# Main points of [2305.06196] for string theorists

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The paper is written as a math paper,
so I'd like to present a summary for string theorists.

It touches upon the following topics:

- non-susy heterotic branes [2303.17623]
- classification of 2d spin holomorphic CFTs [2303.16917]
- discrete part of the Green-Schwarz coupling
- Stolz-Teichner conjecture,
- and more ...

The Segal-Stolz-Teichner conjecture says

$$\mathbf{TMF}_d = \frac{\left\{ \begin{array}{c} \text{2d } \mathcal{N}{=}(0,1) \text{ supersymmetric theory} \\ \text{with } d = 2(c_R - c_L) \end{array} \right\}}{\text{continuous deformation}}$$

[Segal 1988] [Stolz-Teichner 2002] [Stolz-Teichner 1108.0189]

**Question:**

How do we detect the deformation classes?

**General answer:**

Find functions

$$f : \{\text{SQFTs}\} \to \text{numbers}$$

which are invariant under deformations.

**Classic example:**

**Elliptic genus** [Witten 1989]

- the generating function of the Witten index of the system on R-sector $S^1$ for each value of $L_0$:

$$
\begin{aligned}
Z_{\text{elliptic}} &= \mathrm{tr}_{\mathcal{H}^R_{S^1}} (-1)^{F_R} q^{L_0 - c_L/24} \bar{q}^{\bar{L}_0 - c_R/24} \\
&= \mathrm{tr}_{\mathcal{H}^R_{S^1}|_{\text{right-moving vac.}}} (-1)^{F_R} q^{L_0 - c_L/24}
\end{aligned}
$$

- Nonzero only when $d = 2(c_R - c_L) \equiv 0$ mod $4$.

**Another example:**

**Mod-2 elliptic genus** [YT-Yamashita-Yonekura 2302.07548]

- the generating function of the **mod-2** Witten index of the system on R-sector $S^1$ for each value of $L_0$

$$Z_{\text{elliptic}} = \text{“}\operatorname{tr}_{\mathcal{H}^R_{S^1}} (+1)^{F_R}\text{”} q^{L_0 - c_L/24} \bar{q}^{\bar{L}_0 - c_R/24}$$
$$= \text{“}\operatorname{tr}_{\mathcal{H}^R_{S^1}|_{\text{right-moving vac.}}} (+1)^{F_R}\text{”} q^{L_0 - c_L/24}$$

- Nonzero only when $d = 2(c_R - c_L) \equiv 1, 2 \bmod 8$.

**Question:**

Do ordinary and mod-2 elliptic genus characterize deformation classes ?

**Answer:**

**No**, if you believe the Stolz-Teichner conjecture.

# Bunke-Naumann invariant

[Bunke-Naumann 0912.4875]
[Gaiotto,Johnson-Freyd,Witten1902.10249]
[Gaiotto,Johnson-Freyd 1904.05788]
[Yonekura 2207.13858]

considered a subtler invariant, which assigns e.g.

$$\mathcal{N}=(0,1)\ S^3\ \sigma\text{-model with } \int H = k$$

the value

$$k \in \mathbb{Z}_{24}.$$

Can be non-zero when $d = 2(c_R - c_L) \equiv 3 \bmod 24$.

**Question:**

Does the combination of **ordinary or mod-2 elliptic genus** and **Bunke-Naumann invariant** completely detect deformation classes?

**Answer:**

**Still no**, assuming Stolz-Teichner conjecture.

Let $A_d$ be the subgroup of $\mathbf{TMF}_d$
whose ordinary/mod-2 elliptic genus is zero.

In the range $-31 \leq d \leq 9$, the nonzero cases are:

$$\begin{aligned} A_3 = \mathbb{Z}_{24}, \quad A_6 &= \mathbb{Z}_2, \quad A_8 = \mathbb{Z}_2, \quad A_9 = \mathbb{Z}_2, \ldots \\ A_{-28} &= \mathbb{Z}_2, \quad A_{-30} = \mathbb{Z}_2, \quad A_{-31} = \mathbb{Z}_2, \ldots \end{aligned}$$

$A_3 = \mathbb{Z}_{24}$ is detected by Bunke-Naumann invariant,
but what are the others?

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What are $A_{-28,-30,-31}$?

In addition, mathematicians say that

$$A_d \longleftrightarrow A_{-22-d}$$

are Pontryagin dual if $d \not\equiv 3$ mod $24$:

$$\begin{array}{llll} A_3 = \mathbb{Z}_{24}, & A_6 = \mathbb{Z}_2, & A_8 = \mathbb{Z}_2, & A_9 = \mathbb{Z}_2, \ldots \\ & \updownarrow & \updownarrow & \updownarrow \\ & A_{-28} = \mathbb{Z}_2, & A_{-30} = \mathbb{Z}_2, & A_{-31} = \mathbb{Z}_2, \ldots \end{array}$$

What is this pairing, physically?

**Here the classification of spin holomorphic CFTs comes in.**

Stolz-Teichner conjecture concerns $\mathcal{N}=(0,1)$ SQFTs
and $d = 2(c_R - c_L)$.

**Purely left-moving** (i.e. $c_L > 0$, $c_R = 0$) **non-supersymmetric**
modular-invariant spin CFTs are
**actually** $\mathcal{N}=(0,1)$ **SQFTs with** $d = -2c_L$.

These are classified recently in
[Boyle Smith, Lin, YT, Zheng 2303.16917] ($c_L \leq 16$)
[Rayhaun 2303.16921] ($c_L \leq 24$)
[Höhn-Möller 2303.17190] ($c_L \leq 24$)

| $c_L$ | $-2c_L$ | |
|---|---|---|
| $16$ | $-32$ | $E_8 \times E_8$, $so(32)$, $so(16) \times so(16)$ |
| $\frac{31}{2}$ | $-31$ | $(E_8)_2$ |
| $15$ | $-30$ | $su(16)$ |
| $14$ | $-28$ | $E_7 \times E_7$ |
| $12$ | $-24$ | $so(24)$ |
| $\vdots$ | $\vdots$ | $\vdots$ |

- **The red ones** have zero ordinary and/or mod-2 elliptic genus,
- and appear exactly when $A_{-d}$ are nontrivial.
- They are very likely SQFT representatives of $A_{-28,-30,-31}$.

Furthermore, **these spin-CFTs provide the angular part of the non-supersymmetric heteortic $p = 4$-, $6$- and $7$-branes of** [Kaidi-Ohmori-YT-Tachikawa 2303.17623].

$$\underbrace{\mathbb{R}^{p,1} \times \mathbb{R}_{>0}} \quad \times \quad \underbrace{S^{8-p} + \text{current algebra}}$$

$$\downarrow \text{RG}$$

$$\begin{array}{ccccc} A_9 & d = 9 & \leftrightarrow & (E_8)_2 & A_{-31} \\ A_8 & d = 8 & \leftrightarrow & su(16) & A_{-30} \\ A_6 & d = 6 & \leftrightarrow & E_7 \times E_7 & A_{-28} \end{array}$$

This arises **exactly on the places where** the pairing $A_d \leftrightarrow A_{-d-22}$ mathematicians constructed arises.

Concretely, take the pair

$$A_6 \quad d = 6 \quad \leftrightarrow \quad E_7 \times E_7 \quad A_{-28}$$

**Question:**

What would $A_6 \simeq \mathbb{Z}_2$ generated by

$$SU(2) \times SU(2) \text{ with } H \text{ flux}$$

provide for heterotic string compactification with $E_7 \times E_7$?

**Answer:**

$SU(2) \simeq S^3$ is trivial in spin bordism, but
is **not trivial** with $\int H = 1$ in **string bordism**, a bordism theory with $dH = \frac{1}{2}p_1(R)$ appropriate for heterotic string theory.

$S^3 \times S^3$ with $\int H = 1$ on both sides is a $\mathbb{Z}_2$ string bordism class.

**There can be discrete gravitational/$H$-field theta angle**
which assigns $-1$ for this torsion class.

**Once the internal CFT for the heterotic compactification is fixed**, **such discrete gravitational/$H$-field theta angle should be computable**.

For a $d$-dimensional gravitational/$H$-field theta angle, the internal CFT should have

$$c_L = 26 - d, \qquad c_R = \frac{3}{2}(10 - d)$$

therefore it is an element in

$$\mathrm{TMF}_{2(c_R - c_L) = -22-d}$$

which realizes exactly the pairing

$$d \longleftrightarrow -22 - d$$

predicted by algebraic topologists!

So the natural guess is that **the Pontryagin=Anderson dual pairing**

$$A_d \longleftrightarrow A_{-22-d}$$

mathematicians had constructed **is actually the gravitational/$H$-field theta angle which is part of the Green-Schwarz coupling**.

To show this, with Yamashita (and with a lot of help from Yonekura) we developed **the theory of discrete, global part of Green-Schwarz cancellation and coupling** using stable homotopy theory.

Very schematically, the perturbative Green-Schwarz cancellation and coupling goes as:

- $P_{d+2}$ has a factor of $\frac{1}{2}p_1(R) - n(F)$
- therefore can be divided by it and has the form

$$P_{d+2} = (\frac{1}{2}p_1(R) - n(F))X_{d-2}$$

- then we have $\int B \wedge X_{d-2}$

Including the global part, we need to show

- The global version of $P_{d+2}$ has a factor of $\frac{1}{2}p_1(R) - n(F)$. This we already did in [YT-Yamashita 2108.13542].
- **The global version of $P_{d+2}$ can be divided by it in an appropriate sense.**
- **It gives the global version of G.S. coupling**, and furthermore
- **It equals the Anderson duality of $\mathrm{TMF}$.**

Interestingly for me, the step that

- **the global version of $P_{d+2}$ can be divided by $\frac{1}{2}p_1(R) - n(F)$ in an appropriate sense**

was pointed out by Prof. Kawazumi while Yamashita was giving a seminar on [YT-Yamashita 2108.13542] to mathematicians, in the form

> ***A primary invariant vanished***
> ***so there should be a nonzero secondary invariant.***
> ***What's the physics interpretation?***

This was when I was working on non-supersymmetric heterotic strings.
After a while, I realized that this might be the Green-Schwarz coupling.

The rest is history.