Triangle Anomaly
from
Einstein Manifolds

Yuji Tachikawa
(Univ. of Tokyo, Hongo)

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by S. Benvenuti, L. Pando Zayas and YT

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1. Introduction & Summary

AdS/CFT correspondence

Consider large # of D3 branes put on the tip of the CY cone $Y$

$$ds_Y^2 = dr^2 + r^2 ds_X^2$$

- $X_5$ is a 5d base called Sasaki-Einstein.
- $d = 4$, $\mathcal{N} = 1$

- Field theory on the D3 $\Rightarrow$ Some quiver gauge theory
- Near Horizon Limit of the D3 $\Rightarrow$ AdS$_5 \times X_5$

Both should describe the same physics
Examples

- Many explicit metrics for $X_5: S^5, T^{1,1}, Y^{p,q}$ and $L^{p,q,r}$ \[(Gauntlett et al.)\]

  $\Rightarrow$ Central charge $a = (\text{volume})^{-1}$

- Quiver theories known through tiling etc. \[(Hanany et al.)\]

  $\Rightarrow$ Central charge $a$ from $a$-maximization \[(Intriligator-Wecht)\]

They agree and are, in general, irrational.
Generalization

- Toric Sasaki-Einsteins
  ⇒ $Z$-minimization (Martelli-Sparks-Yau) gives the volume

- Corresponding quivers
  ⇒ #(chiral superfields) and #(gauge groups) known
  ⇒ $a$-maximization

They agree (Butti-Zaffaroni)!
Recall
the correspondence of **CS terms** with **Triangle Anomalies**

<table>
<thead>
<tr>
<th>AdS</th>
<th>CFT</th>
</tr>
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<tbody>
<tr>
<td>$\phi$</td>
<td>$\mathcal{O}$</td>
</tr>
<tr>
<td>$Z[\phi(x)</td>
<td>_{x_5=\infty} = \hat{\phi}(x)]$</td>
</tr>
<tr>
<td>$A^I_\mu$</td>
<td>$J_I^\mu$ : current for $Q_I$</td>
</tr>
<tr>
<td>$Z[A^I_\mu(x)</td>
<td><em>{x_5=\infty} = \hat{A}^I</em>\mu(x)]$</td>
</tr>
</tbody>
</table>
\[ S_{CS} = \int \frac{1}{24\pi^2} c_{IJK} A^I \wedge F^J \wedge F^K \]

\[ \Downarrow \]

**Gauge dependence**

\[ \delta \chi \langle e^{-\int A_I^J J^\mu_I} \rangle_{SCFT} = \int_{AdS_5} \frac{1}{24\pi^2} c_{IJK} \delta \chi (A^I) \wedge F^J \wedge F^K \]

\[ = \int_{M^4} \frac{1}{24\pi^2} c_{IJK} \chi^I F^J \wedge F^K \]

\[ \Downarrow \]

\[ c_{IJK} = \text{tr} \, Q_I Q_J Q_K. \]
Our goal today

Calculate both sides and check that they match!

◊ CS terms from Kaluza-Klein reduction on $X$

\[ c_{IJK} = \frac{N^2}{2} \int_X \omega \{ I \wedge \nu_{kJ} \omega_K \} \quad \text{(general)} \]

\[ c_{IJK} = \frac{N^2}{2} | \det(k_I, k_J, k_K) | \quad \text{(toric)} \]

◊ Triangle anomaly from the structure of the quiver

\[ \text{tr} \ Q_I Q_J Q_K = \frac{N^2}{2} | \det(k_I, k_J, k_K) | \]
\[ c_{IJK} \text{ independent of } k_{L \neq I, J, K} \]

⇒ Higgsing through Dibaryon condensation.

⋄ Intricate mixing of angular momenta and baryonic charges

\[
\begin{align*}
0 & \to M \to HG_3(X) \to H_3(X) \to 0 \\
0 & \to H^3(X) \to HG^3(X) \to N \to 0
\end{align*}
\]
1. Introduction & Summary

2. KK reduction

3. Dibaryons and Giant Gravitons

4. Evaluation for the toric SE

5. Matching to quivers

6. Comments & Outlook
2. KK reduction

cf. M on $CY_3$

- M theory on Calabi-Yau threefold $X$

\[ C = \left( A^I \right)_{M^5} \wedge \left( \omega_I \right)_{CY} \]

- $b^2 = \dim H^2(X)$ of gauge fields

\[ \int_{M^{11}} C \wedge dC \wedge dC \Rightarrow \]

\[ c_{IJK} \int_{M^5} A^I \wedge F^J \wedge F^K \quad \text{where} \quad c_{IJK} = \int_{CY} \omega_I \wedge \omega_J \wedge \omega_K \]
Type IIB on Einstein mfd $X$

$X \cong U(1)^\ell$ with $b^3 = \dim H^3(X)$

Expected: $\ell$ gauge fields from $g_{\mu\nu}$, $b^3$ from $F_5$.

Ansatz? $c_{IJK}$?
Metric

\[ ds^2_{\text{AdS}_5} = \eta_{\mu\nu} f^\mu f^\nu \]
\[ ds^2_{X^5} = \sum (e^i)^2 \quad \Rightarrow \quad \sum (\hat{e}^i)^2 = \sum (e^i + k^i_a A^a)^2 \]

N.B. The Hodge star * exchanges

\[ f^1, \ldots, f^5 \quad \Leftrightarrow \quad \hat{e}^1, \ldots, \hat{e}^5 \]

Thus

\[ *\text{vol}_{\text{AdS}_5} = \hat{e}^1 \cdots \hat{e}^5 = \text{vol}_X + A^a \wedge \iota_{k_a} \text{vol}_X + \cdots. \]
Five-form

\[ F_5 = F_{0,5} + F_{1,4} + F_{2,3} + F_{3,2} + F_{4,1} + F_{5,0}. \]

where

\[ F_{0,5} = \frac{2\pi N}{V} \text{vol} X, \quad F_{5,0} = \frac{2\pi N}{V} \text{vol}_{\text{AdS}}, \]

\[ F_{1,4} = \frac{2\pi N}{V} A^a \wedge \iota_{\kappa_a} \text{vol} X, \quad F_{4,1} = 0, \]

\[ F_{2,3} = NF^I \wedge \omega_I, \quad F_{3,2} = N(*F^I) \wedge *\omega_I. \]

which satisfies \( F_5 = *F_5 \) up to first order.

N.B. Other fields \( = 0 \); assume all internal w.f. are \( U(1)^\ell \) inv.
Impose $dF_5 = 0 \implies d_{\text{AdS}} F_{p,q+1} + d_X F_{p+1,q} = 0$:

$$0 = \left( d_{\text{AdS}} \ast F^I \right) \wedge \ast \omega_I,$$

$$0 = \left( d_{\text{AdS}} F^I \right) \wedge \omega_I + \left( \ast F^I \right) \wedge d_X \ast \omega_I,$$

$$0 = 2\pi \left( d_{\text{AdS}} A^a \right) \wedge \iota_{k_a} \text{vol}_X + V F^I \wedge d_X \omega_I.$$ 

↓

$dF^I = d \ast F^I = 0 \iff \exists c^a_I$ s.t.

$$d \ast \omega_I = 0$$

$$-V d\omega_I = c^a_I 2\pi \iota_{k_a} \text{vol}_X$$

$$dA^a = c^a_I F^I$$
Recap.

\[ d \ast \omega_I = 0 \]
\[ d\omega_I + \iota_{k_I} \text{vol}^\circ = 0 \quad \text{where} \quad k_I = 2\pi c_I^a k_a, \quad \text{vol}^\circ = \text{vol}/V. \]

\[ \delta F_5 = N d(A^I \wedge \omega_I) \quad \text{where} \quad A^a = c_I^a A^I. \]

- \( \omega_I \) closed and co-closed \( \Rightarrow b^3 \) of them
- \( \omega_I \) closed-up-to-isom. and co-closed \( \Rightarrow d = b^3 + \ell \) of them

OK, how about \( c_{IJK} \)?
One contribution

\[ F_5 = \ast F_5 \text{ forces at 2nd order} \]

\[
F_{2,3} = F^I \wedge \omega_I \\
F_{3,2} = (\ast F^I) \wedge (\ast \omega_I) + A^a \wedge F^I \wedge \iota_{k_a} \omega_I
\]

\[ \downarrow \]

\[
d_{\text{AdS}} F_{4,1} + d_x F_{3,2} = 0 \text{ gives} \\
(d \ast F^I) \wedge (\ast \omega_I) = F^a \wedge F^I \wedge (\iota_{k_a} \omega_I)
\]
Result

\[ d \ast F^I \int_X (\omega_K \wedge *\omega_I + \frac{1}{16V^2} k_K k_I \text{vol}) = \frac{1}{3\pi} F^I \wedge F^J \int_X \omega\{I \wedge \iota_{k_J} \omega_K\} \]

\[ \downarrow \]

\[ \tau_{IJ} = \frac{N^2}{2\pi} \int_X (\omega_J \wedge *\omega_I + \frac{1}{16V^2} k_J k_I \text{vol}) \]

\[ c_{IJK} = \frac{N^2}{2} \int_X \omega\{I \wedge \iota_{k_J} \omega_K\} \]

where \( d\omega_I + \iota_{k_I} \text{vol}^\circ \equiv 0 \)

cf. \( \tau_{IJ} \) obtained in (Barnes-Gorbatov-Intriligator-Wright)
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3. Giant Gravitons and Normalization of Charges

$A^I$ from:

- $g_{\mu\nu} \Rightarrow$ KK Angular Momenta : $\ell$ of them
- $F_5 \Rightarrow$ Baryonic Charges : $b^3$ of them

Total: $d = \ell + b^3$ of gauge fields

What’s the integral basis?
\[ \delta F_5 = N d (A^I \wedge \omega_I) \Rightarrow \text{D3-brane wrapping on } C \text{ has charge } N \int_C \omega_I \]

\[ \omega_I \text{ is not closed!} \quad d\omega_I + \iota_{k_I} \text{vol} = 0 \]
\[ \Rightarrow \text{Depends more than just its homology class.} \]

\[ \text{Consider branes ‘at rest’ which is charge eigenstate} \]
\[ \Rightarrow \text{C : invariant under } U(1)^{\ell} \]

\[ \text{E.g. } S^5 \subset SO(6) \supset U(1)^3 \]
\[ \Rightarrow \text{Brane wrapping the } S^3 \text{ at equator: Maximal Giant Gravitons!} \]
Define $C \sim C'$ if $C - C' = \partial D$, $D$: invariant

$$\Rightarrow (\int_C - \int_{C'})\omega_I = \int_{\partial D} \omega_I = \int_D \nu_{k_I} \text{vol} = 0$$

Assume there are $d = \ell + b^3$ such three-cycles $\Rightarrow$ Normalize

$$\int_{C^I} \omega_J = \delta^I_J.$$ 

N.B. As we will see, assumption has been checked for toric SEs.
Robustness of $c_{IJK} \propto \int \omega\{I \wedge \iota_{k_J} \omega_K\}$.

- **Shift** $\omega_I \rightarrow \omega_I + d\alpha_I$

  $\Rightarrow \quad \delta c_{IJK} \propto \int d\alpha\{I \wedge \iota_{k_J} \omega_K\} = \int \alpha\{I \wedge \iota_{k_J} \iota_{k_K}\} \text{vol}^o = 0$

- **Shift** $\text{vol}^o \rightarrow \text{vol}^o + d\alpha$.  $d\omega_I + \iota_{k_I} \text{vol}^o = 0 \Rightarrow \omega_I \rightarrow \omega_I + \iota_{k_I} \alpha$

  $\Rightarrow \quad \delta c_{IJK} \propto \int \iota_{k_I} \alpha \wedge \iota_{k_J} \omega_K\} = - \int \alpha \wedge \iota\{I\iota_{k_J} \iota_{k_K}\} \text{vol}^o = 0$

Also

$\delta \int_{C_I} \omega_J = \int_{C_I} \iota_{k_J} \alpha = 0$. 

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Method to obtain $c_{IJK}$

**Step 1.** Find any five-form $\text{vol}^\circ$ s.t. $\int \text{vol}^\circ = 1$.

**Step 2.** Find invariant three-cycles $C^I$.

**Step 3.** Find the dual $\omega_I$ by $\int_{C^I} \omega_J = \delta^I_J$ so that $d\omega_I + \iota_{k_I} \text{vol}^\circ = 0$.

**Step 4.** Plug in to $c_{IJK} = \frac{N^2}{2} \int \omega_{\{I \wedge \iota_{k_J} \omega_K\}}$. 
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4. Evaluation for toric SEs

- The cone over a toric SE $X \backslash U(1)^3$ is a toric Calabi-Yau
- $X \xrightarrow{\text{mom. map}}$ the interior of $d$-gon $B$ with $T^3$ fiber:

Coordinate along $T^3: \theta^{1,2,3}$ with $\theta^i \sim \theta^i + 1$

- $k_I = k_{Ii} \partial_{\theta^i}$ degenerates on $E_I$. 
Take $\mathcal{F}$ supported on $S$ s.t. $\int_B \mathcal{F} = 1$

Take $\mathcal{A}_I$ supported on $R_I$ s.t. $d\mathcal{A}_I = \mathcal{F}$
\[
\text{vol}^\circ = \mathcal{F} \wedge d\theta^1 d\theta^2 d\theta^3 \\
\omega_I = -\mathcal{A}_I \wedge \nu_{kI} d\theta^1 d\theta^2 d\theta^3 \\
d\omega_I + \nu_{kI} \text{vol}^\circ = 0
\]

\[\diamond \quad \omega_I \text{ independent of other edges} \Rightarrow c_{IJK} \text{ independent of } k_{L \neq I,J,K}\]
X: toric SE for the triangle $k_{I,J,K}$

Y: Periodicity changed $\theta_i \sim \theta_i + k_{I,J,K}^i = S^5$

$\Rightarrow X = Y/\Gamma$ where $\Gamma = \mathbb{Z}^3/(\mathbb{Z}k_I + \mathbb{Z}k_J + \mathbb{Z}k_K)$ with

$\#\Gamma = |\det(k_I, k_J, k_K)|$.

$\Rightarrow c_{IJK}$ is $\#\Gamma$ times that of $S^5$, which is $N^2/2$. 
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5. Matching to Quivers

Toric Data $\leftrightarrow$ Quivers

- Tiling $\Rightarrow$ Toric Data: Forward Algorithm
- Toric Data $\Rightarrow$ Tiling: Inverse Algorithm, Rhombi, ...

- Many works
Known Properties

Given the toric data sorted counterclockwise

\[ k_I = (1, \vec{k}_I), (I = 1, \ldots, d) \]

As (p,q)-web: \( v_i = \vec{k}_I - \vec{k}_{I-1} \)

- Gauge groups : all \( SU(N) \)
- \# (Gauge groups) = Twice the area of the toric diagram

- \( d \) global nonanomalous \( U(1) \) symmetries \( Q_I \). Consider \( Q = a^I Q_I \).

- Chiral superfields classified into \( B_{ij} \) s.t.
  - Net \# \( B_{ij} = \det(\vec{v}_i, \vec{v}_j) \)
  - The charge under \( Q \) is \( a_I + a_{I+1} + \cdots + a_{J-1} \)

- Charge of the superpotential = \( \sum a_I \)
\[ c_{IKJ}a^Ia^Ja^K = n_V \left( \frac{1}{2} \sum a^I \right)^3 + \sum_{I<J} n_{IJ} \left( \sum_{K=I}^{J-1} a^K - \frac{1}{2} \sum a^I \right)^3 \]

where

\[ n_V = \sum \det(\vec{k}_I - \vec{k}_1, \vec{k}_{I+1} - \vec{k}_1), \]
\[ n_{IJ} = \det(\vec{k}_J - \vec{k}_{J-1}, \vec{k}_I - \vec{k}_{I-1}) \]

(Butti-Zaffaroni)
How can we simplify?

Denote the quantity for toric data \( \{k_1, \ldots, k_{N-1}\} \) by adding \( \sim \).

\[
\Rightarrow \quad n_{I,N-1} + n_{I,N} = \tilde{n}_{I,N-1} \quad \text{and} \quad n_{V} - n_{N-1,N} = \tilde{n}_{V}.
\]

That is,

\[
\mathcal{B}_{I,N-1} \cup \mathcal{B}_{I,N} \rightarrow \tilde{\mathcal{B}}_{I,N-1} \quad \text{and} \quad \mathcal{G} - \mathcal{B}_{N-1,N} \rightarrow \tilde{\mathcal{G}}.
\]

\[
\Rightarrow \quad c_{IJK} a^I a^J a^K \bigg|_{a_N=0} = \tilde{c}_{IJK} a^I a^J a^K.
\]

\( c_{IJK} \) is independent of \( k_L \neq I, J, K \).

One can check \( c_{IJK} = \frac{N^2}{2} \left| \det(k_I, k_J, k_K) \right| \).
6. Comments and Outlook

- We obtained the formula
  \[ c_{IJK} = \frac{N^2}{2} \int \omega \{ I \wedge \iota_{k_J} \omega_K \} \]

- On both sides of the duality for toric SEs, we obtained
  \[ c_{IJK} = \frac{N^2}{2} | \text{det}(k_I, k_J, k_K) | \]

- So far, so good.

Why is it independent of \( k_L \neq I, J, K \)?
Consider D3-branes wrapping on $L$-th edge

- Dibaryons charged only w.r.t. $L$-th global symmetry
- **Condense**!

$\Rightarrow L$-th global symmetry broken; $L$-th edge shrunk

- $c_{IJK}$ can be evaluated before or after the Higgsing
- Independent of $k_L$!
Outlook

◊ Study more about dibaryon condensation & topology change
  ● Reminiscent of blackhole condensation & topology change in CY

◊ Study more about the charge lattice. $K_G(X)$?

◊ Study the SUSY on 5d, extend the reduction to scalars.

Much to be done!
7. On charge lattice

String theory: assigns to $X$ a lattice of rank $\ell + b^3$.

- What is it mathematically? $\Rightarrow \{\omega_I\}$ at the level of ‘de Rham’

- Integral structure: not the direct sum of KK + D3
$$0 \to M \xrightarrow{\iota} HG_3(X) \to H_3(X) \to 0$$

$$0 \to H^3(X) \to HG^3(X) \xrightarrow{p} N \to 0$$

$N$: the space of Killing vectors.  
$N\mathbb{Z}$: with periodicity $2\pi$  
$M$: dual of $N$.  
$M\mathbb{Z}$: irreps of $U(1)^\ell$

- $p$: defined by $d\omega_I + \iota_{k_I} \text{vol}^\circ = 0$
- $\iota$: See blackboard ...

*Your help appreciated.*