Comments on SUSY Quantum Field Theories

Yuji Tachikawa (U. Tokyo, Dept. Phys & Kavli IPMU)

Group 30, Ghent

Let me start by thanking **the organizers** for the invitation

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and **those of you** who came to the ceremony yesterday:
the time could have been used for more sightseeing!

I won't forget your kindness.

I'm not sure if I really deserved to get a prize named after the great Hermann Weyl.

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named after the great Hermann Weyl.

After all, I didn’t know the **following classic fact**
about the Weyl group until the last weekend,
that I learend from an article I read in the flight to Belgium:

Take your favorite simple Lie group, say $\boldsymbol{E_8}$.

As everybody knows, its “expontents plus one” are:

$$\mathbf{2, 8, 12, 14, 18, 20, 24, 30.}$$

What I didn’t know about was their product, which is

$$\mathbf{696729600.}$$

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As everybody knows, its "expontents plus one" are:

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What I didn't know about was their product, which is

$$696729600.$$

This is, of course, the order of the Weyl group of E_8.

In general,

$$\prod_i (e_i + 1) = |W|.$$

I think I have a lot more to learn about groups.

So, I consider this prize as an admonition
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I also didn't quite understand the concept of this Colloquium
before I came –
this is my first time in this historic conference series.

Is it a **math** conference, or a **physics** conference?

By definition, in a **physics** conference,
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I've heard so many definitions in all the plenary talks so far.

So I conclude this is a **math** conference. QED.

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So I conclude this is a **math** conference. QED.

There's a reason why I brought up this cultural difference.

Note that I **never** wrote proofs in my papers.

So I'm not a **mathematician**.

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So I'm not a **mathematician**.

Note also that I **never** wrote papers about the real world.

So I'm not a **physicist**.

But then, why am I giving a talk here?

Group Theoretical Methods in **Physics**

Quantum Field Theoretical
Methods
in **Mathematics**

Group theory
has been very effective
in (mathematical) **physics**

Quantum field theory
has been very effective
in (physical) **mathematics**

Main Exhibit:

Mirror symmetry (from the late ’80)

- Given a Calabi-Yau manifold X, there is an associated 2d supersymmetric QFT $Q(X)$:

$$X \mapsto Q(X)$$

- In the space of 2d supersymmetric QFTs, there's an automorphism σ

$$Q \mapsto \sigma(Q) \mapsto \sigma^2(Q) = Q$$

- But **nobody has given precise definitions** of what $Q(X)$ or σ means yet!

- Given X, it often happens that there's another Calabi-Yau Y s.t.

$$\sigma(Q(X)) = Q(Y)$$

- Properties of X and Y related in a way mysterious to those who don't know QFT, e.g.

complex structure $\leftrightarrow$ symplectic structure

- Big math industry studying the relation between X and Y.
- But **they don't talk directly** about $Q(X)$, as if it's the **one-who-must-not-be-named …**

There are many other minor such events
where QFT led to new mathematical relations.

For example ...

The expression

$$\oint \frac{\prod_{\pm\pm\pm} \Gamma_{p,q}(t u^{\pm 1} v^{\pm 1} z^{\pm 1}) \prod_{\pm\pm\pm} \Gamma_{p,q}(t x^{\pm 1} y^{\pm 1} z^{\pm 1})}{\prod_{\pm} \Gamma_{p,q}(t^2 z^{\pm 2}) \prod_{\pm} \Gamma_{p,q}(z^{\pm 2})} \frac{dz}{2\pi i z}$$

is obviously symmetric under $u \leftrightarrow v$, $x \leftrightarrow y$.

But it is also symmetric under $u \leftrightarrow x$. Here

$$\Gamma_{p,q}(z) = \prod_{j,k \geq 0} \frac{1 - z^{-1} p^{j+1} q^{k+1}}{1 - z p^j q^k}$$

is the elliptic Gamma function.

The symmetry of

$$\oint \frac{\prod_{\pm\pm\pm} \Gamma_{p,q}(tu^{\pm 1}v^{\pm 1}z^{\pm 1}) \prod_{\pm\pm\pm} \Gamma_{p,q}(tx^{\pm 1}y^{\pm 1}z^{\pm 1})}{\prod_{\pm} \Gamma_{p,q}(t^2 z^{\pm 2}) \prod_{\pm} \Gamma_{p,q}(z^{\pm 2})} \frac{dz}{2\pi i z}$$

under $u \leftrightarrow x$:

- On the physics side, conjectured by [Rastelli et al.] in 2009.
- On the math side, [van de Bult] proved this relation in 2009.
- This was completely independent!

What is this QFT thing, that touches many parts of mathematics?

Those who do QFT (including me) feel they know what that is.

But those feelings are not enough to convey it to
an audience as diverse as today's.

You need some formalism. You want some solid starting point.

So, what's the formalism of QFT?

There are many *formalisms*:

- **Wightman axioms**
- **Algebraic Quantum Field Theories**
- **Topological Quantum Field Theories**
- **Vertex Operator Algebras**

that capture some of the aspects.

But **none of them is comprehensive enough** to even state what the mirror symmetry is, or what the Seiberg-Witten theory is.

Such a formalism hasn't been written down anywhere yet, but I don't think it's impossible to do so.

After all, the axioms of a group, that look so straightforward today, took many years to be straightened out.

Another thing to point out:

In the study of QFT (in whatever formalization I mentioned so far, or in particle physics community in general), people tend to study **each individual QFT**, Q_1, Q_2, ...**one by one**.

For example, the whole experimental high energy particle physics can be said to be the quest to find **exactly which QFT Q_{SM} describes elementary particles, and thus, the universe.**

It's like studying groups one by one.

There's nothing wrong with that, particularly when
there's a few particularly interesting groups / QFTs.

But you should also study group homomorphism $G \to H$, representations of groups, action of groups on spaces, the quotient space G/H, ...

Similarly, we should study the **interrelation** among QFTs,
the relation of QFTs with other mathematical objects, etc.

In the rest of the talk, I'd like to show an example
how this kind of idea led to new mathematical relations.

From now on, a QFT is four-dimensional and $\mathcal{N} = 2$ supersymmetric, unless otherwise specified.

A few basic formal **axioms** of QFTs I use:

- QFT Q and a 4d manifold $X \mapsto$ the partition function $Z_Q(X)$.
- QFT Q_1 and $Q_2 \mapsto$ QFT $Q_1 \times Q_2$.
- This product is commutative, associative, has a unit $\bullet$

$$Q \times \bullet = Q.$$

- The product is compatible with taking the partition function

$$Z_{Q_1 \times Q_2}(X) = Z_{Q_1}(X) Z_{Q_2}(Y).$$

I also need a concept of G-**symmetric QFTs**.

- G-symmetric Q, a 4d manifold X with G connection,
  $\mapsto$ the partition function $Z_Q(X)$.
- G-symmetric Q, a homomorphism $\varphi : H \to G$,
  $\Rightarrow$ one can regard Q as H-symmetric.
- G_1-symmetric Q_1 and G_2-symmetric G_2
  $\Rightarrow$ $Q_1 \times Q_2$ is $G_1 \times G_2$ symmetric, such that

$$Z_{Q_1 \times Q_2}(X) = Z_{Q_1}(X) Z_{Q_2}(X).$$

- $\bullet$ is G-symmetric for any G.

A QFT is a $\{id\}$-symmetric QFT.

Note that the formal properties are
very much like **spaces with G action**:

- A space X with G action, and $\varphi : H \to G$
  $\Rightarrow X$ has H action
- X_1 with G_1 action and X_2 with G_2 action,
  $\Rightarrow X \times Y$ has $G_1 \times G_2$ action
- A point $\bullet$ has trivial G action for any G.

Given a space X with $G \times H$ action,
X/G is a space with H action.

Given a QFT Q that is $G \times H$ symmetric,
$Q /\!/\!/ G$ is a H-symmetric QFT.

Usually this operation is called **coupling to the gauge group G.**

With the formal operations so far, we already have **interesting QFTs**:

$$\bullet /\!\!/\!\!/ G$$

Usually called **pure $\mathcal{N} = 2$ gauge theories with gauge group G**.

For a four-manifold M, (a version of)

$$Z_{\bullet /\!\!/\!\!/ \mathbf{SU(2)}}(M)$$

is the **Donaldson invariant**.

Another basic construction is this.

Given a symplectic representation R of G, there is a G-symmetric QFT:

$$R \mapsto \mathbf{Hyp}(R)$$

usually called the **free hypermultiplet**, with formal properties:

- $\mathbf{Hyp}(R \oplus R') = \mathbf{Hyp}(R) \times \mathbf{Hyp}(R')$
- $\mathbf{Hyp}(\bullet) = \bullet$

Now you can consider

$$\mathbf{Hyp}(R) /\!\!/\!\!/ G$$

usually called $\mathcal{N} = 2$ **supersymmetric gauge theories**.

For example, take $G = \mathbf{U(1)}$, and $R = V \oplus V^*$, where $V \simeq \mathbb{C}$ is a standard 1-dimensional representation of $\mathbf{U(1)}$. Let

$$Q' = \mathbf{Hyp}(R) /\!\!/\!\!/ \mathbf{U(1)}.$$

A version of its partition function,

$$Z_{Q'}(M),$$

is the **Seiberg-Witten invariant.**

These invariants are concrete objects but rather deep,
so I don't discuss that today.

There are easier objects to discuss, too. Given G-symmetric Q, one has

- Hyperkähler space $\mathcal{M}_{\text{Higgs}}(Q)$ with G action
- Superconformal index $\mathbf{SCI}(Q)$
  which is a class function on G that is a formal power series in p, q, t

Again, they preserve formal properties:

- $\mathcal{M}_{\text{Higgs}}(Q_1 \times Q_2) = \mathcal{M}_{\text{Higgs}}(Q_1) \times \mathcal{M}_{\text{Higgs}}(Q_2)$
- $\mathbf{SCI}(Q_1 \times Q_2) = \mathbf{SCI}(Q_1) \times \mathbf{SCI}(Q_2)$.

As for $Q /\!/\!/\!/ G$, we have

$$\mathcal{M}_{\text{Higgs}}(Q /\!/\!/\!/ G) = \mathcal{M}_{\text{Higgs}}(Q) /\!/\!/ G$$

where $/\!/\!/G$ is the hyperkähler quotient construction and

$$\mathbf{SCI}(Q /\!/\!/\!/ G) = \oint \frac{\mathbf{SCI}(Q)(z)}{\prod_\alpha \Gamma_{p,q}(z^\alpha)\Gamma_{p,q}(t^2 z^\alpha)} \prod_{i=1}^{r} \frac{dz_i}{2\pi\sqrt{-1}z_i}$$

where α runs over the roots of G.

Note: $\mathbf{SCI}(Q)$ is a function on G, so I took $z \in \mathbf{U}(1)^r \subset G$.

As for $\boldsymbol{Q} = \mathbf{Hyp}(\boldsymbol{R})$, we have

$$\mathcal{M}_{\text{Higgs}}(\mathbf{Hyp}(\boldsymbol{R})) = \boldsymbol{R}$$

and

$$\mathbf{SCI}(\mathbf{Hyp}(\boldsymbol{R}))(\boldsymbol{z}) = \prod_{\boldsymbol{w}} \Gamma_{p,q}(t\boldsymbol{z}^{\boldsymbol{w}})$$

where $\boldsymbol{z}$ is in the Cartan of $\boldsymbol{G}$ and $\boldsymbol{w}$ runs over the weights of $\boldsymbol{R}$.

So, for given a symplectic representation R of $G \times H$, we have $Q = \mathbf{Hyp}(R) /\!\!/\!\!/ G$ is H-symmetric, and

$$\mathcal{M}_{\text{Higgs}}(\mathbf{Hyp}(R) /\!\!/\!\!/ G) = R/\!\!/\!\!/G$$

is a hyperkähler space with H action, and

$$\mathbf{SCI}(\mathbf{Hyp}(R) /\!\!/\!\!/ G)(y) = \oint \frac{\prod_{w \oplus v} \Gamma_{p,q,t}(z^w y^v)}{\prod_\alpha \Gamma_{p,q}(z^\alpha)\Gamma_{p,q}(t^2 z^\alpha)} \prod_{i=1}^{r} \frac{dz_i}{2\pi z_i}$$

is a so-called elliptic beta integral.

They are both well-studied in mathematics.

Note that we start from a group G and its representation R,
that are both well established.

We then pass to $\mathbf{Hyp}(R) /\!\!/\!\!/ G$, which **isn't well formulated yet**.

Then we pass back to $\mathcal{M}_{\mathrm{Higgs}}(\mathbf{Hyp}(R) /\!\!/\!\!/ G)$ or $\mathbf{SCI}(\mathbf{Hyp}(R) /\!\!/\!\!/ G)$, both of which are again **well-established** mathematical-physical objects.

How do we get something new?

Because there are more constructions.

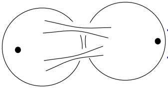
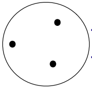
Let G be simply-laced G. There's a **6-dimensional** supersymmetric QFT $\mathcal{S}_G$ with very good properties, called the **6d $\mathcal{N}=(2,0)$ theory.**

Given a k-punctured Riemann surface C and a 4d manifold X, define

$$Z_G[C](X) = \mathcal{S}_G(X \times C).$$

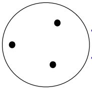
This gives a 4d QFT $\mathcal{S}_G[C]$ depending on C.

With k-points, $\mathcal{S}_G[C]$ is G^k symmetric. For example,

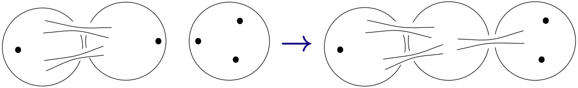
$\mathcal{S}_G[$  $]$, $\mathcal{S}_G[$  $]$

are G^2, G^3 symmetric, respectively.

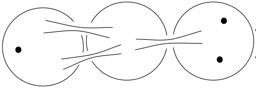
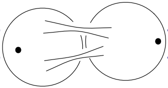
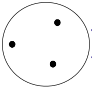
Not only that, $\mathcal{S}_G[C]$ **depends only on the topology of** C:
you can exchange points without changing the theory.

Mathematically, there's an S_k action on G^k,
and $\mathcal{S}_G[$  $]$ is $S_k \ltimes G^k$ symmetric.

You can connect two punctures of two Riemann surfaces:



then

$\mathcal{S}_G[$  $] = (\mathcal{S}_G[$  $] \times \mathcal{S}_G[$  $]) /\!/\!/ G$

where the gauging G is performed w.r.t. $G_{\text{diag}} \to G \times G$ associated to the two punctures connected.

You can say

$$C \mapsto \mathcal{S}_G[C]$$

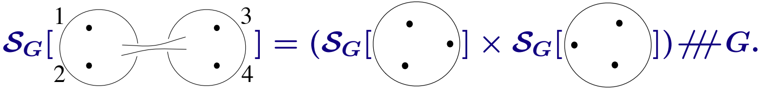
maps **the operations among Riemann surfaces** to **the operations among QFTs.**

For those who know the axioms of 2d TQFT:

A **usual** 2d TQFT takes values in **the monoidal category of vector spaces.**

This TQFT $\mathcal{S}_G$ takes values in **the monoidal category of 4d susy QFTs.**

Consider



Note that on the LHS, G^4 are manifestly interchangeable, by the formal property. There's an action of $S_4 \ltimes G^4$.

On the RHS, we started from two objects with $S_3 \ltimes G^3$ symmetry. But by connecting two G, we only have

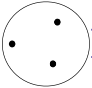
$$S_2 \ltimes ((S_2 \ltimes G^2) \times (S_2 \ltimes G^2)).$$

So, something nontrivial is going on. But this nontriviality happens within the category of QFTs.

We can get something nontrivial happening in something well defined, by applying $\mathcal{M}_{\text{Higgs}}$ or **SCI**.

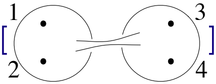
For this, let's take $G = \mathbf{SU(2)}$. Then it's known

$$\mathbf{Hyp}(V_1 \otimes V_2 \otimes V_3) = \mathcal{S}_{\mathbf{SU(2)}}[\;\;]$$

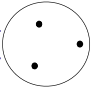
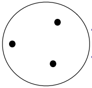


here, $V_i \simeq \mathbb{C}^2$ is the defining representation of $\mathbf{SU(2)}$. The LHS correctly has $\mathbf{SU(2)}^3$ action, with S_3 permuting them.

We have

$$\mathcal{M}_{\text{Higgs}}(\mathcal{S}_{\mathbf{SU(2)}}[$$


$$])$$

$$= \mathcal{M}_{\text{Higgs}}((\mathcal{S}_{\mathbf{SU(2)}}[$$

$$] \times \mathcal{S}_{\mathbf{SU(2)}}[$$

$$]) /\!/\!/ \mathbf{SU(2)})$$

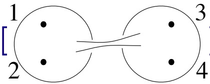
$$= (V_1 \otimes V_2 \otimes V \oplus V \otimes V_3 \otimes V_4)/\!/\!/\mathbf{SU(2)}$$

From the LHS, it should have S_4 permuting four $\mathbf{SU(2)}$s.

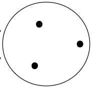
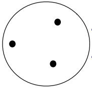
The RHS doesn't obviously have it.

In fact the RHS is the ADHM construction of the minimal nilpotent orbit of $\mathbf{SO(8)}_{\mathbb{C}}$. There's an outer automorphism S_3 acting on $\mathbf{SO(8)}$, that provides S_4 permuting $\mathbf{SU(2)}$s.

We also have

$\mathbf{SCI}(\mathcal{S}_{\mathbf{SU(2)}}[$ 

$])(u, v, x, y)$

$= \mathbf{SCI}((\mathcal{S}_{\mathbf{SU(2)}}[$  $] \times \mathcal{S}_{\mathbf{SU(2)}}[$  $]) /\!\!/\!\!/ \mathbf{SU(2)})(u, v, x, y)$

$$= \mathbf{SCI}((V_1 \otimes V_2 \otimes V \oplus V \otimes V_3 \otimes V_4) /\!\!/\!\!/ \mathbf{SU(2)})(u, v, x, y)$$

$$= \oint \frac{\prod_{\pm\pm\pm} \Gamma_{p,q}(t u^{\pm 1} v^{\pm 1} z^{\pm 1}) \prod_{\pm\pm\pm} \Gamma_{p,q}(t x^{\pm 1} y^{\pm 1} z^{\pm 1})}{\prod_{\pm} \Gamma_{p,q}(t^2 z^{\pm 2}) \prod_{\pm} \Gamma_{p,q}(z^{\pm 2})} \frac{dz}{2\pi i z}$$

From the LHS, the RHS should have S_4 symmetry permuting u, v, x, y. [Rastelli et al.]

Indeed it has, as proved by [van de Bult].

There are more operations, extracting concrete, well-defined mathematical physical objects from QFTs, e.g.

$$G\text{-symmetric } Q \mapsto W(Q)$$

where $W(Q)$ is a VOA with sub $\hat{g}$ affine Lie algebra VOA,

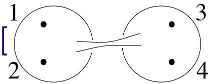
$$W(Q /\!\!/\!\!/ G)$$

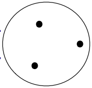
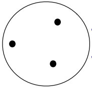
is given by something like Drinfeld-Sokolov reduction of $W(Q)$ with respect to $\hat{g}$, and

$$W(\mathbf{Hyp}(R))$$

is the standard symplectic boson VOA.

Then

$W(\mathcal{S}_{\mathbf{SU(2)}}[$  $])$

$= W((\mathcal{S}_{\mathbf{SU(2)}}[$  $] \times \mathcal{S}_{\mathbf{SU(2)}}[$  $]) /\!\!/\!\!/ \mathbf{SU(2)})$

$$= W((V_1 \otimes V_2 \otimes V \oplus V \otimes V_3 \otimes V_4) /\!\!/\!\!/ \mathbf{SU(2)})$$

is a VOA that can explicitly be written down, and
has four $\widehat{\mathfrak{su}(2)}$ affine Lie subalgebra.

But the existence of S_4 action permuting four $\widehat{\mathfrak{su}(2)}$ hasn't been proved.

In fact the final VOA is believed to be just $\widehat{\mathfrak{so}(8)}_{-2}$.

Yet another one: for a G-symmetric Q,

$$Z_{\text{Nekrasov}}(Q)$$

is an element in the **equivariant cohomology of the moduli space of G instantons.**

$$Z_{\text{Nekrasov}}(\mathbf{Hyp}(R))$$

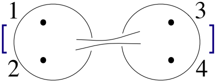
is determined by the index bundle of the Dirac operator associated to the representation R of G in the instanton background.

$$Z_{\text{Nekrasov}}(Q /\!\!/\!\!/ G)$$

is also computable, given $Z_{\text{Nekrasov}}(Q)$.

There are more formal properties satisfied by $\mathcal{S}_G[C]$ that depends on the complex structure of C that I didn't have time to explain. But then, the ability to permute four points on

$$Z_{\text{Nekrasov}}(\mathcal{S}_G[\quad])$$



is essentially equivalent to having a $W(G)$-algebra action on the equivariant cohomology of the moduli space of G instantons.

This was how **L. Fernando Alday**, **Davide Gaiotto** and I conjectured the relation between 4d gauge theory and 2d conformal field theory, although I streamlined the argument with lots of hindsight today.

That conjecutre, which has recently proved by mathematicians, was the main reason I was awarded the prize.

I'd like to thank you again, and to thank **Fernando** and **Davide**.

After all, **I was the last one to join the collaboration**: they needed a Mathematica code to compute Nekrasov's partition function, which I happened to have written as a project for my master's thesis.

Both Fernando and Davide moved on to other projects, and I'm the only one out of three who's still working on it.

I’ve been giving similar talks in many places, so I already have a more detailed write-up.

It’s available on my web page,

http://member.ipmu.jp/yuji.tachikawa/not-on-arxiv.html

So please have a look.

As a conclusion:

Many properties of QFT that are used to derive new mathematical conjectures are formalizable.

Of course there'll remain some 'deus ex machina', for example, the existence of the operation

$$Q \mapsto Q /\!\!/\!\!/ G$$

contains the solution to one of the Clay Millenium Problems: Seiberg and Witten have already shown that given some simple properties of

$$\bullet /\!\!/\!\!/ G,$$

you can easily show that the pure non-supersymmetric G gauge theory has a mass gap, and this part of the argument, again, is formalizable without much problem.

Similarly, the existence of the 6d $\mathcal{N} = (2, 0)$ theory $\mathcal{S}_G$ is another 'deus ex machina'. But, assuming that, the map

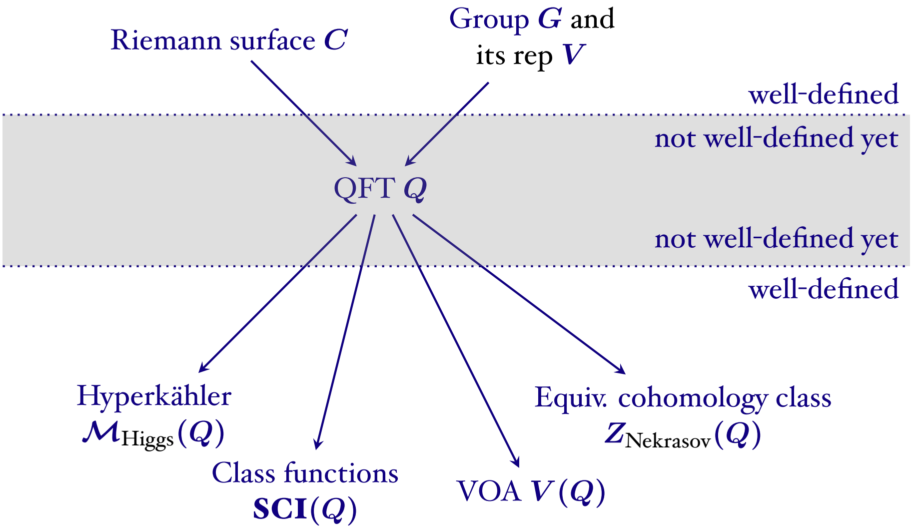
$$C \mapsto \mathcal{S}_G[C]$$

can be constructed rather formally, and many of the properties follow straightforwardly, in a way understandable to mathematicians (and mathematical physicisits).

So, we'll be able to rigorously show that the symmetry $u \leftrightarrow x$ of

$$\oint \frac{\prod_{\pm\pm\pm} \Gamma_{p,q}(tu^{\pm 1}v^{\pm 1}z^{\pm 1}) \prod_{\pm\pm\pm} \Gamma_{p,q}(tx^{\pm 1}y^{\pm 1}z^{\pm 1})}{\prod_{\pm} \Gamma_{p,q}(t^2 z^{\pm 2}) \prod_{\pm} \Gamma_{p,q}(z^{\pm 2})} \frac{dz}{2\pi i z}$$

follows from the existence of $\mathcal{S}_{\mathbf{SU(2)}}$.


Riemann surface C
Group G and its rep V
well-defined
not well-defined yet
QFT Q
not well-defined yet
well-defined
Hyperkähler $\mathcal{M}_{\text{Higgs}}(Q)$
Class functions $\mathbf{SCI}(Q)$
VOA $V(Q)$
Equiv. cohomology class $Z_{\text{Nekrasov}}(Q)$