Physics and Algebraic Topology

Part II: 21st century

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Modern times

2010s

Integer quantum Hall system is an example of

$(n+1)$-dimensional quantum field theory (QFT)
with **unique gapped ground state** with G-symmetry.

Often called

SPT phases

and/or

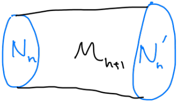
invertible phases.

(SPT= symmetry protected topological)

A more general $(n+1)$-dimensional quantum field theory (QFT) Q assigns a Hilbert space to a spatial manifold N_n:

$$N_n \mapsto \mathcal{H}_Q(N_n),$$

and for



it assigns

$$Z_Q(M_{n+1}) : \mathcal{H}_Q(N_n) \to \mathcal{H}_Q(N'_n).$$

The manifold can be equipped with various structures of your choice, metric, orientation, spin structure, G-bundle with connection, etc., giving rise to different flavors of QFTs.

Corresponding to

we require

$$Z_Q(M')Z_Q(M) = Z_Q(M' \circ M).$$

So, a QFT Q has a functor from a suitable bordism category to the category of vector spaces as part of its data.

We assume $\mathcal{H}_Q(\varnothing) = \mathbb{C}$, then

$Z_Q($  $) : \mathcal{H}_Q(\varnothing) \to \mathcal{H}_Q(\varnothing)$

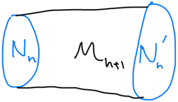
determines a complex number

$Z_Q($  $) \in \mathbb{C},$

called the partition function.

A QFT Q is **SPT/invertible/with unique gapped ground state**
$\Leftrightarrow \mathcal{H}_Q(N)$ **is always 1-dimensional**.

Integer quantum Hall material is a $(2+1)$-dimensional
spin invertible QFT with $U(1)$ **symmetry**:

$Z_Q($ 

$) : \mathcal{H}_Q(N) \to \mathcal{H}_Q(N')$

N, N' are 2-dimensional; M is 3-dimensional;
they come with **spin** structure and $U(1)$ **bundle with connection**,

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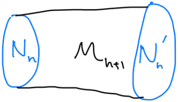
$Z_Q($  $) : \mathcal{H}_Q(N) \to \mathcal{H}_Q(N')$

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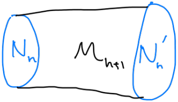
$Z_Q($ 

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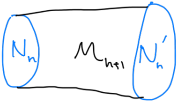
$Z_Q($ 

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N, N' are 2-dimensional; M is 3-dimensional;
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and $\mathcal{H}_Q(N)$ is **always 1-dimensional**.

We would like to understand

$$\mathbf{Inv}^{n+1}_{\mathcal{S},G} := \pi_0(\{ \begin{matrix} (n+1)\text{-dim. } \textbf{invertible} \text{ QFTs} \\ \text{with structure } \mathcal{S} \text{ and symmetry } G \end{matrix} \})$$

Here $\mathcal{S}$ can be spin structure, orientation only, etc.

As invertible QFTs form a group under tensor product

$$\begin{aligned} \mathcal{H}_{Q\times Q'}(N) &= \mathcal{H}_Q(N) \otimes \mathcal{H}_{Q'}(N), \\ Z_{Q\times Q'}(M) &= Z_Q(M) \otimes Z_{Q'}(M), \quad \text{etc.,} \end{aligned}$$

$\mathbf{Inv}^{n+1}_{\mathcal{S},G}$ will be an Abelian group.

Dijkgraaf-Witten (1990)

$$\mathrm{Inv}^{n+1}_{?,G} \overset{\text{proposal}}{=} H^{n+2}(BG, \mathbb{Z})$$

Dependence on $\mathcal{S}$ not appreciated at that time.
Wrong if taken too literally.

Integer quantum Hall effect is the case $n = 2$, $G = U(1)$. Then

$$H^4(BU(1), \mathbb{Z}) \simeq \mathbb{Z}$$

is generated by $(c_1)^2$, but we need $\frac{1}{2}(c_1)^2$ as we saw,
for which the spin structure was crucial.

[Chen-Gu-Liu-Wen 1106.4772]

$$\mathbf{Inv}^{n+1}_{\text{oriented},G} \overset{\text{proposal}}{=} H^{n+2}(BG, \mathbb{Z})$$

An influential paper, which introduced
and popularized the notion of SPT phases.

(The terminology “invertible phases” originates
from [Freed-Moore hep-th/0409135].)

Now known to be wrong for $n \geq 4$.

How about the **spin** case?

[Freed hep-th/0607134], [Gu-Wen 1201.2648]

$$\mathbf{Inv}^{n+1}_{\text{spin},G} \overset{\text{proposal}}{=} E^{n+2}(BG)$$

where E^d is a cohomology theory given by

$$E^d(X) = \frac{\left\{(a,b) \in C^{d-3}(X,\mathbb{Z}/2) \times C^d(X,\mathbb{Z}) \;\middle|\; \begin{array}{l} \delta a = 0, \\ \delta b = \beta \circ \mathrm{Sq}^2 a \end{array} \right\}}{\text{certain equiv. relation}}$$

where

β is the Bockstein for $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0$ and

Sq^2 is the Steenrod square.

(Amazingly, Gu and Wen rediscovered
the cochain-level expression of Sq^2 by themselves!)

Another way to define $\boldsymbol{E^d(X)}$ is to write it as

$$\boldsymbol{E^d(X) = [X, E_d]}$$

where $\boldsymbol{E_d}$ is a two-stage Postnikov tower

$$\boldsymbol{K(\mathbb{Z}, d) \to E_d \to K(\mathbb{Z}/2, d-2)}$$

whose Postnikov invariant

$$\boldsymbol{E_d \to K(\mathbb{Z}/2, d-2) \xrightarrow{x} K(\mathbb{Z}, d+1)}$$

is given by

$$\boldsymbol{x = \beta \circ \mathrm{Sq}^2 \circ \iota}$$

where $\boldsymbol{\iota}$ is the generator of $\boldsymbol{H^{d-2}(K(\mathbb{Z}/2, d-2), \mathbb{Z}/2)}$.

[Schnyder-Ryu-Furusaki-Ludwig 0803.2786], [Kitaev 0901.2686]

$$KO^{n-2}(pt) \to \mathrm{Inv}^{n+1}_{\mathsf{spin},pt}$$

They classified **free** **spin invertible phases without additional symmetry**.

They also considered structures related but not quite **spin**
(such as imposing time reversal, corresponding to considering **pin±**)
so that the classification is $KO^{n+i}(pt)$ for arbitrary i mod 8.

Called the **periodic table** of **free** **topological superconductors**.
(see e.g. a nice lecture by Ryu)
(see also [Gomi-Yamashita 2111.01377])

Kitaev (2015)

$$\mathbf{Inv}^{n+1}_{\mathcal{S},G} = E^{n+2}_{\mathcal{S}}(BG)$$

where $E_{\mathcal{S}}$ **should be a generalized cohomology theory.**

Kitaev only gave a talk and never wrote it up.

Fleshed out in [Xiong 1701.00004] and [Gaiotto, Johnson-Freyd 1712.07950] etc.

[Kapustin-Thorngren-Turzillo-Wang 1406.7329]
[Freed-Hopkins 1604.06527]

$$\mathbf{Inv}^{n+1}_{\mathcal{S},G} \overset{\text{proposal}}{=} (I_{\mathbb{Z}}\Omega^{\mathcal{S}})^{n+2}(BG)$$

where $\Omega^{\mathcal{S}}$ is the $\mathcal{S}$-bordism homology and $I_{\mathbb{Z}}$ is the Anderson dual.

Further discussions in [Yonekura 1803.10796], [Yamashita-Yonekura 2106.09270], [Yamashita, 2110.14828]

People think this is the definitive version.

A generalized (co)homology theory $h^n(X)$, $h_n(X)$ satisfies the Eilenberg-Steenrod axioms for the ordinary (co)homology **except** the dimension axiom.

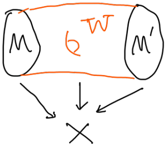
So $h_n(pt) = h^{-n}(pt)$ can be nontrivial for $n \neq 0$.

Bordism group

$$\Omega_n^{\mathcal{S}}(X) = \left\{ \begin{array}{c} \mathcal{S}\text{-structured manifold } M_n \\ \text{together with } f : M_n \to X \end{array} \right\} \Big/ \text{bordism}$$

is an example, where

$$M \overset{\text{bordant}}{\sim} M' \quad \Leftrightarrow$$



For a generalized homology theory $h_*(-)$,
there is the **Anderson dual** cohomology theory $I_{\mathbb{Z}}h^*(-)$
which satisfies the analogue of the universal coefficient theorem:

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(h_{d-1}(X), \mathbb{Z}) \to (I_{\mathbb{Z}}h)^d(X) \to \mathrm{Hom}_{\mathbb{Z}}(h_d(X), \mathbb{Z}) \to 0$$

The universal coefficient theorem of $H(-, \mathbb{Z})$ means that

$$I_{\mathbb{Z}}H(-, \mathbb{Z}) = H(-, \mathbb{Z}).$$

Similarly, $I_{\mathbb{Z}}K = K$ and $I_{\mathbb{Z}}KO^{\bullet} = KO^{\bullet+4}$.

Construction of $I_{\mathbb{Z}}h$ from h

Note that

$$X \mapsto \mathrm{Hom}(\pi_\bullet^S(X), \mathbb{Q}), \qquad X \mapsto \mathrm{Hom}(\pi_\bullet^S(X), \mathbb{Q}/\mathbb{Z})$$

are generalized cohomology theories. Let us say that they are represented by spectra

$$I\mathbb{Q}, \qquad I\mathbb{Q}/\mathbb{Z}$$

and define $I\mathbb{Z}$ to be the homotopy fiber

$$I\mathbb{Z} \to I\mathbb{Q} \to I\mathbb{Q}/\mathbb{Z}.$$

Then,

$$I_{\mathbb{Z}}h := [h, I\mathbb{Z}]$$

represents $I_{\mathbb{Z}}h$ when h is the spectrum representing h.

Classification of fermionic invertible phases

$$\mathbf{Inv}^{n+1}_{\mathrm{spin},G} = (I_{\mathbb{Z}}\Omega^{\mathrm{spin}})^{n+2}(BG)$$

$\Omega^{\mathrm{spin}}_{\bullet}(pt)$ was determined in Anderson-Brown-Peterson (1967) and the Anderson dual was introduced in Anderson (1969).

Physicists now need them!

That's why graduate students in condensed matter physics learn
the Atiyah-Hirzebruch spectral sequence and
the Adams spectral sequence to compute them.

Relation to previous proposals:

[Freed hep-th/0607134], [Gu-Wen 1201.2648]

$$\mathrm{Inv}^{n+1}_{\mathrm{spin},G} \stackrel{?}{=} E^{n+2}(BG)$$

where E^d is a cohomology theory represented by a two-stage Postnikov tower

$$K(\mathbb{Z},d) \to E_d \to K(\mathbb{Z}/2,d-2)$$

such that the associated Postnikov invariant

$$E_d \to K(\mathbb{Z}/2,d-2) \xrightarrow{x} K(\mathbb{Z},d+1)$$

is given by

$$x = \beta \circ \mathrm{Sq}^2 \circ \iota$$

where ι is the generator of $H^2(K(\mathbb{Z}/2,d-2),\mathbb{Z}/2)$.

Its relation to

$$\mathbf{Inv}^{n+1}_{\mathrm{spin},G} = (I_{\mathbb{Z}}\Omega^{\mathrm{spin}})^{n+2}(BG)$$

is that the said two-stage Postnikov tower

$$K(\mathbb{Z}, d) \to E_d \to K(\mathbb{Z}/2, d-2)$$

is the truncation of the spectrum representing $I_{\mathbb{Z}}\Omega^{\mathrm{spin}}$ to its first two nontrivial stages:

d	0	1	2	3	4	$\cdots$
$(I_{\mathbb{Z}}\Omega^{\mathrm{spin}})^d$	$\mathbb{Z}$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	$\cdots$
E^d	$\mathbb{Z}$	0	$\mathbb{Z}/2$	0	0	$\cdots$

In particular, there is a natural transformation

$$E^d(BG) \to (I_{\mathbb{Z}}\Omega^{\mathrm{spin}})^d(BG).$$

Relation to previous proposals:

[Schnyder-Ryu-Furusaki-Ludwig 0803.2786], [Kitaev 0901.2686]

$$KO_G^{n-2}(pt) \to \mathrm{Inv}^{n+1}_{\mathsf{spin},G}$$

classifying **free** **spin invertible phases without additional symmetry**.

Its relation to

$$\mathrm{Inv}^{n+1}_{\mathsf{spin},G} = (I_{\mathbb{Z}}\Omega^{\mathsf{spin}})^{n+2}(BG)$$

is that it is the Anderson dual to the APS orientation

$$(\Omega^{\mathsf{spin}})_d(X) \to KO_d(X)$$

which is

$$KO^{d-4}(X) = (I_{\mathbb{Z}}KO)^d(X) \to (I_{\mathbb{Z}}\Omega^{\mathsf{spin}})^d(X).$$

Relation to previous proposals:

[Chen-Gu-Liu-Wen 1106.4772]

$$\mathbf{Inv}^{n+1}_{\text{oriented},G} \stackrel{?}{=} H^{n+2}(BG,\mathbb{Z})$$

We now believe

$$\mathbf{Inv}^{n+1}_{\text{oriented},G} = (I_{\mathbb{Z}}\Omega^{\text{oriented}})^{n+2}(BG)$$

Again

d	0	1	2	3	4	$\cdots$
$(I_{\mathbb{Z}}\Omega^{\text{oriented}})^d(pt)$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\cdots$
$H^d(pt,\mathbb{Z})$	$\mathbb{Z}$	0	0	0	0	$\cdots$

and there is a homomorphism

$$H^d(X,\mathbb{Z}) \to (I_{\mathbb{Z}}\Omega^{\text{oriented}})^d(X).$$

The homomorphism

$$\tilde{H}^{n+2}(X,\mathbb{Z}) \to (I_{\mathbb{Z}}\widetilde{\Omega^{\text{oriented}}})^{n+2}(X).$$

for $X = BG$ with finite groups G
fails to be surjective starting at $n + 2 = 6$.

SPT phases associated to these points are discussed e.g. in
[Fidkowski-Haah-Hastings 1912.05565] and [Chen-Hsin, 2110.14644]

These correspond to $n + 1 = 4 + 1$ dimensional systems.

Present

2020s

The last topic of the talk is about **physics and elliptic cohomology**.

There are three types of complex curves with Abelian group law:

$$\mathbb{C}, \quad \mathbb{C}^\times, \quad \text{elliptic curves}.$$

Correspondingly, there are three types of cohomology theories:

$$H^*(-,\mathbb{Z}), \quad K^*(-), \quad \text{elliptic cohomologies}.$$

They are all complex orientable: a complex n-fold M_{2n} has the fundamental class $[M_{2n}] \in E_{2n}(M)$.

All these cohomology theories have the 1st Chern class $c_1(\mathcal{L}) \in E^*(X)$ for complex line bundles $\mathcal{L} \to X$.

The group law dictates how $c_1(\mathcal{L} \otimes \mathcal{L}')$ is expressed in terms of $c_1(\mathcal{L})$ and $c_1(\mathcal{L}')$.

Today I would like to discuss their real analogues:

$$H^*(-, \mathbb{Z}), \quad KO^*(-), \quad TMF^*(-).$$

TMF is the topological modular form, constructed by Hopkins et al. in late 1990s. (cf. [Hopkins' talk at ICM 2002, math/0212397])

I hear the construction uses a sheaf of E_∞-ring specta over the moduli stack of elliptic curves over $\mathbb{Z}$.

I don't understand any of the words in the last sentence.

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if M is **spin**. $=$ the trivialization of $w_2(TM)$ is given.

M_n has a fundamental class in $TMF_n(M)$
if M is **string**. $=$ the trivialization of $p_1(TM)$ is given.

Note that the first three nontrivial homotopy group of O is

$$\pi_0(O) = \mathbb{Z}/2, \quad \pi_1(O) = \mathbb{Z}/2, \quad \pi_3(O) = \mathbb{Z}$$

and w_1, w_2, p_1 are the corresponding obstruction classes.

Adams spectral sequences computing them have the form

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}(0)}(H^*(X,\mathbb{Z}/2),\mathbb{Z}/2) \Rightarrow H_{t-s}(X,\mathbb{Z})_{\hat{2}}$$
$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(H^*(X,\mathbb{Z}/2),\mathbb{Z}/2) \Rightarrow ko_{t-s}(X)_{\hat{2}}$$
$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}(2)}(H^*(X,\mathbb{Z}/2),\mathbb{Z}/2) \Rightarrow tmf_{t-s}(X)_{\hat{2}}$$

where $\mathcal{A}(n)$ is the subalgebra of the Steenrod algebra generated by Sq^1, Sq^2, ..., Sq^{2^n}.

TMF is the natural next entry after $H(-,\mathbb{Z})$ and KO.

N.B. there are no cohomology theories such that

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}(n)}(H^*(X, \mathbb{Z}/2), \mathbb{Z}/2) \Rightarrow E_{t-s}(X)_{\hat{2}}$$

for $n \geq 3$. If so, the corresponding spectrum E should have

$$H^\bullet(E, \mathbb{Z}/2) = \mathcal{A}//\mathcal{A}(n),$$

whose first two nonzero elements would be e at degree zero and $\mathrm{Sq}^{2^{n+1}}\, e$. But the latter can be rewritten using lower $\mathrm{Sq}^{2^k}\, e$ in terms of secondary cohomology operations (used in Adams' solution to the Hopf-invariant one problem), leading to a contradiction.

[see this MO answer]

So the sequence H, KO, TMF seems to stop here.

KO is 8-periodic:

$$KO^{n+8}(X) \simeq KO^{n}(X)$$

TMF is $24^2 = 576$-periodic:

$$TMF^{n+576}(X) \simeq TMF^{n}(X)$$

TMF is called the topological **modular form** since there is a homomorphism

$$TMF_* \to MF_*[\Delta^{-1}]$$

where

$$MF = \mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - 1728\Delta).$$

is the ring of integral modular forms, with

$$c_4 = 1 + 240q + \cdots, \quad c_6 = 1 - 504q - \cdots$$

are the Eisenstein series and

$$\Delta = q - 24q^2 + \cdots$$

is the modular disciminant.

$\boldsymbol{TMF_* \to MF_*[\Delta^{-1}]}$ is rationally isomorphic

$$\boldsymbol{TMF_* \otimes \mathbb{Q} \simeq MF_*[\Delta^{-1}] \otimes \mathbb{Q},}$$

and it is isomorphic at degree 0

$$\boldsymbol{TMF_0 = \mathbb{Z}[J]}$$

where $\boldsymbol{J}$ is the modular $\boldsymbol{J}$-invariant,
but not surjective in general.

For example, $\boldsymbol{k\Delta}$ is in the image only when $\boldsymbol{24}$ divides $\boldsymbol{k}$.

$\boldsymbol{TMF_* \to MF_*[\Delta^{-1}]}$ also has a lot of torsion.

$\boldsymbol{KO^n(X)}$ has a geometric realization: for $\boldsymbol{n = 0}$,
it is given by virtual differences of real vector bundles over $\boldsymbol{X}$.

Is there a similarly nice realization of $\boldsymbol{TMF^n(X)}$?

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Is there a similarly nice realization of $TMF^n(X)$?

Segal-Stolz-Teichner conjecture

$$TMF^n(X) = \pi_0 \left\{ \begin{array}{c} \mathbf{2}\text{-dim'l } \textbf{supersymmetric} \text{ QFT} \\ \text{of degree } n \text{ parameterized by } X \end{array} \right\}$$

Segal 1988, Stolz-Teichner 2002, 2011

This is a very difficult conjecture. The RHS isn't even defined yet.

An easier version is:

$$KO^n(X) = \pi_0 \left\{ \begin{array}{c} \textbf{1}\text{-dim'l } \textbf{time-reversal invariant} \\ \textbf{supersymmetric} \text{ QFT} \\ \text{of } \textbf{degree } n \text{ parameterized by } X \end{array} \right\}$$

which was rigorously formulated and proved.

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Degree n means that there is an action of $Cl(n, \mathbb{R})$.

Therefore the statement becomes

$$KO^n(X) \overset{?}{=} \pi_0 \left\{ \begin{array}{c} \text{family of } \textbf{odd self-adjoint operators } Q \\ \text{parameterized over } X \\ \text{on a } \mathbb{Z}/2\text{-}\textbf{graded real} \text{ Hilbert space } \mathcal{H} \\ \text{commuting } \textbf{with } Cl(n, \mathbb{R}) \textbf{ action} \end{array} \right\}$$

and the RHS is more or less the definition of KO in terms of Fredholm operators.

(For a detailed proof,
see e.g. [Cheung 0811.2267] or [Ulrickson 1901.02110]).

The TMF version is much harder:

$$TMF^n(X) = \pi_0 \left\{ \begin{array}{c} \text{2-dim'l supersymmetric QFT} \\ \text{of degree } n \text{ parameterized by } X \end{array} \right\}$$

The LHS involves sheaves of spectra over the moduli stack of elliptic curves over $\mathbb{Z}$.

The RHS involves QFTs, which seem to me a purely characteristic-0 phenomenon.

Still, nontrivial physics motivation and checks.

For example, take

$$TMF_3(pt) = \mathbb{Z}/24,$$

which is naturally isomorphic to

$$\Omega_3^{\text{framed}}(pt) = \pi_3^S(pt) = \lim \pi_{n+3} S^n.$$

In the standard math definition, the computation involves elliptic curves in characteristic 2 and 3.

The same $\mathbb{Z}/24$ also follows from an intricate construction in QFT.

[Gaiotto, Johnson-Freyd 1904.05788]

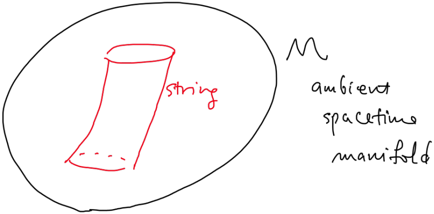
Historically, elliptic cohomologies / TMF came from two strands of ideas.

One is purely from within algebraic topology, called chromatic phenomena, about which I have no clue.

Another is from Witten.

(This part of the story is nicely summarized in Landweber 1988.)

In string theory we consider strings moving in a manifold:



This should be described by a **2-dim'l supersymmetric QFT** on the worldsheet of the string.

It gives rise to a **sequence of Dirac operators** acting on the spinor bundle SM tensored with tensor powers of the tangent bundle TM.

In 1984, Witten asked the property of the index of these operators to Landweber and Stong, who then informed Ochanine about the question.

By 1986, they realized that there is a generalization of the $\hat{A}$ genus

$$\int_M \hat{A} \in \mathbb{Z}$$

which takes the values in modular forms

$$\int_M \phi_W \in MF.$$

Here, M needs to be spin (i.e. $w_2 = 0$) for the former
and string (i.e. $p_1 = 0$) for the latter.

$\hat{A}$ was known to come from KO.
There should be some nice cohomology theory for ϕ_W.
It took about 15 years for mathematicians to construct TMF.

But physicists were almost completely detached from these developments until very recently.

Only in November 2018 papers on this topic appeared (by Gaiotto, Johnson-Freyd and Gukov-Pei-Putrov-Vafa), in which **some physics checks of the Segal-Stolz-Teichner conjecture** were made.

Instead, **assuming the Segal-Stolz-Teichner conjecture**, we can use the known properties of TMF to **deduce the properties** of 2d supersymmetric QFTs and **of string theory**.

In particular, with **Yamashita at RIMS**, I showed that **there is no anomaly in heterotic string theory**. [YT-Yamashita 2108.13542]

Anomalies of heterotic string theories

What is an anomaly?

I said that an n-dim'l QFT Q assigns the partition function

$Z_Q($ 

$) \in \mathbb{C}$,

but the partition function of an anomalous QFT Q is instead given as

$Z_Q($ 

$) \in \mathcal{H}_{\mathcal{A}}(M)$

where $\mathcal{A}$ is an $(n+1)$-dim'l **invertible** QFT and
$\mathcal{H}_{\mathcal{A}}$ is its Hilbert space which is one dimensional.

There are many anomalous QFTs.
Notable examples are free massless fermions, for which $\mathcal{H}_{\mathcal{A}}(M)$ is the determinant line bundle of the Dirac operator.

A n-dim'l possibly-anomalous spin QFT Q has

$$\mathcal{A}_Q\text{: a } (n+1)\text{-dim'l spin invertible QFT}$$

as part of the data.

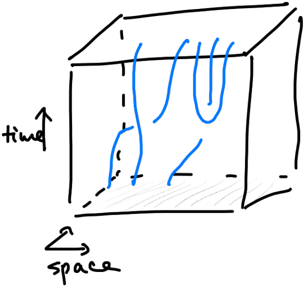
This is given by an element

$$\mathcal{A}_Q \in \mathbf{Inv}^{n+1}_{\text{spin}} = (I_{\mathbb{Z}}\Omega^{\text{spin}})^{n+2}.$$

Now, there is a procedure called the **second quantization** we learn in the basic QFT course.

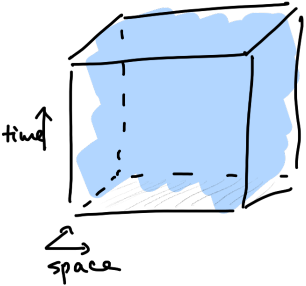
This is a machinery which does

$$\{\text{time-reversal-invariant quantum mechanics of degree } n-2\}$$
$$\downarrow$$
$$\{\text{possibly-anomalous } n\text{-dim'l spin QFT}\}$$



quantum mechanics of particles'

↦



quantum field theory

This is a machinery which does

$$\{\text{time-reversal-invariant quantum mechanics of degree } n-2\}$$
$$\downarrow$$
$$\{\text{possibly-anomalous } n\text{-dim'l spin QFT }\}$$

Applying the **Stolz-Teichner** for the source and the **anomaly** for the target, we have a homomorphism

$$KO^{n-2} \to (I_{\mathbb{Z}}\Omega^{\mathrm{spin}})^{n+2}.$$

This is the Anderson dual to the spin orientation of the KO theory:

$$\Omega^{\mathrm{spin}} \to KO^{n}$$

where we use $I_{\mathbb{Z}}KO^{n+4} = KO^{n}$.

We already encountered this before in a different context.

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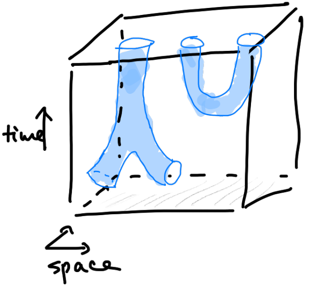
We already encountered this before in a different context.

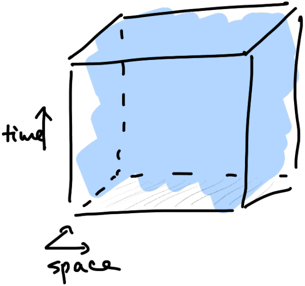
My interest is the anomaly of heterotic string theory, which is a machinery which does

$$\{\text{2-dim'l supersymmetric QFT of degree } \boldsymbol{n+22}\}$$
$$\downarrow$$
$$\{\text{possibly-anomalous } \boldsymbol{n}\text{-dim'l quantum gravity with string structure}\}$$



quantum field theory of strings

$\mapsto$



quantum gravity theory

We'd like to consider

$$\{\text{2-dim'l supersymmetric QFT of degree } \boldsymbol{n+22}\}$$
$$\downarrow$$
$$\{\text{possibly-anomalous } \boldsymbol{n}\text{-dim'l quantum gravity with string structure}\}$$

Again applying the **Stolz-Teichner** for the source and the **anomaly** for the target, we have a natural transformation

$$\boldsymbol{TMF^{n+22}(X)} \to \boldsymbol{(I_{\mathbb{Z}}\Omega^{\text{string}})^{n+2}(X)}.$$

String theory is often non-anomalous from miraculous reasons.
So we would like to know whether this homomorphism is zero.

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$$TMF^{n+22}(X) \to (I_{\mathbb{Z}}\Omega^{\text{string}})^{n+2}(X)$$

The seminal paper of Green and Schwarz (1984),
which started superstring theory as we know it,
showed that the image of a particular element of $TMF^{10+22}(pt)$
is torsion.

The paper by Witten with an appendix by Stong (1986) proved that
the image of this particular element is actually zero.

$$TMF^{n+22}(X) \to (I_{\mathbb{Z}}\Omega^{\text{string}})^{n+2}(X)$$

Lerche-Nilsson-Schellekens-Warner (1988) showed that
the image in general is torsion (although not phrased in this language.)

With **Yamashita at RIMS**, we showed that it is always a zero map [YT-Yamashita 2108.13542].

Physically this means that **there is no anomaly in heterotic string theory**.

Let me give an outline of how it is done.

Physics tells us that

$$\alpha : TMF^{n+22}(X) \to (I_{\mathbb{Z}}\Omega^{\text{string}})^{n+2}(X)$$

comes from a map of spectra

$$\alpha : TMF \to \Sigma^{-20} I_{\mathbb{Z}} M\text{String}$$

or equivalently by

$$\alpha : TMF \wedge M\text{String} \to \Sigma^{-20} I\mathbb{Z}.$$

Physics also tells us that this factors through the natural $M\mathrm{String}$-module structure on TMF:

$$\alpha : TMF \wedge M\mathrm{String} \to TMF \xrightarrow{\gamma} \Sigma^{-20} I\mathbb{Z}.$$

So we need to determine the element

$$\gamma \in [TMF, \Sigma^{-20} I\mathbb{Z}] = (I_{\mathbb{Z}} TMF)^{-20}(pt).$$

Physics paper Lerche-Nilsson-Schellekens-Warner (1988) already showed that

$$\gamma \in [TMF, \Sigma^{-20} I\mathbb{Z}] = (I_{\mathbb{Z}} TMF)^{-20}(pt)$$

is at most torsion.

But $(I_{\mathbb{Z}} TMF)^{-20}(pt)$ is freely generated over $\mathbb{Z}$, because

$$TMF_{-21}(pt) = 0.$$

So γ is zero.

The hard part was

- to translate what I wanted to show physically
  in terms of stable homotopy theory, and
- to find someone who knows stable homotopy theory
  and also is interested in this problem.

It was then immediate for my collaborator **Yamashita**
to show it does vanish.

Today I surveyed the interaction between physics and algebraic topology.

Concrete homotopy groups are useful in studying topological solitons.

(math: 1930s, physics: 1970s)

Chern classes are useful in understanding integer quantum Hall effect.

(math: 1940s, physics: 1980s)

D-branes are classified by K-theory.

(math: 1960s, physics: 2000s)

Anderson duals of bordism homologies classify SPT phases.

(math: 1960s, physics: 2010s)

TMF and 2d supersymmetric field theories.

(math: 2000s, physics: 2020s)

We're trailing behind, but slowly catching up.