On the absence of global anomalies of heterotic string theories

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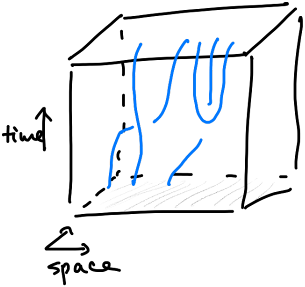
Introduction

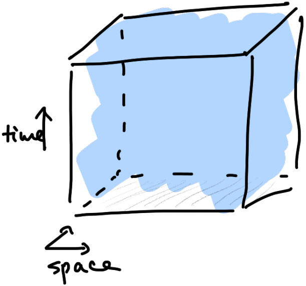
Quantum field theory deals with **particles** moving in the space-time.

In the first quantized approach, what we do is:



quantum mechanics of particles'

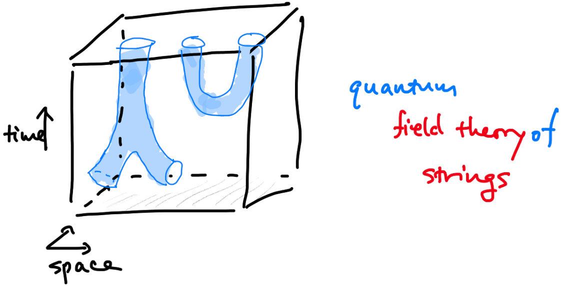
$\mapsto$



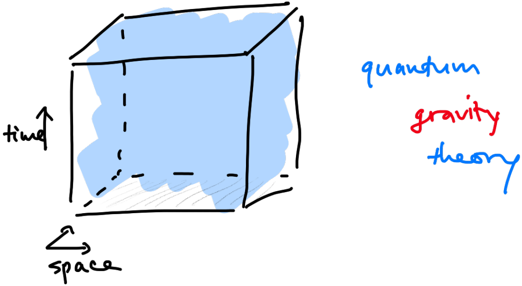
quantum field theory

Instead, **string theory** deals with **strings** moving in the space-time.

What we do is:


time
space
quantum
field theory of
strings

$\mapsto$


time
space
quantum
gravity
theory

Two noteworthy features of string theory:

1. It contains quantized gravity.

String theory wasn't devised to quantize gravity.

But treating relativistic strings quantum mechanically, you're force-fed with quantized gravity.


give me relativistic strings!
???
Quantum Gravity
I didn't order it...
Here you are!
QG
Eat it!
No! No! I'm not ready...

2. It's mysteriously consistent.

If you start analyzing string theory,
you find many places where it can fail to be consistent.

But it somehow manages to remain consistent,
thanks to various mysterious mathematical coincidences.

Today's talk is about one of such instances,

the absence of anomalies in string theory.

Anomalies

A D-dimensional QFT T can have anomalies.

Let $Z_T[M_D, g]$ be its partition function
on the manifold M_D with a metric g.

Let $g' \sim g$ be a metric diffeomorphic to g.

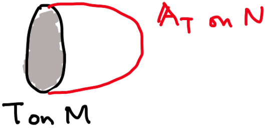
In an anomalous theory,

$$Z_T[M_D; g'] = e^{i\theta(M;g',g)} Z_T[M_D; g],$$

where the phase θ is **computable but nonzero**.

This poses a problem in quantum gravity, since we'd like to perform the path integral of $Z_T[M_D; g']$ over **the diffeomorphism class of g**.

In a more modern perspective, an anomalous theory $\boldsymbol{T}$ on $\boldsymbol{M_D}$ lives on the boundary of another theory $\mathbb{A}_{\boldsymbol{T}}$ on $\boldsymbol{N_{D+1}}$, where $\boldsymbol{M_D = \partial N_{D+1}}$:



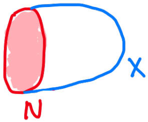
The partition function of the combined system is well-defined, but that of the boundary theory alone is not.

Then you can't perform the path integral of the metric only on the boundary.

String theory is inconsistent unless $\mathbb{A}_{\text{string}}$ is trivial.

The theory $\mathbb{A}_T$ characterizing the anomaly of a theory T is invertible. It consists of two parts, the **perturbative part** and the **global part**.

The **perturbative part** specifies $\mathbb{A}_T$ on $N_{D+1} = \partial X_{D+2}$

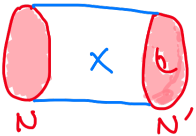


via

$$Z_{\mathbb{A}_T}[N_{D+1}] = \exp(2\pi i \int_{X_{D+2}} I_T)$$

where I is a polynomial of characteristic classes, known as the **anomaly polynomial** of T.

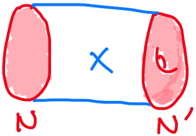
The same $\boldsymbol{I_T}$ gives the following relation: if



then

$$Z_{\mathbb{A}_T}[N_{D+1}] = \exp(2\pi i \int_{X_{D+2}} I_T) Z_{\mathbb{A}_T}[N'_{D+1}].$$

Assuming the perturbative part I_T vanishes, we see that



implies

$$Z_{\mathbb{A}_T}[N_{D+1}] = Z_{\mathbb{A}_T}[N'_{D+1}].$$

This means that $\mathbb{A}_T$ determines a map

$$Z_{\mathbb{A}_T} : \Omega_{D+1}^{\text{structure}} \to U(1)$$

where $\Omega_{D+1}^{\text{structure}}$ is the group of equivalence classes $N_D \sim N'_D$, known as the **bordism group**. This is the **global anomaly**.

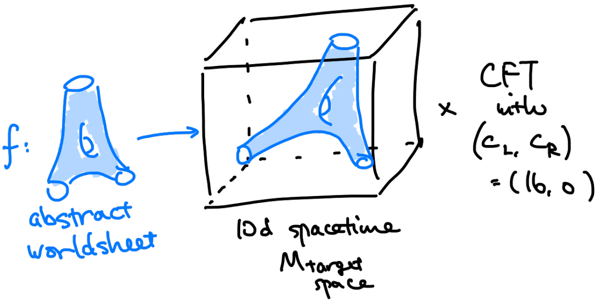
("structure" can be **spin**, **oriented**, ..., depending on your setup.)

Anomalies of 10d heterotic string theory

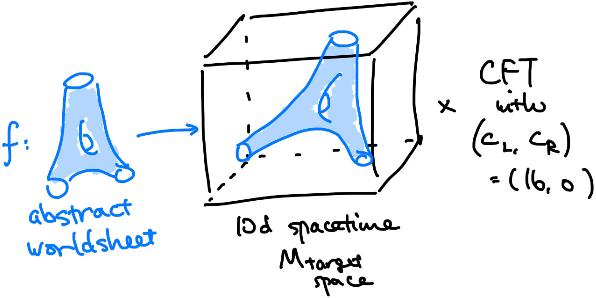
10d heterotic string theory is an elaborate machine with

Input: a modular invariant 2d CFT $T_{\text{worldsheet}}$
with $(c_L, c_R) = (16, 0)$

Output: a 10d quantum gravity $QG[T_{\text{w.s.}}]$



Let us first discuss the anomaly of the **worldsheet theory**.

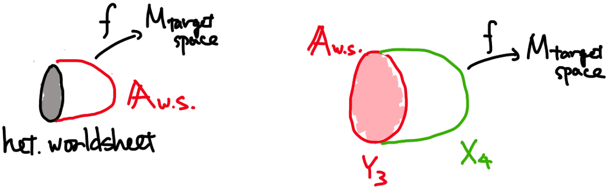


The 10d spacetime is equipped with a three-form $\boldsymbol{H}$ with

$$dH = \frac{p_1(M_{\text{target space}})}{2}.$$

Why?

The heterotic worldsheet has 10 right-moving fermions coming from the pullback $f^*(TM_{\text{target space}})$ of the tangent bundle of the spacetime.



These fermions, taking values in the pullback of $TM_{\text{t.s.}}$, has the anomaly

$$Z_{\mathbb{A}_{\text{w.s.fermion}}}[Y_3] = \exp(-2\pi i \int_{X_4} \frac{f^*(p_1(M_{\text{target space}}))}{2}).$$

which prevents us from doing the path integral over the worldsheet. Luckily, heterotic string theory has a 3-form field H with the coupling

$$Z_{\mathbb{A}_{\text{from } H}}[Y_3] = \exp(2\pi i \int_{Y_3} f^*(H)).$$

and $dH = p_1(M_{\text{target space}})/2$, so

$$Z_{\mathbb{A}_{\text{tot}}} = Z_{\mathbb{A}_{\text{w.s.fermion}}} Z_{\mathbb{A}_{\text{from } H}} = 1.$$

Now you can path-integrate over the worldsheet.

Anomaly of the spacetime theory

10d heterotic string theory is an elaborate machine with

Input: a modular invariant 2d CFT $T_{\text{worldsheet}}$ with $(c_L, c_R) = (16, 0)$

Output: a 10d quantum gravity $QG[T_{\text{w.s.}}]$

The 2d theory $T_{\text{w.s.}}$ has **one state with $L_0 = 0$, i.e. the vacuum**.
Let N be the number of **states with $L_0 = 1$**.

The 2d vacuum state gives **10d gravitino (a spin-$3/2$ fermion)**.
Each 2d state with $L_0 = 1$ gives a **10d fermion (of spin-$1/2$)**.
So there will be N **10d fermions**.

The anomaly polynomial of a **10d gravitino** is

$$I_{\text{gravitino}} = \frac{p_1^3}{3780} - \frac{13p_1p_2}{756} + \frac{31p_3}{3780} = \frac{31p_3}{3780}$$

while that of N **spin-1/2 fermions** is

$$NI_{1/2} = N\left[-\frac{31p_1^3}{967680} + \frac{11p_1p_2}{241920} - \frac{p_3}{60480}\right] = -\frac{Np_3}{16 \cdot 3780}.$$

Here we used the fact that heterotic string theory has 3-form field H such that $dH = p_1/2$. As recalled,
this was required for the anomaly cancellation on the worldsheet.

We see $I_{\text{gravitino}} + NI_{1/2} = 0$ iff $N = 31 \cdot 16 = 496$.

10d heterotic string theory is an elaborate machine with

Input: a modular invariant 2d CFT $T_{\text{worldsheet}}$ with $(c_L, c_R) = (16, 0)$

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There are only two such 2d CFTs,
based on $E_8 \times E_8$ or $SO(32)$ current algebra.
(This is now mathematically proved: [Dong-Mason, math.QA/0203005])

For both,

$$N = \begin{matrix}\text{\# of states} \\ \text{with } L_0 = 1\end{matrix} = \dim G = 496.$$

So $I_{\text{gravitino}} + 496 I_{1/2} = 0$.

We can conclude this without the explicit classification.
For $T_{\text{w.s.}}$ with $c_L = 16$, its torus partition function

$$Z_{T_{\text{w.s.}}}(q) = \operatorname{tr} q^{L_0 - c/24} = q^{-2/3}(1 + Nq + \cdots).$$

should be modular invariant up to a phase.

The theory of modular functions tells us that the unique such function is

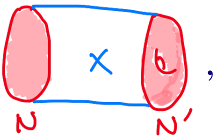
$$\eta(q)^{-16} c_4(q)^2 = q^{-2/3}(1 - q + \cdots)^{-16}(1 + 240q + \cdots)^2$$

where η is the Dedekind eta and c_4 is the normalized 4-th Eisenstein series.

Then N is automatically 496,
guaranteeing the vanishing of the perturbative anomaly.

How about the global anomaly?

We learned that the perturbative anomaly cancels. Therefore, when



we have

$$Z_{\mathbb{A}_{\text{heterotic}}}[N_{11}] = Z_{\mathbb{A}_{\text{heterotic}}}[N'_{11}].$$

This means that we have a homomorphism

$$Z_{\mathbb{A}_{\text{heterotic}}} : \Omega_{11}^{\text{string}} \to U(1)$$

where Ω_d^{string} is the string bordism group,
i.e. the group of equivalence classes $N_d \sim N'_d$ where
every manifold in question is equipped with H solving $dH = p_1/2$.

Somehow $\boldsymbol{\Omega}^{\text{string}}_{d\leq 16}$ was computed already in [Giambalvo 1971]:

d	0	1	2	3	4	5	6	7
Ω^{string}_d	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	0
d	8	9	10	11	12	13	14	15
Ω^{string}_d	$\mathbb{Z}\oplus\mathbb{Z}_2$	$(\mathbb{Z}_2)^2$	$\mathbb{Z}_6$	0	$\mathbb{Z}$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Somehow it's miraculously zero in the required place! Therefore

$$Z_{\mathbb{A}_{\text{heterotic}}} : \Omega^{\text{string}}_{11} \to U(1)$$

is automatically trivial, guaranteeing the vanishing of the global anomaly.

Why was the bordism group of manifolds with $dH = p_1/2$ interesting to mathematicians in 1971?

Consider tangent bundles (or more generally orthogonal bundles) on manifolds M.

The first non-triviality is associated to $\pi_0(O(n)) = \mathbb{Z}_2$, corresponding to the class $w_1 \in H^1(M, \mathbb{Z}_2)$.
If w_1 is trivialized, we have the **orientation**.

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The second non-triviality is associated to $\pi_1(SO(n)) = \mathbb{Z}_2$, corresponding to the class $w_2 \in H^2(M, \mathbb{Z}_2)$.
If w_2 is trivialized, we have the **spin structure**.

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The third non-triviality is associated to $\pi_3(Spin(n)) = \mathbb{Z}$, corresponding to the class $p_1/2 \in H^4(M, \mathbb{Z})$.
If $p_1/2$ is trivialized, we have the **string structure**.

Summary so far

In 10d heterotic string theories,

The perturbative anomaly cancels,
because the theory of modular forms knew
the ratio $\mathbf{496}$ between $\boldsymbol{I}_{\text{gravitino}}$ and $\boldsymbol{I}_{\text{spin-1/2 fermion}}$.

The global anomaly vanishes,
because for some strange reasons $\boldsymbol{\Omega}^{\text{string}}_{\mathbf{11}}$ is trivial.

The rest of the talk: lower-dim'l compactifications

In lower-dimensional heterotic compactifications,

the perturbative anomaly is known to be canceled in a similar manner: the theory of modular forms is known to produce precisely the required number of fermion fields.

This was shown by various subsets of
{Lerche, Nilsson, Schellekens and Warner} in the late 1980s.

How about the global anomaly?

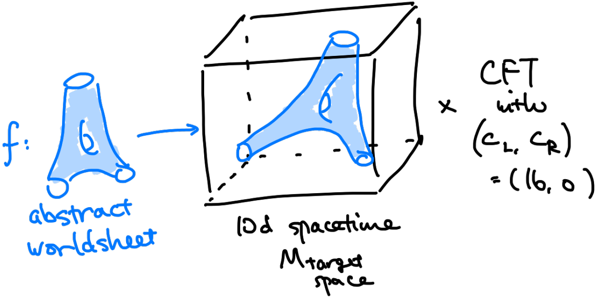
Well, to start with, **no general theory of global anomalies was known until very recently,** (except perhaps implicitly to a very, very small number of people such as Freed or Witten).

Luckily, the study of SPT phases on the cond-mat side from around 2010 sparked a lot of activities in hep-th and math.AT, and **we now have a veritable understanding of it using bordisms**, the viewpoint from which I already used in this talk.

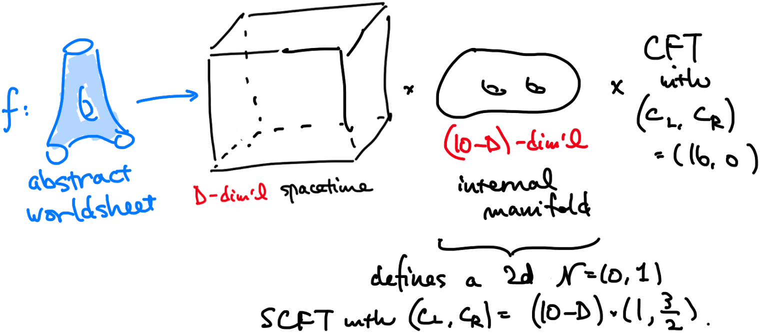
This turns out to be crucial. Let us continue.

Lower dimensional heterotic compactifications

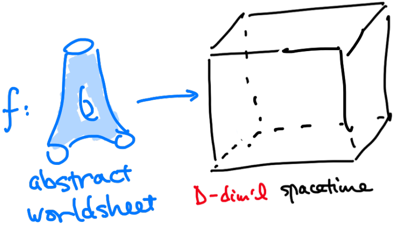
10d heterotic string theory has the following structure



A **geometric** compactifications to ***D*** dimensions have the form



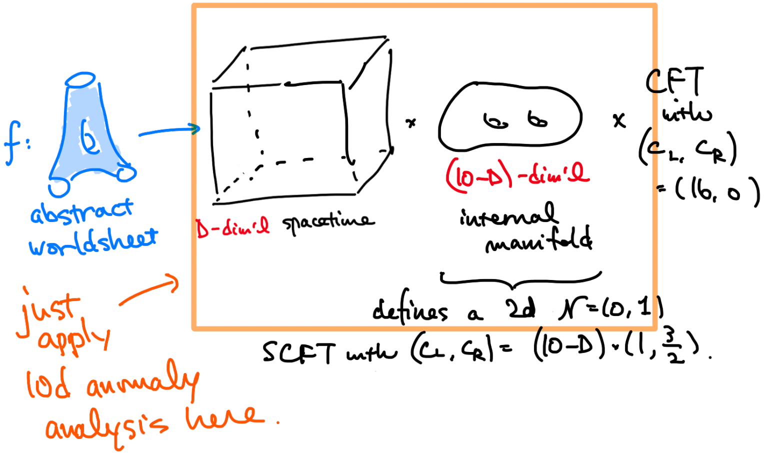
More general nongeometric compactifications have the form



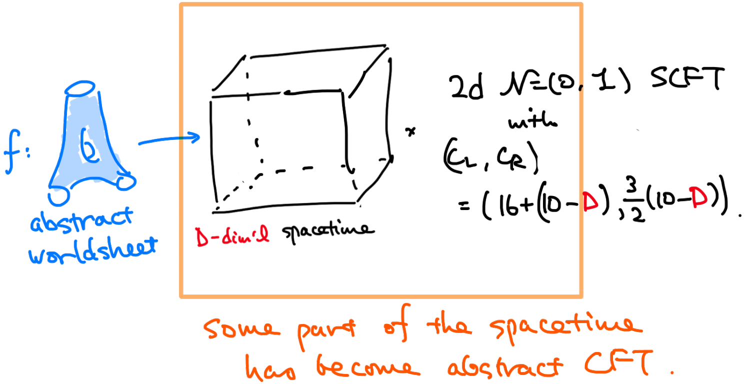
$\times$ 2d $\mathcal{N}=(0,1)$ SCFT with

$$(c_L, c_R) = \left(16 + (10 - D), \tfrac{3}{2}(10 - D)\right)$$

Anomalies are guaranteed to cancel in geometric compactifications, since



This doesn't work with more general compactifications



How should we proceed? Let us first proceed naively…

4d Witten anomaly

Consider the case $D = 4$. The worldsheet theory is a 2d $\mathcal{N}{=}(0,1)$ SCFT $T_{\text{w.s.}}$ with

$$(c_L, c_R) = (16 + (10-4), \frac{3}{2}(10-4)) = (22, 9).$$

If $T_{\text{w.s.}}$ has $su(2)$ flavor symmetry, the resulting 4d quantum gravity theory has $su(2)$ gauge group.

The R-sector states of $T_{\text{w.s.}}$ with $(L_0, \bar{L}_0) = (1, 0)$ give 4d chiral fermions, where 4d chirality is given by the right-moving fermion number $(-1)^{F_R}$ on the worldsheet.

A fermion in the doublet of $su(2)$ has a global anomaly associated to

$$\pi_4(SU(2)) = \Omega_4^{\text{spin}}(BSU(2)) = \mathbb{Z}_2.$$

Let us say the doublet irrep of $su(2)$ appears N_2 times in the R-sector states of $T_{\text{w.s.}}$ with $(L_0, \bar{L}_0) = (1, 0)$.

Then the 4d quantum gravity theory obtained from $T_{\text{w.s.}}$ is afflicted with Witten anomaly unless

$$N_2 \equiv 0 \mod 2.$$

2d $\mathbb{Z}_{24}$ anomaly

Consider the case $D = 2$. The worldsheet theory is a 2d $\mathcal{N}=(0,1)$ SCFT $T_{\text{w.s.}}$ with

$$(c_L, c_R) = (16 + (10 - 2), \frac{3}{2}(10 - 2)) = (24, 12).$$

The R-sector states of $T_{\text{w.s.}}$ with $(L_0, \bar{L}_0) = (1, 0)$ give 2d spacetime chiral fermions, where 2d spacetime chirality is given by the right-moving fermion number $(-1)^{F_R}$ on the worldsheet.

Let us say the net number of chiral fermions is N.

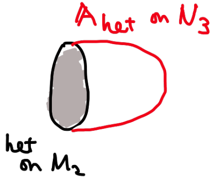
Their combined anomaly polynomial is

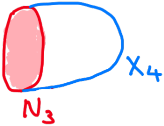
$$N\frac{p_1}{48} = 0$$

since $dH = p_1/2$. The perturbative part of the anomaly is automatically absent.

But this can leave a global anomaly, because this cancellation was achieved by adding a term $NH/24$ in the 3d anomaly theory.

What I mean is the following:





$$\frac{N}{24}\int_{N_3} H = N \int_{X_4} \frac{p_1}{48}$$

thanks to $dH = p_1/2$

$$\Rightarrow \quad Z_{A_{het}}\left[S_3 \text{ with } \int_{S_3} H = 1 \right] = e^{2\pi i \frac{N}{24}} .$$

generator of $\Omega_3^{string} = \mathbb{Z}_{24}$.

That is, the global anomaly of this 2d heterotic system is characterized by

$$Z_{\mathbb{A}_{\text{het}}} : \Omega_3^{\text{string}} = \mathbb{Z}_{24} \to U(1)$$

which is given by

$$\mathbb{Z}_{24} \ni 1 \mapsto \exp(2\pi i \frac{N}{24}) \in U(1).$$

Therefore, the theory is afflicted with the $\mathbb{Z}_{24}$ global anomaly unless the worldsheet theory $T_{\text{w.s.}}$ is such that

$$N := \mathrm{tr}_V (-1)^{F_R} \equiv 0 \mod 24$$

where V is the space of R-sector states with $(L_0, \bar{L}_0) = (1, 0)$.

So the questions are:

Let $T_{\text{w.s.}}$ be a 2d $\mathcal{N}{=}(0,1)$ SCFT with central charge

$$(c_L, c_R) = \begin{cases} (22, 9) & \text{for } D = 4, \\ (24, 12) & \text{for } D = 2. \end{cases}$$

Let V be the space of R-sector states with $(L_0, \bar{L}_0) = (1, 0)$.
Heterotic string theory constructed from $T_{\text{w.s.}}$ has an anomaly unless

V contains an even number of 2-dim'l irrep of $SU(2)$ $\quad (D = 4)$,
$\text{tr}_V(-1)^{F_R}$ is divisible by 24 $\quad (D = 2)$.

These are the kind of questions physicists don't know how to answer at present. Math comes to the rescue!

Topological modular forms

Topological modular forms, $\mathbf{TMF}$, generalize and refine the ring of modular forms.

It was mathematically constructed around 2000 by Hopkins et al., using an amalgam of algebraic topology and algebraic geometry.

[Hopkins math.AT/0212397]

We have Abelian groups $\mathbf{TMF}_{\nu}$, for $\nu \in \mathbb{Z}$.

For our purpose, the **conjecture of Segal-Stolz-Teichner** is crucial:

The Segal-Stolz-Teichner conjecture says

$$\mathbf{TMF}_{\boldsymbol{\nu}} = \frac{\left\{ \begin{array}{c} \text{2d } \boldsymbol{\mathcal{N}{=}(0,1)} \text{ supersymmetric theory} \\ \text{with } \boldsymbol{\nu = 2(c_R - c_L)} \end{array} \right\}}{\text{continuous deformation}}$$

[Segal 1988] [Stolz-Teichner 2002] [Stolz-Teichner 1108.0189]

Here allowed deformations are:

- relevant / marginal / irrelevant
- going up and down RG flows
- adding a sector which spontaneously breaks SUSY

The Segal-Stolz-Teichner conjecture says

$$\mathbf{TMF}_{\nu} = \frac{\left\{ \begin{array}{c} \text{2d } \mathcal{N}{=}(0,1) \text{ supersymmetric theory} \\ \text{with } \nu = 2(c_R - c_L) \end{array} \right\}}{\text{continuous deformation}}$$

[Segal 1988] [Stolz-Teichner 2002] [Stolz-Teichner 1108.0189]

An $\mathcal{N}{=}(0,1)$ SCFT T with $2(c_R - c_L) = \nu$ should then determine an element

$$[T] \in \mathbf{TMF}_{\nu}.$$

Why is this conjecture plausible?

Mathematicians constructed a map ϕ_W,
which for us extracts the elliptic genus

$$\begin{array}{ccccc} \phi_W: & \mathbf{TMF}_\nu & \to & \left\{ \begin{array}{c} \text{modular forms of} \\ \text{weight } \frac{\nu}{2} \text{ with} \\ \text{integer coeff.s and poles} \end{array} \right\} \\ & \Cup & & \Cup \\ & \text{a theory } T & \mapsto & \phi_W(T) = \eta(q)^\nu Z_{\text{ell}}(T;q) \end{array}$$

where

$$Z_{\text{ell}}(T;q) = \mathrm{tr}_R (-1)^{F_R} q^{L_0 - c_L/24}$$

is physicists' elliptic genus;
the factor of $\eta(q)^\nu$ is to improve modular invariance properties.

Mathematicians call ϕ_W the Witten genus.

Mathematicians also constructed a map σ,
which is the quantization map under the conjecture:

$$\begin{array}{cccc} \sigma : & \left\{ \begin{array}{c} \nu\text{-dim'l manifold} \\ \text{with } dH = \frac{p_1}{2} \end{array} \right\} & \to & \mathrm{TMF}_\nu \\ & \Cup & & \Cup \\ & (M, H) & \mapsto & \begin{array}{c} \mathcal{N}{=}(0,1) \text{ sigma model} \\ \text{on } (M, H) \end{array} \end{array}$$

Physicists know that there is a sigma model anomaly unless $dH = p_1/2$. Mathematicians know this condition in their own way.

And the composition

$$
\begin{array}{ccccc}
\left\{ \begin{array}{c} \nu\text{-dim'l manifold} \\ \text{with } dH = \frac{p_1}{2} \end{array} \right\} & \xrightarrow{\sigma} & \mathbf{TMF}_\nu & \xrightarrow{\phi_W} & \left\{ \begin{array}{c} \text{modular forms of} \\ \text{weight } \frac{\nu}{2} \text{ with} \\ \text{integer coeff.s and poles} \end{array} \right\} \\
\Cup & & \Cup & & \Cup \\
(M, H) & \mapsto & \begin{array}{c} \text{sigma model} \\ \text{on } (M, H) \end{array} & \mapsto & \begin{array}{r} \phi_W(\sigma(M, H)) = \\ \eta(q)^\nu Z_{\text{ell}}(\sigma(M, H); q) \end{array}
\end{array}
$$

does what physicists expect.

The computation of this part
goes back to [Witten's elliptic genus paper].

Solution to the 2d $\mathbb{Z}_{24}$ anomaly issue

Let $T_{\text{w.s.}}$ be a 2d $\mathcal{N}{=}(0,1)$ SCFT with central charge

$$(c_L, c_R) = (24, 12).$$

Let V be the space of R-sector states with $(L_0, \bar{L}_0) = (1, 0)$.
Heterotic string theory constructed from $T_{\text{w.s.}}$ has an anomaly
unless $\mathrm{tr}_V(-1)^{F_R}$ is divisible by 24.

In other words, unless the elliptic genus of $T_{\text{w.s.}}$

$$\begin{aligned} Z_{\text{ell}}(T_{\text{w.s.}}; q) &= \mathrm{tr}_R(-1)^{F_R} q^{L_0 - c_L/24} \\ &= a q^{-1} + b + O(q^1) \end{aligned}$$

is such that b **is divisible by 24.**

Let $T_{\text{w.s.}}$ be a 2d $\mathcal{N}=(0,1)$ SCFT with central charge

$$(c_L, c_R) = (24, 12).$$

It determines a class $[T_{\text{w.s.}}] \in \mathbf{TMF}_{2(c_R-c_L)} = \mathbf{TMF}_{-24}$, and

$$\phi_W([T_{\text{w.s.}}]) = \eta(q)^{24} Z_{\text{ell}}(T_{\text{w.s.}}; q) = \eta(q)^{-24}(aq^{-1} + b + \cdots).$$

This is a modular form
of weight -12
with integer coefficients
and poles of order at most 2.

Using standard facts about modular functions, we can conclude

$$\phi_W(T_{\text{w.s.}}; q) = a\Theta_{E_8}^3\Delta^{-2} + (-744a + b)\Delta^{-1}$$

where

$$\Theta_{E_8} = 1 + 240q + \cdots$$

is the theta function of the E_8 lattice and

$$\Delta = \eta(q)^{24}$$

is the modular discriminant.

Now, a theorem of Hopkins concerning ϕ_W says that

> $b\Delta^k$ is in the image of ϕ_W from TMF_{24k}
> iff b is a multiple of $\dfrac{24}{\gcd(24, k)}$.

Here $k = -1$, so b is a multiple of 24. Done.

4d question?

Let $T_{\text{w.s.}}$ be a 2d $\mathcal{N}=(0,1)$ SCFT with central charge

$$(c_L, c_R) = (22, 9)$$

with $su(2)$ symmetry.
Let V be the space of R-sector states with $(L_0, \bar{L}_0) = (1, 0)$.

Heterotic string theory constructed from $T_{\text{w.s.}}$ has an anomaly
unless V contains an even number of 2-dim'l irrep of $SU(2)$.

$\mathbf{TMF}$ should be able to answer that, since such a $T_{\text{w.s}}$ should determine a class

$$[T_{\text{w.s.}}] \in \mathbf{TMF}^{SU(2)}_{2(9-22)}$$

whose image under ϕ_W should know this mod-2 behavior, just as in the case of $\mathbb{Z}_{24}$ anomaly we discussed.

But this approach hasn't worked yet, since

- $\mathbf{TMF}^{SU(2)}_{-26}$ has not been computed.
- The image under ϕ_W has not been determined either.

Even if it worked, you would then be forced to analyze different spacetime dimensions D with different symmetry groups G one by one.

General solution

Again, math comes to the rescue: there is a way to solve the anomaly question, once and for all, for every (D, G) at once.

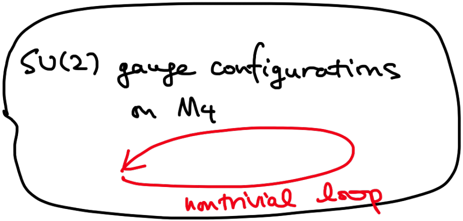
This is where the help from my wonderful collaborator **Mayuko Yamashita** was essential.

It uses a lot of algebraic topology.

Let me first try to give some rough ideas behind the derivation.

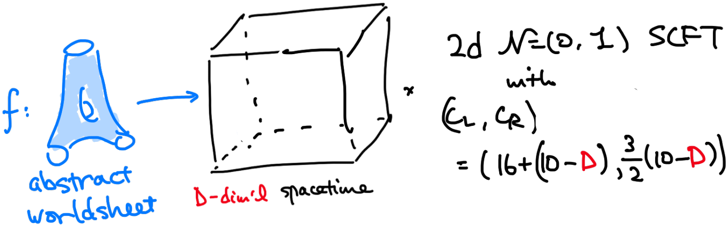
Rough ideas behind the general solution

Recall that Witten's $SU(2)$ global anomaly in 4d is associated to a nontrivial loop in the space of gauge configurations:



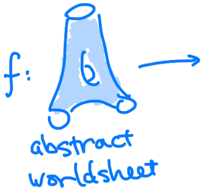
In particular, there is a loop associated to $\pi_4(SU(2)) = \mathbb{Z}_2$, which was responsible for the gauge anomaly.

Now consider heterotic compactifications:



The global anomaly (of traditional type) is associated to nontrivial loops in such configurations.

In fact, this is just a special case of



2d $\mathcal{N}=(0,1)$ SCFT

with

$$(c_L, c_R) = \left(16 + (10 - 0), \tfrac{3}{2}(10 - 0)\right) = (26, 15).$$

and the global anomaly (of traditional type) is associated to nontrivial loops in such configurations.

So, what can cause the global anomaly (of traditional type) is

$$\pi_1\left(\text{space of } \boxed{\begin{array}{l}\text{2d } \mathcal{N}=(0,1) \text{ SCFTs} \\ \quad\text{with} \\ (c_L, c_R) \\ \quad = \left(16+(10-0), \tfrac{3}{2}(10-0)\right). \\ \quad = (26, 15).\end{array}}\right)$$

Now, the Stolz-Teichner conjecture in the form I quoted was

$$\mathrm{TMF}_\nu = \frac{\left\{\begin{array}{c}\text{2d } \mathcal{N}{=}(0,1) \text{ supersymmetric theory} \\ \text{with } \nu = 2(c_R - c_L)\end{array}\right\}}{\text{continuous deformation}}$$

But this can also be written as

$$\mathrm{TMF}_\nu = \pi_0 \left(\begin{array}{c}\text{space of} \\ \text{2d } \mathcal{N}{=}(0,1) \text{ supersymmetric theories} \\ \text{with } \nu = 2(c_R - c_L)\end{array}\right)$$

whose generalized form is

$$\mathrm{TMF}_{\nu+k} = \pi_k \left(\begin{array}{c}\text{space of} \\ \text{2d } \mathcal{N}{=}(0,1) \text{ supersymmetric theories} \\ \text{with } \nu = 2(c_R - c_L)\end{array}\right)$$

What can cause the global anomaly of heterotic strings is

$$\mathbf{TMF}_{\nu+1} = \pi_1 \left(\begin{array}{c} \text{space of} \\ \text{2d } \mathcal{N}\!=\!(0,1) \text{ supersymmetric theories} \\ \text{with } \nu = 2(c_R - c_L) \end{array} \right)$$

where

$$\nu = 2(15 - 26) = -22.$$

But it is known that $\mathbf{TMF}_{-21} = 0$, so **there is no such nontrivial loop in the configuration space of heterotic string theories**, and therefore **there can be no global anomaly**.

A more precise description

Recall the modern general theory of anomalies:
the anomaly of a D-dimensional theory with spacetime structure S and internal symmetry G is characterized by

$$I_{\mathbb{Z}}\Omega^{S}_{D+2}(BG)$$

which fits into

$$0 \to \overbrace{\mathrm{Hom}(\Omega^{S}_{D+1}(BG)|_{\mathrm{tor}}, U(1))}^{\text{global anomaly}} \to (I_{\mathbb{Z}}\Omega^{S})^{D+2}(BG) \to \underbrace{\mathrm{Hom}(\Omega^{S}_{D+2}(BG)|_{\mathrm{free}}, \mathbb{Z})}_{\text{perturbative anomaly}} \to 0$$

This is the Anderson dual of the bordism group.

So, the anomalies of D-dimensional string theories are characterized via

$$\left\{ \begin{array}{c} D\text{-dimensional heterotic} \\ \text{string theories} \\ \text{with gauge symmetry } G \end{array} \right\} \to (I_{\mathbb{Z}}\Omega^{\text{string}})^{D+2}(BG).$$

Recall also that 2d SCFTs determine classes in $\mathbf{TMF}$:

$$\left\{ \begin{array}{c} \text{2d } \mathcal{N}{=}(0,1) \text{ SCFTs with} \\ (c_L, c_R) = (26 - D, \frac{3}{2}(10 - D)) \\ \text{with flavor symmetry } G \end{array} \right\} \to \mathrm{TMF}_G^{2((26-D)-\frac{3}{2}(10-D))}$$

$$= \mathrm{TMF}_G^{D+22}$$

$$\to \mathrm{TMF}^{D+22}(BG).$$

Recall that heterotic string theory is the machinery

$$\left\{\begin{array}{c}\text{2d } \boldsymbol{\mathcal{N}=(0,1)} \text{ SCFTs with} \\ \boldsymbol{(c_L, c_R) = (26-D, \tfrac{3}{2}(10-D))} \\ \text{with flavor symmetry } \boldsymbol{G}\end{array}\right\} \to \left\{\begin{array}{c}\boldsymbol{D}\text{-dimensional heterotic} \\ \text{string theories} \\ \text{with gauge symmetry } \boldsymbol{G}\end{array}\right\}.$$

We combine the ingredients:

$$\left\{\begin{array}{c} \text{2d } \mathcal{N}=(0,1) \text{ SCFTs with} \\ (c_L, c_R) = (26-D, \tfrac{3}{2}(10-D)) \\ \text{with flavor symmetry } G \end{array}\right\} \xrightarrow[\text{construction}]{\text{heterotic}} \left\{\begin{array}{c} D\text{-dimensional heterotic} \\ \text{string theories} \\ \text{with gauge symmetry } G \end{array}\right\}$$

$$\text{take} \Big\downarrow \text{TMF class} \qquad\qquad\qquad \text{extract} \Big\downarrow \text{anomaly}$$

$$\mathrm{TMF}^{D+22}(BG) \xrightarrow{\quad \mathbb{A} \quad} (I_{\mathbb{Z}}\Omega^{\text{string}})^{D+2}(BG)$$

The homomorphism $\mathbb{A}$ characterizes anomalies (both perturbative and global) of heterotic string theories. We would like to show it vanishes.

Physically, we expect $\mathbb{A}$ to have various natural properties.

For example, with $X \to Y$, the square

$$\begin{array}{ccc} \mathrm{TMF}^{D+22}(Y) & \xrightarrow{\mathbb{A}} & (I_{\mathbb{Z}}\Omega^{\text{string}})^{D+2}(Y) \\ \downarrow & & \downarrow \\ \mathrm{TMF}^{D+22}(X) & \xrightarrow{\mathbb{A}} & (I_{\mathbb{Z}}\Omega^{\text{string}})^{D+2}(X) \end{array}$$

should commute. Mathematically, this means that

$\mathbb{A}$ is a natural transformation of two generalized cohomology theories, TMF and $I_{\mathbb{Z}}\Omega^{\text{string}}$.

As such, it is represented by an element

$$\begin{aligned} \mathbb{A} &\in [\mathrm{TMF}, \Sigma^{-20} I_{\mathbb{Z}}\Omega^{\text{string}}] \\ &= [\mathrm{TMF} \wedge \Omega^{\text{string}}, \Sigma^{-20} I\mathbb{Z}]. \end{aligned}$$

Also, D-dim'l heterotic string theory with $T_{w.s.}$,
further compactified on a d-dim'l manifold (M, H),
is $(D-d)$-dim'l heterotic string theory with $T_{w.s.} \times \sigma(M, H)$.

This determines a multiplication

$$[T_{w.s.}] \in \mathbf{TMF}, \qquad [(M, H)] \in \Omega^{\text{string}}$$
$$\longrightarrow [T_{w.s.}] \times [\sigma(M, H)] \in \mathbf{TMF}.$$

Compatibility of this multiplication with

$$\mathbb{A} : \mathbf{TMF} \wedge \Omega^{\text{string}} \to \Sigma^{-20} I\mathbb{Z}$$

means that $\mathbb{A}$ is determined by a single element $\mathbb{B}$ as in

$$\mathbb{A} : \mathbf{TMF} \wedge \Omega^{\text{string}} \xrightarrow{\text{multiplication}} \mathbf{TMF} \xrightarrow{\mathbb{B}} \Sigma^{-20} I\mathbb{Z}.$$

This element $\mathbb{B}$ takes values in

$$\mathbb{B} \in [\mathbf{TMF}, \Sigma^{-20} I\mathbb{Z}] = (I_{\mathbb{Z}}\mathbf{TMF})^{-20}(pt),$$

which is given by

$$0 \to \mathrm{Hom}(\mathbf{TMF}_{-21}(pt)|_{\text{tor}}, U(1)) \xrightarrow{i} (I_{\mathbb{Z}}\mathbf{TMF})^{-20}(pt) \xrightarrow{p} (\mathrm{Hom}(\mathbf{TMF}_{-20}(pt)|_{\text{free}}, \mathbb{Z}) \to 0.$$

Now, $p(\mathbb{B})$ characterizes the perturbative part of the anomaly, and is therefore known to be zero.

Therefore, $\mathbb{B}$ is in the image of i.

But mathematicians has computed that $\mathbf{TMF}_{-21}(pt) = 0$.

So $\mathbb{B} = 0$. Done.

Comments

Clearly we are not really done.

What I did was to transfer

the question of global anomalies of heterotic strings

to

the validity of the Segal-Stolz-Teichner conjecture

$$\mathrm{TMF}_\nu = \frac{\left\{ \begin{array}{c} \text{2d } \mathcal{N}{=}(0,1) \text{ supersymmetric theory} \\ \text{with } \nu = 2(c_R - c_L) \end{array} \right\}}{\text{continuous deformation}}$$

$$\mathbf{TMF}_\nu = \frac{\left\{\begin{array}{c}\text{2d } \mathcal{N}=(0,1) \text{ supersymmetric theory} \\ \text{with } \nu = 2(c_R - c_L)\end{array}\right\}}{\text{continuous deformation}}$$

It will be very hard to get a mathematically rigorous proof.
The RHS isn't even defined yet!

Instead, let us consider what it tells us, assuming its validity.

Many subtle properties on the LHS are known.

They translate to **many subtle properties of 2d theories** which are not at all apparent to us.

One example is the crucial input we used :

$$\mathbf{TMF_{-21} = 0.}$$

This means that all 2d $\mathcal{N}=(0,1)$ theories with $2(c_R - c_L) = -21$ can be continuously connected.

How do we even begin to understand this in our own way?

As another example, let us take $\mathbf{TMF_3} = \mathbb{Z}_{24}$.

This means that 2d $\mathcal{N}=(0,1)$ theories with $2(c_R - c_L) = 3$ can be classified by $\mathbb{Z}_{24}$.

Examples in each class $k \in \mathbb{Z}_{24}$ are believed to be given by

$\mathcal{N}=(0,1)$ sigma models on $S^3 = SU(2)$ with WZW level k.

How to see the mod-24 behavior in k was discussed in [Gaiotto, Johnson-Freyd, Witten 1902.10249].

How to extract a $\mathbb{Z}_{24}$ invariant from such a theory was discussed in [Gaiotto, Johnson-Freyd 1904.05788].

Also, consider the theorem of Hopkins we used, concerning the image of

$$\begin{array}{cccc} \phi_W : & \mathbf{TMF}_\nu & \to & \left\{ \begin{array}{c} \text{modular forms of} \\ \text{weight } \frac{\nu}{2} \text{ with} \\ \text{integer coeff.s and poles} \end{array} \right\} \\ & \Uplus & & \Uplus \\ & \text{a theory } T & \mapsto & \phi_W(T) = \eta(q)^\nu Z_{\text{ell}}(T; q). \end{array}$$

Namely,

$b\Delta^k$ is in the image of ϕ_W from $\mathbf{TMF}_{24k}$
iff b is a multiple of $\dfrac{24}{\gcd(24, k)}$.

In our language it is given as follows.

Consider a 2d $\mathcal{N}=(0,1)$ *theory with* $2(c_R - c_L) = 24k$.

If its elliptic genus is constant, it is a multiple of $24/\gcd(24,k)$.

The theories with $Z_{\text{ell}} = 24/\gcd(24,k)$ were constructed for $1 \leq k \leq 5$ in [Gaiotto, Johnson-Freyd 1811.00589].

But even in that case, it isn't understood why Z_{ell} can't be a smaller integer.

In particular,

If the elliptic genus of 2d $\mathcal{N}=(0,1)$ *theory is simply* 1*,*
then $c_L - c_R$ *is divisible by* 288*.*

Conversely, there should be a 2d $\mathcal{N}=(0,1)$ *theory*
whose elliptic genus is 1 *and* $c_L - c_R = \pm 288$*.*

Again the question is open.

Summary

Today, I considered **global anomalies in heterotic string theories.**

Such questions can be answered using the **mathematical theory of $\boldsymbol{TMF}$**, using the **Segal-Stolz-Teichner conjecture**:

$$\mathrm{TMF}_\nu = \frac{\left\{ \begin{array}{c} \text{2d } \mathcal{N}{=}(0,1) \text{ supersymmetric theory} \\ \text{with } \nu = 2(c_R - c_L) \end{array} \right\}}{\text{continuous deformation}}$$

This conjecture predicts **many unexplored properties of 2d theories**, which I think are worth pursuing.

The list of hep-th papers on **TMF** is not very long.

The exhaustive list is

Gaiotto, Johnson-Freyd	1811.00589
Gukov, Pei, Putrov, Vafa	1811.07884
Gaiotto, Johnson-Freyd, Witten	1902.10249
Gaiotto, Johnson-Freyd	1904.05788
Johnson-Freyd	2006.02922
YT	2108.13542
Lin, Pei	2112.10724

It's a young field and newcomers are welcomed...