Physics and Algebraic Topology

Part I: 20th century

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Mathematics is unreasonably effective for us physicists.

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. XIII, 001–14 (1960)

The Unreasonable Effectiveness of Mathematics in the Natural Sciences

Richard Courant Lecture in Mathematical Sciences delivered at New York University, May 11, 1959

EUGENE P. WIGNER
Princeton University

"and it is probable that there is some secret here which remains to be discovered." (*C. S. Peirce*)

There is a story about two friends, who were classmates in high school, talking about their jobs. One of them became a statistician and was working on population trends. He showed a reprint to his former classmate. The reprint started, as usual, with the Gaussian distribution and the statistician explained to his former classmate the meaning of the symbols for the actual population, for the average population, and so on. His classmate was a bit incredulous and was not quite sure whether the statistician was pulling his leg. "How can you know that?" was his query. "And what is this symbol here?" "Oh," said the statistician, "this is π." "What is that?" "The ratio of the circumference of the circle to its diameter." "Well, now you are pushing your joke too far," said the classmate, "surely the population has nothing to do with the circumference of the circle."

https://doi.org/10.1002/cpa.3160130102

But the usefulness depends on the subfields of math.

Ordinary/partial differential equations are obviously effective.

Group theory is also obviously effective to describe **symmetry**.

Differentiable manifolds are the basis of **general relativity**.

Algebraic geometry?

Heavily used in some parts of string theory,
but real physicists won't call it physics.

(I am a string theorist, so it's OK to disparage string theory.)

Number theory?

Appears here and there, but will it ever be integral to physics?

Mathematical Logic?

Will some theoretical physics question be undecidable within ZFC?

cf. [Shiraishi-Matsumoto 2012.13889]
[Cubitt, 2105.09854]

(If you download the slides, texts in purple are linked to journal webpages etc.)

How about algebraic topology?

Some use have been made in the past.

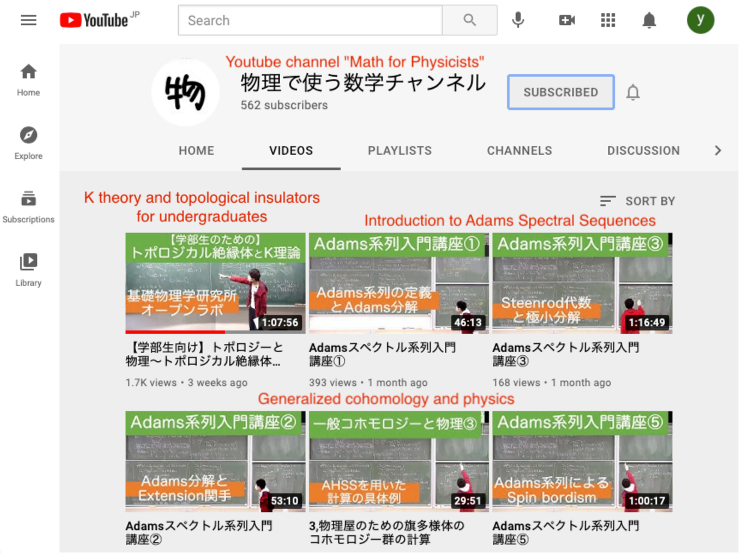
Notably, **homotopy groups** were used
to understand **topological solitons in 1970s.**

Not much else has been used until late 1990s,
when string theorists started to use K-theory.

(We can debate whether string theory is physics, though.)

More recently, in the last 10 years,
physicists started to use algebraic topology **more fully**.

A youtube channel run by a grad student in Kyoto:



https://www.youtube.com/channel/UCi4ZotOnAla-loruLQkeyMw/videos

Today I would like to review the relationship between physics and algebraic topology.

Concrete homotopy groups are useful in studying topological solitons.

(math: 1930s, physics: 1970s)

Chern classes are useful in understanding integer quantum Hall effect.

(math: 1940s, physics: 1980s)

D-branes are classified by K-theory.

(math: 1960s, physics: 2000s)

Anderson duals of bordism homologies classify SPT phases.

(math: 1960s, physics: 2010s)

TMF and 2d supersymmetric field theories.

(math: 2000s, physics: 2020s)

We're trailing behind, but slowly catching up.

Pre-history

up to 1970s

Math side

Hopf invariant / fibration (1931)

$$S^3 = \{(a, b) \in \mathbb{C}^2 \mid |a|^2 + |b|^2 = 1\} \\ \to S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\},$$

where

$$(a, b) \mapsto (2\,\mathrm{Re}\,a\bar{b}, 2\,\mathrm{Im}\,a\bar{b}, |a|^2 - |b|^2)$$

(a, b) and $e^{i\theta}(a, b)$ map to the same point on S^2.

S^3 is an S^1 bundle over S^2 with $\int_{S^2} c_1 = 1$.

Physics side

Dirac's quantization condition (1931)

The magnetic charge of a magnetic monopole is an integer multiple of a fixed constant.

Modern paraphrase of Dirac's argument:

Wavefunction of an electron is a section of a complex line bundle $\mathcal{L}$ *over space.*

Electromagnetic field is the $U(1)$ *connection of this line bundle, and the magnetic field strength* F *is its curvature . Therefore,*

$$\int_{S^2} \frac{F}{2\pi} = \int_{S^2} c_1(\mathcal{L}) \in \mathbb{Z}.$$

Math side

Steady progress in algebraic topology.

Stiefel-Whitney / Pontryagin / Chern classes	('30s – '40s)
Eilenberg-Steenrod axiom for (co)homology	(1945)
$H^*(G) := H^*(BG)$ for finite G	(Eilenberg-Mac Lane 1947)
Bordism groups	(Pontryagin, Thom '50s)
Adams spectral sequence	(1958)
K-theory	(Atiyah-Hirzebruch 1959, 1961)

Physics side

Not much happens in this area until 1970s,

when some concrete homotopy groups were used

to study topological solitons.

What are topological solitons?

Solitons are 'solitary' wave configurations
which are stable for some reasons:

- **Non-topological** solitons

  KdV equations, integrable systems …

- **Topological** solitons

  Today's focus.

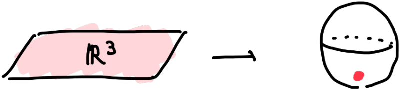
Topological solitons

Consider a configuration

$$\phi : \mathbb{R}^3 \to S^3$$

where $\mathbb{R}^3$ is our space, and S^3 is the target space of the field ϕ.

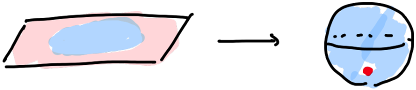
When ϕ is basically a constant map plus a fluctuation,



its quantization describes **pions**, a type of elementary particles.

More generally, for a finite energy configuration,
most of the region of $\mathbb{R}^3$ needs to map to a single point on S^3.

But the rest can map to anywhere:



effectively describing a map

$$\phi : S^3 \to S^3$$

characterized by the degree $\in \pi_3(S^3) = \mathbb{Z}$.

Skyrme (1961) suggested to identify a configuration

$$\phi : S^3 \to S^3$$

whose degree is $1 \in \pi_3(S^3) = \mathbb{Z}$ as a **proton**.

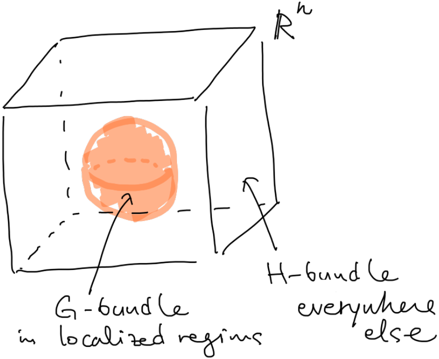
This would explain proton's stability:
the degree can't change continuously.

Not the best model of protons now;
but it is the first example of topological solitons.

Now known as **skyrmions**.

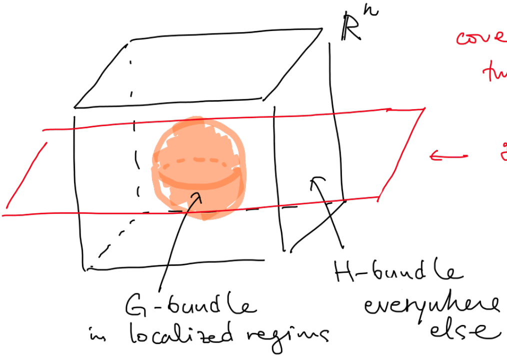
Another large class of topological solitons arises as follows.

A G-symmetric system can come with a G-bundle.

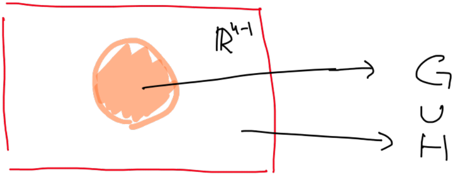


There are situations where having an H-bundle for $H \subset G$ is energetically more favorable.

G is said to be "spontaneously broken to H" in physics.



cover by
two patches.
and
glue in the
middle.



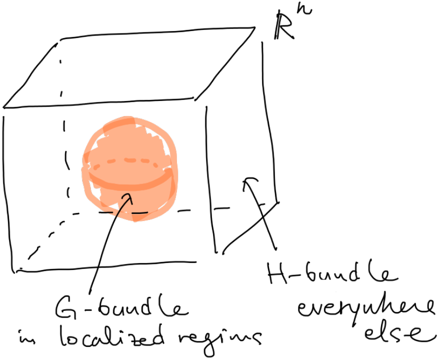
i.e.

$$\begin{array}{ccc} D^{n-1} & \rightarrow & G \\ \cup & & \cup \\ S^{n-2} & \rightarrow & H \end{array}$$

which determines a class in

$$\pi_{n-1}(G/H).$$

A topological soliton

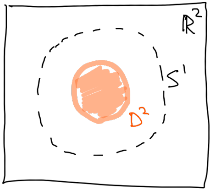


gives a class in

$$\pi_{n-1}(G/H).$$

Example 1

In a superconducting material,
the electromagnetic $G = U(1)$ symmetry is broken to $H = \{\pm 1\}$.



$U(1)$-bundle in the interior; $\{\pm 1\}$-bundle outside.

Measured by $n \in \pi_1(U(1)/\{\pm 1\}) = \mathbb{Z}$, which translates to the magnetic flux

$$\int_{D^2} \frac{F}{2\pi} = \int_{D^2} c_1 = \frac{n}{2}.$$

Known as Abrikosov vortex (1957) in condensed matter physics and Nielsen-Olsen vortex (1973) in high energy physics.

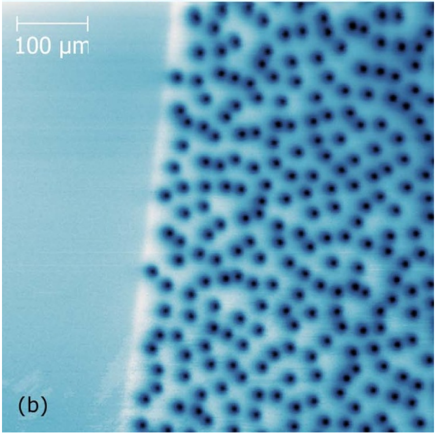
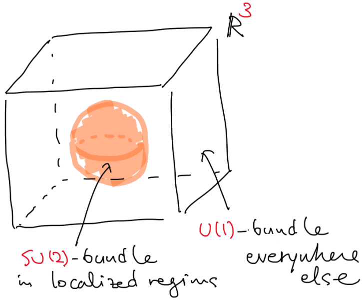


Figure 1 | Images of vortices in 200 nm thick YBCO film taken by Scanning SQUID Microscopy after field cooling at 6.93 μT to 4 K. (b) is taken after heating above T_c and re-cooling. The sample edge at the left side of the images is used as a reference for scan location.

Wells, Pan, Wang, Fedoseev, Hilgenkamp (2015)

Example 2

Taking $G = SU(2)$ and $H = U(1) \subset SU(2)$, you can consider



which is classified by

$$\pi_2(SU(2)/U(1)) = \pi_2(S^2) = \mathbb{Z}.$$

Known as the 't Hooft-Polyakov monopole (1974).

Example 3

The A-phase of the superfluid helium-3 (Osheroff-Richardson-Lee 1972) is characterized by

$$G = SO(3) \times SO(3) \times U(1) \curvearrowright \mathbb{C}^3 \otimes \mathbb{C}^3$$

and

$$H = \text{stabilizer at } \mathbf{e_1} \otimes (\mathbf{e_2} + i\mathbf{e_3})$$

so we have

$$\begin{aligned} \text{vortices}: &\quad \pi_1(G/H) = \mathbb{Z}/4\mathbb{Z}, \\ \text{“monopoles”}: &\quad \pi_2(G/H) = \mathbb{Z}. \end{aligned}$$

Furthermore, $\pi_1(G/H)$ acts nontrivially on $\pi_2(G/H)$.

Volovik-Mineev (1976)

Instantons

Instantons are a somewhat different class of topological solitons.

In the Standard Model of particle physics,

the **strong force** and the **weak force**

are described by

$$\boldsymbol{SU(3)} \quad \text{and} \quad \boldsymbol{SU(2)}$$

gauge theories.

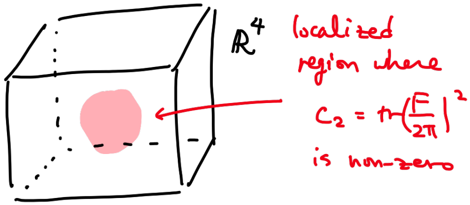
This means that our spacetime $\mathbb{R}^4$ is equipped with
an $\boldsymbol{SU(3)}$ bundle and an $\boldsymbol{SU(2)}$ bundle with connections.

The Chern-Weil representative of c_2 of an $SU(n)$ bundle is

$$c_2 \propto \mathrm{tr}(\frac{F}{2\pi})^2$$

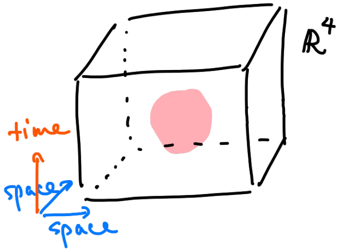
where F is the curvature of the bundle.

There are spacetime configurations where c_2 is localized



and integrates to $0 \neq \int c_2 \in \mathbb{Z}$.

These are known as **instantons**, since it is localized also on an **instant**



and not just localized along the spatial direction.

Physical interpretations are rather different,
but it can be treated very similarly to other solitons.

First discussed by [Belavin-Polyakov-Schwarz-Tyupkin (1975)].

It is natural to impose an additional constraint

$$\boldsymbol{F} = - * \boldsymbol{F}$$

where $*$ is the Hodge star on $\mathbb{R}^{4}$.

This is the famous anti-self-dual (ASD) equation.

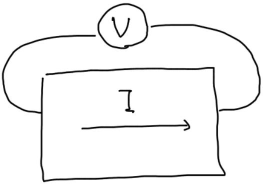
On $\mathbb{R}^{4}$, solutions and their moduli spaces are completely known [Atiyah-Drinfeld-Hitchin-Manin (1978)]

On closed 4-manifolds, the study of the moduli space led to the Donaldson theory, starting from [Donaldson (1983)].

Middle ages

1980s–2000s

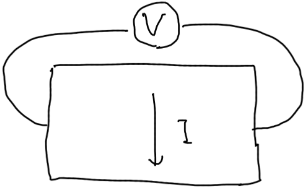
What you learn in high school:



$$I = \sigma V$$

σ is called the conductivity.

In a two-dimensional material, this can also happen:



$$I = \sigma_H V$$

$\boldsymbol{\sigma_H}$ is called the Hall conductivity.

Surprising discovery of von Klitzing, Dorda, Pepper (1980):

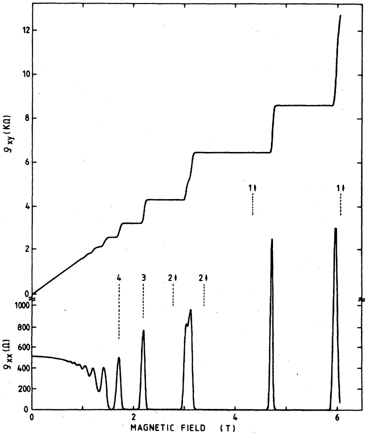


FIG. 14. Experimental curves for the Hall resistance $R_H = \rho_{xy}$ and the resistivity $\rho_{xx} \sim R_x$ of a heterostructure as a function of the magnetic field at a fixed carrier density corresponding to a gate voltage $V_g = 0$ V. The temperature is about 8 mK.

Figure is taken from a slightly later review, von Klitzing (1986)

When the ordinary conductivity σ vanishes, i.e. the system is **gapped**, the Hall conductivity has the universal value

$$\sigma_H = \nu \frac{e^2}{h}, \qquad \nu \in \mathbb{Z}$$

where e is the electric charge of the electron and h is the Planck constant.

Called the **integer quantum Hall effect**.

This is now the accepted method to calibrate the experimental apparatus against the declared value of e^2/h.

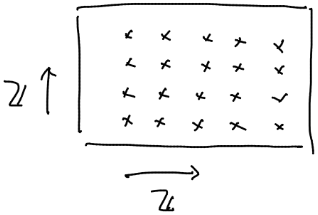
Why is ν an integer?

There are both **microscopic** understanding
and **macroscopic** understanding.

Let's start with the **microscopic** understanding.
In quantum mechanics, the Hamiltonian H acts on the Hilbert space $\mathcal{H}$.

Microscopic understanding

Two-dimensional materials have a lattice structure:



Therefore

$$\mathbb{Z}^2 \curvearrowright \mathcal{H}$$

which allows us to decompose $\mathcal{H}$ in terms of the character

$$T^2 = \mathrm{Hom}(\mathbb{Z}^2, U(1)).$$

This means that $\mathcal{H}$ is the space of sections

$$\psi : T^2 \to \mathcal{H}'$$

of a trivial Hilbert space bundle

$$T^2 \times \mathcal{H}'$$

and the Hamiltonian H has the form

$$(H\psi)(p) = h(p)(\psi(p)) \qquad p \in T^2$$

where $h(p) : \mathcal{H}' \to \mathcal{H}'$ is a self-adjoint operator.

The gapped condition says that the lowest eigenvalue of $h(p)$ is non-degenerate, which determines a one-dimensional subspace

$$L(p) \subset \mathcal{H}'.$$

It forms a line bundle $\mathcal{L} \to T^2$ which is a sub-bundle of $T^2 \times \mathcal{H}'$.

A standard computation using the Kubo formula says that the Hall conductivity is

$$\sigma_H = \frac{e^2}{h} \int_{T^2} c_1(\mathcal{L})$$

and therefore it is an integer multiple of e^2/h.

Thouless-Kohmoto-Nightingale-den Nijs (1982)

Macroscopic understanding

Consider an idealized situation where the quantum Hall material **fills the entire $2+1$ dimensional spacetime M.**

M comes with a $U(1)$ bundle $\mathcal{L}$ with connection A describing the electromagnetic field.

The integer quantum Hall material is gapped with unique ground state.

This means that the **system determines the partition function**

$$Z(M, A) \in U(1).$$

When the $U(1)$ bundle is topologically trivial, A is a one-form. The standard Kubo formula says that the coefficient ν in

$$\sigma_H = \nu \frac{e^2}{h}$$

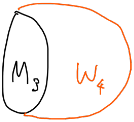
appears in the partition function as

$$Z(M, A) = \exp(i\frac{\nu}{4\pi}\int_M AdA).$$

How do we know that ν is an integer?

We use the fact that AdA is not well-defined for a topologically non-trivial $U(1)$-bundle $\mathcal{L}$.

Given



we have

$$i\frac{\nu}{4\pi}\int_{M_3} AdA = i\frac{\nu}{4\pi}\int_{W_4} FF = \pi i\nu\int_{W_4} c_1(\mathcal{L})^2.$$

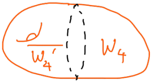
(Note $F = dA$ and $c_1 = F/(2\pi)$.)

The RHS makes sense for topologically nontrivial $\mathcal{L}$,
but looks like it depends on W_4.

Let us compare the two different choices W_4 and W_4':

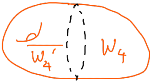
The difference is

$$\frac{\exp\left(\pi i \nu \int_{W_4} c_1(\mathcal{L})^2\right)}{\exp\left(\pi i \nu \int_{W_4'} c_1(\mathcal{L})^2\right)} = \exp\left(\pi i \nu \int c_1(\mathcal{L})^2\right)$$



So we need to ask:

$$\exp\Big(\pi i \nu \int c_1(\mathcal{L})^2\Big) \stackrel{?}{=} 1$$



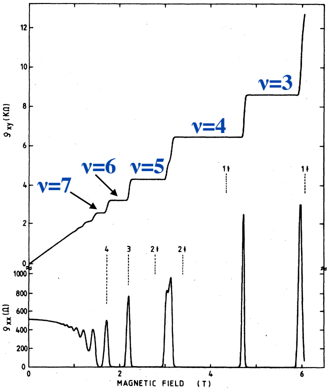
This seems to require $\nu \in 2\mathbb{Z}$,
but odd ν has been experimentally observed.

The resolution: **electrons are spinors**,
and therefore M_3, W_4 etc. require **spin structure**.

The intersection form on a spin 4-manifold is **even**,
and therefore $\nu \in \mathbb{Z}$.

This argument was implicitly known for a long time since late 80s, but the crucial factor of two related to spin structure was not appreciated very much until around 2000.

I think it is quite amazing that we see these facts experimentally in



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D-branes and K-theory

2000s

Reconciling quantum mechanics and gravity is a big question in theoretical physics.

String theory is a framework where it can be done.

But it requires 9+1 dimensions. Some like it, some hate it.

There are several different types of string theories:

Type IIA, Type IIB, Type I, and heterotic

It started in 1980s as a theory of strings (1+1 dimensional objects) moving in spacetime.

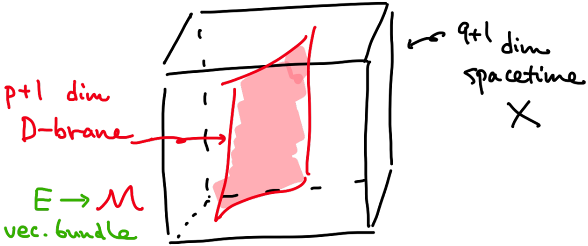
A spacetime is given by M_{9+1} together with metric, spin structure, and a closed 3-form H such that $[H] \in H^3(M_{9+1}, \mathbb{Z})$.

In mid-1990s, it was realized that it also has various other objects known as D-branes [Polchinski hep-th/9510017].

The relation to K-theory was understood by [Minasian-Moore hep-th/9710230] and [Witten hep-th/9810188].

Those which extend along $p + 1$ dimensions are called Dp-branes.

They also come with vector bundles on them.



It was found that various string theories have the following set of D-branes:

$d := 9 - p$	0	1	2	3	4	5	6	7	8	9	10
Type IIB	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
Type IIA	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
Type I	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$

The entry $\mathbb{Z}/2$ means that if we take two Dp-branes, they can pair-annihilate.

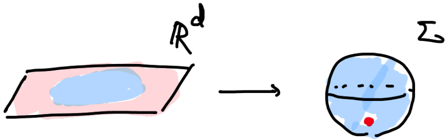
They should look familiar to algebraic topologists: they are

$$K^{-d}(pt), \quad K^{1-d}(pt), \quad KO^{-d}(pt),$$

respectively.

A Dp-brane extends along $p+1$ dimensions within the $(9+1)$-dimensional spacetime. Its codimension is $d := 9 - p$.

For a scalar field taking values in Σ, topological solitons with codimension d



were classified by $\pi_d(\Sigma)$.

D-branes of Type IIB, Type IIA, Type I string theories are instead classified by

$$\widetilde{K}^{0}(S^{d}), \quad \widetilde{K}^{1}(S^{d}), \quad \widetilde{KO}^{0}(S^{d}),$$

respectively.

Comparing these with $\pi_{d}(\Sigma)$, we find that
string theories have the classifying spaces of K-theories as Σ.

On a more general manifold X , D-branes are classified by

$$\widetilde{K}^0(X), \quad \widetilde{K}^1(X), \quad \widetilde{KO}^0(X)$$

for Type IIB, Type IIA, and Type I.

As I said, the string theory spacetime X can come with a class $[H] \in H^3(X, \mathbb{Z})$. Then D-branes are classified by twisted K-theories:

$$\widetilde{K}^0_H(X), \quad \widetilde{K}^1_H(X), \quad \widetilde{KO}^0_H(X).$$

This finding rekindled math interests to
twisted K-theories originally found by [Donovan-Karoubi (1970)].
E.g. see the review [Gomi (2012)] in Japanese.

The spacetime (X, H) and another spacetime (X', H') can be **T-dual** to each other, meaning that
Type IIB on (X, H) and Type IIA on (X', H') are the same, and
Type IIA on (X, H) and Type IIB on (X', H') are the same.

Then we should have

$$K^0_H(X) = K^1_{H'}(X'),$$
$$K^1_H(X) = K^0_{H'}(X').$$

Many T-dual pairs of (X, H) and (X', H') were known. So the equality above might be worth checking.

(T-duality is also known as mirror symmetry when $H = H' = 0$.)

A simplest class of pairs have the following form. Take

$$S^1 \to X \to B$$

whose first Chern class is $c \in H^2(B, \mathbb{Z})$. Then the spacetimes

$$(X, 0)$$

and

$$(B \times S^1, c \cup \theta)$$

are known to be T-dual. Here θ is the generator of $H^1(S^1, \mathbb{Z}) = \mathbb{Z}$.

Therefore we should have

$$K^\bullet(X) = K^{\bullet+1}_{c\cup\theta}(B \times S^1)$$

This can be checked. [Bouwknegt-Evslin-Mathai hep-th/0306062]

It is also interesting to recall
the geometric K-homology of [Baum and Douglas (1982)].

An element of $K_d(X)$ in their description is given by

- a manifold M_d with a spin-c structure
- a map $M_d \to X$
- a virtual vector bundle $E \ominus F$ on M_d

with various equivalence relations.

This is exactly what string theorists expect for a D-brane;
the spin-c condition was understood in [Freed-Witten hep-th/9907189].

So far I reviewed the relationship
between physics and algebraic topology in 20c:

Concrete homotopy groups are useful in studying topological solitons.

(math: 1930s, physics: 1970s)

Chern classes are useful in understanding integer quantum Hall effect.

(math: 1940s, physics: 1980s)

D-branes are classified by K-theory.

(math: 1960s, physics: 2000s)

In the second half, I would explain:

Anderson duals of bordism homologies classify SPT phases.

(math: 1960s, physics: 2010s)

TMF and 2d supersymmetric field theories.

(math: 2000s, physics: 2020s)

We're trailing behind, but slowly catching up.