussions are in [NP12]. For classical G, see an older review [Sha05]. A more conceptual review from a more physical point of view is given in [Tac13].

4.2 Nekrasov's partition function for class S theories

Now we would like to study $Z^{\text{Nek}}(S_\Gamma[C])$. Its $\epsilon_1, \epsilon_2 \to 0$ limit determines $DW(S_\Gamma[C]) = \mathcal{M}_{\text{Hitchin}}(C)$. Therefore it should be some kind of a quantization of the Hitchin system.

First we consider the case when all the punctures are with the label $e = 0$. With n punctures the theory $S_\Gamma[C_{g,n}]$ is G^n symmetric. We write

$$\mathrm{Frac}\, H^*_{G^n \times \mathrm{U}(1)^2}(pt) = \mathrm{Frac}\, H^*_{G \times \mathrm{U}(1)^2}{}^{\otimes n} \tag{4.2.1}$$

in the understanding that each of G appearing on the right hand side refers to an isomorphic but different groups, and that the tensor product is with respect to the base field

$$K = \mathbb{C}(\epsilon_1, \epsilon_2) = \mathrm{Frac}\, H^*_{\mathrm{U}(1)^2}(pt). \tag{4.2.2}$$

In the following we regard that we fixed an evaluation homomorphism $K \to \mathbb{C}$ which sends $\epsilon_{1,2}$ to generic complex numbers.

First let us consider the three-punctured sphere:

$$Z^{\mathrm{Nek}}(S_\Gamma[\;]) \in V_G^{\otimes 3} \otimes X_G \tag{4.2.3}$$

where

$$V_G = H^*_{G \times \mathrm{U}(1)^2}(\mathcal{M}_G) \otimes \mathrm{Frac}\, H^*_G(pt) \tag{4.2.4}$$

and

$$X_G = \mathbb{C}(a_1, \ldots, a_x) \tag{4.2.5}$$

with the coordinates $a_1,\ldots, a_x$ of a patch of $\mathcal{M}_{\mathrm{Coulomb}}(S_\Gamma[\;])$. Therefore

$$x = \dim_{\mathbb{C}} \mathcal{M}_{\mathrm{Coulomb}}(S_\Gamma[\;]) = \frac{1}{2}\dim G - \frac{3}{2}\mathrm{rank}\, G. \tag{4.2.6}$$

We then have, from (3.7.1) and (4.1.22),

$$\begin{aligned} Z^{\mathrm{Nek}}(S_\Gamma[\;]) &= Z^{\mathrm{Nek}}(S_\Gamma[\;] \times S_\Gamma[\;] /\!\!/\!\!/ G_{\mathrm{diag}} \mid \tau) && (4.2.7) \\ &= \Big(\prod \frac{1}{\Gamma_B \Gamma_B}\Big)\langle q^{\mathsf{N}}[M_G], Z^{\mathrm{Nek}}(S_\Gamma[\;]) Z^{\mathrm{Nek}}(S_\Gamma[\;])\rangle && (4.2.8) \end{aligned}$$

where the product of $(\Gamma_B\Gamma_B)^{-1}$ stands for the factor in (4.1.22). This takes values in

$$V_G^{\otimes 4} \otimes X_G^{\otimes 2} \otimes \mathrm{Frac}\, H^*_G(pt) \tag{4.2.9}$$

In more generality, we have

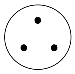
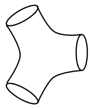
$$Z^{\mathrm{Nek}}(S_\Gamma[C_{g,n}]) \in V_G^{\otimes n} \otimes X_G^{\otimes 2(g-1)+n} \otimes \mathrm{Frac}\, H^*_G(pt)^{\otimes 3(g-1)+n}. \tag{4.2.10}$$

This can be thought of as defining a 2d holomorphic generalized QFT Q_Γ on the Riemann surface C via

$$Z_{Q_\Gamma}[C_{g,n}] := Z^{\mathrm{Nek}}(S_\Gamma[C_{g,n}]). \tag{4.2.11}$$

As Z^{Nek} is basically the partition function on $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$ as explained in (4.1.2), we regard

$$Q_\Gamma = S_\Gamma[\mathbb{R}^4_{\epsilon_1,\epsilon_2}]. \tag{4.2.12}$$

To study Q_Γ, first let us discuss the properties of a 2d holomorphic QFT in general. Regard a three-punctured sphere  to be equipped with three local coordinates $z_{1,2,3}$ so that the punctures are at $z_i = 0$, respectively. Now let us assume that the local coordinates are such that the circles $|z_i| = 1$ do not intersect and do not contain each other. Therefore this is now a sphere with three holes as in . A sphere with two holes with parameter q, in this description, has two local coordinates z and z' and $zz' = q$ with two circles $|z| = 1$ and $|z'| = 1$. The gluing operation in this language is always done by identifying two local coordinates z_1 and z_2 associated to two punctures by $z_1 z_2 = 1$, so that the circles at $|z_1| = 1$ and $|z_2| = 1$ are identified.

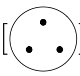
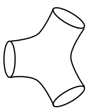
This 2d theory Q_Γ should have a space of states $\mathcal{H}_{Q_\Gamma}(S^1)$. We take it to be

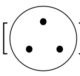
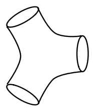
$$\mathcal{H}_{Q_\Gamma}(S^1) = V_G = H^*_G(\mathcal{M}_G) \otimes \operatorname{Frac} H^*_G(pt) \tag{4.2.13}$$

with the inner product

$$V_G \ni v, w \mapsto (v, w) = \langle [\mathcal{M}_G], v \wedge w \rangle \in H^*_G(pt). \tag{4.2.14}$$

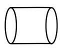
Then we have

$Z^{\text{Nek}}(S_\Gamma[$  $]) = Z_{Q_\Gamma}($  $) : \mathcal{H}_{Q_\Gamma}(S^1) \to \mathcal{H}_{Q_\Gamma}(S^1)^{\otimes 2} \otimes X_G,$ (4.2.15)

$Z^{\text{Nek}}(S_\Gamma[$  $]) = Z_{Q_\Gamma}($  $) : \mathcal{H}_{Q_\Gamma}(S^1)^{\otimes 2} \to \mathcal{H}_{Q_\Gamma}(S^1) \otimes X_G$ (4.2.16)

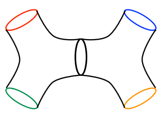
where X_G was introduced in (4.2.5). Here, $\mathcal{H}_{Q_\Gamma}(S^1) = H^*_G(\mathcal{M}_G) \otimes \operatorname{Frac} H^*_G(pt)$ appearing on the right hand side of each equation are considered with respect to three copies of distinct but isomorphic groups G.

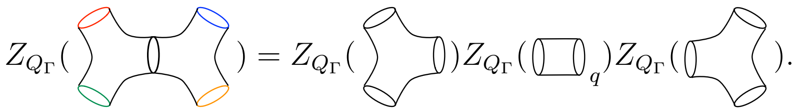
Furthermore, we introduce

$q^{\mathsf{N}} = Z_{Q_\Gamma}($  $_q) : \mathcal{H}_{Q_\Gamma}(S^1) \to \mathcal{H}_{Q_\Gamma}(S^1),$ (4.2.17)

Here two $\mathcal{H}_{Q_\Gamma}(S^1) = H^*_G(\mathcal{M}_G) \otimes \operatorname{Frac} H^*_G(pt)$ appearing in the right hand side are considered with respect to the same group G.

Then the gluing formula (4.2.8) can be understood as the decomposition of

$Z_{Q_\Gamma}($  $) : \mathcal{H}_{Q_\Gamma}(S^1)^{\otimes 2} \to \mathcal{H}_{Q_\Gamma}(S^1)^{\otimes 2} \otimes K_G^{\otimes 2} \otimes \operatorname{Frac} H^*_G(pt)$ (4.2.18) to



(4.2.19)

What is this 2d holomorphic extended QFT Q_Γ? There are two immediate clues:

- For a genus-g surface C_g with no puncture, we have

$$Z_{Q_\Gamma}(C_g) = Z^{\mathrm{Nek}}(S_\Gamma[C_g]) \in X_G^{\otimes 2(g-1)} \otimes (\mathrm{Frac}\, H_G^*(pt))^{\otimes 3(g-1)} \tag{4.2.20}$$

  which has transcendental degree $(g-1)\dim G$, as easily follows from (4.2.5). This is the dimension of the conformal block of the W_G algebra on a genus g Riemann surface.

- Also, the anomaly polynomial of $Q_\Gamma = S_\Gamma[\mathbb{R}^4_{\epsilon_1,\epsilon_2}]$ can be obtained by integrating the anomaly polynomial $A(S_\Gamma)$ of the 6d theory S_Γ, (3.2.3), over $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$ in the equivariant sense. As Q_Γ is a 2d holomorphic QFT, it should have an action of the Virasoro algebra on its space of states $\mathcal{H}_{Q_\Gamma}(S^1)$. The central charge c of this Virasoro algebra is encoded in the anomaly polynomial, and we find

$$c = \mathrm{rank}\, G + \frac{(\epsilon_1+\epsilon_2)^2}{\epsilon_1\epsilon_2} h^\vee(G)\dim G. \tag{4.2.21}$$

  This is closely related to the formula of the central charge c of the W_G algebra in the free field representation:

$$c = \mathrm{rank}\, G + (b+\frac{1}{b})^2 h^\vee(G)\dim G. \tag{4.2.22}$$

  where b is the background charge.

These two points strongly suggests that Q_Γ is in fact the theory of W_G conformal blocks itself, with the identification

$$b^2 = \frac{\epsilon_1}{\epsilon_2}. \tag{4.2.23}$$

4.3 W-algebras and Drinfeld-Sokolov reduction

Before continuing let us recall the basics of the W-algebras [FBZ04]. Given a finite-dimensional group G, we consider the affine Lie algebra $\hat{\mathfrak{g}}$. For simplicity we assume $\mathfrak{g}$ to be simply-laced. There is a way to construct $\hat{\mathfrak{g}}$ as a subalgebra of tensor products of $r = \mathrm{rank}\,\mathfrak{g}$ free bosons, with background charge $b\rho$, which is related to the level k of the affine algebra via

$$k = -h^\vee(G) + \frac{1}{b^2}. \tag{4.3.1}$$

Given a nilpotent element e, one can construct from $\hat{\mathfrak{g}}$ a vertex operator algebra $W(\mathfrak{g}, e)$ by a method called Drinfeld-Sokolov reduction. The central charge of the Virasoro subalgebra is

$$c = \dim \mathfrak{g}_{h=0} + \frac{1}{2}\dim \mathfrak{g}_{h=1} + 24(\frac{\rho}{b}+\frac{bh}{2})\cdot(\frac{\rho}{b}+\frac{bh}{2}) \tag{4.3.2}$$

where h is the Cartan element so that (e, h, f) is an SL(2) triple, and ρ is the Weyl vector. Let $\mathfrak{f} \subset \mathfrak{g}^e$ is the centralizer of (e, h, f). Denote by $\rho_{\mathfrak{f}}$ the Weyl vector of $\mathfrak{f}$. $W(\mathfrak{g}, e)$ has a subalgebra $\hat{\mathfrak{f}}$. For a simple component $\mathfrak{f}_0 \subset \mathfrak{f}^e$ the level is

$$k^{2d}_{\mathfrak{f}_0} = -\sum c_2(R_d) + b^2 \frac{1}{h^\vee(\mathfrak{f}_0)} \sum_d d c_2(R_d) \tag{4.3.3}$$

where

$$\mathfrak{g}_{\mathbb{C}} = \oplus_d R_d \otimes V_d \tag{4.3.4}$$

as before. In particular, $W(\mathfrak{g}, e = 0) = \hat{\mathfrak{g}}$ and $W_G = W(\mathfrak{g}, e_{\text{principal}})$. Note that in the latter case $h_{\text{principal}}/2 = \rho$ and many of the formulas below simplify. We note that the W_G algebra has Virasoro quasi-primary fields

$$W_{d_a}, \quad (a = 1, \ldots, \operatorname{rank} G) \tag{4.3.5}$$

of dimension d_a, where d_a is the a-th exponent of G plus one. In particular, $W_2 = T$ is the energy momentum tensor.

There is a functor which sends a highest-weight $\hat{\mathfrak{g}}$ representation to a highest-weight $W(\mathfrak{g}, e)$ representation. A highest weight irreducible representation of $\hat{\mathfrak{g}}$ is labeled by the level k and an element $\lambda \in \mathfrak{h}$ where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$. We denote it by $\mathcal{L}_\lambda$. Let us denote its image under the functor by $\mathcal{W}_{b\lambda}$. All highest weight irreducible representation of $W(\mathfrak{g}, e)$ is obtained in this manner. In particular, the vacuum representation is the image of the vacuum representation $\mathcal{L}_0$ and therefore is $\mathcal{W}_0$. The operator L_0 in the Virasoro subalgebra of $W(\mathfrak{g}, e)$ acts on the highest weight vector of $\mathcal{W}_a$ by a scalar multiplication by

$$L_0 = -\frac{1}{2} a \cdot a + a \cdot (\frac{\rho}{b} + \frac{bh}{2}) \tag{4.3.6}$$

The important feature is the shifted Weyl invariance of V_a:

$$\mathcal{W}_{a + \rho/b + bh/2} = \mathcal{W}_{wa + \rho/b + bh/2} \tag{4.3.7}$$

where w is a Weyl group element of $\mathfrak{f}$. The invariance of (4.3.6) is just one consequence.

We mainly consider the case when b is real. When

$$a = \sqrt{-1} m + (\frac{\rho}{b} + \frac{bh}{2}), \qquad m \in \mathfrak{h}_{\mathbb{R}} \tag{4.3.8}$$

the eigenvalues of L_0 on $\mathcal{W}_a$ is manifestly nonnegative. In this case there is a unitary structure on it and furthermore $\mathcal{W}_a$ is just the Verma module.

For $W(\mathfrak{g}, e_{\text{principal}})$, given another SL(2) triple (e, h, f), we also consider representations $\mathcal{W}_a$ where a is of the form

$$a = \sqrt{-1} m + (b + \frac{1}{b})(\rho - \frac{h}{2}), \quad m \in \mathfrak{h}^e_{\mathbb{R}}. \tag{4.3.9}$$

This is again a unitary representation. Note that the case (4.3.8) is when $e = 0$. We call these representations semi-degenerate.

4.4 Class S theories and W-algebras

Let us come back to the study of $Q_\Gamma = S_\Gamma[\mathbb{R}^4_{\epsilon_1,\epsilon_2}]$, which we guess is the theory of $W_G = W(\mathfrak{g}, e_{\text{principal}})$ algebra, with the parameter b given as in (4.2.23). Its space of states $\mathcal{H}_{Q_\Gamma}(S^1) = V_G$ was given in (4.2.4). This involved

$$H^*_G(pt) = \mathbb{C}[m_1, \ldots, m_{\operatorname{rank} G}]^W. \tag{4.4.1}$$

We consider an evaluation

$$m : H^*_G(pt) \to \mathbb{C} \tag{4.4.2}$$

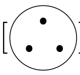
which we regard as an element $m \in \mathfrak{h}$ in the Cartan subalgebra. We thus obtain an infinite dimensional space $\mathcal{V}_m$ from $\mathcal{V}_G$. Our conjecture is that this $\mathcal{V}_m$ is, when m is generic, the Verma module of the W_G algebra, under the following matching of parameters:

$$\mathcal{V}_m = \mathcal{W}_{m'}, \quad m' = \frac{m}{\sqrt{\epsilon_1 \epsilon_2}} + (\sqrt{\frac{\epsilon_1}{\epsilon_2}} + \sqrt{\frac{\epsilon_2}{\epsilon_1}})\rho. \tag{4.4.3}$$

We now have a proof of this statement when G is of type A [SV12, MO12].

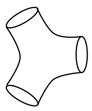
Nekrasov's partition function of a three-punctured sphere gives the following element:

$$Z^{\text{Nek}}(S_\Gamma[\;\;]) \in \mathcal{V}_{m_1} \otimes \mathcal{V}_{m_2} \otimes \mathcal{V}_{m_3} \otimes X_G \tag{4.4.4}$$



which define an intertwiner

$$Z_{Q_\Gamma}(\;\;) \in \mathcal{V}_{m_1} \otimes \mathcal{V}_{m_2} \to \mathcal{V}_{m_3} \otimes X_G. \tag{4.4.5}$$



Here $m_{1,2,3}$ are three evaluations of $H^*_G(pt)$. In the theory of W_G algebras, it is known that the space of intertwiners among three generic Verma modules has transcendental degree

$$\frac{1}{2}(\dim G - 3 \operatorname{rank} G). \tag{4.4.6}$$

which is equal to the transcendental degree of X_G as shown in (4.2.5). For a closed Riemann surface C_g of genus g without puncture, we already saw that

$$Z^{\text{Nek}}(S_\Gamma[C_g]) = Z_{Q_\Gamma}(C_g) \in X_G^{\otimes 2(g-1)} \otimes \operatorname{Frac} H^*_G(pt)^{3(g-1)}. \tag{4.4.7}$$

has the correct transcendental degree as the space of the conformal blocks of W_G algebra with generic c on the Riemann surface of genus $g > 1$. Therefore, our conjecture is that Nekrasov's partition function of class S theory provides the space of conformal blocks of W_G algebras.

So far we only considered Riemann surfaces with punctures with label $e = 0$ only. For other regular punctures labeled by (e, m), let us denote the space we obtain from the consideration of the S_Γ theory by $\mathcal{V}^e_m$. Again, when m is a generic element in $\mathfrak{g}^e$, we conjecturally identify it as a semi-degenerate representation of the W_G algebra as defined in (4.3.9):

$$\mathcal{V}^e_m = \mathcal{W}_{m'}, \quad m' = \frac{m}{\sqrt{\epsilon_1\epsilon_2}} + (\sqrt{\frac{\epsilon_1}{\epsilon_2}} + \sqrt{\frac{\epsilon_2}{\epsilon_1}})(\rho - \frac{h}{2}). \tag{4.4.8}$$

As an example, consider a puncture labeled by the principal element $e_{\text{principal}}$. It is equivalent to not having a puncture. m is necessarily 0, and

$$\mathcal{V}^e_0 = \mathcal{W}_0 \tag{4.4.9}$$

which is the vacuum representation of the W_G algebra. This agrees with the idea that without any puncture in the 2d QFT, the only operation doable on a Riemann surface is to insert a vacuum representation.

As another example, let us recall that we have, for $G = A_{N-1}$,

$$S_\Gamma[\;; e = 0, e = 0, e = [N-1, 1]] = \text{Hyp}(V_1 \otimes \bar{V}_2 \oplus V_2 \otimes \bar{V}_1). \tag{4.4.10}$$

where $V_i \simeq \mathbb{C}^N$. Then

$$Z^{\text{Nek}}(S_\Gamma[\;; (e = 0, m_1), (e = 0, m_2), (e = [N-1, 1], \mu)]) : \mathcal{V}_{m_1} \otimes \mathcal{V}^{[N-1,1]}_\mu \to \mathcal{V}_{m_2} \tag{4.4.11}$$

and μ is the equivariant parameter $H^*_{G[N-1,1]}(pt) \simeq \mathbb{C}[\mu]$. The intertwiner here is uniquely determined, as $\mathcal{M}_{\text{Coulomb}}(\text{Hyp}(V_1 \otimes \bar{V}_2 \oplus V_2 \otimes \bar{V}_1))$ is a point. It was given in (4.1.11) as the Euler class of the determinant line bundle of the Dirac operator associated to $V_1 \otimes \bar{V}_2$. It is satisfying to know that the space of the intertwiner (4.4.11) above, under the identification (4.4.3) and (4.4.8), is known to be unique.

In general, we can consider the theory

$$Q = S_\Gamma[\;; (e_1, m_1), (e_2, m_2), (e_3, m_3)] \tag{4.4.12}$$

and the element

$$Z^{\text{Nek}}(Q) : \mathcal{V}^{e_1}_{m_1} \otimes \mathcal{V}^{e_2}_{m_2} \to \mathcal{V}^{e_3}_{m_3} \otimes X_{G,e_1,e_2,e_3}. \tag{4.4.13}$$

Here we have

$$X_{G,e_1,e_2,e_3} = \mathbb{C}(a_1, \ldots, a_{\operatorname{rank} Q}) \tag{4.4.14}$$

is the algebra of holomorphic functions on a patch of $\mathcal{M}_{\text{Coulomb}}(Q)$, and $\operatorname{rank} Q$ was given in (3.8.3). When $G = A_{N-1}$, the transcendental dimension of the space of the intertwiner of W_G algebra among the representations $\mathcal{V}^{e_i}_{m_i}$, $(i = 1, 2, 3)$ is known and it agrees with $\operatorname{rank} Q$. This is another check of our proposed identification (4.4.8).

We can also consider irregular punctures. The only irregular puncture discussed in this review is the one introduced in Sec. 2.11.2. There, we saw that the Donagi-Witten integrable system of triv$/\!\!/\!\!/G$ for simply-laced G is given by a G-Hitchin system on a sphere with two irregular punctures at $z = 0, \infty$. Correspondingly, we expect that Nekrasov's partition function has the form

$$(\psi, q^{\mathsf{N}}\psi) = (q^{\mathsf{N}/2}\psi, q^{\mathsf{N}/2}\psi) \tag{4.4.15}$$

where ψ is a state in the representation corresponding to the irregular puncture. The formula for Nekrasov's partition function (4.1.22), when applied to the pure theory $\mathrm{triv}/\!\!/\!\!/G$, gives

$$Z^{\mathrm{Nek}}(\mathrm{triv}/\!\!/\!\!/G) \sim \langle [\mathcal{M}_G], q^{\mathsf{N}} \cdot 1\rangle. \tag{4.4.16}$$

Therefore, we find the representation to be $\mathcal{V}_a$ we already discussed, where

$$a : H^*_G(pt) \to \mathbb{C} \tag{4.4.17}$$

is a point on the Coulomb branch $\mathcal{M}_{\mathrm{Coulomb}}(\mathrm{triv}/\!\!/\!\!/Q) \simeq \mathbb{C}[\mathfrak{g}_{\mathbb{C}}]^{G_{\mathbb{C}}}$, and

$$\psi = [\mathcal{M}_G] = \oplus_{n\geq 0}[\mathcal{M}_{G,n}]. \tag{4.4.18}$$

The boundary condition of the Hitchin field, after the application of the Hitchin map, is in general given by

$$u_{d_a} \sim O(1)(\frac{dz}{z})^{d_a},\ (d_a \neq h^\vee(G)), \qquad u_{h^\vee(G)} \sim \frac{\Lambda^{h^\vee(G)}}{z}(\frac{dz}{z})^{h^\vee(G)}. \tag{4.4.19}$$

We propose in general that u_{d_a} is the expectation value of W_G quasiprimary fields $W_{d_a}(z)$ (4.3.5). In terms of Fourier modes, the standard convention is

$$W_{d_a}(z) \sim \sum \frac{W_{d_a,i}}{z^{d_a+i}} dz^{d_a} \tag{4.4.20}$$

which means that the state $\psi' = q^{\mathsf{N}/2}\psi$ corresponding to the pure theory is given by the condition

$$W_{d_a,i}\psi' = 0,\ ((d_a \neq h^\vee(G) \text{ and } i \geq 1) \text{ or } i \geq 2), \quad W_{h^\vee(G),1}\psi' = \Lambda^{h^\vee(G)}\psi'. \tag{4.4.21}$$

This is the condition of a Whittaker state in the representation. Note that $q = \Lambda^{2h^\vee(G)}$ as seen in (2.6.5), and recall that we identified N and L_0. Then the conditions (4.4.21) boils down to the conditions

$$W_{d_a,i}\psi = 0,\ ((d_a \neq h^\vee(G) \text{ and } i \geq 1) \text{ or } i \geq 2), \quad W_{h^\vee(G),1}\psi = \psi. \tag{4.4.22}$$

Indeed, when G is type A, this statement that ψ given geometrically by (4.4.18) is a Whittaker state given by these conditions is already proved.

4.5 Nekrasov's partition function with surface operator

So far we considered the 6d theory S_Γ on $\mathbb{R}^4_{\epsilon_1,\epsilon_2}\times C$. Let us pick a subspace $\mathbb{R}^2_{\epsilon_1}\times\{0\}\subset\mathbb{R}^4_{\epsilon_1,\epsilon_2}$ and introduce a 4d operator with the label (e,m) on $\mathbb{R}^2_{\epsilon_1}\times\{0\}\times C$. Then we can repeat our analysis above, and there should be a 2d theory

$$Q_{\Gamma,(e,m)} = S_\Gamma[\mathbb{R}^4_{\epsilon_1,\epsilon_2}\supset\mathbb{R}^2_{\epsilon_1};(e,m)] \tag{4.5.1}$$

satisfying the defining relation

$$Z_{Q_{\Gamma,(e,m)}}(C) = Z^{\mathrm{Nek}}(S_\Gamma[C])(\mathbb{R}^4_{\epsilon_1,\epsilon_2}\supset\mathbb{R}^2_{\epsilon_1};(e,m)). \tag{4.5.2}$$

The questions then are

- What is the theory $Q_{\Gamma,(e,m)}$?
- What is the 2d operator labeled by (e,m) on $\mathbb{R}^2_{\epsilon_1}\times\{0\}\subset\mathbb{R}^4_{\epsilon_1,\epsilon 2}$ of the 4d theory $S_\Gamma[C]$?

For the former question, an obvious guess is the W-algebra $W(\mathfrak{g},e)$ given by the Drinfeld-Sokolov reduction, briefly recalled in Sec. 4.3. From the formula of the central charge (4.3.2), we see that

$$\begin{aligned}c(W(\mathfrak{g},e)) &- c(W(\mathfrak{g},e_{\mathrm{principal}}))\\ &= (\dim\mathfrak{g}_{h=0}-\mathrm{rank}\,G)-\frac{1}{2}\dim\mathfrak{g}_{h=1}-12\rho\cdot(\rho-\frac{h}{2})+\frac{\epsilon_2}{\epsilon_1}(\frac{h}{2}\cdot\frac{h}{2}-\rho\cdot\rho).\end{aligned} \tag{4.5.3}$$

where we used the relation (4.2.23). This should be given by the integral of the anomaly polynomial of the 4d operator of label e integrated over $\mathbb{R}^2_{\epsilon_1}$. Note that this is given by a linear combination of terms

$$\dim\mathfrak{g}_{h=0}-\mathrm{rank}\,G,\quad \dim\mathfrak{g}_{h=1},\quad \rho\cdot(\rho-\frac{h}{2}),\quad \frac{h}{2}\cdot\frac{h}{2}-\rho\cdot\rho. \tag{4.5.4}$$

The quantities $n_{v,h}(e)$ given in (3.6.4), which are contributions of a 4d operator to the central charges $n_{v,h}$, are also given as linear combinations of the same four terms. This is consistent to the idea that both $n_{v,h}(e)$ and $c(W(\mathfrak{g},e))-c(W(\mathfrak{g},e_{\mathrm{principal}}))$ are given by integrating the anomaly polynomials of the 4d operator of type e. Note that the equivariant integrals

$$\int_{\mathbb{R}^4_{\epsilon_1,\epsilon_2}} 1 = \frac{1}{\epsilon_1\epsilon_2},\quad \int_{\mathbb{R}^2_{\epsilon_1}} 1 = \frac{1}{\epsilon_1} \tag{4.5.5}$$

would naturally provide coefficients of the form $1/(\epsilon_1\epsilon_2)$ or $1/\epsilon_1$ in the linear combination. Here the fact that the formula (4.5.3) has terms of the form $1/\epsilon_1$ and no terms of the form $1/(\epsilon_1\epsilon_2)$ agrees with the fact that the 4d operator is on $\mathbb{R}^2_{\epsilon_1}\times C$.

The algebra $W(\mathfrak{g},e)$ contains the affine subalgebra $\hat{\mathfrak{g}}^e$. For a simple component $\mathfrak{f}\subset\mathfrak{g}^e$, its level $k^{2d}_{\mathfrak{f}}$ is given by (4.3.3). Similarly, a 4d operator of type e gave rise to a G^e-symmetric

4d theory, whose k^{4d} is given in (3.6.6). Again, we see that these two expressions are rather similar, and in terms of $b^2 = \epsilon_2/\epsilon_1$ we only see the coefficients of the form $1/\epsilon_1$. This again gives a small piece of evidence to our general proposal.

To answer the latter question, let us recall the discussions in Sec. 3.8. There, we considered the 4d operator with label (e, m) on

$$X^4 \times C^2 \supset X^4 \times \{pt\}. \tag{4.5.6}$$

There, we saw that the Hitchin field ϕ had the residue of the form

$$\phi \sim \alpha \frac{dz}{z} \tag{4.5.7}$$

where α was given by the formula (3.8.2). In particular, consider the case when α is semisimple. Let $\mathfrak{l}$ be the Levi subalgebra commuting with α. Then e is given by a principal nilpotent element of $\mathfrak{l}$.

The setup here just has a different four-dimensional subspace

$$\mathbb{R}^4 \times C^2 \supset \mathbb{R}^2 \times \{pt\} \times C^2. \tag{4.5.8}$$

Therefore the behavior of the fields transverse to the 4d subspace should be the same. Then, a natural generalization of the conjecture is that there is a natural action of $W(\mathfrak{g}, e)$ on

$$H^*_{G\times \mathrm{U}(1)^2}(\mathcal{M}_{ASD,G,\alpha}) \tag{4.5.9}$$

where $\mathcal{M}_{ASD,G,\alpha}$ is the moduli space of the ASD connection on $\mathbb{R}^4$ with a singularity transverse to $\mathbb{R}^2 \subset \mathbb{R}^4$ given by a semisimple conjugacy class α. When there is no singularity, $\alpha = 0$, and e is the principal nilpotent element of $\mathfrak{g}$. Then $W(\mathfrak{g}, e)$ is just W_G, and we come back to the original conjecture. When the singularity α is a regular semisimple element, i.e. when the Levi subalgebra $\mathfrak{l}$ is Abelian of maximal rank , then e is zero. Then $W(\mathfrak{g}, e)$ is just the affine Lie algebra $\hat{\mathfrak{g}}$. The action of $\mathfrak{g}$ with the level (4.3.1) on the space (4.5.9) has been constructed [Bra04].

4.6 S^4 partition function

Recall that in 2d WZW model for the affine Lie algebra $\mathfrak{g}$ of positive integral level k, we first constructed a finite-dimensional vector bundle over the moduli of the Riemann surface. This vector bundle had a finite number of natural sections $\chi_i(\tau)$, where i labels the sections and τ denotes the complex structure of the surface. These are the conformal blocks of $\mathfrak{g}$ at level k. The mapping class group naturally acts on the space of sections.

The 2d conformal field theory on T^2 is a modular invariant combination

$$\sum c_{i\bar{j}} \chi_i(\tau) \overline{\chi_j(\tau)} \tag{4.6.1}$$

where $c_{i\bar{j}}$ is an integer valued matrix. Usually one of the modular invariant choice is

$$\sum_i \chi_i(\tau) \overline{\chi_i(\tau)} \tag{4.6.2}$$

which is called the diagonal modular invariant.

The 2d WZW models of $\mathfrak{g}$ at level k are called rational CFTs. Here rationality refers to the finite dimensionality of the space of conformal blocks. In the case of W_G algebra at generic c, the dimension of the space of the conformal blocks is infinite dimensional, but we can still form a diagonal invariant. We see in the following that such a diagonal invariant naturally arises by considering the partition function of $S_\Gamma[C]$ on the sphere. These are the simplest examples of irrational CFTs.

Let Q an $\mathcal{N}=2$ supersymmetric theory. Consider the following squashed four-sphere

$$S^4_b := \{(x,z,w) \in \mathbb{R}\times\mathbb{C}\times\mathbb{C} \mid x^2 + b|z|^2 + \frac{1}{b}|w|^2 = 1\}. \tag{4.6.3}$$

This only specifies the metric. The $\mathcal{N}=2$ supersymmetric extension of the concept of the metric has a complex function in it, and we choose it appropriately so that the supermetric has a superisometry. It is known that

$$Z_Q(S^4_b) = \int_\Gamma Z^{\mathrm{Nek}}(Q)(a)\overline{Z^{\mathrm{Nek}}(Q)(a)}da_1\ldots da_{\mathrm{rank}\,Q} \tag{4.6.4}$$

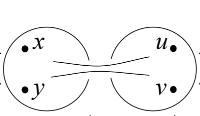
where Γ is a specific real $\mathrm{rank}\,Q$ dimensional cycle in $\mathcal{M}_{\mathrm{Coulomb}}$ [Pes07, HH12].

When $Q = S_\Gamma[C_{g,n}]$, $Z_Q(S^4_b)$ determines a function on the moduli space $\mathcal{M}_{g,n}$ of genus-g Riemann surface with n marked punctures , and is the diagonal invariant of the W_G conformal block, if we assume our conjecture that $Z^{\mathrm{Nek}}(Q)$ gives a natural section of the conformal blocks.

This 2d CFT is called the Toda theory for general G, and the Liouville theory in the simplest case $G = A_1$. The cycle Γ in this case is determined as follows: on $\mathcal{V}_m$ with $m \in \mathfrak{h}_\mathbb{C}$, the Virasoro subalgebra acts with

$$L_0 = -\langle m,m\rangle + \frac{h^\vee(G)\dim G}{24}(b+\frac{1}{b})^2 \tag{4.6.5}$$

as already discussed in (4.3.6). We only pick unitary representations where $L_0 \geq 0$. Then it is natural to take $m \in \sqrt{-1}\mathfrak{h}_\mathbb{R}$.

In particular, for $G = A_1$ and $Q = S_{A_1}[$ 

$]$, using Gaiotto's gluing (3.7.1) and the formula for Nekrasov's partition function (4.1.22), we have

$$Z_Q(S^4_b) = \int_\mathbb{R} da \frac{\prod_{\pm\pm\pm}\Gamma_B(\pm m_1 \pm m_2 \pm a)\prod_{\pm\pm\pm}\Gamma_B(\pm m_3 \pm m_e \pm a)}{\prod_{pm}\Gamma_B(\pm 2a)\Gamma_B(\epsilon_1+\epsilon_2 \pm 2a)} \times e^{-4\pi\,\mathrm{Im}\,\tau\langle a\rangle} Z_{\mathrm{inst}}(a,m_i;\tau)\overline{Z_{\mathrm{inst}}(a,m_i;\tau)} \tag{4.6.6}$$

where

$$Z_{\mathrm{inst}}(a,m_1,m_2;\tau) = \langle[\mathcal{M}_{A_1}], q^{\mathsf{N}}Z^{\mathrm{Nek}}(\mathrm{Hyp}(V_a\otimes V_{m_1}\otimes V_{m_2}))\rangle. \tag{4.6.7}$$

where $V_x \simeq \mathbb{C}^2$ has an action of SU(2) with $H^*_{\mathrm{SU}(2)}(pt) = \mathbb{C}[x]$. As Z^{Nek} is given in (4.1.11), this is a explicitly computable quantity, and is known as the Liouville four-point functions in the 2d CFT literature. For an account on the Liouville theory readable for mathematicians, see e.g. [Tes01].

5 Superconformal indices and Macdonald polynomials

The content of this section is based on a series of papers [GPRR10, GRRY10, GRRY11b, GRRY11a, GRR12].

5.1 Definition

For a G-symmetric $\mathcal{N}=2$ supersymmetric theory Q with $\mathrm{U}(1)_R$ symmetry, let us consider its partition function on $S^1 \times S^3$ with the following flat bundle on it. Namely, we start from $\mathbb{R}\times S^3$, and when we identify $\{x\}\times S^3$ and $\{x+\beta\}\times S^3$, we use the transformations

$$g \in G, \quad s \in \mathrm{U}(1), \quad t \in \mathrm{U}(1) \subset \mathrm{SU}(2), \quad (p,q) \in \mathrm{U}(1)^2 \subset \mathrm{Spin}(4) \tag{5.1.1}$$

where $\mathrm{U}(1)\times\mathrm{SU}(2)$ is the R-symmetry and $\mathrm{Spin}(4)$ is the isometry of S^3. Then we have

$$Z_Q(S^1\times S^3;\beta,p,q,s,t,g) = \mathrm{tr}_{\mathcal{H}_Q(S^3)}(-1)^F e^{-\beta H} pqtsg \tag{5.1.2}$$

where on the left hand side p, q, t and s are considered as complex numbers with absolute number one, and on the right hand side they are considered elements of the groups acting on $\mathcal{H}_Q(S^3)$. The space of states $\mathcal{H}_Q(S^3)$ is $\mathbb{Z}_2$ graded, and $(-1)^F$ is this $\mathbb{Z}_2$ grading. Also,

$$e^{-\beta H} : \mathcal{H}_Q(S^3) \to \mathcal{H}_Q(S^3) \tag{5.1.3}$$

is the operator defined by $Z_Q([0,\beta]\times S^3)$. This supertrace becomes computable when the background has a superisometry. This translates to the condition that two specific linear combinations of β, $\log t$, $\log s$, $\log p$ and $\log q$ should vanish. We write β and s in terms of p, q and t, and write the resulting partition function as $Z^{\mathrm{SCI}}_{p,q,t}(Q)$; we leave the dependence on g implicit in the notation. This is called the superconformal index of the theory Q. We use physicists normalization of t, so that $\mathrm{tr}_{\mathbb{C}^2}\, t = t^{1/2}+t^{-1/2}$. Therefore the expressions below are Laurent polynomials of p, q, $t^{1/2}$.

Let us view the superconformal index from a slightly different point. We first note that for general d-dimensional conformal QFT Q, there is the identification

$$\mathcal{H}_Q(S^{d-1}) = \mathcal{V}_Q \tag{5.1.4}$$

where the left hand side is the state of states on S^{d-1} and the right hand side is the space of point operators. The element e^{-H} defined in (5.1.3) acting on $\mathcal{H}_Q(S^3)$ can be identified with the grading on $\mathcal{V}_Q$. This is called the state-operator correspondence.

When $d=4$, Q is $\mathcal{N}=2$ supersymmetric and conformal with G symmetry, Q is called $\mathcal{N}=2$ superconformal with G symmetry. In this case $\mathcal{H}_Q(S^3)$ has a natural action of the superconformal group

$$\mathrm{SU}(2,2|2) \tag{5.1.5}$$

times G. The corresponding super Lie algebra is $\mathfrak{su}(2,2|2)\times\mathfrak{g}$. The character of the $\mathfrak{su}(2,2|2)\times\mathfrak{g}$ representation $\mathcal{H}_Q(S^3)$ is extremely hard to compute. An easier quantity to compute is obtained as follows. Pick an odd element $\delta \in \mathfrak{su}(2,2|2)$ with $\delta^2 = 0$. The centralizer of δ in $\mathrm{SU}(2,2|2)$ is $\mathrm{SU}(1,1|2)$. Then the cohomology $H(\mathcal{H}_Q(S^3),\delta)$ has an action of $\mathrm{SU}(1,1|2)\times G$, and the superconformal index is the graded virtual character of the $\mathrm{SU}(1,1|2)\times G$ representation $H(\mathcal{H}_Q(S^3),\delta)$:

$$Z^{\mathrm{SCI}}_{p,q,t}(Q) = \mathrm{tr}_{H(\mathcal{H}_Q(S^3),\delta)}(-1)^F pqtg = \mathrm{tr}_{\mathcal{H}_Q(S^3)}(-1)^F pqtg \tag{5.1.6}$$

where $(p,q,t)\in \mathrm{SU}(1,1|2)$ is taken from the Cartan subgroup of $\mathrm{SU}(1,1|2)$. This explains why we have *three* parameters p,q,t.

5.2 Basic properties

For $Q = \mathrm{Hyp}(V)$ for a pseudoreal representation V of a group G, we have

$$Z^{\mathrm{SCI}}_{p,q,t}(\mathrm{Hyp}(V)) = \prod_{w:\text{weights of } V} \Gamma_{p,q}(t^{1/2}z^w) \tag{5.2.1}$$

where $\Gamma_{p,q}(x)$ is the elliptic gamma function

$$\Gamma_{p,q}(x) = \prod_{m,n\geq 0} \frac{1-x^{-1}p^{m+1}q^{n+1}}{1-xp^mq^n} \tag{5.2.2}$$

and we regard $z\in G$ as an element in the Cartan torus $z=(z_1,\ldots,z_r)\in T^r$ and $z^w = \prod_i z_i^{w_i}$ for a weight $w=(w_1,\ldots,w_r)$. This can be checked by recalling that a hypermultiplet consists of a free boson and a free fermion Sec. 2.5, and that the state of states $\mathcal{H}_Q(S^3)$ of a free boson and a free fermion is given by the spectrum of the Laplacian and the Dirac operator, respectively, as we saw in Sec. 1.16 and in Sec. 1.17. In more detail, we have $\mathcal{H}_Q(S^3) = \mathcal{H}_{B_4(V)}(S^3)\otimes \mathcal{H}_{F_4(V)}(S^3)$ where

$$\begin{aligned} \mathcal{H}_{B_4(V)}(S^3) &= \mathbb{C}\oplus\mathcal{A}\oplus \mathrm{Sym}^2\mathcal{A}\oplus\cdots \\ \mathcal{H}_{F_4(V)}(S^3) &= \mathbb{C}\oplus\mathcal{B}^+\oplus \Lambda^2\mathcal{B}^+\oplus\cdots \end{aligned} \tag{5.2.3}$$

with

$$\mathcal{A} = \Gamma(S^3, \underline{V\oplus\bar{V}}), \quad \mathcal{B} = \Gamma(S^3, \underline{V\otimes S\oplus V\otimes S}) \tag{5.2.4}$$

and $\mathcal{B}^+$ is the subspace where the Dirac operator has positive eigenvalue. The superconformal group $\mathrm{SU}(2,2|2)$ contains the conformal group $\mathrm{Spin}(4,2)\simeq \mathrm{SU}(2,2)$ as the subgroup, and it is a fact that

$$\Gamma(S^{d-1},\underline{\mathbb{C}}), \quad \Gamma(S^{d-1},\underline{S\oplus S})^+ \tag{5.2.5}$$

are natural irreducible representations of the conformal group $\mathrm{Spin}(d,2)$. When V is irreducible as pseudoreal representations, the combination $\mathcal{A}\oplus\mathcal{B}^+$ appearing in (5.2.3) is an irreducible representation of $\mathrm{SU}(2,2|2)\times G$. Then $\mathcal{H}_Q(S^3)$ is naturally a $\mathbb{Z}_2$-graded polynomial algebra over $\mathcal{A}\oplus\mathcal{B}^+$, which inherits the action of $\mathrm{SU}(2,2|2)\times G$.

More explicitly, under the compact subgroup $\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2 \times \mathrm{SU}(2)_R \times G$ of $\mathrm{SU}(2,2|2) \times G$, we have

$$\mathcal{A} = \left[\bigoplus_{d \geq 0} V_d^{(1)} \otimes V_d^{(2)} \right] \otimes R \otimes V, \tag{5.2.6}$$

$$\mathcal{B}^+ = \left[\bigoplus_{d \geq 0} (V_d^{(1)} \otimes V_{d+1}^{(2)} \oplus V_{d+1}^{(1)} \otimes V_d^{(2)}) \right] \otimes V \tag{5.2.7}$$

where $V_d^{(i)}$ is the d-dimensional irreducible representation of $\mathrm{SU}(2)_i$, R is the two-dimensional irreducible representation of $\mathrm{SU}(2)_R$. From this we find that

$$[\mathcal{H}_{\mathrm{Hyp}(V)}(S^3)] = \bigotimes_{m,n \geq 0} [\mathrm{Sym}^\bullet(T^{\otimes 1/2} \otimes P^{\otimes m} Q^{\otimes n} \otimes V) \otimes \wedge^\bullet(T^{\otimes -1/2} \otimes P^{\otimes (m+1)} Q^{\otimes (n+1)} \otimes V)] \tag{5.2.8}$$

as an element in the representation ring of $G \times \mathrm{U}(1)^3$, where T, P, Q are the one-dimensional representations for $(t,p,q) \in \mathrm{U}(1)^3 \subset \mathrm{SU}(1,1|2) \subset \mathrm{SU}(2,2|2)$.

Next, the superconformal index behaves multiplicatively under the multiplication of QFTs:

$$Z^{\mathrm{SCI}}_{p,q,t}(Q \times Q') = Z^{\mathrm{SCI}}_{p,q,t}(Q) Z^{\mathrm{SCI}}_{p,q,t}(Q'). \tag{5.2.9}$$

Also, for a $G \times F$-symmetric theory Q, $Q /\!\!/\!\!/ G \,\big|_\tau$ is F-symmetric and its superconformal index is independent of τ and is given by

$$Z^{\mathrm{SCI}}_{p,q,t}(Q /\!\!/\!\!/ G \,\big|_\tau) = \left(\frac{1}{\Gamma_{p,q}(t)\Gamma'_{p,q}(1)}\right)^r \frac{1}{|W|} \int_{T^r} \prod_{i=1}^r \frac{dz_i}{2\pi\sqrt{-1} z_i} \left(\prod_{\alpha:\text{roots of } G} \frac{1}{\Gamma_{p,q}(z^\alpha)\Gamma_{p,q}(tz^\alpha)} \right) Z_{\mathrm{SCI}}(Q). \tag{5.2.10}$$

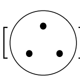
where $z \in T^r \subset G$ and $|W|$ is the order of the Weyl group. At the level of the representation ring the operation

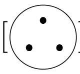
$$|W|^{-1} \int_{T^r} \prod \frac{dz_i}{2\pi\sqrt{-1} z_i} \prod_\alpha z^\alpha : \mathrm{Rep}(G \times F) \ni [V] \mapsto [V^G] \in \mathrm{Rep}(F) \tag{5.2.11}$$

which extracts the invariant part under G.

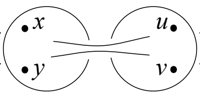
5.3 Application to the theories of class S

Recall

$S_{A_1}[$$] = \mathrm{Hyp}(V_1 \otimes V_2 \otimes V_3)$ (5.3.1) where $V_i \simeq \mathbb{C}^2$ is the defining representation of A_1. Then

$$Z^{\mathrm{SCI}}_{p,q,t}(S_{A_1}[$$

$$]) = \prod_{\pm\pm\pm} \Gamma_{p,q}(t^{1/2}u^{\pm}v^{\pm}z^{\pm}) \tag{5.3.2}$$

where $u, v, w \in \mathrm{U}(1)^3 \subset \mathrm{SU}(2)^3$. Then, from the gluing axiom, we have

$$Z^{\mathrm{SCI}}_{p,q,t}(S_{A_1}[$$


$$\begin{aligned}]) = {}& \frac{1}{\Gamma_{p,q}(t)\Gamma'_{p,q}(1)} \frac{1}{2} \oint \frac{dz}{2\pi\sqrt{-1}z} \prod_{\pm} \frac{1}{\Gamma_{p,q}(z^{\pm 2})\Gamma_{p,q}(tz^{\pm 2})} \\ &\times \prod_{\pm\pm\pm} \Gamma_{p,q}(t^{1/2}u^{\pm}v^{\pm}z^{\pm}) \prod_{\pm\pm\pm} \Gamma_{p,q}(t^{1/2}x^{\pm}y^{\pm}z^{\pm}). \end{aligned} \tag{5.3.3}$$

It should be symmetric under the exchange $u \leftrightarrow x$, which is not apparent from the integral form on the right hand side.

The measure appearing in (5.2.10) is an elliptic generalization of the Macdonald inner product. When $p = 0$, it becomes

$$(\prod_{n\geq 0} \frac{1-q^{n+1}}{1-tq^n})^r \frac{1}{|W|} \prod_{i=1}^{r} \frac{dz_i}{2\pi\sqrt{-1}z_i} \prod_{\alpha} \prod_{n\geq 0} \frac{1-q^n z^{\alpha}}{1-tq^n z^{\alpha}} K(z)^{-2} \tag{5.3.4}$$

where

$$K(z) = (\prod_{n\geq 0} \frac{1}{1-tq^n})^r \prod_{\alpha} \prod_{n\geq 0} \frac{1}{1-tq^n z^{\alpha}}. \tag{5.3.5}$$

and

$$\frac{1}{|W|} \prod_{i=1}^{r} \frac{dz_i}{2\pi\sqrt{-1}z_i} \prod_{\alpha} \prod_{n\geq 0} \frac{1-q^n z^{\alpha}}{1-tq^n z^{\alpha}} \tag{5.3.6}$$

is the standard measure appearing in the theory of Macdonald polynomials. This means that the orthonormal polynomials under (5.3.4) are

$$K(z)\underline{P}_{\lambda}(z) \tag{5.3.7}$$

where

$$\underline{P}_{\lambda}(z) = (\prod_{n\geq 0} \frac{1-q^{n+1}}{1-tq^n})^{-r/2} N_{\lambda}^{-1/2} P_{\lambda}(z). \tag{5.3.8}$$

Here, $P_{\lambda}(z)$ is the standard Macdonald polynomial and

$$N_{\lambda} = \frac{1}{|W|} \int_{T^r} \prod_{i=1}^{r} \frac{dz_i}{2\pi\sqrt{-1}z_i} \prod_{\alpha} \prod_{n\geq 0} \frac{1-q^n z^{\alpha}}{1-tq^n z^{\alpha}} P_{\lambda}(z) P_{\lambda}(z^{-1}) \tag{5.3.9}$$

is the norm of the Macdonald polynomial, which has an explicit infinite-product form.

Consider a class $\mathcal{S}$ theory $Q = S_{\Gamma}[C_g, e_1, \ldots, e_n]$ associated to a curve C of genus g with n punctures labeled by $e_1, \ldots, e_n$. This is a $\prod_i G^{e_i}$ symmetric theory. Then the superconformal index is a function of p, q, t and z_i, where z_i is an element of the Cartan torus of G^{e_i}, which we further regard as an element of the Cartan torus of G.

Then the superconformal index of Q , when $p=0$, is conjecturally given by

$$Z^{\rm SCI}_{p=0,q,t}(Q)(\{z_i\}) = \frac{\prod_{i=1}^n K_{e_i}(z)}{{K_\rho}^{2g-2+n}} \sum_\lambda \frac{\prod_{i=1}^n \underline{P}_\lambda(z_i t^{h_i/2})}{\underline{P}_\lambda(t^\rho)^{2g-2+n}} \tag{5.3.10}$$

Here, at each puncture labeled by e_i, we pick an SL(2) triple (e_i,h_i,f_i). We then used the map

$$G^e \times \rho_e(\mathrm{SU}(2)) \to G \tag{5.3.11}$$

to define

$$(z,t) \mapsto z t^{h/2}. \tag{5.3.12}$$

To define $K_e(z)$, let us make the decomposition

$$\mathfrak{g}_\mathbb{C} = \oplus_d R_d \otimes V_d \tag{5.3.13}$$

as always, where V_d is an irreducible representation of dimension d of $\rho_e(\mathrm{SU}(2))$. Then

$$K_e(z) = \prod_d \prod_{n=0}^\infty \prod_{w:\text{weights of } R_d} \frac{1}{1-t^{(d+1)/2} q^n z^w}. \tag{5.3.14}$$

Note that $K_{e=0}(z) = K(z)$ defined above. The form (5.3.10) makes the associativity transparent.

When the class S theory becomes just $\mathrm{Hyp}(V)$, the general formula (5.3.10) gives conjectural formula rewriting an infinite product determined by the weights of V into a sum over λ. We discussed many such cases in Sec. 3.11. Let us consider the simplest case (5.3.1). We now have an identity

$$\prod_{\pm\pm\pm} \prod_{n\geq 0} \frac{1}{1-t^{1/2} a_1^\pm a_2^\pm a_3^\pm q^n} = \\ \prod_{n\geq 0}\prod_{i=1}^3 \left(\frac{1}{1-ta_i^2 q^n} \frac{1}{1-t} \frac{1}{1-ta_i^{-2}q^n}\right) \sum_{\lambda=0}^\infty \frac{\prod_{i=1}^3 \underline{P}_\lambda(a_i,a_i^{-1};q,t)}{\underline{P}_\lambda(t^{1/2},t^{-1/2};q,t)} \tag{5.3.15}$$

where $\underline{P}_\lambda$ is the A_1 Macdonald polynomial in a nonconventional normalization (5.3.8).

When $p\neq 0$ the generalization of (5.3.10) will be to set

$$K_e(z) = \prod_d \prod_{m,n\geq 0} \prod_{w:\text{weights of } R_d} \frac{1-t^{(d-1)/2} p^{m+1} q^{n+1} z^w}{1-t^{(d+1)/2} p^m q^n z^w}. \tag{5.3.16}$$

and replace $\underline{P}_\lambda$ by $\underline{\Psi}_\lambda$ which is orthonormal under the elliptic measure

$$\left(\frac{\prod_{m,n\geq 0,\ (m,n)\neq(0,0)} (1-p^m q^n)}{\prod_{m,n\geq 0} (1-tp^m q^n)(1-t^{-1}p^{m+1}q^{n+1})}\right)^r \\ \times \frac{1}{|W|} \prod_{i=1}^r \frac{dz_i}{2\pi\sqrt{-1} z_i} \prod_\alpha \prod_{m,n\geq 0} \frac{1-p^m q^n z^\alpha}{(1-tp^m q^n z^\alpha)(1-t^{-1}p^{m+1}q^{n+1}z^\alpha)} \tag{5.3.17}$$

The problem is that the existence and the properties of $\underline{\Psi}_\lambda$ is not quite known in the mathematical literature yet. At least the associativity of the case $\Gamma = A_1$, (5.3.3), is shown by a different method [vdB11].

When $q = t$, the Macdonald polynomial just becomes the character, and the formula (5.3.10) becomes the partition function of a 2d theory $\mathrm{YM}_2^q(G)$ called q-deformed Yang-Mills theory on C:

$$Z_{S_\Gamma[C,e_i]}(S^1 \times S^3) = Z_{\mathrm{YM}_2^q(G)}(C). \tag{5.3.18}$$

This means that

$$S_\Gamma[S^1 \times S^3{}_{q=t,p=0}] = \mathrm{YM}_2^q(G). \tag{5.3.19}$$

When $q \to 0$, the right hand side is just the 2d QFT $\mathrm{triv}_2 /\!\!\!/ G$ discussed in Sec. 1.8.

5.4 A limit and the generators of the Coulomb branch

One interesting limit of the superconformal index is when $u = pq/t$ is fixed and the limit $p, q \to 0$ is taken. We have

$$Z^{\mathrm{SCI}}_{u=pq/t,p\to 0,q\to 0}(\mathrm{Hyp}(V)) = 1 \tag{5.4.1}$$

and

$$Z^{\mathrm{SCI}}_{u=pq/t,p\to 0,q\to 0}(\mathrm{Hyp}(V) /\!\!/\!\!/ G) = \prod_{i=1}^{\mathrm{rank}\, G} \frac{1}{1-u^{d_i}} \tag{5.4.2}$$

where d_i is one plus the i-th exponent of G; this follows from the explicit formula given in Sec. 5.2. In broad generality, it is believed that

$$Z^{\mathrm{SCI}}_{u=pq/t,p\to 0,q\to 0}(Q) = \mathrm{tr}_{\mathbb{C}[\mathcal{M}_{\mathrm{Coulomb}}(Q)]}\, u \tag{5.4.3}$$

where $u \in \mathbb{C}^\times$ is the natural U(1) action on the Coulomb branch of Q, discussed in Sec. 2.4. Once the superconformal index with general p, q and t is understood, we can take this limit of the generalization of (5.3.10), and obtain full information necessary to reconstruct $V(e_i)$ discussed in Sec. 3.9.

5.5 Another limit and the Hilbert series of the Higgs branch

Another interesting subcase is the limit $p = q = 0$, keeping t fixed. Then

$$Z^{\mathrm{SCI}}_{p=q=0,t=\tau^2}(\mathrm{Hyp}(V)) = \prod_w \frac{1}{1-\tau z^w}. \tag{5.5.1}$$

This is the graded character of $\mathbb{C}[V]$. Note also that

$$Z^{\mathrm{SCI}}_{p=q=0,t=\tau^2}(\mathrm{Hyp}(V) /\!\!/\!\!/ G) = \frac{1}{|W|}\int \prod \frac{dz}{2\pi\sqrt{-1}z} \prod_\alpha (1-z^\alpha)(1-\tau^2)^r \prod_\alpha (1-\tau^2 z^\alpha) \prod_w \frac{1}{1-\tau z^w} \tag{5.5.2}$$

is the graded character of $\mathbb{C}[V///G]$ under favorable conditions. Note that the factor $(1-\tau^2)^r\prod_\alpha(1-\tau^2 z^\alpha)$ provides the relation imposed by $\mu_{\mathbb{C}}=0$ in the hyperkähler quotient. The conjecture is that in general

$$Z^{\text{SCI}}_{p=q=0,t=\tau^2}(Q)=\operatorname{ch}\mathbb{C}[\mathcal{M}_{\text{Higgs}}(Q)]=\operatorname{tr}_{\mathbb{C}[\mathcal{M}_{\text{Higgs}}(Q)]}\tau z \tag{5.5.3}$$

under favorable conditions. Here τ is the grading on the Higgs branch and z is in the Cartan torus of G. When $Q=S_G(X;e_i)$, $\eta_G(X;e_i)=\mathcal{M}_{\text{Higgs}}(S_G(X;e_i))$ was discussed at length in Sec. 3.10.

In the formula (5.3.10) in this limit, $K_e(z)$ becomes

$$K_e(z)=\prod_d\prod_{w:\text{weights of }R_d}\frac{1}{1-\tau^{d+1}z^w}. \tag{5.5.4}$$

and $\underline{P}_\lambda$ is replaced by $\underline{H}_\lambda$ which is orthonormal with respect to

$$(\frac{1}{1-\tau^2})^r\frac{1}{|W|}\prod_{i=1}^r\frac{dz_i}{2\pi\sqrt{-1}z_i}\prod_\alpha\frac{1-z^\alpha}{1-\tau^2z^\alpha}. \tag{5.5.5}$$

The standard Hall-Littlewood polynomial is orthogonal with respect to this measure.

This can be used to obtain a conjectural formula of the graded character of the centered instanton moduli spaces of E_r gauge group, since we believe that these spaces arise as the Higgs branch of particular class S theories, as we saw in Sec. 3.10. For the instanton number 1, we just have

$$Z^{\text{SCI}}_{p=q=0,t=\tau^2}(S_{A_2}[S^2;[1^3],[1^3][1^3]])=\operatorname{ch}\mathbb{C}[\tilde{\mathcal{M}}_{E_6,n=1}] \tag{5.5.6}$$
$$Z^{\text{SCI}}_{p=q=0,t=\tau^2}(S_{A_3}[S^2;[2^2],[1^4][1^4]])=\operatorname{ch}\mathbb{C}[\tilde{\mathcal{M}}_{E_7,n=1}] \tag{5.5.7}$$
$$Z^{\text{SCI}}_{p=q=0,t=\tau^2}(S_{A_5}[S^2;[3^2],[2^3][1^6]])=\operatorname{ch}\mathbb{C}[\tilde{\mathcal{M}}_{E_8,n=1}]. \tag{5.5.8}$$

On the right hand side the character is with respect to $\mathbb{C}^\times\times E_r$, and on the left hand side it is with respect to $\mathbb{C}^\times\times\mathrm{SU}(3)^2$, $\mathbb{C}^\times\times\mathrm{SU}(2)\times\mathrm{SU}(4)^2$, $\mathbb{C}^\times\times\mathrm{SU}(2)\times\mathrm{SU}(3)\times\mathrm{SU}(6)$. Note that the rank of the both sides agree.

Although we believe that the instanton moduli spaces are obtained as in (3.10.22) for general n, they are not in favorable conditions where the equality of the superconformal indices and the graded character of the Higgs branch is applicable. A seemingly related fact is that $\mathcal{M}_{E_r,n}$ with $n>1$ has a nontrivial triholomorphic action of $\mathrm{SU}(2)\times E_r$, where $\mathrm{SU}(2)$ comes from a triholomorphic action of $\mathrm{SU}(2)$ on $\mathbb{R}^4$ preserving its hyperkähler structure. Instead, we have the relation

$$\mathcal{M}_{\text{Higgs}}(S_{A_{3n-1}}[S^2;[n^2,n-1,1],[n^3],[n^3]])=\mathbb{C}^2\times\tilde{\mathcal{M}}_{E_6,n}=\mathcal{M}_{E_6,n}, \tag{5.5.9}$$
$$\mathcal{M}_{\text{Higgs}}(S_{A_{4n-1}}[S^2;[2n,2n-1,1],[n^4],[n^4]])=\mathbb{C}^2\times\tilde{\mathcal{M}}_{E_7,n}=\mathcal{M}_{E_7,n}, \tag{5.5.10}$$
$$\mathcal{M}_{\text{Higgs}}(S_{A_{6n-1}}[S^2;[3n,3n-1,1],[2n^3],[n^6]])=\mathbb{C}^2\times\tilde{\mathcal{M}}_{E_8,n}=\mathcal{M}_{E_8,n} \tag{5.5.11}$$

where $\mathcal{M}_{E_r,n}$ is the noncentered moduli space. Then we have

$$Z^{\mathrm{SCI}}_{p=q=0,t=\tau^2}(S_{A_{3n-1}}[S^2;[n^2,n-1,1],[n^3],[n^3]) = \mathrm{ch}\,\mathcal{M}_{E_6,n}, \tag{5.5.12}$$
$$Z^{\mathrm{SCI}}_{p=q=0,t=\tau^2}(S_{A_{4n-1}}[S^2;[2n,2n-1,1],[n^4],[n^4]) = \mathrm{ch}\,\mathcal{M}_{E_7,n}, \tag{5.5.13}$$
$$Z^{\mathrm{SCI}}_{p=q=0,t=\tau^2}(S_{A_{6n-1}}[S^2;[3n,3n-1,1],[2n^3],[n^6]) = \mathrm{ch}\,\mathcal{M}_{E_8,n} \tag{5.5.14}$$

On the right hand side the character is with respect to $\mathbb{C}^\times \times \mathrm{SU}(2) \times E_r$, and on the left hand side it is with respect to $\mathbb{C}^\times \times \mathrm{U}(1)^2 \times \mathrm{SU}(2) \times \mathrm{SU}(3)^2$, $\mathbb{C}^\times \times \mathrm{U}(1)^2 \times \mathrm{SU}(4)^2$, $\mathbb{C}^\times \times \mathrm{U}(1)^2 \times \mathrm{SU}(3) \times \mathrm{SU}(6)$. Note that the rank of the both sides agree. These relations have been put to some test in [GR12, KS12, HMR12].

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